Journal Pre-proof

A new analytical method to optimise the preventive maintenance interval by using a semi-Markov process and z-transform with an application to marine diesel engines

Antonio Sánchez-Herguedas , Angel Mena-Nieto ,
Francisco Rodrigo-Muñoz

PII: S0951-8320(20)30881-4
DOI: https://doi.org/10.1016/j.ress.2020.107394
Reference: RESS 107394

To appear in: *Reliability Engineering and System Safety*

Received date: 23 May 2020
Revised date: 23 November 2020
Accepted date: 2 December 2020

Please cite this article as: Antonio Sánchez-Herguedas , Angel Mena-Nieto , Francisco Rodrigo-Muñoz , A new analytical method to optimise the preventive maintenance interval by using a semi-Markov process and z-transform with an application to marine diesel engines, *Reliability Engineering and System Safety* (2020), doi: https://doi.org/10.1016/j.ress.2020.107394

Highlights

- A new analytical method to calculate the preventive interval is proposed.
- The preventive interval is the time that maximises the returns (income minus costs).
- A semi-Markov process approach has been developed to model the returns.
- The z-transform is used as a tool to solve the resulting difference equations system.
- A real case study of the marine diesel engine is used to apply this analytical method.

A new analytical method to optimise the preventive maintenance interval by using a semi-Markov process and z-transform with an application to marine diesel engines

Antonio Sánchez-Herguedas [a], Angel Mena-Nieto [b], Francisco Rodrigo-Muñoz [c]

[1] *Department of Industrial Management. School of Engineering. University of Seville, Camino de los Descubrimientos s/n, 41092 Seville, Spain*
[2] *Department of Electrical and Thermal Engineering, Design, and Projects. School of Engineering. University of Huelva. Campus El Carmen, 21819 Huelva, Spain*
[3] *Department of Applied Mathematics II. School of Engineering. University of Seville. Camino de los Descubrimientos s/n, 41092 Seville, Spain*

antoniosh@us.es, mena@uhu.es, frodrigo@us.es

ABSTRACT

This article presents a novel method based on a semi-Markov process approach and the z transform to calculate the optimal time interval until the preventive maintenance intervention. For this, a mathematical expression that has its origin in the evolution of the maintenance cost of a patrol boat's propulsion engine is developed, represented mathematically by a system of difference equations. This method can be applied to all assets that are subject to wear, which are very common on industrial facilities. This mathematical expression for the preventive interval is a powerful tool for maintenance managers because this saves him from resorting to complicated mathematical methods. In addition to the proposed mathematical formula, the manager needs to know the assets' failure history data and the costs of maintenance interventions. These data are usually available, making their practical application fast and easy to develop. Due to the method being analytical, it could also serve as an excellent decision support tool. It can be applied to the desired time horizon guaranteeing the obtention of the optimal interval. In many other cases, applying the model can simplify the process of obtaining the asset maintenance programme, from an economic perspective.

Keywords: Maintenance Interval; Maintenance Model; Semi-Markov Process; Finite Horizon; Maintenance Cost;

1. Introduction

Maintenance tasks involve preventive and corrective actions carried out to keep the physical asset in the desired operating condition or to restore it to this condition in case of failure. Maintenance policies aim to provide optimal system reliability, availability and safety performance at the lowest possible maintenance costs providing the highest possible benefits (returns) to the organization. When assets have repetitive failures, maintenance managers realise that the operating and maintenance conditions of this asset are different from the conditions under which the manufacturer designed it. Each asset operates under its own conditions. Updating preventive maintenance intervals to the particular conditions of each asset must be an activity addressed by the maintenance manager. In our case, we show how to update the periodic preventive maintenance programme for a component of the propeller engine installed in patrol boats in charge of coastal control in a southern European country. The component is subject to thermal stress causing a rapid wear effect and leads to repetitive failure. Due to its surveillance function, the patrol boat must be ready for its operation and is financially compensated for services performed, so that the state of fault makes it difficult to carry out its mission and reduces potential income for the Coast Control Service. In the article, we establish a method to obtain the mathematical formula that accurately determines the preventive interval τ, from the maintenance costs and operating income of the patrol boat. The maintenance manager or his engineers, without the need for modelling or mathematical knowledge, can apply this formula to obtain the optimal value of the maintenance interval for that component.

The basic idea is to replace or repair the component at its age τ or failure, whichever occurs first. The periodic PM policy (a type of preventive maintenance policy) can be considered as the most common maintenance policy in which an asset is preventively maintained at fixed time intervals, but this interval depends on the asset's failure history in its operating environment.

Nevertheless, in many industrial businesses it is also very important to consider the operational horizon of assets. The total remaining time of operation according to the business, as demonstrated, influences the length of the preventive interval. The model developed in this work considers the total remaining service time of the equipment, finding the optimal interval τ once the remaining time (planning horizon) has been established. If the horizon is modified, the optimal interval will have a different length.

One could then ask this question. Would it be possible to find a simple mathematical expression that helps those responsible for the maintenance of boat engines (or any industrial equipment) to establish the optimal preventive interval under a finite horizon planning setting that may vary at a certain time (project completion)?

The practical case presented is suitable for the failure mode due to deterioration of O-rings belonging to the refrigerated exhaust system of a marine diesel engine. For four years, failure data and programmed substitutions of 32 engines were collected, this allowed to elaborate the mathematical function that represents their behaviour and use that information together with the costs of the interventions and the operating income to determine the correct location of the intervention of preventive replacement of the O-rings. The information collection process went on for four more years, demonstrating the drastic reduction of the failures and the considerable increase in the reliability of these marine diesel engines.

Through the development of a semi-Markov model, which transition to transition accumulates the returns in the form of costs or income, we demonstrate that the calculation of the average accumulated return for finite number of transitions can be calculated by means of difference equations system. The z transform is used to solve this difference equations system. Deriving the equation of the average accumulated return with respect to the preventive interval τ and equating to zero, the optimal value of the interval is obtained for the different transitions. This allows obtaining the optimal interval for the total operating time of the asset until the completion of a project.

The rest of the paper is organised as follows: Section 2 contains a summarizes the previous literature. Section 3 explains the method followed on paper. In section 3.1, we use the failure data of an element to analyse its behaviour and mathematically define the cumulative distribution function that describes its behaviour in the event of failure. In this case the Weibull distribution is used. For the functions of distribution of maintenance intervention times, whether it is repairs by breakdown or preventive replacements, we consider the average intervention times. Sections 3.2 and 3.3 show the mathematical model (three-state semi-Markov process), which, by evolving in a probabilistic way, allows us to accumulate the returns generated in each state as time goes on. This accumulation of returns is expressed as a difference equations system. In sections 3.4 to 3.5, that system of equations is solved using the z-transform, reaching a solution explicitly. Likewise, the average return is optimised in m steps, which has been obtained as a solution to the difference equation, for the time to preventive maintenance. In section 4, the results obtained are presented and discussed. Finally, section 5 is devoted to present the conclusions of the work.

2. Literature review

There is vast literature on maintenance optimisation models. Crespo et al. (Crespo Marquez and Sánchez Herguedas, 2002) established a classification of the problems to be solved by models: a) determining the optimal maintenance time intervals τ, b) determining the frequency of inspections and maintenance based on the condition, c) determining the optimal resources to comply with maintenance requirements, and d) finding the economic life-cycle of a device in the face of the dilemma between repair and replacement. Our model can be classified in "group a".

Reliability engineering, among others, addresses how to maintain reliability in systems through maintenance, fault diagnosis, and prognosis. Maintenance must be planned in such a way that a balance is achieved between expected benefits and expected consequences for the business, establishing an optimal maintenance plan. One of the solutions to this problem is based on the search for potentially optimal strategies, including the calculation of the preventive maintenance interval (Zio, 2009). In a recent publication (Mizutani S., Zhao X., 2021) various reliability engineering techniques and tools are used to select the best policies that are suitable for objective systems in actual fields for some types of replacements, including periodic. The method developed in this article fits between these techniques as it finds the optimal preventive interval, establishing the frequency of interventions. To this end, it starts from the proven reliability of the equipment, expressed through its survival function, and the income and cost values incurred during operation and maintenance interventions. Other authors such as (Sembiring et al., 2018) use reliability engineering techniques to determine the preventive interval and complete the design of the maintenance plan for a group of rubber production machines. Also (Legát et al., 2016) present a method to determine the optimal periodic preventive interval of optimisation, establishing the relationships between these intervals and the reliability functions. They present some examples to demonstrate the suitability of the method and use the Weibull distribution.

Therefore our model will be categorised under different criteria employed in literature such as: as periodic and sequential PM, component or system perspective, finite or infinite planning horizon, and exact or heuristic optimisation method. Finally, some examples of semi-Markov models and z-transform with applications to maintenance issues are shown.

Regarding the “group a” problems, a sample of studies which have calculated the maintenance preventive interval is presented. (Lin et al., 2015) considered an adjusted preventive maintenance interval in a system that is replaced after a fixed number of preventive interventions and that is minimally repaired at failure. (Coria et al., 2015) developed an analytical optimisation method for a system with post-failure repair and periodic preventive maintenance presenting some examples involving different failure time data. (Hu et al., 2020) consider a one-component system working under time-varying operating condition, where the evolution process of the operating condition is governed by a Markov process. (Laggoune et al., 2010) considered a setting in which each asset has its own preventive maintenance interval, when a unit fails the policy prescribes whether other units should be maintained as well. (Dong et al., 2019) use a parallel redundant system with three types of dependencies: stochastic, structural and economic. (Wang et al., 2017) considered a mathematical reliability model that allows analysing the sensitivity of reliability to redundancy and maintenance interval. (Zhao et al., 2020b) develop a model where they balance the total expected time durations between early or late replacement and the random failure using two variable types of shortage and excess costs. (Zhao et al., 2020a) study the replacement policies obtaining the optimum replacement times and mission durations for the models of replacement first and last, respectively. (Młynarski et al., 2020) present results of the analysis of the developed models for complex technical objects preventive maintenance scheduling and (Li and Zhang, 2020) find the optimal PM interval or both binary-state systems and discrete multistate systems.

Other authors look for the interval for more specific situations: (Zhong et al., 2019) designed a scheduling for offshore wind farms. (Han et al., 2019) ensured the integrity of safety barriers and process safety on offshore installations by determining the maintenance interval. (Semaan and Yehia, 2019) developed a detailed stochastic schedule for a periodic preventive of a military aircraft. (Jung and Park, 2003) determined policies during the post-warranty period. (Huang and Yuan, 2010) for a multi-state deterioration system and (Jung et al., 2010) calculated the cost under warranty policy renewal.

Besides, for much of the equipment in industrial facilities, PM policy is still the dominant maintenance because it is easy to implement, and many assets cannot either be monitored or this task is not profitable for them. In fact, in the literature, PM policies are commonly classified as periodic and sequential PM (Sheu and Chang, 2009). Periodic PM is executed at fixed time interval, τ. On the other hand, sequential PM is implemented at intervals of unequal time lengths more suitable when the asset requires more frequent maintenance as it ages (Iranpoor and Fatemi

Ghomi, 2019). This paper addresses the problem of optimal periodic PM (best periodic preventive interval τ) policy for systems with repairs at failures between preventive actions.

While modelling these problems, the engineer faces two specific decisions (Van Horenbeek et al., 2010): adopting a component or system perspective and selecting a finite or infinite planning horizon. In this work, we focus on a multi-component system (engine), where one of the elements is subject to a single failure mode (dominant failure mode: degradation of O-rings) and where the only possible maintenance action is to recover the element by means of a preventive or corrective task. This implies that there can only be one standard failure as a consequence of the dominant failure mode, and that the entire system is not renewed after the failure (Wang, 2008). Thus, we consider a single component system (O-rings) with a single failure mode, as it is isolated from the rest of the components that make up the asset.

