Analytic integrability problem for perturbations of cubic Kolmogorov systems

Antonio Algaba, Cristóbal García, Manuel Reyes
Dept. Matemáticas, Facultad de Ciencias, Univ. of Huelva, Spain.
e–mails: `algaba@uhu.es`, `cristoba@uhu.es`, `colume@uhu.es`

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Abstract

We solve, by using normal forms, the analytic integrability problem for differential systems in the plane whose first homogeneous component is a cubic Kolmogorov system being the origin an isolated singularity. As an application, we give the analytically integrable systems of a class of systems $\dot{x} = x(P_2 + P_3),\ \dot{y} = y(Q_2 + Q_3)$, being P_i, Q_i homogeneous polynomials of degree i. We also prove that for any $n \geq 3$, there are analytically integrable perturbations of $\dot{x} = xP_n,\ \dot{y} = yQ_n$ which are not orbital equivalent to its first homogeneous component.

1 Introduction

We deal with analytic perturbations of homogeneous cubic Kolmogorov systems

$$\dot{\mathbf{x}} = \mathbf{F}_3 + \text{h.o.t.}, \quad \mathbf{F}_3(x, y) = (xP_2(x, y), yQ_2(x, y))^T \tag{1.1}$$

with P_2 and Q_2 homogeneous polynomials of degree two (vector field with null linear part) being the origin an isolated singular point of $\dot{\mathbf{x}} = \mathbf{F}_3(\mathbf{x})$.

We focus on the study of the analytic integrability of system (1.1). We notice that the condition of polynomial integrability of $\mathbf{F}_3$, the lowest degree homogeneous term of the vector field, is a necessary condition in order to be $\mathbf{F}$ analytically integrable.

Here, we solve the analytic integrability problem for these systems. More specifically, we prove that, under the condition of polynomial integrability of $\mathbf{F}_3$, the vector field $\mathbf{F}$ is analytically integrable if, and only if, it is orbitally equivalent to its lowest degree component (Theorem 3.16). As consequence, we give an expression of the first integral, if it exists (Theorem 3.17).

We characterize its analytic integrability through the existence of a Lie symmetry (Theorem 3.18) and an inverse integrating factor (Theorem 3.19).

We also prove that the existence of an analytic first integral of the systems

$$\dot{\mathbf{x}} = \mathbf{F}_n + \text{h.o.t.}, \quad \mathbf{F}_n(x, y) = (xP_{n-1}(x, y), yQ_{n-1}(x, y))^T, \tag{1.2}$$

with P_{n-1} and Q_{n-1} homogeneous polynomials of degree $n - 1$ and $n \geq 4$, is not equivalent to the orbital equivalence with its lowest degree component, Theorem 3.20.

Thus, in view of the already known results, systems (1.2) for $n \leq 3$ are analytically integrable if, and only if, they are orbitally equivalent to their lowest degree components, see [12, 13, 14, 15, 17] for $n = 1$, [3] for $n = 2$. In the case $n \geq 4$, this property does not hold. Consequently, for $n \geq 4$, it is not known an expression of its first integral. So, the problem remains open. Theorem 3.21 summarizes these results.

Finally, in Section 4, we calculate the analytically integrable Kolmogorov systems of the family

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(-x^2 + (1-s)xy + (2+s)y^2 + P_3(x,y)) \\ y((2+s)x^2 + (1-s)xy - y^2 + Q_3(x,y)) \end{pmatrix}$$

with P_3, Q_3 polynomials of degree 3 and $s = 1$ or $s = 2$. These systems are cuartic perturbations of a vector field with four invariant lines. In this case, its first integral is of the form $xy(x-y)(x+y)^s +$ h.o.t.. We also calculate the analytically integrable Kolmogorov systems of the family

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(-x^2 - 3y^2 + P_3(x,y)) \\ y(3x^2 + y^2 + Q_3(x,y)) \end{pmatrix}$$

with P_3, Q_3 polynomials of degree 3. In this case, they are perturbations of a vector field with three invariant lines. The leader term of a first integral is $xy(x-y)(x^2+y^2)$. We emphasize that all of them have a polynomial inverse integrating factor.

1.1 Invariant curves and first integrals of vector fields

First we give the definition of invariant curve and its associated cofactor.

We deal with a vector field $\mathbf{F} = (P, Q)^T$ with P, Q analytic at the origin and $P(\mathbf{0}) = Q(\mathbf{0}) = 0$. Throughout the paper, we denote by F the differential operator associated to the vector field $\mathbf{F}$, that is, $F := P\partial_x + Q\partial_y$. We recall the concept of invariant curve and its associated cofactor.

Definition 1.1 *It is said that $C \in \mathbf{C}[[x,y]]$ (algebra of formal power series in x, y over $\mathbf{C}$), with $C(\mathbf{0}) = 0$, is an invariant curve of the vector field $\mathbf{F}$, if there exists $K \in \mathbf{C}[[x,y]]$, named cofactor of C, such that $F(C) = KC$.*
Moreover, if $K \equiv 0$, it is said that $\mathbf{F}$ is formally integrable and C is a first integral of $\mathbf{F}$ and if $K = div(\mathbf{F})$ (divergence of $\mathbf{F}$), it is said that C is an inverse integrating factor of $\mathbf{F}$.

Let note that any formal function C with $C(\mathbf{0}) \neq 0$, satisfies $F(C) = KC$ with $K = F(C)/C \in \mathbf{C}[[x,y]]$.

We denote by $\mathcal{P}_k$ the vector space of homogeneous scalar polynomials of degree k, and by $\mathcal{H}_k$ the vector space of polynomial homogeneous vector fields of degree k. We use Taylor expansion of functions and vector fields without to consider questions of convergence. We note that analytic integrability is equivalent to formal integrability, see Mattei & Moussu [16].

Throughout the paper, we denote by $\mathbf{D} = (x,y)^T \in \mathcal{H}_1$ (dissipative homogeneous vector field) and by $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to the polynomial h).
The following splitting of a homogeneous vector field plays a main role in our study.

Proposition 1.2 *[2, Prop.2.7] Every $\mathbf{F}_n \in \mathcal{H}_n$ can be uniquely written as $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D}$ with $h := \frac{1}{n+1}(\mathbf{D} \wedge \mathbf{F}_n) \in \mathcal{P}_{n+1}$ (product wedge of both vector fields) and $\mu := \frac{1}{n+1} div(\mathbf{F}_n) \in \mathcal{P}_{n-1}$.*

In what follows, h is called conservative part of $\mathbf{F}_n$ and μ dissipative part of $\mathbf{F}_n$.

We provide the first homogeneous component of Taylor expansion of a formal invariant curve of a formal vector field.

Proposition 1.3 *Consider $\mathbf{F} = \mathbf{F}_n + h.o.t.$, $\mathbf{F}_n \in \mathcal{H}_n$ with $\mathbf{F}_n \not\equiv \mathbf{0}$. Let C a formal invariant cuve of $\mathbf{F}$ with cofactor K. Then, $C = C_s + h.o.t.$, $C_s \in \mathcal{P}_s$ and $K = K_n + h.o.t.$, $K_n \in \mathcal{P}_n$, being the polynomial C_s an invariant curve of the polynomial vector field $\mathbf{F}_n$ with cofactor K_n.*

Proof. It is enough to consider the lowest degree homogeneous term of the equality $F(C) - KC = 0$. ■

The following two results show the invariant curves of a homogeneous vector field.

Proposition 1.4 *Every homogeneous polynomial invariant curve of a homogeneous vector field* $\mathbf{F}_n$ *is given by* $g_1^{n_1} g_2^{n_2} \ldots g_m^{n_m}$ *being each* g_j *an irreducible polynomial invariant curve of* $\mathbf{F}_n$.
Moreover, its cofactor is $n_1 K_1 + \cdots + n_m K_m$, *being* K_j *the cofactor of* g_j.

Proof. We suppose that $g = g_1 p$, (g_1 an irreducible homogeneous polynomial), is an invariant curve of $\mathbf{F}_n$ with K_n cofactor of g. It has that $F_n(g_1 p) = g_1 F_n(p) + pF_n(g_1) = K_n g_1 p$, that is, $g_1(pK_n - F_n(p)) = pF_n(g_1)$. From the irreducibility of g_1, it has two situations: either g_1 is an irreducible invariant curve of $\mathbf{F}_n$, in such case, p is also an invariant curve of $\mathbf{F}_n$ and we repeat the process for p. Or $p = qg_1$, i.e. $g = g_1^2 q$. We now have that $F_n(g_1^2 q) = g_1^2 F_n(q) + 2qg_1 F_n(g_1) = K_n g_1^2 q$. Thus, $g_1(qK_r - F_r(q)) = 2qF_r(g_1)$. Reasoning of similar way, it completes the proof.
The second part is obtained easily. ■

Proposition 1.5 *Given* $\mathbf{F}_n \in \mathcal{H}_n$, *any factor of* h *(conservative part of* $\mathbf{F}_n$*) is an invariant curve of* $\mathbf{F}_n$. *Conversely, any homogeneous polynomial invariant curve of* $\mathbf{F}_n$ *is a factor of* h.
Moreover, if I *is a polynomial first integral of* $\mathbf{F}_n$, *then* $I = g_1^{n_1} g_2^{n_2} \cdots g_m^{n_m}$ *where* $g_1, \ldots, g_m$ *are all the irreducible factors of* h *in* $\mathbb{R}[x, y]$ *and* $n_i > 0$.

Proof. We know that $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D}$ with $\mu = \frac{1}{n+1}\mathrm{div}(\mathbf{F}_n)$. Let $f \in \mathcal{P}_s$ a factor of h then $h = fg$ and $F_n(f) = X_{fg}(f) + \mu D(f) = fX_g(f) + s\mu f = (X_g(f) + s\mu)f$ Therefore, f is an invariant curve of $\mathbf{F}_n$.
If $f \in \mathcal{P}_s^{\mathbf{t}}$ is an irreducible invariant curve of $\mathbf{F}_n$ with cofactor K_n then $K_n f = F_n(f) = X_h(f) + \mu D(f) = X_h(f) + s\mu f$. Thus, $X_h(f) = (K_n - s\mu)f$ and f is an invariant curve of $\mathbf{X}_h$. So, f divides to h.
Last on, if I is a first integral of $\mathbf{F}_n$, it is an invariant curve of $\mathbf{F}_n$, that is, from Proposition 1.4, a factorization of I is formed by the irreducible factors of h. On the other hand, any first integral satisfying $I(0, 0) = 0$, is zero on every invariant curve. So, $n_i > 0$. ■

1.2 Necessary condition of analytic integrability

Now we study the integrability problem for a vector field whose first homogeneous component is cubic Kolmogorov type. The following result determines the expression of the lowest degree component in the case of polynomial integrability of this class of vector fields.