Regarding the planning horizon of the models, according to Nicolai et al. (Nicolai and Dekker, 2008), those of infinite horizon have been studied extensively in the literature, analysing maintenance policies for stationary situations over time. According to these authors, finite horizon models must be developed, in addition to adaptable horizon models, which take into account short-term analysis (Nicolai and Dekker, 2007). Other authors also agree that this effort is necessary because in most real cases, the horizons are finite (Nakagawa et al., 2004). Besides, on many occasions, the nature of the system clearly defines a finite horizon (Percy, 2008). This happens, for example, when an asset has its useful life limited to that of a system of which it is part. (Baklouti et al., 2020) propose PM strategy which suggests systematically replacing panels with their respective wiring system every time units τ over a finite operating time span.

Considering now the models of finite time horizon, the optimisation methods that can be found in the literature can be exact or heuristic (Lust et al., 2009). Although exact methods always find the optimal global solution, the complexity of the optimisation problem is usually high, and the calculation time usually increases exponentially with the size of the problem. Therefore, heuristic search methods should be further developed to find an optimal value close to the solution at a reasonable time. The developed method in this paper finds an exact value as a solution. (Nakagawa and Mizutani, 2009) made an interesting effort to develop exact methods, converting the usual maintenance models into finite maintenance models by using the partition method. By using this method, three usual models of periodic replacement with minimal repair, block replacement and simple replacement are transformed into finite replacement models that can be solved analytically. The optimal policies for each model are derived analytically and calculated numerically. (Pandey et al., 2015) proposed a model which includes a finite planning horizon and limited available resources to perform maintenance scheduling; (Schutz et al., 2011) proposed and model periodic and sequential preventive maintenance policies for a system that performs various missions over a finite planning horizon. Our model aims to achieve the exact value for optimal preventive interval τ, adopting a component perspective in a finite planning horizon.

On the other hand, semi-Markov processes have previously been successfully applied to maintenance, see (Grabski, 2014). (Lyubchenko et al., 2018) presented an approach for assessment of recommended preventive maintenance intervals of radio communication devices and use a semi-Markov model. (Kumar and Varghese, 2018) used a Semi-Markov process to determine system availability for a two component series system undergoing a preventive maintenance policy. The system model is developed by considering a Weibull failure distribution and a Log-normal repair distribution. The system undergoes preventive maintenance at fixed intervals of time. (Zhang and Gao, 2012) presented a study in which they obtain the optimal maintenance policies that minimise the cost of the life cycle of a road network. The surface deterioration is modelled as a semi-Markov process. (Crespo Marquez and Sánchez Herguedas, 2002) selected a similar approach to formulate the dynamic state of the system, in this case, the optimisation criterion to solve the problem is to minimise the total cumulative cost expected of the system for a finite number of transitions, by using a recurring relationship and dynamic programming backwards. Different corrective or preventive actions that could take place at different times are optimised. Our work represents an evolution of this research by solving the system of equations exposed.

In the literature, z transforms have been previously used to solve very diverse problems, such as retirement planning (Savoye, 2009), electrical circuit resolution (Roy, 2011), solving differential equations (Anake et al., 2014) and determining the limit probability of a Markov process in a machine maintenance problem (Shakenova, 2012). In our case we use it to calculate the average accumulated return in each transition, solving the difference equation system proposed from the semi-Markov model.

Our paper differs from prior works in the following ways. First, our method uses the z transform to find the average accumulated return on each transition. Second, by using the z transform applying other methods previously used in the literature is avoided, such as dynamic programming. Third, the model incorporates income per hour of operation which has not been included in previous studies. Furthermore, many of the methods previously developed to obtain the optimal preventive interval and found in the literature are heuristic, complex, need many data, or assume many simplifying hypotheses and restrictions that are not always applicable in the day-to-day business (de Jonge and Scarf, 2020). As far as we know, there is no simple formula that responds to these challenges. On the contrary, there are methods that in many cases do not guarantee that the solution obtained is the best. These require a lot of data to simulate, and excellent mathematical understanding by whoever applies it, or wishes to invest a great deal of time in its application in an engine room. This results in many of the studies never being applied by the majority of the maintenance engineers.

3. Material and methods

The objective of the work is to find a mathematical expression that determines the optimal preventive interval. This interval depends 1) on the remaining time of use of the asset, 2) on its failure statistics, 3) on the costs of maintenance tasks and the penalty for its inactivity and 4) on income from its operation. The asset should verify two hypotheses for this process to be applied successfully. First, that the asset is always in one of these three states: operational, corrective or preventive. Furthermore, the second hypothesis is that it shows a clear tendency to wear. These hypotheses are met by many types of industrial equipment. Throughout the paper we will find the formula and show the application process. So that a maintenance manager, without specialised knowledge in calculus, can apply the procedure to the assets of the plant without it consuming too much time.

The different steps followed to develop the method are represented in **Fig. 1**. The method begins with the selection of all the necessary data for the model, economic and maintenance management data (green) and failure and preventive data (blue), and with the elaboration of the observed distribution functions and their possible adjustment to several theoretical distribution functions. Next (red), a mathematical optimisation model based on semi-Markov processes is developed, the accumulated returns in the successive transitions between states are calculated, the results in an analytical expression the average accumulated return by applying the z transforms, as well as the temporary determination of the location of the preventive maintenance task, in the case of a finite horizon.

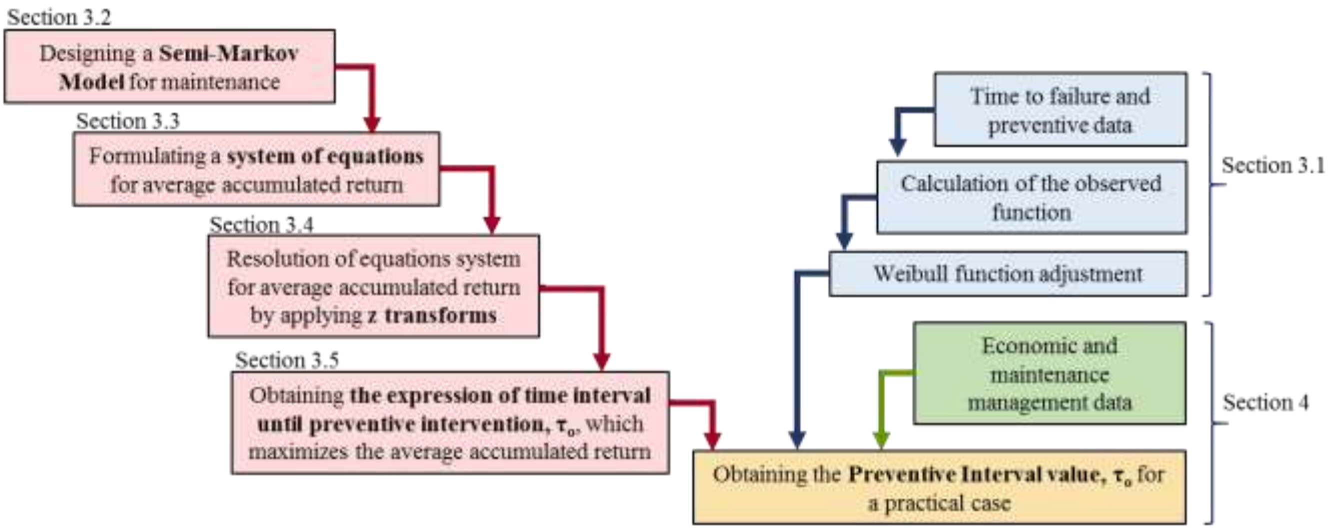


Fig. 1. Methodological proposal.

3.1. Real case. Data selection and information processing

We analysed the behaviour in the event of failure of the O-rings located at the end connections of the cooling pipe (from now on referred to as crossover) that joins the two exhausts of a 12-cylinder V-shaped alternative diesel engine. As a result of the high temperatures reached by the exhaust gases inside the crossover, even though it is subjected to forced cooling using a cooling liquid, these O-rings often deteriorate, despite the thermal protection applied to them during their assembly phase. This results in coolant leaks both to the outside of the engine and to the inside of the crossover. The high temperatures are due to the operation of the engine itself and their prolonged effect over time explains the deterioration of the O-rings.

Besides, there are also specific situations where these high temperatures reach peaks that deteriorate the O-rings in a few minutes. In our case, since we are dealing with an engine installed in a high-speed coastal patrol boat, the abrupt turning at high speeds during a chase (when the engine is being most requested) can cause, under certain conditions, the propeller to come out of the water with the consequent increase in revolutions, causing the injection to be cut off and the engine to stop immediately. In this situation, the cooling is cancelled, and the crossover reaches a maximum temperature that is prolonged in time until the engine is restarted, and the temperatures are normalised. In most cases, the damage is already done. Then we have the same failure mode caused by two different causes. The first one is due to a continuous high temperature, which causes deterioration in the medium term, and the second one is due to a temperature peak, which causes the failure at short term. This second cause has a solution related to the learning of the steering of the patrol boat. In fact, it was not a problem generalised to all the boats but limited to some of them, so its elimination was focused on the correct learning of the steering technique.

The need to study the available data and determine the optimal time for preventive replacement of the O-rings to avoid repetitive failures was raised regarding the problem of continuous high temperature. Initially, the preventive replacement intervention was designed to be carried out every 4000 hours of engine operation. This fact, along with occasional deterioration due to very high temperature, had caused a distribution of failures in 32 engines from 16 boats, as shown in **Fig. 2**. This failure mode was responsible for an excessive number of hours of unavailability of the patrol boat, allowing "situations" that were precisely tried to avoid with the active presence of the patrol boat.

Each red cell represents a failure and the orange cell an O-ring preventive replacement; the green and blue cells indicate availability before and after the change of preventive interval. All the engine preventive interventions are expressed by the letter R followed by the number of hours in multiples of 100. This situation remained unchanged until the fourth year of data collection, when a new replacement time of the O-rings was introduced, every 1000 hours. This fact coincides in **Fig. 3** with the orange cells that serve as the beginning of a new scenario in which availability is indicated by blue cells. In this new maintenance scenario, the number of failures was reduced to 10.8% during the next four years. Note the decrease in red cells on the right side of **Fig. 3**.

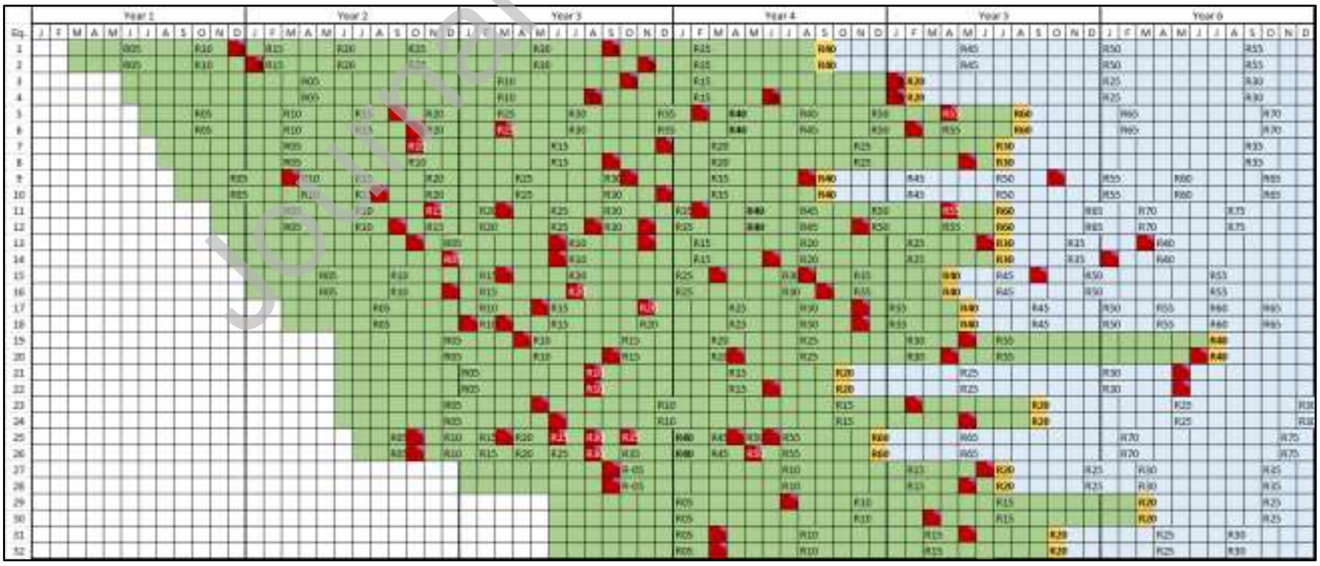


Fig. 2. One scenario. Crossover O-rings failures (red cells). O-ring preventive replacement change (orange cells). The green and blue cells indicate availability before (green cells) and after (blue cells) the preventive interval change. All engine revisions are expressed in multiples of 100 hours (R15 = revision 1500 hours).