Proposition 1.6 (Necessary condition of analytic integrability) *Consider* $\mathbf{F} = \mathbf{F}_3 + h.o.t.$ *where* $\mathbf{F}_3 = (xP_2, yQ_2)^T$ *with* P_2 *and* Q_2 *quadratic homogeneous polynomials. If* $\mathbf{F}$ *is formally integrable, then there exist a linear change of variables and a re-parameterization linear of the time such that* $\mathbf{F}$ *is transformed into one of the following vector fields:*
(a) $\mathbf{F} = \mathbf{F}_{3,a} + h.o.t.$, *being*

$$\mathbf{F}_{3,a} = (-qxy^2, py^3)^T, \tag{1.3}$$

($\mathbf{F}_{3,a}$ *is a reducible polynomial vector field with two invariant lines) with* $gcd(p, q) = 1$ *and* $I_M = x^p y^q$ *is a polynomial first integral of* $\mathbf{F}_{3,a}$ *of degree* $M = p + q$.
(b) $\mathbf{F} = \mathbf{F}_{3,b} + h.o.t.$, *being*

$$\mathbf{F}_{3,b} = (x^2(-qx + (q + r)y), xy((p + r)x - py))^T, \tag{1.4}$$

($\mathbf{F}_{3,b}$ is a reducible polynomial vector field with three invariant lines) with gcd$(p,q,r)=1$ and $I_M=x^py^q(x-y)^r$ is a polynomial first integral of $\mathbf{F}_{3,b}$ of degree $M=p+q+r$.
(c) $\mathbf{F}=\mathbf{F}_{3,c}+h.o.t.$, being $\mathbf{F}_{3,c}$ the vector field

$$(x(-qx^2+[c(q+s)+r+q]xy-c(q+r+s)y^2),y((p+r+s)x^2-[c(p+r)+s+p]xy+pcy^2))^T, \tag{1.5}$$

($\mathbf{F}_{3,c}$ is an irreducible polynomial vector field with four real invariant lines) with gcd$(p,q,r,s)=1, c\in[-1,1), c\neq 0$ and $I_M=x^py^q(x-y)^r(x-cy)^s$ is a polynomial first integral of $\mathbf{F}_{3,c}$ of degree $M=p+q+r+s$.
(d) $\mathbf{F}=\mathbf{F}_{3,d}+h.o.t.$, being $\mathbf{F}_{3,d}$ the vector field

$$(x(-qx^2-2b(q+r)xy-(b^2+1)(q+2r)y^2),y((p+2r)x^2+2b(p+r)xy+p(b^2+1)y^2))^T, \tag{1.6}$$

($\mathbf{F}_{3,d}$ is an irreducible polynomial vector field with two real invariant lines and two complex invariant lines) with gcd$(p,q,r)=1$ and $I_M=x^py^q[(x+by)^2+y^2]^r$ is a polynomial first integral of $\mathbf{F}_{3,d}$ of degree $M=p+q+2r$.

Proof. Let $I=I_M+$ h.o.t. be a formal first integral of $\mathbf{F}$. Equation $F(I)=0$ for degree $M+1$ is $F_3(I_M)=0$, i.e. $\mathbf{F}_3$ is polynomially integrable and I_M is a first integral of $\mathbf{F}_3$. By [2, Prop.2.7], $\mathbf{F}_3=\mathbf{X}_h+\mu\mathbf{D}$ with $h=xyp_2$ and $\mu=q_2$ being p_2 and q_2 quadratic homogeneous polynomials.
We write $q_2:=Ax^2+Bxy+Cy^2$ and distinguish the following cases by according the factors of p_2:
- Assume $p_2=((ax+by)+dy)((ax+by)-dy)$.

For $a=0$, if $b=\pm d$, it has that $\mathbf{F}_3=\mu\mathbf{D}$ and it is not formally integrable; otherwise $b\neq\pm d$, it has that the conservative part is $h=xy^3$. From Proposition 1.5, if there exists a first integral of $\mathbf{F}_3$, it has the expression $I_M=x^py^q$ with gcd$(p,q)=1$. By imposing $F_3(I_M)=0$, it arrives to $A=0, B=0$ and $C=\frac{3p-q}{p+q}$. By performing the reparameterization of the time $4t=(p+q)\tau$, $\mathbf{F}_3$ turns on the vector field (1.3).
Assume now $a\neq 0$. For $b-d\neq 0$, by performing the scaled $x=\frac{(d-b)}{a}x$, we get $p_2=(x-y)(x-(b+d)y)$.
We distinguish three cases:
Suppose $b+d=0$, i.e., $h=x^2y(x-y)$. Now, by Proposition 1.5, the expression of a first integral of $\mathbf{F}_3$ is $I_M=x^py^q(x-y)^r$ with gcd$(p,q,r)=1$. Of $F_3(I_M)=0$, it has that $A=\frac{a}{M}(p+r-3q), B=\frac{2b}{M}(-q-r+p)$ and $C=0$ with $M=p+q+r$. By reparameterizing the time $4t=M\tau$, $\mathbf{F}_3$ turns on vector field (1.4).
Suppose $b+d=1$, i.e., $h=xy(x-y)^2$. In such a case, the change x by $x-y$ and y by $-y$, it transforms h onto $x^2y(x-y)$.
Suppose $b+d\neq 1$ and $b+d\neq 0$. By performing the scaled $x=\frac{(d-b)}{a}x$, we can assume $p_2=(x-y)(x-cy)$, with $c\neq 0$ and $c\neq 1$. Moreover, if $|c|>1$, we change y by cy so $|c|<1$. From Proposition 1.5, if there would exist a first integral of $\mathbf{F}_3$, it would have the expression $I_M=x^py^q(x-y)^r(x-cy)^s$ with gcd$(p,q,r,s)=1$. By imposing $F_3(I_M)=0$, it arrives to

$$A=\frac{1}{M}(r+s-3q+p), B=\frac{2}{M}(qc+r-pc-rc-p-s+q+sc), C=\frac{c}{M}(3p-q-r-s),$$

being $M=p+q+r+s$. By performing the reparameterization of the time $4t=M\tau$, $\mathbf{F}_3$ turns on (1.5) being $I_M=x^py^q(x-y)^r(x-cy)^s$ a first integral.
Assume now $a\neq 0$ and $b-d=0$. If $b+d=0$, it has that $h=x^3y$. So by changing x by y it arrives to (1.3). And if $b+d\neq 0$, by changing d by $-d$ it is the case $b-d\neq 0$ and $b+d=0$.
- Assume $p_2=(ax+by)^2+d^2y^2$ with $ad\neq 0$. By means of a linear change, we can assume $a=d=1$. Now, by Proposition 1.5, the expression of a first integral of $\mathbf{F}_3$ would be $I_M=x^py^q[(x+by)^2+y^2]^r$ with gcd$(p,q,r)=1$. By imposing $\mathbf{F}_3(I_M)=0$,

it has that

$$A = \frac{1}{M}(-3q + p + 2r), B = \frac{4b}{M}(p - q), C = \frac{(b^2 + 1)}{M}(3p - q - 2r),$$

with $M = p + q + 2r$. By reparameterizing the time $4t = M\tau$, $\mathbf{F}_3$ turns on vector field (1.6). ■

2 Normal form for perturbations of homogeneous cubic Kolmogorov systems

We do not consider questions of convergence in the normal forms because the formal integrability is equivalent to the analytical integrability for the vector fields analyzed, see [16].

Orbital normal forms of linear and quadratic perturbations have been provided by several authors, see [17, 20, 5, 14, 18, 6, 7, 8, 9, 19]. For homogeneous cubic systems, as far as we know, Basov [11] analyzes the linear equivalence. In [10] an orbital normal form of the perturbations of $(y^3, -x^3)$ is given.

An orbital normal form of vector fields, whose leader homogeneous term is fixed, is described in Section 5. As a particular case, we provide a suitable expression of the orbital normal form for a class of vector fields $\mathbf{F} = \mathbf{F}_3$+h.o.t. with $\mathbf{F}_3 = \mathbf{X}_h + \mu\mathbf{D} \in \mathcal{H}_3$.

Theorem 2.7 *Let* $\mathbf{F} = \mathbf{F}_3 + h.o.t.$, $\mathbf{F}_3 \in \mathcal{H}_3$. *If* $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{k+4}\right) = \{0\}$ *for all* $k \in \mathbb{N}$ *then* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G} = \mathbf{F}_3 + \sum_{j>3} \mathbf{G}_j, \ \text{with } \mathbf{G}_j = \mathbf{X}_{g_{j+1}} + \eta_{j-1}\mathbf{D} \in \mathcal{H}_j,$$

where $g_{j+1} \in \mathrm{Cor}\left(\ell^{\mathrm{c}}_{j+1}\right)$ *and* $\eta_{j-1} \in \mathrm{Cor}\left(\ell_{j-1}\right)$. *(where Cor(·) is a complementary subspace to Range(·)), being* ℓ_k *(Lie operator of* $\mathbf{F}_3$*) and* ℓ^{c}_k *(Lie operator of* $\mathbf{F}_3$ *moved) defined in (5.15) and (5.16) with* $n = 3$, *respectively, and* Δ_k *for all* $k > 2$, *the subspaces such that* $\mathcal{P}_k = \Delta_k \bigoplus h\mathcal{P}_{k-3}$, *(such subspaces must be considered as fixed).*

Polynomially integrable cubic Kolmogorov vector fields, whose origin is an isolated singularity, are $\mathbf{F}_{3,c}$ and $\mathbf{F}_{3,d}$, modulo a linear change of variables, see Proposition 1.6. Therefore, from Proposition 1.3, the analytically integrable perturbations of cubic Kolmogorov systems whose origin is an isolated singular point can only be of the form $\mathbf{F} = \mathbf{F}_{3,c}$ + h.o.t. or $\mathbf{F} = \mathbf{F}_{3,d}$ + h.o.t., modulo a linear change of variables.

Next result is referred to vector fields whose first homogeneous component is $\mathbf{F}_3 = \mathbf{F}_{3,c}$.

Lemma 2.8 *Consider* $\mathbf{F}_{3,c}$ *with* $gcd(p, q, r, s) = 1, c \in [-1, 1), c \neq 0$. *It has that for all* $k \in \mathbb{N}$, $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{k+4}\right) = \{0\}$. *Moreover,* $\mathrm{Cor}\left(\ell^{\mathrm{c}}_{k+4}\right) = \{0\}$.

Proof. Vector field $\mathbf{F}_{3,c} = \mathbf{X}_h + \mu\mathbf{D}$ with $h = \frac{M}{4}xy(x-y)(x-cy)$, $M = p+q+r+s$ and $\mu = \frac{1}{4}((M-4q)x^2 + 2((q+s-p-r)c+q+r-p-s)xy + (4p-M)cy^2)$. We choose the bases $\Delta_{k+2} = \langle x^{k+2}, x^{k+1}y, x^k y^2, y^{k+2}\rangle$ and $\Delta_{k+4} = \langle x^{k+4}, x^{k+3}y, x^{k+2}y^2, y^{k+4}\rangle$. We have that

$$\begin{aligned}
\ell^{\mathrm{c}}_{k+4}(x^{k+2}) &= A_1 x^{k+4} + B_1 x^{k+3}y + C_1 x^{k+2}y^2, \\
\ell^{\mathrm{c}}_{k+4}(x^{k+1}y) &= A_2 x^{k+3}y + B_2 x^{k+2}y^2, \\
\ell^{\mathrm{c}}_{k+4}(x^k y^2) &= A_3 x^{k+3}y + B_3 x^{k+2}y^2, \\
\ell^{\mathrm{c}}_{k+4}(y^{k+2}) &= A_4 x^{k+3}y + B_4 x^{k+2}y^2 + C_4 y^{k+4},
\end{aligned}$$

with

$$\begin{array}{rcl}
A_1 &=& -\frac{2+k}{4+k}(M+qk),\\
A_2 &=& \frac{2+k}{4+k}(2M+(r+s)k),\\
B_2 &=& -\frac{2+k}{4+k}((c+1)M+(cr+s)k),\\
A_3 &=& -\frac{1}{c}B_2,\\
B_3 &=& -\frac{2+k}{4+k}((c^2+1)M+k(c^2r+s)),\\
C_4 &=& \frac{2+k}{4+k}c(M+pk).
\end{array}$$

In this way, the determinant of the matrix of the operator ℓ^{c}_{k+4} is

$$\frac{(k+2)^4}{(k+4)^4}(M+pk)(M+qk)(M+rk)(M+sk),$$

which is different from zero. Therefore, both $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{k+4}\right)$ and $\mathrm{Cor}\left(\ell^{\mathrm{c}}_{k+4}\right)$ are trivial subspaces. ■

For perturbations of $\mathbf{F}_3 = \mathbf{F}_{3,d}$ we have a similar result which enunciate without the proof.

Lemma 2.9 *Consider* $\mathbf{F}_{3,d}$ *with* $gcd(p,q,r)=1$. *It has that for all* $k\in\mathbb{N}$, $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{k+4}\right)=\{0\}$. *Moreover,* $\mathrm{Cor}\left(\ell^{\mathrm{c}}_{k+4}\right)=\{0\}$.

For computing $\mathrm{Cor}\left(\ell_k\right)$ with $k>n$, we need the following two technical lemmas.

Lemma 2.10 *Consider* $\mathbf{F}_n\in\mathcal{H}_n$ *irreducible and* $f\in\mathbb{C}[x,y]$ *an irreducible invariant curve of* $\mathbf{F}_n$. *If* $F_n(p_k)\in\langle f\rangle$ *with* $p_k\in\mathcal{P}_k$, *then* $p_k\in\langle f\rangle$.

Proof. If $F_n(p_k)=0$ then p_k is a first integral of $\dot{\mathbf{x}}=\mathbf{F}_n$. A first integral of $\mathbf{F}_n$ vanishes on any invariant curve of it, i.e., $p_k(\mathbf{x})=0$ when $f(\mathbf{x})=0$. Therefore, by Hilbert‘s Nullstellensatz $p_k\in rad\langle f\rangle$. Since $\langle f\rangle$ is a prime ideal, then $\langle f\rangle=rad\langle f\rangle$, in consequence $p\in\langle f\rangle$.

If $F_n(p_k)\neq 0$, let $\nu\in\mathbb{C}[x,y]\setminus\{0\}$ such that $f\nu=F_n(p_k)$. Consider $\gamma(t)$, real or complex, a solution curve of $\dot{\mathbf{x}}=\mathbf{F}_n(\mathbf{x})$ which is a parametrization of $f(\mathbf{x})=0$. We assume that $\lim_{t\to-\infty}\gamma(t)=\mathbf{0}$, (the other case $\lim_{t\to+\infty}\gamma(t)=\mathbf{0}$ is proved in a similar way). Taking into account that $p_k(\mathbf{0})=0$ then

$$\begin{array}{rcl}
p_k(\gamma(t)) &=& p_k(\gamma(t))-p_k(\mathbf{0})=\displaystyle\int_{-\infty}^{t}\frac{dp_k(\gamma(s))ds}{ds}=\int_{-\infty}^{t}\nabla_{\mathbf{x}}p_k\cdot\mathbf{F}_n(\gamma(s))ds\\
&=& \displaystyle\int_{-\infty}^{t}F_n(p_k)(\gamma(s))ds=\int_{-\infty}^{t}f(\gamma(s))\nu(\gamma(s))ds=0.
\end{array}$$

Recalling that $f(\mathbf{x})=0$ is the union of orbits, we have that $p_k(\mathbf{x})=0$ when $f(\mathbf{x})=0$. Therefore, by Hilbert‘s Nullstellensatz $p_k\in rad\langle f\rangle$. Since $\langle f\rangle$ is a prime ideal, then $\langle f\rangle=rad\langle f\rangle$, in consequence $p_k\in\langle f\rangle$. ■

Remark. The hypothesis of the irreducibility of the invariant curve is fundamental. For instance, if we consider $\mathbf{F}_2:=(-2x^2,-3x^2-2xy+3y^2)^T\in\mathcal{H}_2$ irreducible and the invariant curve $(y-x)^2$, for $p_3=x^2(y-x)$ we have that $F_2(p_3)=3x^2(y-x)^2\in\langle(y-x)^2\rangle$ and nevertheless $p_3\notin\langle(y-x)^2\rangle$.

For vector fields whose first homogeneous component is integrable with four invariant real lines, we provide the following result.

Lemma 2.11 *Consider* $\mathbf{F}_{3,c}$ *with* p,q,r,s *natural numbers. Let* k *and* m *natural numbers with* $p+q+r+s\neq p\frac{k}{j}$, $p+q+r+s\neq q\frac{k}{j}$, $p+q+r+s\neq r\frac{k}{j}$, $p+q+r+s\neq s\frac{k}{j}$, *for each* $j=1,\ldots,m-1$. *If* $p_k\in\mathcal{P}_k$ *verifies* $F_{3,c}(p_k)\in\langle f_i^m\rangle$, *being* $f_1=x$, $f_2=y$, $f_3=x-y$, $f_4=x-cy$, *invariant irreducible curves of* $\mathbf{F}_{3,c}$, *then* $p_k\in\langle f_i^m\rangle$, $i=1,2,3,4$.

Proof. We prove the case $i = 1,\ (f_1 = x)$, the cases $i = 2, 3, 4$ are analogous. Lemma 2.10 proves the statement for $m = 1$.

We first consider the case $m = 2$. We denote by $K_1 = -qx^2 + (c(q+s)+r+q)xy - c(q+r+s)y^2$ the cofactor of x. If $F_{3,c}(p_k) \in \langle x^2 \rangle$ then $F_{3,c}(p_k) \in \langle x \rangle$ and by Lemma 2.10 we have that there exists $p_{k-1} \in \mathcal{P}_{k-1}$ such that $p_k = xp_{k-1}$, therefore

$$\begin{aligned} F_{3,c}(p_k) &= F_{3,c}(xp_{k-1}) = p_{k-1}F_{3,c}(x) + xF_{3,c}(p_{k-1}) = p_{k-1}K_1x + xF_{3,c}(p_{k-1}) \\ &= x\left(\tfrac{K_1}{k-1}D(p_{k-1}) + F_{3,c}(p_{k-1})\right) = x(F_{3,c} + \tfrac{K_1}{k-1}D)(p_{k-1}) \in \langle x^2 \rangle. \end{aligned}$$

Hence $(F_{3,c} + \frac{K_1}{k-1}D)(p_{k-1}) \in \langle x \rangle$. Vector field $\mathbf{F}_{3,c} + \frac{K_1}{k-1}\mathbf{D}$ is irreducible if, and only if, $p+q+r+s \neq pk$. Applying Lemma 2.10 we have that $p_{k-1} \in \langle x \rangle$ and consequently $p_k \in \langle x^2 \rangle$.

Consider now the case $m = 3$. If $F_{3,c}(p_k) \in \langle x^3 \rangle$ then $F_{3,c}(p_k) \in \langle x^2 \rangle$ and by the previous paragraph we have that there exists $p_{k-2} \in \mathcal{P}_{k-2}$ such that $p_k = x^2p_{k-2}$, therefore

$$\begin{aligned} F_{3,c}(p_k) &= F_{3,c}(x^2p_{k-2}) = p_{k-2}F_{3,c}(x^2) + x^2F_{3,c}(p_{k-2}) = 2p_{k-2}K_1x^2 + x^2F_{3,c}(p_{k-2}) \\ &= x^2\left(\tfrac{2K_1}{k-2}D(p_{k-2}) + F_{3,c}(p_{k-2})\right) = x^2(F_{3,c} + \tfrac{2K_1}{k-2}D)(p_{k-2}) \in \langle x^3 \rangle. \end{aligned}$$

Hence $(F_{3,c} + \frac{2K_1}{k-2}D)(p_{k-2}) \in \langle x \rangle$ and as $\mathbf{F}_{3,c} + \frac{2K_1}{k-2}\mathbf{D}$ is irreducible if, and only if, $p+q+r+s \neq p\frac{k}{2}$, applying Lemma 2.10 we have that $p_{k-2} \in \langle x \rangle$ and consequently $p_k \in \langle x^3 \rangle$. Reasoning by induction we get the result for $m \in \mathbb{N}$.
In a similar way, it is easy to prove that for $f_2 = y,\ f_3 = x - y$ and $f_4 = x - cy$, the conditions are $p+q+r+s \neq q\frac{k}{j},\ p+q+r+s \neq r\frac{k}{j}$ and $p+q+r+s \neq s\frac{k}{j},\ j = 1, \ldots, m-1$, respectively. ■

Analogously, we have the following result for $\mathbf{F}_3 = \mathbf{F}_{3,d}$. We have also omitted its proof.

Lemma 2.12 *Consider* $\mathbf{F}_{3,d}$ *with* p, q, r *natural numbers. Let* k *and* m *natural numbers with* $p+q+2r \neq p\frac{k}{j},\ p+q+2r \neq q\frac{k}{j},\ p+q+r \neq r\frac{k}{j}, j = 1, \ldots, m-1$. *If* $p_k \in \mathcal{P}_k$ *such that* $F_{3,d}(p_k) \in \langle f_i^m \rangle$, *being* $f_1 = x,\ f_2 = y,\ f_3 = (x+by)^2 + y^2$, *invariant curves of* $\mathbf{F}_{3,d}$, *then* $p_k \in \langle f_i^m \rangle,\ i = 1, 2, 3$.

Next statement establishes a cyclicity relation between the co-ranges of the operators ℓ_k.