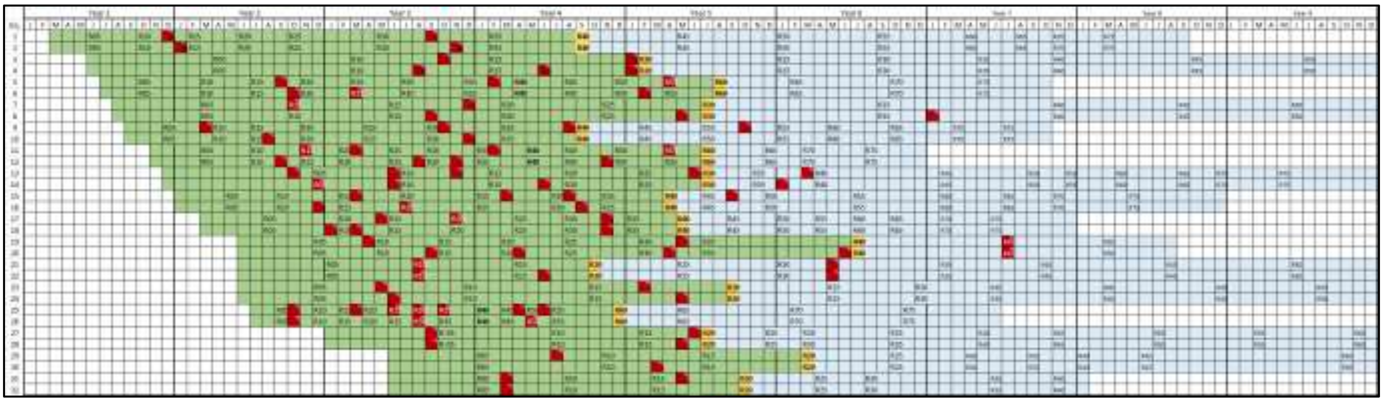


Fig. 3. Two scenarios. Crossover O-rings failures (red cells). O-ring preventive replacement change (orange cells). The green and blue cells indicate availability before (green cells) and after (blue cells) the preventive interval change. All engine revisions are expressed in multiples of 100 hours (R15 = revision 1500 hours).

Table 1. Relationship of times until failure (F) and other times (R) for each equipment.

No. of order	Equipment	Reason for stop	Start of operation (hours)	End of operation (hours)	Hours of operation
1	1	F	0	1,274	1,274
2	1	F	1,274	3,234	1,960
3	1	R	3,234	4,000	766
4	2	F	0	1,413	1,413
5	2	F	1,413	3,339	1,926
6	2	R	3,339	4,000	661
7	3	F	0	1,304	1,304
8	3	F	1,304	1,916	612
9	3	R	1,916	2,000	84
10	4	F	0	1,211	1,211
⋮	⋮	⋮	⋮	⋮	⋮
120	32	F	0	604	604
121	32	R	604	2.000	1.396
Times	F	83			
Times	R	38			

We then order the operating hours in increasing order and apply the expressions (1) and (2) that follow (see chapter 10 of Maintenance Engineering) (Crespo Márquez et al., 2004) only the failure data (F) in **Table 1**, which gives us the results of **Table 2**.

$$New\ increase = \frac{N + 1 - (new\ order\ number\ previous\ element\ that\ failed)}{N + 1 - (No.\ of\ elements\ above)} \quad (1)$$

where N is the total number of elements (121).

$$New\ order\ number = new\ increase + new\ previous\ order\ number \quad (2)$$

Table 2. Columns 1 and 3 are the 1st and 6th columns of Table 1, in increasing order. Columns 4 and 5 have been calculated according to the Eqs. (1) y (2). Failure (F), Preventive replace (R).

No. of order	Type	Hours of operation	New Increment	New No. of order
81	R	45		
9	R	84		
13	R	84		
109	R	103		

33	R	121		
29	R	176		
71	F	190	1.05172414	1.051724
112	R	199		
92	R	217		
16	R	259		
12	F	276	1.07989532	2.131619
51	F	296	1.07989532	3.211515
89	R	324		
119	R	371		
11	F	429	1.12074583	5.442433
⋮	⋮	⋮	⋮	⋮

The medians range method (Johnson, 1964) calculates the observed distribution function F_i that estimates the theoretical distribution function (Weibull, Log-normal) for the failure data. In our case, we do it using Benard's formula (Campbell and Jardine, 2001), **Eq. (3)**:

$$F_i = \frac{i - 0{,}3}{N + 0{,}4}\,, \tag{3}$$

where i (column 2 of **Table 3**) is the new order number calculated using **Eq. (2)**. The results are presented in the last column of **Table 3**.

Table 3. Values of the observed function ($\boldsymbol{F_i}$) calculated using the Benard´s formula.

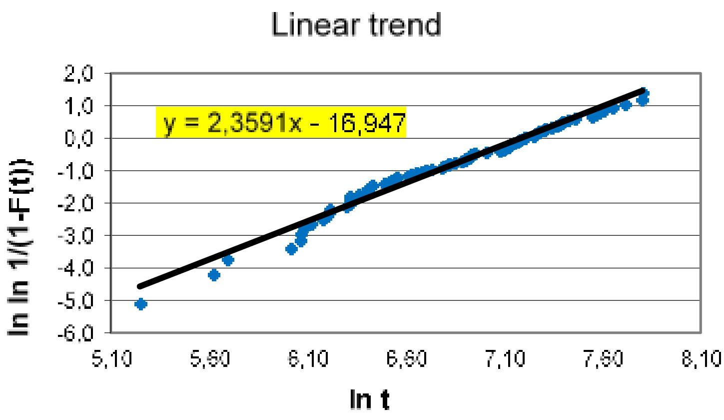
Operation hours until failure(t_i)	New order number (i)	F_i
190	1.051724	0.006192126
276	2.131619	0.015087475
296	3.211515	0.023982824
409	4.321688	0.033127574
429	5.442433	0.042359418
430	6.563179	0.051591262
437	7.683925	0.060823106
454	8.804671	0.070054950

Next, we determine the probability distribution function that best suits the observed function. To do this, we compare the adjustment between the known functions that could be adapted: Weibull and Log-normal functions.

First, we try to find the Weibull function of two parameters (α, β), using columns 1 and 3 of **Table 3**. We represent the function observed against the operation hours until failure in a graph with logarithmic scales on the axis vertical ($\ln(1/(1 - F_i))$) and in the horizontal axis $\ln(t_i)$.

Using Excel, we adjust a straight line to the points of this graph using the least-squares method. We estimate the shape parameter (α) by the slope of the adjustment line, scale parameter (β) by the exponential of the quotient between the ordinate at the origin and the slope of the adjustment line. See **Fig. 4**.

Hours to failure, t	Observed Distribution, F(t)	ln t	ln ln (1/(1-F(t)))
190	0.00619	5.247024	-5.081373
276	0.01509	5.620401	-4.186299
296	0.02398	5.690359	-3.718304
409	0.03313	6.013715	-3.390592
429	0.04236	6.061457	-3.140001
430	0.05159	6.063785	-2.938035
437	0.06082	6.079933	-2.768574
454	0.07005	6.118097	-2.622380
481	0.07929	6.175867	-2.493665
492	0.08852	6.198479	-2.378558
498	0.09775	6.210600	-2.274346
499	0.10698	6.212606	-2.179051
⋮	⋮	⋮	⋮

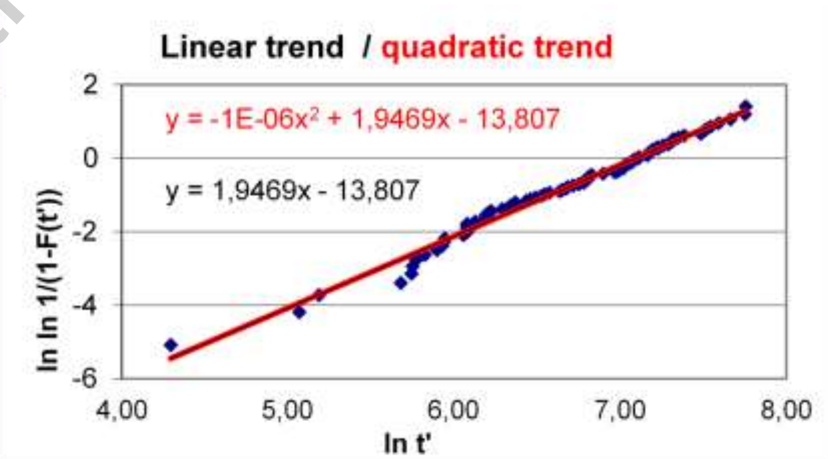


α =	2.36
β =	1317.47
γ =	0.00

Fig. 4. Obtaining the parameters of the Weibull function (two parameters).

The Weibull function supports a third parameter called guaranteed life (γ). To estimate it, we adjust again using the least-squares method a parabola to the same data in **Table 3**. Then we modify the horizontal axis, subtracting a number (x) from time, $\ln(t_i - x)$. Considering various values of x, the one that manages to make the coefficient of the second-degree term of the parabola very close to zero, is used as an estimate of the location parameter γ. **Fig. 5** shows the result. The whole procedure can be consulted in (Crespo Márquez et al., 2004).

Hours to failure, t	Observed Distribution, F(t)	ln t'	ln ln (1/(1-F(t')))
190	0.00619	4.292745	-5.081373
276	0.01509	5.069954	-4.186299
296	0.02398	5.188319	-3.718304
409	0.03313	5.677326	-3.390592
429	0.04236	5.743538	-3.140001
430	0.05159	5.746737	-2.938035
437	0.06082	5.768843	-2.768574
454	0.07005	5.820579	-2.622380
481	0.07929	5.897613	-2.493665
492	0.08852	5.927371	-2.378558
498	0.09775	5.943238	-2.274346
499	0.10698	5.945858	-2.179051
⋮	⋮	⋮	⋮



With $t' = (t - \gamma)$

α =	1.95
β =	1,202.36
γ =	116.83

Fig. 5. Obtaining the parameters of the Weibull function (three parameters).

With the same data and a similar procedure, we can estimate the average values (μ = 6.9551) and the standard deviation (σ = 0.570) of the Log-normal from the observed distribution. With the values of the estimated distribution functions in **Table 4** the graphs of **Fig. 6** have been drawn, observing the adjustment of the different functions.

Table 4. Estimated failure time distributions.

Hours until failure, t	Observed Distribution, F(t)	Weibull F(t;α,β,γ)	Weibull F(t;α,β)	Log-Normal
190	0.00619	0.00428783	0.01032166	0.00136495
276	0.01509	0.01932312	0.02472417	0.009601445
296	0.02398	0.02426952	0.02909559	0.013249031
409	0.03313	0.06167301	0.06135405	0.049314371
429	0.04236	0.06985513	0.06841268	0.058466244
430	0.05159	0.07027575	0.06877622	0.058944579
437	0.06082	0.07325026	0.07134924	0.062347073
454	0.07005	0.08069097	0.07780290	0.070996406
481	0.07929	0.09312081	0.08864488	0.085803491
⋮	⋮	⋮	⋮	⋮

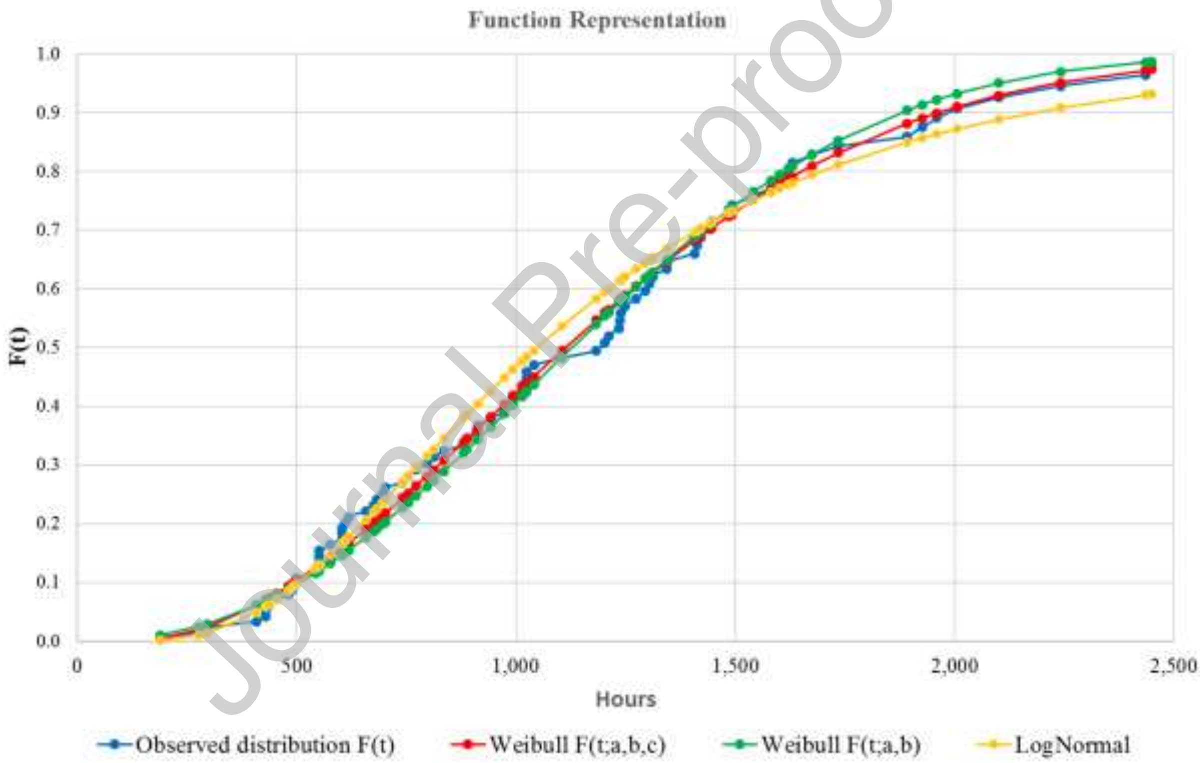


Fig. 6. Comparison of the adjustment to the experimental function of the Weibull and Log-normal functions.

To measure the quality of the adjustments, we use the Root Mean Square Error (RMSE):

$$\sum (F_{observed} - F_{theoretical})^2$$

The lowest value corresponds to the Weibull function of three parameters as can be seen in **Fig. 7**.

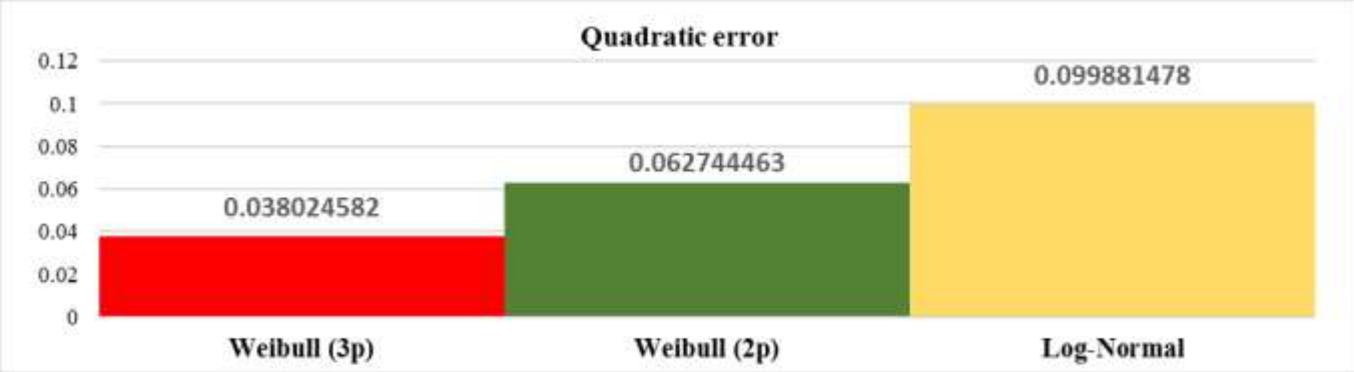


Fig. 7. Comparison of the Root Mean Square Errors of the adjustments of the theoretical functions regarding the measurements.

The reader can find other ways to use the Weibull parameters related to the preventive interval in (Sgarbossa et al., 2018), (Sgarbossa et al., 2019).

In this work, the least squares method has been chosen to estimate the parameters of the failure time distributions, because it is a simple method well known by practitioners. An alternative method with higher mathematical content is the method of maximum likelihood (Li et al., 2020), (Balakrishnan and Mitra, 2011), (Hong et al., 2009).

3.2. Semi-Markov maintenance model with returns for a finite period of time

A semi-Markov process has been chosen for the construction of the model because it is a powerful tool for the optimisation of reliability and maintenance problems (Hu and Yue, 2003), (Kim and Makis, 2009), (Kim, M. J., & Makis, 2010), (Zhang and Gao, 2012). This type of tool is widely used to model maintenance systems with a finite number of operating periods (White and Gertsbakh, 2006).

A detailed explanation of the model proposed below can be found in (Crespo Marquez and Sánchez Herguedas, 2002). It is composed of three states (1. *Operational*, 2. *Corrective* and 3. *Preventive*), which represent the behaviour of the system under study in the event of failure. In the *Operational* state, the equipment will operate, producing some income. The time that the equipment is operating follows a three-parameter Weibull distribution function, according to the results of section 3.1. If after a time τ, the equipment does not fail, the system makes a transition to the *Preventive* state, in which the maintenance task is performed, characterised by the cost of spare parts, labour, and the cost derived from inactivity. Once the preventive task is completed, the system would return to the *Operational* state with probability 1. If the failure due to the deterioration of the crossover O-rings occurs before reaching the time τ, the system would go directly from the *Operational* to the *Corrective* state, accumulating some labour costs of repair, spare parts and unavailability. The system composed of the three states evolves during the time considered appropriate (finite), through the mechanism switching from one state to another, according to certain probabilities of transition. These transitions are expressed graphically in **Fig. 8**. The time until the preventive τ will be calculated by optimising the average accumulated return, as detailed in the following sections.

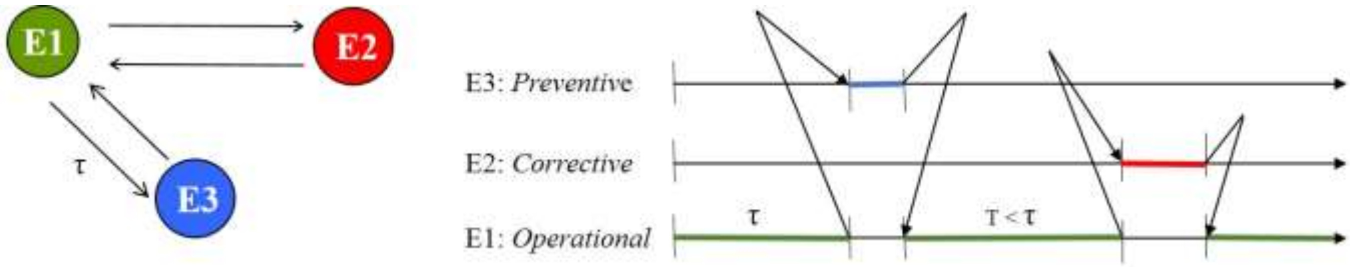


Fig. 8. Transition process between states.

3.3. Formulating a difference equation system for average accumulated return

In our model, each transition is associated with a return that can be either a reward in the form of income, such as profits made while the system is in *operating* condition, or a loss in the form of costs, such as those incurred by repair after a failure or preventive replacement. Suppose that r_{ij} denotes the return on the transition from state *i* to *j* (from here on we will often refer to states by their index), which will be positive if it is a income, or negative if it is a cost.

To make the model more general, we will consider that the system has n states. Later we will particularise for our case $n = 3$. In m transitions or successive steps, starting from state i, the system will accumulate returns, which, added to their respective signs, will constitute the accumulated return $Q_i(m)$ in m steps from state *i.* For each value of *i* and *m* the return is random variable. The random character is because once an initial state is determined, the next one is unpredictable and so are those that follow it, so in m transitions, the system can evolve in many ways. This is the reason why we are interested in knowing the average accumulated return, and not the accumulated return itself. In the following, for brevity, the accumulated return will simply be called return.

Let be $v_i(m) = E\big(Q_i(m)\big)$ the average return, in m steps, when the system starts from state i. To calculate it, the m steps are separated in two stages. The first is the transition from the initial state i to the next j. As the latter can be any of the system states, the return $Q_i(1)$ in a single step is a random variable that can reach the values $r_{i1}(1),\ r_{i2}(1),\ \cdots,\ r_{in}(1)$, with the respective probabilities $p_{i1},\ p_{i2},\ \cdots\ ,p_{in}$. Then, the average return in one step can be formulated as follows:

$$v_i(1) = \sum_{j=1}^{n} r_{ij}(1)\, p_{ij} \tag{4}$$

We now proceed with the remaining $m-1$ steps. Once the first step is taken, the system is found in a state called j, where j takes one of the values $1, 2, 3, \cdots, n$.. Therefore, the average return in the following $m-1$ steps is a random variable that can reach the values $v_1(m-1),\ v_2(m-1),\ \cdots, v_n(m-1)$, with respective probabilities, $p_{i1}, p_{i2}, \cdots, p_{in}$, which remains constant along with the m steps, since the process is homogeneous. Therefore, the expected value of the return of the remaining $m-1$ steps can be formulated as

$$\sum_{j=1}^{n} v_j(m-1) \cdot\ p_{ij} \tag{5}$$

We conclude that the total average return of the system in m steps is calculated according to **Eq. (6)**:

$$v_i(m) = v_i(1) + \sum_{j=1}^{n} v_j(m-1)\, p_{ij} \tag{6}$$

If the vector $V(m) = (v_1(m),\ v_2(m),\ \cdots, v_n(m))^t$, is defined, where the superscript *t* indicates transposition in the matrix sense, **Eq. (6)** is written as a matrix difference equation:

$$V(m) = V(1) + P\, V(m-1) \tag{7}$$

Where P is the transition probability matrix. **Eq. (7)** provides the average accumulated return in *m* steps from any possible starting state i. At this point, we proceed to solve the difference equation system.

Although our study focuses on a finite horizon process, it may be interesting to analyse the long-term behavior that we carried out in section 3.4.

3.4. Solving the system of difference equations by applying transforms z

The z-transform of an $x(m)$ sequence, usually denoted as $\mathcal{Z}[x(m)]$, is a complex variable function that is defined as the following Laurent series:

$$\mathcal{Z}[x(m)] = \sum_{n=0}^{\infty} x(n) z^{-n}, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{8}$$

The z-transform of the vector is another vector whose components are the z-transforms of the components. It is convenient to rewrite the difference equation with the index increased by one unit, that is $V(m+1) = V(1) +$

$P\,V(m)$, and apply the z-transform to both members of the equality. To do this, we take into account the linearity of the transform and the fact that the matrix P is constant (meaning it does not depend on m):

$$\mathcal{Z}[V(m+1)] = \mathcal{Z}[V(1)] + P\,\mathcal{Z}[V(m)] \tag{9}$$

Using the properties of the z-transform, the **Eq. (10)** is reached:

$$z\big(\mathcal{Z}[V(m)] - V(0)\big) = V(1)\frac{z}{z-1} + P\,\mathcal{Z}[V(m)] \tag{10}$$

Multiplying by the inverse of the $I - z^{-1}P$ matrix (which is regular), dividing by z and reordering the terms, we obtain:

$$\mathcal{Z}[V(m)] = \tfrac{1}{z-1}(I - z^{-1}P)^{-1}V(1) + (I - z^{-1}P)^{-1}V(0) \tag{11}$$

The initially average return vector $V(0)$ is the zero vector since there is no return before the initial moment in which the evolution of the chain begins. Therefore:

$$\mathcal{Z}[V(m)] = \frac{1}{z-1}(I - z^{-1}P)^{-1}V(1) \tag{12}$$

Now, it only remains to reverse this transformation to obtain the average accumulated return in m steps $V(m)$.