Lemma 2.13 *Consider* $\mathbf{F}_{3,c}$ *with* p, q, r, s *natural numbers and* $M = p+q+r+s$. *For* $k \geq 2$, *it is always possible to choose* $\mathrm{Cor}(\ell_{k+M})$, *a complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M\mathrm{Cor}(\ell_k)$ *being* $I_M = x^py^q(x-y)^r(x-cy)^s$.

Proof. We first see that both subspaces have the same dimension. Indeed, by Lemma 2.11, $Ker(\ell_k) = \langle I_M^l \rangle$ if $k-2 = lM$. Otherwise, $Ker(\ell_k) = \{0\}$. Thus, $\dim(\mathrm{Cor}(\ell_k)) = 2$ if $k = lM$ and $\dim(\mathrm{Cor}(\ell_k)) = 1$, otherwise; i.e., $\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{k+M}))$.

For completing the proof it is enough to prove that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M\mathrm{Cor}(\ell_k) \cap \mathrm{Range}(\ell_{k+M}) = \{0\}$ by *reductio ad absurdum.* Let $p_k \in \mathrm{Cor}\,(\ell_k) \setminus \{0\}$ such that $p_kI_M \in \mathrm{Range}\,(\ell_{k+M})$, then there exists $p_{k+M-2} \in \mathcal{P}^{\mathbf{t}}_{k+M-2} \setminus \{0\}$ such that $\ell_{k+M}(p_{k+M-2}) = p_kI_M$, that is, $\ell_{k+M}(p_{k+M-2})$ is multiple of I_M. As $\frac{p(k+M-2)}{j} > \frac{pM}{j} > M,\ j = 1, \ldots, p-1;\ \frac{q(k+M-2)}{j} > M,\ j = 1, \ldots, q-1;\ \frac{r(k+M-2)}{j} > M,\ j = 1, \ldots, r-1;\ \frac{s(k+M-2)}{j} > M,\ j = 1, \ldots, s-1$, by applying Lemma 2.11 we have that $p_{k+M-2} = p_{k-2}I_M$ with $p_{k-2} \in \mathcal{P}^{\mathbf{t}}_{k-2} \setminus \{0\}$ and consequently

$$p_kI_M = F_{3,c}(p_{k+M-2}) = F_{3,c}(p_{k-2}I_M) = I_MF_{3,c}(p_{k-2}).$$

Hence $p_k = F_{3,c}(p_{k-2})$, that is, $p_k \in \mathrm{Range}\,(\ell_k) \cap \mathrm{Cor}\,(\ell_k)$ which gives a contradiction. ■

For vector fields $\mathbf{F}_{3,d}$ + h.o.t., it has the following cyclicity.

Lemma 2.14 *Consider* $\mathbf{F}_{3,d}$ *with* p, q, r *natural numbers and* $M = p + q + 2r$. *For* $k \geq 2$, *it is always possible to choose* $\mathrm{Cor}(\ell_{k+M})$, *a complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$ *being* $I_M = x^p y^q ((x + by)^2 + y^2)^r$.

Next result provides an orbital normal form of vector fields whose first homogeneous component is integrable and cubic Kolmogorov type and the origin is an isolated equilibrium. This normal form depends on an integral of the first homogeneous component of the vector field.

Theorem 2.15 *Vector field* $\mathbf{F} = \mathbf{F}_3 +$ *h.o.t. with* $\mathbf{F}_3 = \mathbf{F}_{3,c}$ *(or* $\mathbf{F}_{3,d}$*, resp.) is orbitally equivalent to*

$$\dot{\mathbf{x}} = \mathbf{F}_3 + \sum_{j=3}^{M+1} \eta_j^{(0)} \mathbf{D} + \sum_{i=1}^{\infty} \sum_{j=3}^{M+1} \eta_j^{(i)} I_M^i \mathbf{D},$$

with $\eta_j^{(i)} \in \mathrm{Cor}(\ell_j)$ *and* $I_M = x^p y^q (x - y)^r (x - cy)^s$, $M = p + q + r + s$, *(*$I_M = x^p y^q ((x + by)^2 + y^2)^r$, $M = p + q + 2r$, *resp.).*

Proof. Applying Theorem 5.32 and Lemma 2.8 for $\mathbf{F}_3 = \mathbf{F}_{3,c}$ (or Lemma 2.9 for $\mathbf{F}_3 = \mathbf{F}_{3,d}$), we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{F}_3 + \sum_{j\geq 3} \eta_j \mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$. In order to finish the proof it is sufficient to apply Lemma 2.13 for $\mathbf{F}_3 = \mathbf{F}_{3,c}$ (or Lemma 2.14 for $\mathbf{F}_3 = \mathbf{F}_{3,d}$) for the components of the normal form of degree greater than $M + 1$. ■

3 Main results

Our purpose is to characterize the analytically integrable vector fields that are perturbations of cubic Kolmogorov systems whose origin is an isolated singular point. For that, we will assume that the lowest degree component of the vector field satisfies the necessary condition of polynomially integrability given in Proposition 1.6, i.e. we deal with the vector field $\mathbf{F} = \mathbf{F}_{3,c}$ + h.o.t. and $\mathbf{F} = \mathbf{F}_{3,d}$ + h.o.t.. Next, we give the main result of our study. It solves the analytic integrability problem for the vector fields addressed. It also gives the expression of a first integral.

Theorem 3.16 *Consider* $\mathbf{F} = \mathbf{F}_{3,c}$ + *h.o.t. (*$\mathbf{F} = \mathbf{F}_{3,d}$ + *h.o.t., resp.). The vector field* $\mathbf{F}$ *is analytically integrable if, and only if, it is orbitally equivalent to* $\mathbf{F}_{3,c}$ *(*$\mathbf{F}_{3,d}$*, resp.).*
Moreover, in such a case, $\mathbf{F}$ *has an analytic first integral of the form* $I = I_M$ + *h.o.t. being* $I_M = x^p y^q (x - y)^r (x - cy)^s$ *a primitive first integral of* $\mathbf{F}_{3,c}$ *(* $I_M = x^p y^q ((x + by)^2 + y^2)^r$ *a primitive first integral of* $\mathbf{F}_{3,d}$*, resp.).*

Proof. We do the proof for the case $\mathbf{F}_3 = \mathbf{F}_{3,c}$ only. The other case is similar.

We see the sufficiency. The polynomial $I_M = x^p y^q (y - x)^r (x - cy)^s$ is a first integral of $\mathbf{F}_{3,c}$ which it is transformed into a formal first integral $I = I_M$ + h.o.t. of $\mathbf{F}$ and from [Theorem A,[16]] $\mathbf{F}$ is analytically integrable.
We see the necessity of the condition. Applying Theorem 5.32 and Lemma 2.8, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_{3,c} + \sum_{j\geq 3} \eta_j \mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$. Let note that $\mathbf{F}$ has an analytic first integral is equivalent to $\mathbf{G}$ has a formal first integral. Assume that $\mathbf{G}$ is formally integrable and not all the η_j are zero. Let N

defined by $N = \min\{j > 1 : \eta_j \not\equiv 0\}$. A formal first integral of $\mathbf{G}$ is of the form $I = I_M^l + \sum_{j>Ml} I_j$ with $I_j \in \mathcal{P}_j$. Imposing the integrability condition we have

$$\begin{aligned} 0 &= (G(I))_{N+Ml} = (\eta_N D)(I_M^l) + F_{3,c}(I_{Ml+N-2}) \\ &= Ml\eta_N I_M^l + \ell_{Ml+N}\left(I_{Ml+N-2}\right). \end{aligned}$$

But this equation is incompatible since by Lemma 2.13 $Ml\eta_N I_M^l \in \mathrm{Cor}\,(\ell_{Ml+N})$ and $\ell_{Ml+N}\left(I_{Ml+N-2}\right) = -Ml\eta_N I_M^l \in \mathrm{Range}\,(\ell_{Ml+N})$ which is a contradiction. Consequently, $\mathbf{G} = \mathbf{F}_{3,c}$, i.e. $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_{3,c}$.

We now see the second part. First integrals of $\mathbf{F}_{3,c}$ are $\Psi(I_M)$ for any formal function Ψ. So, first integrals of $\mathbf{F}$ are $\Psi(I_M + \text{h.o.t.})$ since $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_{3,c}$. Thus, $I_M + \text{h.o.t.}$ is also a first integral of $\mathbf{F}$. ■

A consequence of Theorem 3.16 is the following result.

Theorem 3.17 *Consider* $\mathbf{F} = \mathbf{F}_{3,c} + h.o.t.$ *(*$\mathbf{F} = \mathbf{F}_{3,d} + h.o.t.$*, resp.). The vector field* $\mathbf{F}$ *is analytically integrable if, and only if,* $\mathbf{F}$ *has a first integral of the form* $I = (x + h.o.t.)^p(y + h.o.t.)^q(x - y + h.o.t.)^r(x - cy + h.o.t.)^s$ *(*$I = (x + h.o.t.)^p(y + h.o.t.)^q((x + by)^2 + y^2 + h.o.t.)^r$*, resp.).*

Proof. We prove the necessary condition. Consider $\mathbf{F} = \mathbf{F}_{3,c} + \text{h.o.t.}$. From Theorem 3.16 there exist a change of variables and a reparemeterization of the time such that $\mathbf{F}_{3,c}$ is transformed into $\mathbf{F}$. These changes transform the first integral $I_M = x^p y^q (y - x)^r (x - cy)^s$ of $\mathbf{F}_{3,c}$ into a first integral of $\mathbf{F}$, it which has the expression $I = (x + \text{h.o.t.})^p(y + \text{h.o.t.})^q(x - y + \text{h.o.t.})^r(x - cy + \text{h.o.t.})^s$.
For $\mathbf{F} = \mathbf{F}_{3,d} + \text{h.o.t.}$ the proof is analogous.

Reciprocally, if $\mathbf{F}$ has a formal first integral, from [16] $\mathbf{F}$ has an analytic first integral. ■

The following theorem characterizes the analytic integrability of a vector field whose first homogeneous component is cubic Kolmogorov type through the existence of a Lie symmetry.

Theorem 3.18 *Consider* $\mathbf{F} = \mathbf{F}_3 + h.o.t.$ *with* $\mathbf{F}_3 = \mathbf{F}_{3,c}$ *or* $\mathbf{F}_3 = \mathbf{F}_{3,d}$*. Then* $\mathbf{F}$ *is analytically integrable if, and only if, there exist a formal vector field* $\mathbf{G} = \sum_{j\geq 1} \mathbf{G}_j$*,* $\mathbf{G}_j \in \mathcal{H}_j$*,* $\mathbf{G}_1 = (x, y)^T$ *and a formal scalar function* ν*,* $\nu(\mathbf{0}) = 2$ *such that* $[\mathbf{F}, \mathbf{G}] = \nu\mathbf{F}$*, i.e.* $\mathbf{F}$ *has a Lie symmetry.*

The proof of Theorem 3.18 follows from Theorem 3.16 and applying [4, Theorem 1.3].