In our model of $n = 3$ states, the functions and variables involved are:

Time-related probabilistic functions:

- $F(t)$, is the cumulative distribution function (CDF) of failures of the equipment, $f(t)$ is the probability density function (PDF). After the discussion of section 3.1, we will use the three-parametric Weibull distribution.
- $G(t_c)$, is the distribution function of the time the equipment remains under corrective maintenance and $g(t_c)$ its probability density function.
- $H(t_p)$, is the distribution function of the time that the equipment remains under preventive maintenance and $h(t_p)$ its probability density function.

Returns (costs and incomes):

- R_1, income per time unit that the system remains in state 1 (E_1: Operational).
- R_2, cost per time unit that the system remains on state 2 (E_2: Corrective)
- R_3, cost per time unit that the system remains on state 3 (E_3: Preventive).

- R_{12}, cost of transition from state 1 to state 2.
- R_{21}, cost of transition from state 2 to state 1.
- R_{13}, cost of transition from state 1 to state 3.
- R_{31}, cost of transition from state 3 to state 1.

Matrices to describe the process:

- P, the transition probability matrix between states (p_{ij} is the probability of going from state i to j)
- F, stay time matrix (F_{ij} is the average time in state i before the system goes to state j).
- R, returns matrix, where $r_{ij} = R_i + R_{ij}$.

In our case of a system with three states, the transition probability matrix P and stay time F in each state are:

$$P = \begin{pmatrix} 0 & F(\tau) & 1 - F(\tau) \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & A & \tau \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix}, \tag{13}$$

where,

$$A = \frac{1}{F(\tau)} \int_0^{\tau} t \cdot f(t)\ dt\,; \qquad B = \int_0^{\infty} t_c\, g(t_c) dt_c\,; \qquad C = \int_0^{\infty} t_p\, h(t_p) dt_p\,.$$

Starting from F, we obtain R as follows:

$$R = \begin{pmatrix} 0 & r_{12} & r_{13} \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & R_1 A + R_{12} & R_1\, \tau + R_{13} \\ R_2 B + R_{21} & 0 & 0 \\ R_3 C + R_{31} & 0 & 0 \end{pmatrix}.$$

To simplify the notation, we will call $p = F(\tau)$.

The matrix P has the following remarkable property:

$$P^m = \begin{cases} P & \text{if } m \text{ is odd} \\ P^2 & \text{if } m \text{ is even} \end{cases} \qquad m = 1,2,3, \dots$$

From P^m it follows that the process lacks a limiting distribution, since if π is said distribution, it must be $\pi = \lim_{m\to\infty}\left(\pi^{(0)} P^m\right)$, where $\pi^{(0)}$ is any initial distribution vector, but this limit does not exist due to P^m. However the process has steady state distribution since the fixed point equation $\pi = \pi P$ together with the condition $\sum_{i=1}^{3} \pi_i = 1$, has the single solution $\pi = \begin{pmatrix} \frac{1}{2} & \frac{p}{2} & \frac{1-p}{2} \end{pmatrix}$, which is easy to prove.

Taking into account that $V(0) = 0$ because there is no return before the first step and that the vector $V(1)$, according to **Eq. (4)** is:

$$\begin{pmatrix} v_1(1) \\ v_2(1) \\ v_3(1) \end{pmatrix} = \begin{pmatrix} r_{12}p + r_{13}(1-p) \\ r_{21} \\ r_{31} \end{pmatrix}$$

we can develop the matrix $(I - z^{-1}P)^{-1}$, resulting:

$$\mathcal{Z}[V(m)] = \frac{1}{(z-1)^2(z+1)} \begin{pmatrix} \big(r_{12}p + r_{13}(1-p)\big)z^2 + \big(r_{21}p + r_{31}(1-p)\big)z \\ \big(r_{12}p + r_{13}(1-p)\big)z + r_{21}z^2 + (r_{31} - r_{21})(1-p) \\ \big(r_{12}p + r_{13}(1-p)\big)z + r_{31}z^2 + (r_{21} - r_{31})p \end{pmatrix} \tag{14}$$

The three components of **Eq. (14)** are linear combinations of rational functions in z. For each of them we can find their serial development by Laurent and from there look for their inverse transformations.

It is reasonable to observe the evolution of the system from the *operational* state. Therefore, we will only deal with the first component of $V(m)$. The result would be the following:

$$v_1(0) = 0,$$
$$v_1(1) = r_{12}F(\tau) + r_{13}\big(1 - F(\tau)\big),$$
$$v_1(m) = \frac{1}{4}\left[\begin{array}{c} \big(2m + 1 + (-1)^{m-1}\big)\left(R_1 \int_0^\tau tf(t)dt + R_{12}F(\tau) + (R_1\tau + R_{13})\big(1 - F(\tau)\big)\right) \\ +\big(2m - 1 - (-1)^{m-1}\big)\begin{pmatrix} \left(R_2 \int_0^\infty t_c\, g(t_c)dt_c + R_{21}\right)F(\tau) \\ +\left(R_3 \int_0^\infty t_p\, h(t_p)dt_p + R_{31}\right)\big(1 - F(\tau)\big) \end{pmatrix} \end{array}\right] \tag{15}$$
$$for\ values\ of\ m = 2, 3, 4, \ldots$$

3.5. Optimisation of the expected performance when starting from the operational state

Once we reach the explicit expression for $v_1(m)$, we proceed to calculate the operation time up to the preventive one, τ, which makes maximum $v_1(m)$. To do this, we use the direct method of deriving **Eq. (15)** with respect to τ, equalise the derivative to zero and obtain the corresponding value from τ.

$$\begin{aligned} \frac{dv_1(m)}{d\tau} = \frac{1}{4}\Big[f(\tau)\big(&(2m + 1 + (-1)^{m-1})(R_{12} - R_{13}) \\ &+ (2m - 1 - (-1)^{m-1})(R_2B + R_{21} - R_3C - R_{31})\big)\Big] \\ &+ (2m + 1 + (-1)^{m-1})\frac{R_1}{4}\big(1 - F(\tau)\big). \end{aligned} \tag{16}$$

In section 3.1 we have found that the three-parameter Weibull distribution is the one that best fits the observed distribution. That is why in the development that follows we will use it in our calculations:

$$f(t) = \frac{\alpha}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\alpha-1} e^{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}}, \qquad F(t) = 1 - e^{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}}, \qquad \alpha, \beta, \gamma > 0;$$

Substituting them in **Eq. (16)** gives an equation which, when equaled to zero, allows to clear the value of τ that makes $v_1(m)$ maximum:

$$(\tau-\gamma)^{\alpha-1} = \frac{\beta^{\alpha}}{\alpha} \cdot \frac{-R_1}{R_{12} - R_{13} + \frac{2m-1-(-1)^{m-1}}{2m+1+(-1)^{m-1}}(R_2B + R_{21} - R_3C - R_{31})} \tag{17}$$

The range of values of the function $(\tau-\gamma)^{\alpha-1}$ is the interval (0, ∞) if 0 < α < 1 and the interval [0, ∞) if $\alpha \geq 1$, so **Eq. (17)** will have a solution if and only if the denominator is verified to be greater than zero. Since R_1, α and β are strictly positive, we can show that in cases where $\alpha > 1$ and $0 < \alpha < 1$, we can always reach the optimal solution.

In the particular case of $\alpha < 1$ it is shown that the optimum is in infinite, which means that preventive maintenance would not be indicated either. Only corrective maintenance would be indicated. On the other hand, for the case $\alpha = 1$, either there is no solution, or any positive number is, depending on whether the second term of the expression is different or equal to 1, respectively. Nor would the preventive one be indicated.

In other cases, such as the model presented in (Zhao et al., 2020b), where only costs are considered, the existence of a finite preventive interval is demonstrated. In this case they balance the total expected time durations between early or late replacement and the random failure using two variable types of shortage and excess costs.

4. Results and Discussion

Once we have estimated the parameters of the Weibull distribution that we are going to use according to section 3.1, we need to estimate the distribution functions of the repair time $G(t_c)$ and the preventive time $H(t_p)$. Since these are small time intervals (less than 1%) compared to operating times, we can use the average time of both interventions as estimators. In our case, these values are $\mu_1 = 8$ and $\mu_2 = 7$ hours, respectively. On the other hand, there are other returns associated with the study, such as the estimated income per hour of equipment operation R_1, which has been valued at 6.0 €/hour; the cost associated with the onset of the failure R_{12}, estimated at 4,320 €, mainly as a result of the loss in availability; the cost incurred on ordering the preventive R_{13}, estimated at 1 €; the cost of the corrective R_2, 95.0 €/hour; the cost of the preventive hour R_3, 82.0 €/hour and the cost of the spare part necessary to make the intervention R_{21} and R_{31}, 620 €, as shown in **Table 5**.

Table 5. Parameters and data input to the model.

Failure distribution function			F. D. Corrective times		F. D. Preventive time	
Weibull (α, β, γ)			Normal (μ_1)		Normal (μ_2)	
α =1.95	β = 1,202.36	γ =116.83	μ_1 = 8		μ_2 = 7	
Operation Returns			**Corrective Returns**		**Preventive Returns**	
R_1 (€/hour)	R_{12} (€)	R_{13} (€)	R_2 (€/hour)	R_{21} (€)	R_3 (€/hour)	R_{31} (€)
6.0	-4,320	-1.0	-95.0	-620	-82.0	-620

By replacing the parameters and data of **Table 5** in **Eq. (17)**, the value of the time until the preventive τ_o is obtained, which makes the average accumulated return for different steps maximum (in this figure, for m from 1 to 10). These values are shown numerically and graphically in **Fig. 9**.

m	τ_o
1	1,139.59
2	1,095.04
3	1,116.83
4	1,095.04
5	1,109.46
6	1,095.04
7	1,105.82
8	1,095.04
9	1,103.65
10	1,095.04

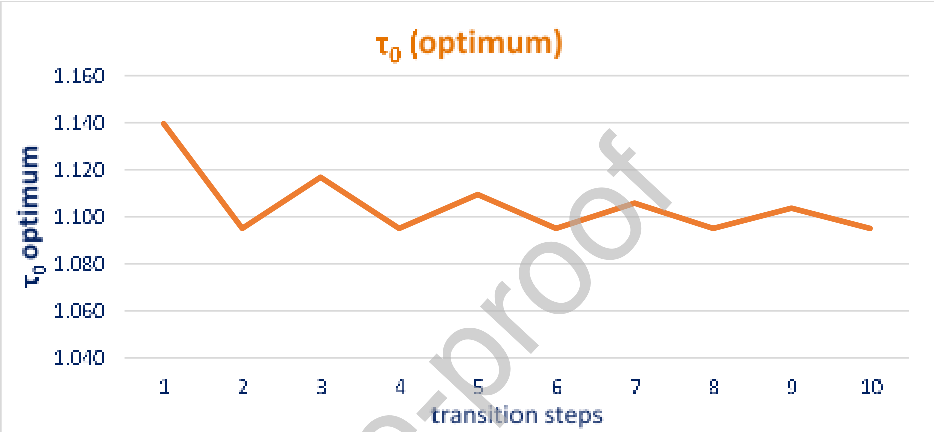


Fig. 9. Optimal values of τ_o and their graphic representation for a succession of steps.

The model we are analyzing is simple but reflects a real situation that occurs in many industrial equipment. Every two steps, the described process starts an operational stage once again, closing a cycle. Hence the optimal value (1,095 hours) is repeated for the even values of the steps. In other models with more states this particular fact does not occur.