We solve the analytic integrability problem through the existence of a formal inverse integrating factor.

Theorem 3.19 *Consider* $\mathbf{F} = \mathbf{F}_{3,c} + h.o.t.$ *(*$\mathbf{F} = \mathbf{F}_{3,d} + h.o.t.$*, resp.). Then* $\mathbf{F}$ *is analytically integrable if, and only if, it has a formal inverse integrating factor of the form* $V = xy(x - y)(x - cy) + h.o.t.$ *(* $V = xy((x + by)^2 + y^2) + h.o.t.$*, resp.).*

Proof. We do the proof for the case $\mathbf{F}_3 = \mathbf{F}_{3,c}$ only. Other case is analogous.

We prove that the condition is necessary. We assume that $\mathbf{F}$ is analytically integrable. From Theorem 3.16, it is orbitally equivalent to $\mathbf{F}_{3,c} = \mathbf{X}_h + \mu\mathbf{D}$ being $h = \frac{M}{4}xy(x - y)(x - cy)$ with $M = p + q + r + s$ and $\mu = \frac{1}{4}((M - 4q)x^2 + 2((q + s - p - r)c + q + r - p - s)xy + (4p - M)cy^2)$, which has the inverse integrating factor h. Undoing the change, it has that $\mathbf{F}$ has a formal inverse integrating $V = h + \text{h.o.t.}$.

Now we will see the sufficiency of the condition. Let $V = h + \text{h.o.t.}$ a formal inverse integrating factor of $\mathbf{F}$ with h a conservative part of $\mathbf{F}_{3,c}$. Since Theorem 5.32 and Lemma 2.8, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_{3,c} + \sum_{j\geq 3} \eta_j \mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$. Therefore, $\mathbf{F}$ has a formal inverse integrating factor if, and only if, $\mathbf{G}$ has it too. Moreover, the formal inverse integrating factor W of $\mathbf{G}$ is also of the form $W = h + \text{h.o.t.}$. On the other hand, the unique invariant curves of $\mathbf{G}$ are $x, y, x-y, x-cy$. So, we get $W = hu$ being u formal and $u(\mathbf{0}) = 1$. Equation $G(W) - W\text{div}(\mathbf{G}) = 0$ is

$$0 = uG(h) + hG(u) - hu\text{div}(\mathbf{G}).$$

As $G(h) = 4h\mu + \sum_{j>3} 4h\eta_j$ and $\text{div}(\mathbf{G}) = 4\mu + \sum_{j>3}(j+2)\eta_j$, it has that

$$0 = h(G(u) - u\sum_{j>3}(j-2)\eta_j).$$

Expanding $u = 1 + \sum_{i\geq 1} u_i$, the above equation to degree $i+2$ becomes

$$0 = F_{3,c}(u_i) - i\eta_{i+2} + \sum_{k=1}^{i}(2k-i)\eta_{i-k+2}u_k. \tag{3.7}$$

We see that $\eta_j = 0$ for all j. Indeed, otherwise, let $j_0 = \min\{j \in \mathbb{N} : \eta_{j+2} \not\equiv 0\}$. Expression (3.7) to degree $j_0 + 2$ is

$$F_{3,c}(u_{j_0}) = j_0\eta_{j_0+2} - \sum_{k=1}^{j_0}(2k-j_0)\eta_{j_0-k+2}u_k.$$

As $\eta_{j_0-k+2} = 0$ for $1 \leq k \leq j_0 - 1$, we get $F_{3,c}(u_{j_0}) = j_0\eta_{j_0+2}$, i.e. $\eta_{j_0+2} \in \text{Cor}\,(\ell_{j_0+2})$ and $\eta_{j_0+2} \in \text{Range}\,(\ell_{j_0+2})$. We conclude that $\eta_{j_0+2} = 0$. ■

3.1 Integrable Kolmogorov systems not orbital equivalent to a homogeneous systems

Consider the vector fields $\mathbf{F} = \mathbf{F}_n + \text{h.o.t.}$ with $\mathbf{F}_n = (xP_{n-1}, yQ_{n-1})^T$ polynomially integrable and whose origin is an isolated singular point, being P_{n-1}, Q_{n-1} homogeneous polynomials of degree $n-1 \geq 0$. For $n = 1, 2$ or 3 it is proved that the vector fields $\mathbf{F}$ are analytically integrable if and only if are orbital equivalent to $\mathbf{F}_n$, i.e. they can be transformed by means of a change of variables and a reparameterization of the time into a homogeneous system, see [15, 17] for $n = 1$, [3] for $n = 2$ and Theorem 3.16 for $n = 3$.

Next, we show that for $n \geq 4$, this property is not satisfied and therefore the integrability problem through normal form is a harder problem to solve than for $n \leq 3$.

Theorem 3.20 *Vector fields*

$$(x(2ny^{2n-1} - x^{2n-1} - (n+1)x^n y^n), y(2nx^{2n-1} - y^{2n-1} + (n+1)x^n y^n))^T \tag{3.8}$$

and

$$(x((2n+1)y^{2n} - x^{2n} - (n+1)x^{n+1}y^n), y((2n+1)x^{2n} - y^{2n} + (n+2)x^{n+1}y^n))^T \tag{3.9}$$

with $n \geq 2$, have an analytic first integral but they are not orbital equivalent to their lowest degree homogeneous term.

Proof. Both vector fields (3.8) and (3.9) are Hamiltonian whose Hamiltonian functions are $x^{2n}y - xy^{2n} + x^{n+1}y^{n+1}$ and $x^{2n+1}y - xy^{2n+1} + x^{n+2}y^{n+1}$, respectively. Thus, they are polynomially integrable.
Vector field (3.8) is $\mathbf{F}_{2n} + \mathbf{F}_{2n+1}$ with $\mathbf{F}_{2n} = x(2ny^{2n-1} - x^{2n-1}), y(2nx^{2n-1} - y^{2n-1}))^T$ and $\mathbf{F}_{2n+1} = (-x((n+1)x^n y^n), y((n+1)x^n y^n))^T$. We are going to prove that (3.8)

is not orbitally equivalent to $\mathbf{F}_{2n}$. Indeed, from [4, Theorem 1.3], if (3.8) would be orbitally equivalent to $\mathbf{F}_{2n}$, there would be a formal vector field $\mathbf{G} = (x + u_{20}x^2 + u_{11}xy + u_{02}y^2 + \text{h.o.t.}, y + v_{20}x^2 + v_{11}xy + v_{02}y^2 + \text{h.o.t.})^T$ and a formal scalar function $\nu = (2n-1) + d_{10}x + d_{01}y + \text{h.o.t.}$ such that $[\mathbf{F}_{2n} + \mathbf{F}_{2n+1}, \mathbf{G}] = \nu(\mathbf{F}_{2n} + \mathbf{F}_{2n+1})$, i.e.,

$$(D\mathbf{F}_{2n+1})\mathbf{G} - (D\mathbf{G})\mathbf{F}_{2n+1} - \nu\mathbf{F}_{2n+1} + (D\mathbf{F}_{2n})\mathbf{G} - (D\mathbf{G})\mathbf{F}_{2n} - \nu\mathbf{F}_{2n} = 0. \quad (3.10)$$

For $n = 2$, the coefficient of y^5 of the first component of (3.10) is $6u_{02}$ and the coefficients of x^5 and x^2y^3 of the second component are $6v_{20}$ and $3 - 12v_{20} + 12u_{02}$, respectively. So, equation (3.10) is not satisfied and (3.8) for $n = 2$ is not orbitally equivalent to $\mathbf{F}_4$.

For $n \geq 3$, the coefficient of $x^{n+1}y^n$ of the first component of (3.10) is $-(n+1)$. So, equation (3.10) is not satisfied and (3.8) for $n \geq 3$ is not orbitally equivalent to $\mathbf{F}_{2n}$. Analogously, it is easy to prove that (3.9) for $n \geq 2$ is not orbitally equivalent to its lowest degree term. ■

At its current rate, the link between orbital equivalence of a system (1.2) and its lowest degree component and its analytic integrability is shown in the following result.

Theorem 3.21 *Consider* $\mathbf{F} = \mathbf{F}_n + h.o.t.$ *with* $\mathbf{F}_n = (xP_{n-1}, yQ_{n-1})^T$ *with* P_{n-1} *and* Q_{n-1} *homogeneous polynomials of degree* $n-1$ *and assume that the origin of* $\dot{\mathbf{x}} = \mathbf{F}_n(\mathbf{x})$ *is an isolated singular point. It has that:*

(1) For $n = 1$, $\mathbf{F}$ *is analytically integrable if, and only if, it is orbitally equivalent to* $\mathbf{F}_1$ *with*

$$\mathbf{F}_1 = (-qx, py)^T, \; p, q \in \mathbb{N}, \; gcd(p,q) = 1.$$

Moreover, in such a case, $\mathbf{F}$ *has an analytic first integral of the form* $I = I_M + h.o.t.$ *being* $I_M = x^p y^q$ *a primitive first integral of* $\mathbf{F}_1$.

(2) For $n = 2$, $\mathbf{F}$ *is analytically integrable if, and only if, it is orbitally equivalent to* $\mathbf{F}_2$ *with*

$$\mathbf{F}_2 = (x(-qx + (q+r)y), y((p+r)x - py))^T, \; p, q, r \in \mathbb{N}, \; gcd(p,q,r) = 1.$$

Moreover, in such a case, $\mathbf{F}$ *has an analytic first integral of the form* $I = I_M + h.o.t.$ *being* $I_M = x^p y^q (x-y)^r$ *a primitive first integral of* $\mathbf{F}_2$.

(3) For $n = 3$, $\mathbf{F}$ *is analytically integrable if, and only if, it is orbitally equivalent to to* $\mathbf{F}_{3,c}$ *(*$\mathbf{F}_{3,d}$*, resp.).*

Moreover, in such a case, $\mathbf{F}$ *has an analytic first integral of the form* $I = I_M + h.o.t.$ *being* $I_M = x^p y^q (x-y)^r (x-cy)^s$ *a primitive first integral of* $\mathbf{F}_{3,c}$ *(* $I_M = x^p y^q ((x+by)^2 + y^2)^r$ *a primitive first integral of* $\mathbf{F}_{3,d}$*, resp.).*

Proof. Case $n = 1$ it is a well-known result, see [14, 15, 17, 21]. Case $n = 2$, it is proved in [3] and case $n = 3$ is Theorem 3.16. ■

4 Applications

Consider the analytic integrability problem of the Kolmogorov system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(-x^2 + (1-s)xy + (2+s)y^2) \\ y((2+s)x^2 + (1-s)xy - y^2) \end{pmatrix} + \begin{pmatrix} x(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) \\ y(b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3) \end{pmatrix} \quad (4.11)$$

with $s \in \mathbb{N}$ and $a_{ij}, b_{ij} \in \mathbb{R}$. Vector field associated is $\mathbf{F} = \mathbf{F}_3 + \mathbf{F}_4$ being $\mathbf{F}_3 = \mathbf{F}_{3,c}$ given by (1.5) with $(p,q,r,s) = (1,1,1,s)$ and $c = -1$.