In **Fig. 9** it is possible to observe the process' behaviour for a finite period of time, knowing that, for this case, the step has an average duration of $p_{12}\, f_{12} + p_{13}\, f_{13} = 912$ hours, with p_{1j} and f_{1j} $(j = 2{,}3)$ elements of the matrices in **Eq. (13)**. We can determine the optimal moment τ_o of the preventive intervention according to the time until the end of the project. For example, if 4,560 hours remained to finish the project, the process would advance to step number 5 and τ_o would be 1,109 hours.

This tool allows finding τ_o for different situations of finite time and also the one of infinite time. It can be seen that, as the time horizon lengthens, the values of τ_o will get closer to the corresponding value of the infinite horizon, as would be expected.

Also, throughout the process of obtaining the τ_o we have deduced how the average accumulated return varies as a function of time until preventive maintenance, $\frac{dv_1(m)}{d\tau}$. This behaviour can be observed in **Fig. 10**, the maximum value of v_1 is found for the value of τ equal to 1,095 hours. Furthermore, for high values of τ, which could correspond to a situation where no preventive maintenance is done, the average accumulated return, predictably, is not affected by small variations of τ.

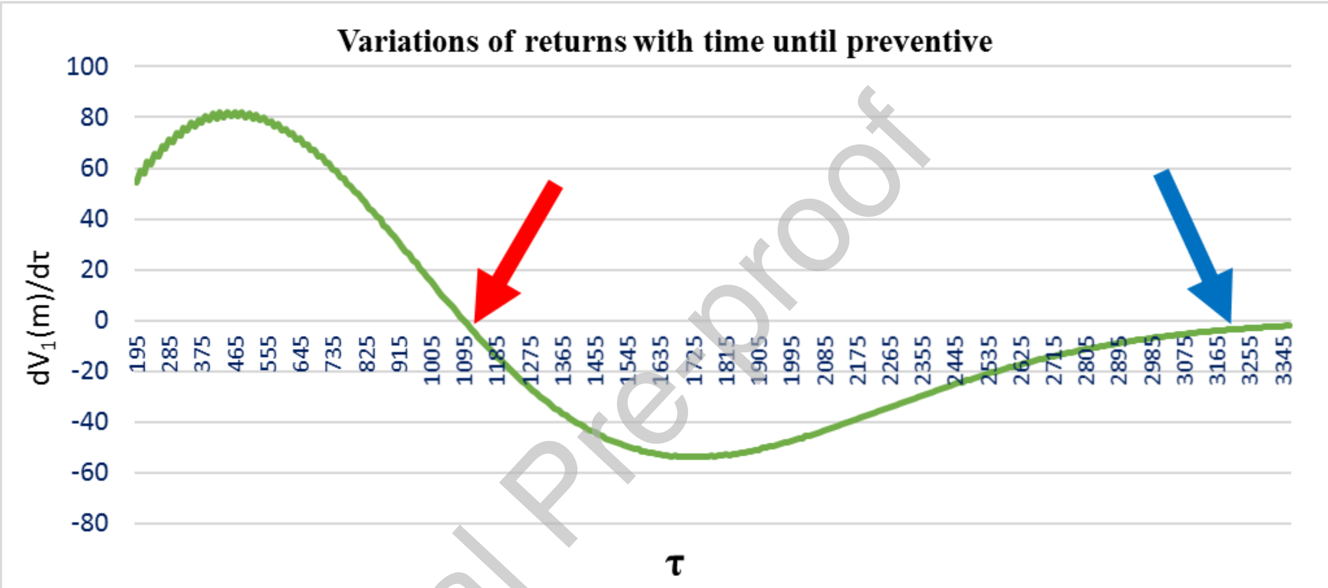


Fig. 10. Graphical representation of the behaviour of the $\frac{dv_1(m)}{d\tau}$ compared to the time until preventive maintenance, τ. Using ***Eq. (16)***.

The value of τ is traditionally deduced by comparison with similar elements and by applying the procedures recommended by the RCM method. It has also been deducted from a required reliability value. These methods are not very precise, as the former requires testing and analysis of the results and the latter does not assess the costs of the chosen option and may, therefore, deviate from the optimal economic solution.

The calculation of the interval of preventive maintenance value τ is one of the most discussed issues, both from an academic and business point of view. When a company promotes the realisation of RCM to its critical equipment and the CBM is not possible, the default maintenance acquires remarkable significance. Phase 8 of the RCM method described in (Rausand and Vatn, 2008) refers to the methods for optimising the maintenance interval. These methods consist of the construction of the cost function and its minimisation for values of τ. In this context, our work can be considered as a contribution and an innovation to this phase 8 of the RCM method.

In other cases, it is desired to find the value τ that maximises availability (Kumar and Varghese, 2018) or the average time until failure (Zhong and Jin, 2014). To reach this value of τ, numerical techniques are used, such as the golden section technique. On the contrary, our method explicitly obtains the value of τ, and achieves it in a much simpler way.

This newly proposed method assesses the situation economically and looks for the most profitable solution. The novelty arises when returns are described as the solution of the difference equation system resulting from a semi-Markov process. This avoids calculating them through dynamic programming, as it has been done traditionally. Later on, the fact that this type of equation system admits a direct solution through z transforms is profited. The result is beneficial for maintenance management because it allows automating the calculation of optimal frequencies for preventive interventions without the need for expert knowledge.

Compared to the mathematical expressions obtained for models with infinite horizons, this expression allows us to calculate the optimal τ for any time horizon we see fit, and in most cases that value is different from the value for an infinite time horizon.

If we did not have this novel method and we wanted to find the optimal value of τ (without using the mathematical expression for τ) we should:

1) Find the value of the average accumulated return. For that, we should use some simulation technique. This technique must determine the value of this variable for each step, based on the data of the problem and for different values of τ.

2) Among the values obtained for each step m, the value of τ that maximises v_i should be found. This process is simple since it consists in finding the largest value of v_i in a list of values.

For example, by using dynamic programming we can simulate our system of equations, **Eq. (6)**, using different values of m and τ. After the simulation, there is a large table of values for v_i, from which we could obtain the values of τ that maximise v_i for each value of m. A brief summary of this table is shown in **Table 6**.

Table 6. Returns values $v_1(m)$ for different values of m and τ. The shaded cells represent the maximum v_1

	$v_1(m)$										
τ	**$v_1(5)$**	**$v_1(6)$**	**$v_1(7)$**	**$v_1(8)$**	**$v_1(9)$**	**$v_1(10)$**	**$v_1(11)$**	**$v_1(12)$**	**$v_1(13)$**	**$v_1(14)$**	**$v_1(15)$**
1,095	7,510.427	6,225.682	9,585.655	8,300.909	11,660.882	10,376.136	13,736.109	12,451.363	15,811.336	14,526.590	17,886.564
1,096	7,510.550	6,225.677	9,585.776	8,300.903	11,661.001	10,376.129	13,736.227	12,451.354	15,811.453	14,526.580	17,886.678
1,097	7,510.663	6,225.664	9,585.885	8,300.885	11,661.106	10,376.106	13,736.327	12,451.328	15,811.548	14,526.549	17,886.770

1,098	7,510.768	6,225.642	9,585.982	8,300.855	11,661.196	10,376.069	13,736.409	12,451.283	15,811.623	14,526.497	17,886.837
1,099	7,510.863	6,225.611	9,586.067	8,300.814	11,661.270	10,376.018	13,736.474	12,451.221	15,811.677	14,526.425	17,886.881
1,100	7,510.950	6,225.571	9,586.140	8,300.761	11,661.331	10,375.951	13,736.521	12,451.141	15,811.711	14,526.331	17,886.901
1,101	7,511.028	6,225.522	9,586.202	8,300.696	11,661.376	10,375.870	13,736.550	12,451.044	15,811.724	14,526.218	17,886.898
1,102	7,511.097	6,225.464	9,586.252	8,300.619	11,661.407	10,375.774	13,736.562	12,450.929	15,811.716	14,526.084	17,886.871
1,103	7,511.157	6,225.398	9,586.290	8,300.531	11,661.423	10,375.664	13,736.556	12,450.796	15,811.688	14,525.929	17,886.821
1,104	7,511.209	6,225.323	9,586.317	8,300.431	11,661.424	10,375.539	13,736.532	12,450.646	15,811.640	14,525.754	17,886.748
1,105	7,511.252	6,225.239	9,586.331	8,300.319	11,661.411	10,375.399	13,736.491	12,450.479	15,811.571	14,525.559	17,886.651
1,106	7,511.286	6,225.147	9,586.335	8,300.196	11,661.384	10,375.245	13,736.433	12,450.294	15,811.482	14,525.343	17,886.531
1,107	7,511.311	6,225.046	9,586.326	8,300.062	11,661.342	10,375.077	13,736.357	12,450.092	15,811.373	14,525.108	17,886.388
1,108	7,511.328	6,224.936	9,586.306	8,299.915	11,661.285	10,374.894	13,736.264	12,449.873	15,811.243	14,524.852	17,886.222
1,109	7,511.336	6,224.818	9,586.275	8,299.758	11,661.214	10,374.697	13,736.154	12,449.636	15,811.093	14,524.576	17,886.033
1,110	7,511.335	6,224.691	9,586.232	8,299.588	11,661.129	10,374.486	13,736.026	12,449.383	15,810.923	14,524.280	17,885.820
1,111	7,511.326	6,224.556	9,586.178	8,299.408	11,661.030	10,374.260	13,735.882	12,449.112	15,810.734	14,523.964	17,885.586
1,112	7,511.308	6,224.412	9,586.112	8,299.216	11,660.916	10,374.020	13,735.720	12,448.824	15,810.524	14,523.628	17,885.328
1,113	7,511.281	6,224.260	9,586.034	8,299.013	11,660.788	10,373.766	13,735.541	12,448.519	15,810.294	14,523.272	17,885.047
1,114	7,511.246	6,224.099	9,585.946	8,298.798	11,660.645	10,373.498	13,735.345	12,448.197	15,810.045	14,522.897	17,884.744

The method presented in the paper has been compared with the utilization of Monte Carlo simulation. A dynamic simulation model has been built using continuous time stochastic simulation. The methodology used can be consulted in (Crespo Márquez, 2010). The variability is introduced by three seeds, one for the preventive time, a second for the corrective time and a third for the random number generating failures according to the Weibull distribution. Difference equations are used in the model and the simulation time step considered is one hour. 200 simulations were done per scenario and only three of the many scenarios studied are referred ($\tau = 500, \tau = 1{,}095 \; and \; \tau = 2{,}000 \; hours$). A total number of 15,000 hours of simulation were considered. The results from this simulation are shown in **Table 7** and can be compared with the ones presented in **Fig. 11** from the analytical method.

Table 7. Monte Carlo simulation. Returns values $v_1(m)$ for different values of m and τ.