Leader component of the vector field can be expressed as $\mathbf{F}_3 = \mathbf{X}_h + \mu\mathbf{D}$ with $h = \frac{1}{4}xy(x-y)(x+y)$ and $\mu = \frac{1}{4}(-4xy + y^2 + x^2)(-1+s)$. From Proposition 1.6, the vector field $\mathbf{F}_3$ is polynomially integrable and a primitive first integral is $I_{s+3} = xy(x-y)(x+y)^s$.

The following result solves the integrability problem of this family for $s = 1$ and $s = 2$.

Theorem 4.22 *System (4.11) for $s = 1$ or $s = 2$ is analytically integrable if, and only if, one of the following conditions holds:*
For $s = 1$,
(1) $a_{03} + 4b_{03} = 2a_{12} + 3b_{12} = 3a_{21} + 2b_{21} = 4a_{30} + b_{30} = 0$,
(2) $b_{21} + 3b_{03} = 2a_{12} - 3b_{30} - 3b_{12} = a_{21} + a_{03} + 2b_{03} = 2a_{30} + b_{30} + b_{12} = a_{03}b_{30} + a_{03}b_{12} + 2b_{30}b_{03} = 0$.
For $s = 2$,
(3) $b_{30} + 3b_{21} + b_{12} + 11b_{03} = a_{12} + 4b_{21} + 15b_{03} = a_{21} + a_{03} - b_{21} - b_{03} = a_{30} - b_{21} - 4b_{03} = a_{03}b_{21} + 4a_{03}b_{03} + 3b_{21}b_{03} + b_{12}b_{03} + 11b_{03}^2 = 0$,
(4) $5b_{30} + 6b_{21} + 8b_{12} + 16b_{03} = a_{03} + 6b_{03} = 3a_{12} + 4b_{12} - 7b_{03} = 15a_{21} + 6b_{21} - 7b_{12} - 14b_{03} = 15a_{30} - 3b_{21} - 4b_{12} - 8b_{03} = 0$,
(5) $3b_{30} - 5b_{21} + 10b_{12} - 30b_{03} = a_{03} + 5b_{03} = a_{12} + 2b_{12} - 3b_{03} = a_{21} + b_{21} - b_{12} + 3b_{03} = 3a_{30} + b_{21} - 2b_{12} + 6b_{03} = 0$.

Proof. To prove the necessary condition, it has obtained the expression of $F(I)$ being $I = xy(x - y)(x + y)^s +$ h.o.t. up to order $s + 9$. Equation $F(I) = 0$ provides twelve compatibility's conditions, two by each degree of the expression of $F(I)$, but we do not write them here. Their vanishing leads to the systems (4.11) for cases 1–5.

We prove the sufficiency. System (4.11) for case 1 is a Hamiltonian system whose Hamiltonian function is $xy(x - y)(x + y) - \frac{1}{2}xy(2a_{30}x^3 + a_{21}x^2y - b_{12}xy^2 - 2b_{03}y^3)$. System (4.11) for cases 2 and 3 has an inverse integrating factor $xy(x - y)(x + y)(1 - xa_{30} - yb_{03})$ whose lowest degree term is h. Applying Theorem 3.19, the vector field is analytically integrable.
System (4.11) for case 4 has a polynomial first integral

$$xy(x - y)(-15x - 15y + 3b_{21}x^2 + 10b_{03}xy + 8b_{03}x^2 + 5b_{12}xy + 4b_{12}x^2 + 15b_{03}y^2)^2.$$

System (4.11) for case 5 has a polynomial first integral

$$xy(x + y)^2(6x - 6y + 12b_{03}x^2 - 4b_{12}x^2 + 2b_{21}x^2 + 3b_{12}xy - 9b_{03}xy + 6b_{03}y^2).$$

This concludes the proof. ■

We have the following result.

Theorem 4.23 *Analytically integrable systems (4.11) for $s = 1$ or $s = 2$ have a polynomial inverse integrating factor of degree less or equals than 5.*

Proof. From Theorem 4.22, the analytically integrable systems (4.11) for $s = 1$ or $s = 2$ are the systems given in Theorem 4.22 and Table 1 provides their inverse integrating factors. ■

Sys (4.11)	Inverse integrating factor
Case 1	$xy(x - y)(x + y) - \frac{1}{2}xy(2a_{30}x^3 + a_{21}x^2y - b_{12}xy^2 - 2b_{03}y^3)$
Cases 2-3	$xy(x - y)(x + y)(1 - a_{30}x - b_{03}y)$
Case 4	$xy(x - y)((4b_{12} + 8b_{03} + 3b_{21})x^2 + (10b_{03} + 5b_{12})xy + 15b_{03}y^2 - 15x - 15y)$
Case 5	$xy(x + y)((-4b_{12} + 2b_{21} + 12b_{03})x^2 + (3b_{12} - 9b_{03})xy + 6b_{03}y^2 + 6x - 6y)$

Table 1: Polynomial inverse integrating factors of analytically integrable systems (4.11).

Next, we study the analytic integrability of the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x(x^2+3y^2) \\ y(3x^2+y^2) \end{pmatrix} + \begin{pmatrix} x(a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3) \\ y(b_{30}x^3+b_{21}x^2y+b_{12}xy^2+b_{03}y^3) \end{pmatrix}. \qquad (4.12)$$

Vector field is $\mathbf{F} = \mathbf{F}_3 + \mathbf{F}_4$ being $\mathbf{F}_3 = \mathbf{F}_{3,d}$ given by (1.6) with $(p,q,r) = (1,1,1)$ and $b = 0$.

Leader component of vector field is the Hamiltonian vector field $\mathbf{F}_3 = \mathbf{X}_h$ with $h = \frac{1}{4}xy(x^2+y^2)$. Therefore, vector field $\mathbf{F}_3$ is polynomially integrable and a primitive first integral is $I_4 = xy(x^2+y^2)$.

The following result solves the integrability problem of this family.

Theorem 4.24 *System (4.12) is analytically integrable if, and only if, one of the following conditions holds:*
(1) $b_{12}-7b_{03} = b_{21}-15b_{03} = b_{30}-b_{03} = a_{03}+b_{03} = a_{12}+15b_{03} = a_{21}+7b_{03} = a_{30}+b_{03} = 0$,
(2) $b_{12}+7b_{03} = b_{21}-15b_{03} = b_{30}+b_{03} = a_{03}+b_{03} = a_{12}-15b_{03} = a_{21}+7b_{03} = a_{30}-b_{03} = 0$,
(3) $b_{12}-6b_{03} = b_{21}+3b_{03} = b_{30} = a_{03} = a_{12}+3b_{03} = a_{21}-6b_{03} = a_{30}-b_{03} = 0$,
(4) $b_{12}+6b_{03} = b_{21}+3b_{03} = b_{30} = a_{03} = a_{12}-3b_{03} = a_{21}-6b_{03} = a_{30}+b_{03} = 0$,
(5) $a_{03}+4b_{03} = 2a_{12}+3b_{12} = 3a_{21}+2b_{21} = 4a_{30}+b_{30} = 0$,
(6) $b_{21}-3b_{03} = 2a_{12}+3b_{30}-3b_{12} = a_{21}-a_{03}-2b_{03} = 2a_{30}+b_{30}-b_{12} = a_{03}b_{30} - a_{03}b_{12}+2b_{30}b_{03} = 0$.

Proof. To prove the necessary condition, it has obtained the expression of $F(I)$ being $I = xy(x^2+y^2) + \text{h.o.t.}$ up to order 9. Its vanishing arrives to systems (4.12) for cases 1–6.

We prove the sufficiency.

System (4.12) for case 1 has an analytic first integral

$$\begin{aligned} &xy(x^2+y^2+4b_{03}x^2y+4b_{03}xy^2-2b_{03}^2x^3y-4b_{03}^2x^2y^2-2b_{03}^2xy3) \\ &\qquad /(1+b_{03}x+b_{03}y+4b_{03}^2xy-4b_{03}^3x^2y-4b_{03}^3xy^2)^2. \end{aligned}$$

System (4.12) for case 2 is transformed into previous system under the involution $(x,y) \leftrightarrow (-x,y)$.
In case 3 the system (4.12) has an analytic first integral

$$xy(4b_{03}^2x^2y^2-6b_{03}x^2y+6b_{03}xy^2+3x^2+3y^2)/(1-b_{03}x+b_{03}y+2b_{03}^2xy)^3.$$

Case 4 is dual to the previous one under the involution $(x,y) \leftrightarrow (-x,y)$.
System (4.12) for case 5 is a Hamiltonian system whose Hamiltonian function is

$$xy(x^2+y^2) - a_{30}x^4y + (1/3)b_{21}x^3y^2 + 1/2b_{12}x^2y^3 + b_{03}xy^4.$$

In case 6, system (4.12) for $b_{30} \neq 0$ has an inverse integrating factor

$$yx(x^2+y^2)(xb_{30}^2x - b_{12}b_{30}x + 2b_{30} - a_{03}b_{30}y + a_{03}b_{12}y).$$

whose first component is the conservative part of $\mathbf{F}_3$. Applying Theorem 3.19, the vector field is analytically integrable.
Otherwise, $b_{30} = 0$, system (4.12) has an inverse integrating factor

$$xy(x^2+y^2)(2+2b_{03}y-b_{12}x)$$

whose lowest degree term is $2h$, applying Theorem 3.19, system (4.12) for this case is analytically integrable. This concludes the proof. ■

Theorem 4.25 *Analytically integrable systems (4.12) has a polynomial inverse integrating factor of degree less or equals than 9.*

Proof. From Theorem 4.24, the analytically integrable systems (4.11) are the systems given in Theorem 4.24 and Table 2 provides their inverse integrating factors. ■

Sys (4.12)	Inverse integrating factor
Case 1	$xy(x^2+y^2+4b_{03}x^2y+4b_{03}xy^2-2b_{03}^2x^3y-4b_{03}^2x^2y^2-2b_{03}^2xy^3)$ $\times(1+b_{03}x+b_{03}y+4b_{03}^2xy-4b_{03}^3x^2y-4b_{03}^3xy^2)$
Case 2	$xy(2b_{03}^2x^3y-4b_{03}^2x^2y^2+2b_{03}^2xy^3+4b_{03}x^2y-4b_{03}xy^2+x^2+y^2)$ $\times(1-b_{03}x+b_{03}y-4b_{03}^2xy-4b_{03}^3x^2y+4b_{03}^3xy^2)$
Case 3	$xy(4b_{03}^2x^2y^2-6b_{03}x^2y+6b_{03}xy^2+3x^2+3y^2)(1-b_{03}x+b_{03}y+2b_{03}^2xy)$
Case 4	$xy(2b_{03}^2xy-b_{03}x-1-b_{03}y)(4b_{03}^2x^2y^2-6b_{03}x^2y-6b_{03}xy^2+3x^2+3y^2)$
Case 5	$xy(6a_{30}x^3-2b_{21}x^2y-6x^2-3b_{12}xy^2-6b_{03}y^3-6y^2)$
Case 6	$xy(x^2+y^2)(b_{30}^2x-b_{12}b_{30}x+2b_{30}-a_{03}b_{30}y+a_{03}b_{12}y),\ b_{30}\neq 0$
	$xy(x^2+y^2)(2+2b_{03}y-b_{12}x),\ b_{30}=0.$

Table 2: Polynomial inverse integrating factors of analytically integrable systems (4.12).

5 Appendix: Normal Form of vector fields whose leader term is a homogeneous vector field.

The results provided in this section have been included for completeness. In this section, we determine an orbital normal form for a general vector field $\mathbf{F}=\mathbf{F}_n+\cdots$, where the leader term of $\mathbf{F}$, $\mathbf{F}_n=\mathbf{X}_h+\mu\mathbf{D}\in\mathcal{H}_n$, with $h\in\mathcal{P}_{n+1}$ and $\mu\in\mathcal{P}_{n-1}$, conservative-dissipative decomposition of $\mathbf{F}_n$, that is, through the section h will represent the conservative part of $\mathbf{F}_n$ and μ will be the dissipative part of $\mathbf{F}_n$, see Proposition 1.2.

The following subspaces of $\mathcal{H}_k$ will be useful in the study of the homological operator under orbital equivalence that we will see later.

$$\begin{aligned}\mathcal{C}_k &:= \left\{\mathbf{X}_{g_{k+1}} : g_{k+1}\in\Delta_{k+1}, \text{ a complementary subspace to } h\mathcal{P}_{k-n}\right\},\\ \mathcal{D}_k &:= \left\{\eta_{k-1}\mathbf{D} : \eta_{k-1}\in\mathcal{P}_{k-1}\right\},\\ \mathcal{F}_k &:= \left\{\lambda_{k-n}\mathbf{F}_n : \lambda_{k-n}\in\mathcal{P}_{k-n}\right\}.\end{aligned}$$

We give the following technical lemma, providing the conservative-dissipative splitting of the vector fields $\lambda_{k-n}\mathbf{F}_n$.

Lemma 5.26 *Consider $\lambda_{k-n}\in\mathcal{P}_{k-n},\ k>n$. Then,*

$$\lambda_{k-n}\mathbf{F}_n \quad = \quad \mathbf{X}_{\frac{n+1}{k+1}\lambda_{k-n}h}+\left(\tfrac{1}{k+1}X_h(\lambda_{k-n})+\lambda_{k-n}\mu\right)\mathbf{D}. \tag{5.13}$$

Proof. From Proposition 1.2, the conservative-dissipative splitting of a homogeneous vector field $\lambda_{k-n}\mathbf{X}_h$ is $\lambda_{k-n}\mathbf{X}_h=\frac{n+1}{k+1}\mathbf{X}_{\lambda_{k-n}h}+\frac{1}{k+1}X_h(\lambda_{k-n})\mathbf{D}$ and as $\lambda_{k-n}\mathbf{F}_n=\lambda_{k-n}\mathbf{X}_h+\lambda_{k-n}\mu\mathbf{D}$, the result follows. ∎

We now see some properties of the above subspaces.

Lemma 5.27 *It has that:*

(a) $\mathcal{C}_k\cap\mathcal{D}_k=\{\mathbf{0}\}$,

(b) $\mathcal{C}_k\cap\mathcal{F}_k=\{\mathbf{0}\}$,

(c) $\mathcal{D}_k\cap\mathcal{F}_k=\{\mathbf{0}\}$.

Proof. Item [(a)] follows trivially.

We prove [(b)]. Otherwise, there is some $\mathbf{P}_k \in \mathcal{D}_k \cap \mathcal{F}_k \setminus \{\mathbf{0}\}$. Then, one can find $\lambda_{k-n} \in \mathcal{P}_{k-n} \setminus \{0\}$ such that $\mathbf{P}_k = \lambda_{k-n}\mathbf{F}_n$, and there exists $\mu_{k-1} \in \mathcal{P}_{k-1}$ such that $\mathbf{P}_k = \mu_{k-1}\mathbf{D}$. Hence, $0 = \mathbf{D} \wedge (\mu_{k-1}\mathbf{D}) = \mathbf{D} \wedge \mathbf{P}_k = \mathbf{D} \wedge (\lambda_{k-n}\mathbf{F}_n) = (n+1)\lambda_{k-n}h$, which is a contradiction.

We see that [(c)] holds, because otherwise, there is some $\mathbf{P}_k \in \mathcal{C}_k \cap \mathcal{F}_k \setminus \{\mathbf{0}\}$. Then, $\mathbf{P}_k = \mathbf{X}_g = \lambda_{k-n}\mathbf{F}_n$ where $\lambda_{k-n} \in \mathcal{P}^{\mathbf{t}}_{k-n} \setminus \{0\}$ and $g \in \Delta_{k+1} \setminus \{0\}$. So, from (5.13) and the uniqueness of the conservative-dissipative splitting, we get $g = \frac{n+1}{k+1}\lambda_{k-n}h$. Hence $g \in \Delta_{k+1} \cap h\mathcal{P}_{k-n} = \{0\}$, which is a contradiction. ■

We provide a new decomposition of homogeneous vector fields.

Lemma 5.28 *$\mathcal{H}_k = \mathcal{C}_k \bigoplus \mathcal{D}_k \bigoplus \mathcal{F}_k$, for every $k \geq n$.*

Proof. It is obvious that $\mathcal{C}_k + \mathcal{D}_k + \mathcal{F}_k \subseteq \mathcal{H}_k$. We show that $\mathcal{H}_k \subseteq \mathcal{C}_k + \mathcal{D}_k + \mathcal{F}_k$. Let us consider $\mathbf{P}_k \in \mathcal{H}_k$. From Proposition 1.2, we can write $\mathbf{P}_k = \mathbf{X}_{\widehat{g}} + \mu_{k-1}\mathbf{D}$, for some $\widehat{g} \in \mathcal{P}_{k+1}$, $\mu_{k-1} \in \mathcal{P}_{k-1}$. As $\mathcal{P}_{k+1} = \Delta_{k+1} \bigoplus h\mathcal{P}_{k-n}$, we can also write $\widehat{g} = g + \tilde{g}h$ for some $g \in \Delta_{k+1}$, $\tilde{g} \in \mathcal{P}_{k-n}$. Then, $\mathbf{P}_k = \mathbf{X}_g + \mathbf{X}_{\tilde{g}h} + \mu_{k-1}\mathbf{D}$. Using (5.13) for $\tilde{g}\mathbf{F}_n$, we obtain $\tilde{g}\mathbf{F}_n = \mathbf{X}_{\frac{n+1}{k+1}\tilde{g}h} + \left(\frac{1}{k+1}X_h(\tilde{g}) + \tilde{g}\mu\right)\mathbf{D}$. So, $\mathbf{X}_{\tilde{g}h} = \frac{k+1}{n+1}\tilde{g}\mathbf{F}_n - \left(\frac{1}{n+1}X_h(\tilde{g}) + \frac{k+1}{n+1}\tilde{g}\mu\right)\mathbf{D}$, and then

$$\mathbf{P}_k \quad = \quad \mathbf{X}_g + \left(\mu_{k-1} - \tfrac{1}{n+1}X_h(\tilde{g}) - \tfrac{k+1}{n+1}\tilde{g}\mu\right)\mathbf{D} + \tfrac{k+1}{n+1}\tilde{g}\mathbf{F}_n.$$

Applying Lemma 5.27, the proof follows. ■

The above result allows to define the corresponding projectors Π^{c}, Π^{d} and Π^{f}. Also, we can identify $\mathcal{H}_k = \mathcal{C}_k \bigoplus \mathcal{D}_k \bigoplus \mathcal{F}_k \equiv \mathcal{C}_k \times \mathcal{D}_k \times \mathcal{F}_k$. We will denote $\Pi^{\mathrm{d}}(\mathbf{P}_k) = \mathbf{P}^{\mathrm{d}}_k := \mathrm{Proy}_{\mathcal{D}_k}(\mathbf{P}_k)$, $\Pi^{\mathrm{c}}(\mathbf{P}_k) = \mathbf{P}^{\mathrm{c}}_k := \mathrm{Proy}_{\mathcal{C}_k}(\mathbf{P}_k)$, and $\Pi^{\mathrm{f}}(\mathbf{P}_k) = \mathbf{P}^{\mathrm{f}}_k := \mathrm{Proy}_{\mathcal{F}_k}(\mathbf{P}_k)$.

Lemma 5.29 *Given $\mathbf{P}_k \in \mathcal{H}_k$, then there exist $g \in \Delta_{k+1}$, $\eta \in \mathcal{P}_{k-1}$ and $\lambda \in \mathcal{P}_{k-n}$ such that*

$$\mathbf{P}_k = \mathbf{X}_g + \eta\mathbf{D} + \lambda\mathbf{F}_n,$$

where

$$g = \frac{\mathrm{Proy}_{\Delta_{k+1}}\mathbf{D}\wedge\mathbf{P}_k}{k+1}, \quad \lambda = \frac{\mathrm{Proy}_{h\mathcal{P}_{k-n}}\mathbf{D}\wedge\mathbf{P}_k}{(n+1)h}, \quad \eta = \frac{\mathrm{div}(\mathbf{P}_k) - F_n(\lambda) - \lambda\mathrm{div}(\mathbf{F}_n)}{k+1}. \tag{5.14}$$

Proof. Only remains to find the expressions of g, η and λ. On the one hand,

$$\mathbf{D} \wedge \mathbf{P}_k = \mathbf{D} \wedge (\mathbf{X}_g + \eta\mathbf{D} + \lambda\mathbf{F}_n) = (k+1)g + (n+1)\lambda h.$$

Therefore g and λ is given by (5.14). On the other hand, as $\mathrm{div}(\mathbf{P}_k) = (k+1)\eta + F_n(\lambda) + \lambda\mathrm{div}(\mathbf{F}_n)$, it obtains η. ■

Next, we are interested in obtaining an expression of the homological operator under equivalence (see [1]) defined as,

$$\begin{array}{rcl} \mathcal{L}_{n+k-1} : \mathcal{H}_k \times \mathcal{P}_{k-1} & \to & \mathcal{H}_{n+k-1} \\ \mathcal{L}_{n+k-1}(\mathbf{P}_k, \nu_{k-1}) & = & -[\mathbf{F}_n, \mathbf{P}_k] - \nu_{k-1}\mathbf{F}_n. \end{array}$$

The following linear operator also plays an important role in this study (see also [2]).

$$\begin{array}{rcl} \ell_{k-1} & : & \mathcal{P}_{k-n} \longrightarrow \mathcal{P}_{k-1} \\ & & \mu_{k-n} \longrightarrow F_n(\mu_{k-n}), \end{array} \tag{5.15}$$

We give some properties of Lie bracket related with the projectors above.