Variable	Count	Min	Max	Mean	Median	StDev	(Norm)
Returs for $m = 10$							
$\tau = 500$	200	-8,706	9,007	5,970	8,679	3,521	0.5898
$\boldsymbol{\tau = 1{,}095}$	**200**	**-10,819**	**26,693**	**10,477**	**10,313**	**6,790**	**0.6481**
$\tau = 1{,}500$	200	-10,819	34,041	9,552	8,137	8,733	0.9142
Returs for $m = 8$							
$\tau = 500$	200	-5,212	7,208	4,867	6,962	3,132	0.6434
$\boldsymbol{\tau = 1{,}095}$	**200**	**-11,477**	**21,324**	**8,125**	**8,241**	**6,196**	**0,7626**
$\tau = 1{,}500$	200	-11.477	30,962	7,411	6,616	7,778	1.050
Returs for $m = 6$							
$\tau = 500$	200	-7,011	5,409	5,535	5,245	2,836	0.8022
$\boldsymbol{\tau = 1{,}095}$	**200**	**-8,032**	**16,119**	**6,177**	**6,991**	**5,653**	**0.9152**
$\tau = 1{,}500$	200	-8,032	23,245	5,482	5,170	6,572	1.199

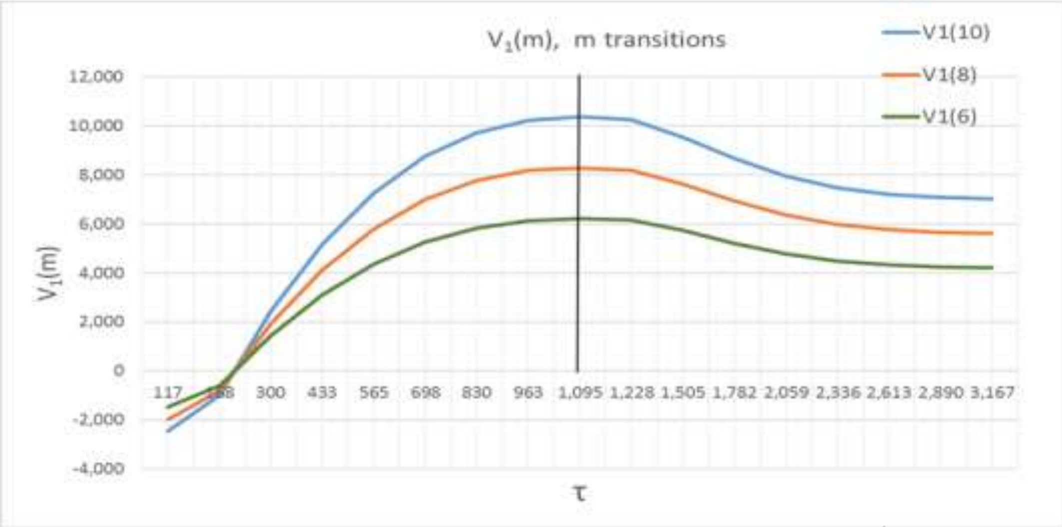


Fig. 11. Analytical method. Returns values $v_1(m)$ for different values of m and τ.

In the scenario $\tau = 1{,}095$, looking at the mean returns values to a certain number of transitions, it is possible to check how accurate the results of the analytical model are. The analytical approach turned out to be much more straightforward and less time consuming, while Monte Carlo is laborious and requires testing many different scenarios.

This new method also allows the maintenance manager to do an analysis of the result against the input data. This analysis becomes a very basic study since it only requires verifying the variation of τ by varying an input data using the mathematical expression of τ_o. As an example, the value of each of the data has been varied by 10%. β, R_1 y R_{12} are the ones that have most influence. This is represented in **Table 8**.

Table 8. Variation of the optimal interval τ_o, if each of the input data is varied by a percentage (±10%).

Rewards	**Quantity**	$\boldsymbol{\tau_o}$	**Quantity**	**Failure Data**	**Quantity**	$\boldsymbol{\tau_o}$	**Quantity**
R_1 (€/hora)	10%	1198.63	**9,46%**	R_{31} (€)	10%	1109.46	1.32%
	-10%	992.03	**-9,41%**		-10%	1081.02	-1.28%
R_2 (€/hora)	10%	1077.91	-1.56%	α	10%	1048.91	-4.21%
	-10%	1112.78	1.62%		-10%	1183.61	8.09%
R_3 (€/hora)	10%	1108.38	1.22%	**β**	10%	1306.82	**19.34%**
	-10%	1082.05	-1.19%		-10%	904.51	**-17.40%**
R_{12} (€)	10%	1004.87	**-8.23%**	γ	10%	1106.73	1.07%
	-10%	1204.93	**10.04%**		-10%	1083.36	-1.07%
R_{13} (€)	10%	1095.07	0%	μ_1	10%	1077.91	-1.56%
	-10%	1095.02	0%		-10%	1112.78	1.62%
R_{21} (€)	10%	1081.02	-1.28%	μ_2	10%	1108.38	1.22%
	-10%	1109.46	1.32%		-10%	1082.05	-1.19%

5. Conclusions

Preventive maintenance (PM) actions carried out at fixed time intervals are very frequent in equipment maintenance. The correct selection of the interval can mean saving costs for companies. The maintenance manager knows the equipment failure data and the cost of interventions. Having a mathematical formula that relates interval, failure data, costs and planning horizon can be a competitive advantage for that company. It allows one to optimise the interval at a very low price, since managers do not need to resort to complicated methods that consume time and resources of the maintenance department.

This work develops a practical mathematical tool that allows calculating the optimal preventive interval, in an analytical way, just by having data from the fault history and the preventive tasks (costs and income). No simulations or development of complicated studies are needed. Its simplicity can make it suitable for general use by being applied, to many assets, since the model, it comes from is very general and is not subject to significant restrictions (remaining in operational, corrective or preventive state and showing a clear tendency to wear).

In this article, we have presented a new method to obtain the value of time until preventive maintenance that maximises the average accumulated return in a finite horizon of m steps. A semi-Markov process formed by three states has been considered, specified in a real example of a failure mode in a diesel engine's exhaust system.

It has been possible to determine the average accumulated return, showing that it can be expressed as the solution of a difference equations system. For its resolution, we use the technique of the z -transform.

To maximise this average accumulated return in m steps $v_1(m)$, the direct derivation procedure has been used concerning the time until preventive maintenance τ, equalising this derivation to zero and studying the conditions of existence of the optimal solution, τ_0. The method calculates τ_0 from **Eq. (17)**, for which it needs the time to failure data and the returns of each maintenance task. The method was implemented with real data collected over years. These real data are transformed into input values using the medians range method with Weibull plots and least-squares method.

Among the advantages of this method over previous methods, we can highlight:

- It allows direct calculation with an analytical expression of the average accumulated return for m steps. Therefore, it could be adapted to any project with a finite time horizon.

- The maintenance technician in the company can, with little information available, obtain the optimal value of τ without having to know any mathematical development, merely applying the expression for τ_o. This fact makes this method highly recommended, since obtaining τ_o requires no other knowledge except maintenance.
- It is possible to make τ_o updates at any time, as this method is easy to apply, adapting its value to the horizon required by the needs of the industrial plant.
- It directly obtains the value of time until the preventive that maximises the average accumulated return, without the need to use more complex numerical methods such as the dynamic programming which is often used.
- Provide the optimum value $\tau_0(m)$ for situations where the operation-repair cycle is not completed and other situations with a finite time horizon.
- By providing specific solutions, this method allows comparing different scenarios by merely changing any parameters in the Weibull distribution or return values in each state. This simplifies the calculation tasks with respect to other models that would have to be solved numerically.
- This method does not need any simulations for its verification since the obtained result is exact.

As a result of years of experience in a leading multinational in the large engine maintenance sector, we consider that the proposal described in this paper will be of great interest to those responsible for maintaining ships with high-power engines and other equipment on board.

In this work, only economic data from maintenance activities have been used, but the method also admits being able to use other factors relevant to the business such as availability, safety, environment, energy consumption, emissions, etc.

Based on this work, further investigations can be undertaken concerning the expansion of the number of states, for example, other operational states with a certain level of degradation, the inclusion of imperfect maintenance or the implementation of control actions for various states.

Author Statement

Sánchez-Herguedas, A.: Conceptualization, Investigation, Methodology, Validation, Software, Data Curation.

Mena-Nieto, A.: Investigation, Supervision, Visualization, Writing- Reviewing and Editing.
Rodrigo-Muñoz, F.: Investigation, Formal analysis, Writing-Original draft preparation.

Declaration of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Anake, T., Edeki, S., Oguntunde, P., Odetunmibi, O., 2014. On a Z-Transformation Approach to a Continuous-Time Markov Process with Nonfixed Transition Rates. Gen. Math. Notes 24, 85–96.

Baklouti, A., Mifdal, L., Dellagi, S., Chelbi, A., Al Madina, B., Tillila, Q., 8143 Agadir, B.P., Ma, M.M.-P., 2020. An Optimal Preventive Maintenance Policy for a Solar Photovoltaic System. Sustain. 12, 13. https://doi.org/10.3390/su12104266

Balakrishnan, N., Mitra, D., 2011. Likelihood inference for lognormal data with left truncation and right censoring with an illustration. J. Stat. Plan. Inference 141, 3536–3553. https://doi.org/10.1016/j.jspi.2011.05.007

Campbell, J., Jardine, A., 2001. Maintenance Excellence: optimizing equipment life-cycle decisions. M. Dekker, New York.

Coria, V.H., Maximov, S., Rivas-Dávalos, F., Melchor, C.L., Guardado, J.L., 2015. Analytical method for optimization of maintenance policy based on available system failure data. Reliab. Eng. Syst. Saf. 135, 55–63. https://doi.org/10.1016/j.ress.2014.11.003

Crespo Márquez, A., 2010. Dynamic modelling for supply chain management: Dealing with front-end, back-end and integration issues. Springer London. https://doi.org/10.1007/978-1-84882-681-6

Crespo Márquez, A., Moreu de León, P., Sánchez Herguedas, A., 2004. Ingeniería de mantenimiento. Técnicas y métodos de aplicación a la fase operativa de los equipos. Ed. Aenor, Madrid.

Crespo Marquez, A., Sánchez Herguedas, A., 2002. Models for maintenance optimization: A study for repairable systems and finite time periods. Reliab. Eng. Syst. Saf. 75, 367–377. https://doi.org/10.1016/S0951-8320(01)00131-4

de Jonge, B., Scarf, P.A., 2020. A review on maintenance optimization. Eur. J. Oper. Res. 285, 805–824. https://doi.org/10.1016/j.ejor.2019.09.047

Dong, W., Liu, S., Du, Y., 2019. Optimal periodic maintenance policies for a parallel redundant system with component dependencies. Comput. Ind. Eng. 138, 106133. https://doi.org/10.1016/j.cie.2019.106133

Grabski, F., 2014. Semi-Markov Processes: Applications in System Reliability and Maintenance. Elsevier Inc., Waltham. https://doi.org/10.1016/C2013-0-14260-2

Han, Y., Zhen, X., Huang, Y., Vinnem, J.E., 2019. Integrated methodology for determination of preventive maintenance interval of safety barriers on offshore installations. Process Saf. Environ. Prot. 132, 313–324. https://doi.org/10.1016/j.psep.2019.09.035

Hong, Y., Meeker, W.Q., McCalley, J.D., 2009. Prediction of remaining life of power transformers based on left truncated and right censored lifetime data. Ann. Appl. Stat. 3, 857–879. https://doi.org/10.1214/00-AOAS231

Hu, J., Shen, J., Shen, L., 2020. Periodic preventive maintenance planning for systems working under a Markovian operating condition. Comput. Ind. Eng. 142, 106291. https://doi.org/10.1016/j.cie.2020.106291

Hu, Q., Yue, W., 2003. Optimal replacement of a system according to a semi-Markov decision process in a semi-Markov environment. Optim. Methods Softw. 18, 181–196. https://doi.org/10.1080/1055678031000111803

Huang, C.C., Yuan, J., 2010. A two-stage preventive maintenance policy for a multi-state deterioration system. Reliab. Eng. Syst. Saf. 95, 1255–1260. https://doi.org/10.1016/j.ress.2010.07.001

Iranpoor, M., Fatemi Ghomi, S.M.T., 2019. Periodic flexible maintenance planning in a single-machine production environment. J. Ind. Eng. Int. 15, 627–635. https://doi.org/10.1007/s40092-019-0314-x

Johnson, L., 1964. The statistical treatment of fatigue experiments. Elsevier, Oxford (UK).

Jung, G.M., Park, D.H., 2003. Optimal maintenance policies during the post-warranty period. Reliab. Eng. Syst. Saf. 82, 173–185. https://doi.org/10.1016/S0951-8320(03)00144-3

Jung, K.M., Park, M., Park, D.H., 2010. System maintenance cost dependent on life cycle under renewing warranty policy. Reliab. Eng. Syst. Saf. 95, 816–821. https://doi.org/10.1016/j.ress.2010.02.010

Kim, M. J., & Makis, V., 2010. A maintenance model with minimal and general repair. IMA J. Manag. Math. https://doi.org/10.1093/imaman/dpn016

Kim, M., Makis, V., 2009. Optimal maintenance policy for a multi-state deteriorating system with two types of failures under general repair. Comput. Ind. Eng. 57, 298–303. https://doi.org/10.1016/j.cie.2008.11.023

Kumar, G., Varghese, J.P., 2018. Optimum Preventive Maintenance Policy for a Mechanical System Using Semi-markov Method and Golden Section Technique, in: 2018 IEEE International Conference on Industrial Engineering and Engineering Management (IEEM). Bangkok, pp. 232–236. https://doi.org/10.1109/IEEM.2018.8607343

Laggoune, R., Chateauneuf, A., Aissani, D., 2010. Impact of few failure data on the opportunistic replacement policy for multi-component systems. Reliab. Eng. Syst. Saf. 95, 108–119. https://doi.org/10.1016/j.ress.2009.08.007

Legát, V., Mošna, F., Aleš, Z., Jurča, V., 2016. Preventive maintenance models – higher operational reliability. Eksploat. i Niezawodn. - Maint. Reliab. 19, 134–141. https://doi.org/10.17531/ein.2017.1.19

Li, Q., Li, D., Huang, B., Jiang, Z., Mingcheng, M.M.E., Ma, J., 2020. Failure Analysis for Truncated and Fully Censored Lifetime Data with a Hierarchical Grid Algorithm. IEEE Access 8, 34468–34480. https://doi.org/10.1109/ACCESS.2020.2974266

Li, R., Zhang, X., 2020. Preventive Maintenance Interval Optimization for Continuous Multistate Systems. https://doi.org/10.1155/2020/2942940

Lin, Z.L., Huang, Y.S., Fang, C.C., 2015. Non-periodic preventive maintenance with reliability thresholds for complex repairable systems. Reliab. Eng. Syst. Saf. 135, 145–156. https://doi.org/10.1016/j.ress.2014.12.010

Lust, T., Roux, O., Riane, F., 2009. Exact and heuristic methods for the selective maintenance problem. Eur. J. Oper. Res. 197, 1166–1177. https://doi.org/10.1016/j.ejor.2008.03.047

Lyubchenko, A., Pacheco, J., Kopytov, E., Lutchenko, S., Maystrenko, V., Bartosh, S., 2018. An approach for optimal maintenance planning of radio communication devices considering reliability and operational costs. Moscow Work. Electron. Netw. Technol. MWENT 2018 - Proc. 2018-March, 1–5. https://doi.org/10.1109/MWENT.2018.8337301

Mizutani S., Zhao X., N.T., 2021. WIB (Which-Is-Better) Problems in Maintenance Reliability Policies, in: Misra K.B. (Ed.), Handbook of Advanced Performability Engineering. Springer, Cham, pp. 523–547. https://doi.org/https://doi.org/10.1007/978-3-030-55732-4_23

Młynarski, S., Pilch, R., Smolnik, M., Szybka, J., Wiązania, G., 2020. A model of an adaptive strategy of preventive maintenance of complex technical objects. Eksploat. i Niezawodn. 22, 35–41. https://doi.org/10.17531/ein.2020.1.5

Nakagawa, T., Mizutani, S., 2009. A summary of maintenance policies for a finite interval. Reliab. Eng. Syst. Saf. 94, 89–96. https://doi.org/10.1016/j.ress.2007.04.004

Nakagawa, T., Yasui, K., Sandoh, H., 2004. Note on optimal partition problems in reliability models. J. Qual. Maint. Eng. 10, 282–287. https://doi.org/10.1108/13552510410564909

Nicolai, R.P., Dekker, R., 2008. Optimal Maintenance of Multi-component Systems: A Review, in: Khariy A. H. Kobbacy; D. N. Prabhakar Murthy (Ed.), Complex System Maintenance Handbook. Springer, London, pp. 263–286. https://doi.org/10.1007/978-1-84800-011-7_11

Nicolai, R.P., Dekker, R., 2007. A comparison of two non-stationary degradation processes, in: Proceedings of the European Safety and Reliability Conference, 2007, ESREL 2007. TAYLOR & FRANCIS LTD, London, Stavanger, NORWAY, pp. 469–476.

Pandey, M., Zuo, M.J., Moghaddass, R., 2015. Selective maintenance scheduling over a finite planning horizon. Proc. Inst. Mech. Eng. Part O J. Risk Reliab. 230, 162–177. https://doi.org/10.1177/1748006X15598914

Percy, D.F., 2008. Preventive Maintenance Models for Complex Systems, in: Khariy A. H. Kobbacy; D. N. Prabhakar Murthy (Ed.), Complex System Maintenance Handbook. Springer, London, pp. 179–208. https://doi.org/10.1007/978-1-84800-011-7_8

Rausand, M., Vatn, J., 2008. Reliability Centred Maintenance, in: Khariy A. H. Kobbacy; D. N. Prabhakar Murthy (Ed.), Complex System Maintenance Handbook. Springer, p. 97. https://doi.org/10.1007/978-1-84800-011-7_4

Roy, D., 2011. Corrections to "Difference Equations, Z-transforms and Resistive Ladders." IETE J. Educ. 52, 86. https://doi.org/https://doi.org/10.1080/09747338.2011.10876085

Savoye, P., 2009. On the inclusion of difference equation problems and Z transform methods in sophomore differential equation classes. Primus 19, 491–499. https://doi.org/https://doi.org/10.1080/10511970802105010

Schutz, J., Rezg, N., Léger, J.-B., 2011. Periodic and sequential preventive maintenance policies over a finite planning horizon with a dynamic failure law. J. Intell. Manuf. 22, 523–532. https://doi.org/10.1007/s10845-009-0313-7

Semaan, N.M., Yehia, N., 2019. A stochastic detailed scheduling model for periodic maintenance of military rotorcraft. Aircr. Eng. Aerosp. Technol. 91, 1195–1204. https://doi.org/10.1108/AEAT-09-2018-0254

Sembiring, N., Panjaitan, N., Angelita, S., 2018. Design of preventive maintenance system using the reliability engineering and maintenance value stream mapping methods in PT. XYZ. IOP Conf. Ser. Mater. Sci. Eng. 309. https://doi.org/10.1088/1757-899X/309/1/012128

Sgarbossa, F., Zennaro, I., Florian, E., Calzavara, M., 2019. Age replacement policy in the case of no data: the effect of Weibull parameter estimation. Int. J. Prod. Res. 0, 1–19. https://doi.org/10.1080/00207543.2019.1660824

Sgarbossa, F., Zennaro, I., Florian, E., Persona, A., 2018. Impacts of weibull parameters estimation on preventive maintenance cost. IFAC-PapersOnLine 51, 508–513. https://doi.org/10.1016/j.ifacol.2018.08.369

Shakenova, R., 2012. The Limiting Probabilities in the Process of Servicing. Appl. Math. 2, 184–186. https://doi.org/10.5923/j.am.20120206.01

Sheu, S.H., Chang, C.C., 2009. An extended periodic imperfect preventive maintenance model with age-dependent failure type. IEEE Trans. Reliab. 58, 397–405. https://doi.org/10.1109/TR.2009.2020103

Van Horenbeek, A., Pintelon, L., Muchiri, P., 2010. Maintenance optimization models and criteria. Int. J. Syst. Assur. Eng. Manag. 1, 189–200. https://doi.org/10.1007/s13198-011-0045-x

Wang, B., Wang, X., Bie, Z., Judge, P.D., Wang, X., Green, T.C., 2017. Reliability Model of MMC Considering Periodic Preventive Maintenance. IEEE Trans. Power Deliv. 32, 1535–1544. https://doi.org/10.1109/TPWRD.2016.2602888

Wang, W., 2008. Delay Time Modelling, in: Springer Series in Reliability Engineering. https://doi.org/10.1007/978-1-84800-011-7_14

White, J.F., Gertsbakh, I.B., 2006. Models of Preventive Maintenance. J. Oper. Res. Soc. https://doi.org/10.2307/3009462

Zhang, X., Gao, H., 2012. Road maintenance optimization through a discrete-time semi-Markov decision process. Reliab. Eng. Syst. Saf. 103, 110–119. https://doi.org/10.1016/j.ress.2012.03.011

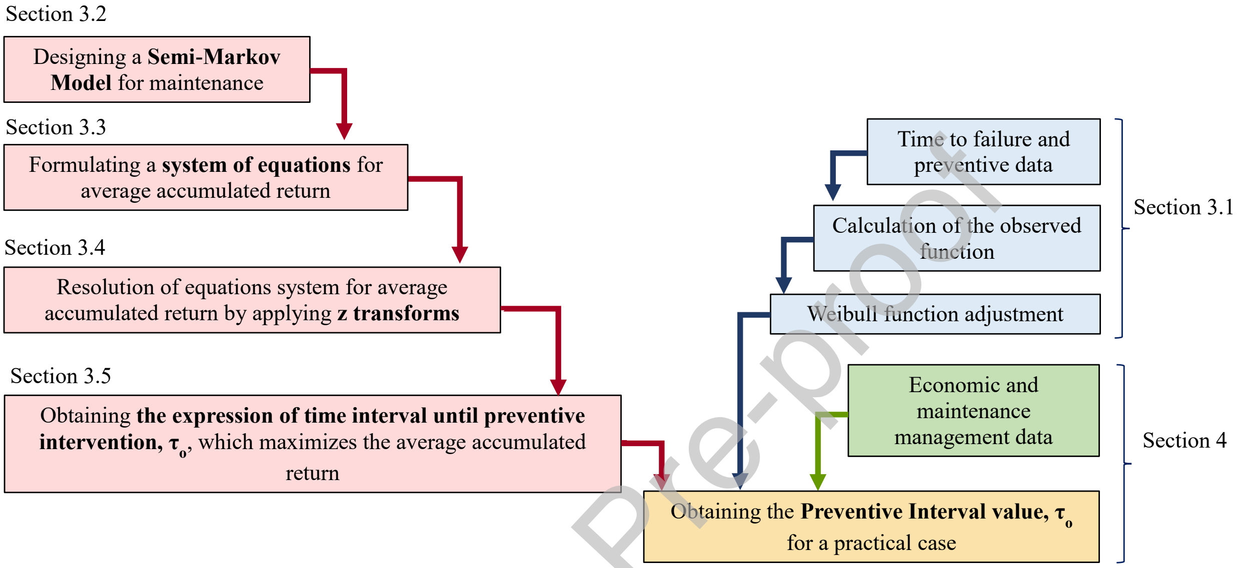
Zhao, X., Cai, J., Mizutani, S., Nakagawa, T., 2020a. Preventive replacement policies with time of operations, mission durations, minimal repairs and maintenance triggering approaches. J. Manuf. Syst. 1–11. https://doi.org/10.1016/j.jmsy.2020.04.003

Zhao, X., Chen, M., Nakagawa, T., 2020b. Periodic replacement policies with shortage and excess costs. Ann. Oper. Res. https://doi.org/10.1007/s10479-020-03566-z

Zhong, C., Jin, H., 2014. A novel optimal preventive maintenance policy for a cold standby system based on semi-Markov theory. Eur. J. Oper. Res. 232, 405–411. https://doi.org/10.1016/j.ejor.2013.07.020

Zhong, S., Pantelous, A.A., Goh, M., Zhou, J., 2019. A reliability-and-cost-based fuzzy approach to optimize preventive maintenance scheduling for offshore wind farms. Mech. Syst. Signal Process. 124, 643–663. https://doi.org/10.1016/j.ymssp.2019.02.012

Zio, E., 2009. Reliability engineering: Old problems and new challenges. Reliab. Eng. Syst. Saf. 94, 125–141. https://doi.org/10.1016/j.ress.2008.06.002


Section 3.2
Designing a Semi-Markov Model for maintenance
Section 3.3
Formulating a system of equations for average accumulated return
Section 3.4
Resolution of equations system for average accumulated return by applying z transforms
Section 3.5
Obtaining the expression of time interval until preventive intervention, τ_o, which maximizes the average accumulated return
Time to failure and preventive data
Calculation of the observed function
Weibull function adjustment
Section 3.1
Economic and maintenance management data
Obtaining the Preventive Interval value, τ_o for a practical case
Section 4