Lemma 5.30 *Let us consider* $\lambda_{k-n}\mathbf{F}_n \in \mathcal{F}_k$, $\eta_{k-1}\mathbf{D} \in \mathcal{D}_k$, $\mathbf{X}_g \in \mathcal{C}_k$*. Then:*

(a) $[\mathbf{F}_n, \lambda_{k-n}\mathbf{F}_n] = -\ell_{k-1}(\lambda_{k-n})\,\mathbf{F}_n \in \mathcal{F}_{n+k-1}$.

(b) $[\mathbf{F}_n, \eta_{k-1}\mathbf{D}] = -\ell_{n+k-2}(\eta_{k-1})\,\mathbf{D} + (n-1)\eta_{k-1}\mathbf{F}_n \in \mathcal{D}_{n+k-1} \bigoplus \mathcal{F}_{n+k-1}$.

(c) $[\mathbf{F}_n, \mathbf{X}_g] = -\mathbf{X}_{(F_n - \frac{n+1}{n+k}\mu D)(g)} + \frac{n+1}{n+k}X_g(\mu)\mathbf{D}$.

Proof. Items **(a)**, **(b)**, are consequences of following properties $[\mu\mathbf{F}, \mathbf{G}] = G(\mu)\mathbf{F} + \mu[\mathbf{F}, \mathbf{G}]$, $[\mathbf{F}_k, \mathbf{D}_0] = (k-1)\mathbf{F}_k$, respectively. From the properties of the Lie bracket and the previous properties we deduce

$$\begin{aligned}
[\mathbf{F}_n, \mathbf{X}_g] &= [\mathbf{X}_h, \mathbf{X}_g] + [\mu\mathbf{D}, \mathbf{X}_g] = \mathbf{X}_{-X_h(g)} + X_g(\mu)\mathbf{D} + \mu\,[\mathbf{D}, \mathbf{X}_g] \\
&= \mathbf{X}_{-X_h(g)} + X_g(\mu)\mathbf{D} - (k-1)\mu\mathbf{X}_g \\
&= \mathbf{X}_{-X_h(g)} + X_g(\mu)\mathbf{D} - \tfrac{(k-1)(k+1)}{k+n}\mathbf{X}_{\mu g} - \tfrac{k-1}{k+n}X_g(\mu)\mathbf{D} \\
&= -\mathbf{X}_{X_h(g) + \frac{(k-1)(k+1)}{n+k}\mu g} + \tfrac{n+1}{n+k}X_g(\mu)\mathbf{D} \\
&= -\mathbf{X}_{\left(F_n - \frac{n+1}{n+k}\mu D\right)(g)} + \tfrac{n+1}{n+k}X_g(\mu)\mathbf{D},
\end{aligned}$$

which implies item **(c)**. ■

Let note that $[\mathbf{F}_n, \mathbf{X}_g]^{\mathrm{c}} = -\mathbf{X}_{\left(\ell^{\mathrm{c}}_{n+k}(g)\right)}$ where $\ell^{\mathrm{c}}_{n+k} : \Delta_{k+1} \longrightarrow \Delta_{n+k}$ is defined by

$$\ell^{\mathrm{c}}_{n+k}(g) = \mathrm{Proy}_{\Delta_{n+k}}\left(F_n - \tfrac{n+1}{n+k}\mu D\right)(g). \tag{5.16}$$

We prove the following result.

Lemma 5.31 *Given an integer* $k > n$*, it verifies that*

$$\mathit{Range}(\mathcal{L}_{n+k-1}(\mathcal{H}_k \times \mathit{Cor}(\ell_{k-1}))) = \mathit{Range}(\mathcal{L}_{n+k-1}(\mathcal{H}_k \times \mathcal{P}_{k-1}))$$

being $\mathit{Cor}(\ell_k)$ *a complementary subspace to the range of the linear operator* ℓ_k*.*

Proof. Any function $\nu_{k-1} \in \mathcal{P}_{k-1}$ is expressed as $\nu_{k-1} = \tau_{k-1} + \tau^r_{k-1}$ with $\tau_{k-1} \in \mathrm{Cor}(\ell_{k-1})$, being $\mathrm{Cor}(\ell_{k-1})$ a complementary subspace to the range of the linear operator ℓ_{k-1}, and $\tau^r_{k-1} \in \mathrm{Range}(\ell_{k-1})$, i.e. there exists $\tau_{k-n} \in \mathcal{P}_{k-n}$ such that $\ell_{k-1}(\tau_{k-n}) = \tau^r_{k-1}$. So, from Lemma 5.30 (item (**a**)),

$$\begin{aligned}
\mathcal{L}_{n+k-1}(\mathbf{P}_k, \tau_{k-1}) &= -[\mathbf{F}_n, \mathbf{P}_k] - \tau_{k-1}\mathbf{F}_n - \ell_{k-1}(\tau_{k-n})\mathbf{F}_n \\
&= -[\mathbf{F}_n, \mathbf{P}_k] - \tau_{k-1}\mathbf{F}_n + [\mathbf{F}_n, \tau_{k-n}\mathbf{F}_n] \\
&= -[\mathbf{F}_n, \mathbf{P}_k - \tau_{k-n}\mathbf{F}_n] - \tau_{k-1}\mathbf{F}_n.
\end{aligned}$$

Consequently, both operator $\mathcal{L}_{n+k-1}$ restricted to $\mathcal{H}_k \times \mathrm{Cor}(\ell_{k-1})$ and operator $\mathcal{L}_{n+k-1}$ have the same range. ■

In this way, by keeping the notation, we consider $\mathcal{L}_{n+k-1}$ restricted to $\mathcal{H}_k \times \mathrm{Cor}(\ell_{k-1})$ as the homological operator under orbital equivalence.

With this notation the homological operator under equivalence can be written as

$$\mathcal{L}_{n+k-1} : \mathcal{C}_k \times \mathcal{D}_k \times \mathcal{F}_k \times \mathrm{Cor}(\ell_{k-1}) \longrightarrow \mathcal{C}_{n+k-1} \times \mathcal{D}_{n+k-1} \times \mathcal{F}_{n+k-1},$$

such as

$$\mathcal{L}_{n+k-1}\left(\mathbf{P}^{\mathrm{c}}_k, \mathbf{P}^{\mathrm{d}}_k, \mathbf{P}^{\mathrm{f}}_k, \nu_{k-1}\right) = -\left([\mathbf{F}_n, \mathbf{P}^{\mathrm{c}}_k]^{\mathrm{c}}, \left[\mathbf{F}_n, \mathbf{P}^{\mathrm{c}}_k + \mathbf{P}^{\mathrm{d}}_k\right]^{\mathrm{d}}, \left[\mathbf{F}_n, \mathbf{P}^{\mathrm{c}}_k + \mathbf{P}^{\mathrm{d}}_k + \mathbf{P}^{\mathrm{f}}_k\right]^{\mathrm{f}} + \nu_{k-1}\mathbf{F}_n\right),$$

where $\mathbf{P}^{\mathrm{c}}_k = \mathbf{X}_g$ with $g \in \Delta_{k+1}$, $\mathbf{P}^{\mathrm{d}}_k = \mu_{k-1}\mathbf{D}$ with $\mu_{k-1} \in \mathcal{P}_{k-1}$, $\mathbf{P}^{\mathrm{f}}_k = \lambda_{k-n}\mathbf{F}_n$ with $\lambda_{k-n} \in \mathcal{P}_{k-n}$ and $\nu_{k-1} \in \mathrm{Cor}(\ell_{k-1})$.

Next result provides us an orbital normal form of the vector field $\mathbf{F}$ with leader term non-conservative.

Theorem 5.32 *Let* $\mathbf{F} = \mathbf{F}_n + h.o.t.$ *with* $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D} \in \mathcal{H}_n$. *If* $\operatorname{Ker}\left(\ell^{\mathrm{c}}_{n+k}\right) = \{0\}$ *for all* $k \in \mathbb{N}$ *then* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G} = \mathbf{F}_n + \sum_{j>n} \mathbf{G}_j, \ \textit{with } \mathbf{G}_j = \mathbf{X}_{g_{j+1}} + \eta_{j-1}\mathbf{D} \in \mathcal{H}_j,$$

where $g_{j+1} \in \operatorname{Cor}\left(\ell^{\mathrm{c}}_{j+1}\right)$ *and* $\eta_{j-1} \in \operatorname{Cor}\left(\ell_{j-1}\right)$, *the operators* ℓ^{c}_{n+j} *and* ℓ_{j-1} *are defined in (5.16) and (5.15), respectively.*

Proof. It is sufficient to show that

$$\operatorname{Cor}\left(\mathcal{L}_{n+k-1}\right) \quad = \quad \mathbf{X}_{\operatorname{Cor}\left(\ell^{\mathrm{c}}_{n+k}\right)} \bigoplus \operatorname{Cor}\left(\ell_{n+k-2}\right)\mathbf{D},$$

is a complementary subspace to the range of $\mathcal{L}_{n+k-1}$.

From Lemma 5.30, we get $\mathcal{L}_{n+k-1}\left(\mathbf{P}^{\mathrm{c}}_k, \mathbf{P}^{\mathrm{d}}_k, \mathbf{P}^{\mathrm{f}}_k, \nu_{k-1}\right) =$

$$\left(\mathbf{X}_{\left(\ell^{\mathrm{c}}_{n+k}(g)\right)}, -\left[\mathbf{F}_n, \mathbf{X}_g\right]^{\mathrm{d}} + \ell_{n+k-2}\left(\mu_{k-1}\right)\mathbf{D}_0, -\left[\mathbf{F}_n, \mathbf{X}_g\right]^{\mathrm{f}} - \mu_{k-1}\mathbf{F}_n + \ell_{k-1}\left(\lambda_{k-n}\right)\mathbf{F}_n - \nu_{k-1}\mathbf{F}_n\right).$$

By using $\mathcal{F}^{\mathbf{t}}_{n+k-1} = \operatorname{Range}\left(\ell_{k-1}\right)\mathbf{F}_n \bigoplus \operatorname{Cor}\left(\ell_{k-1}\right)\mathbf{F}_n$, and $\mathcal{P}^{\mathbf{t}}_{k-1} = \operatorname{Range}\left(\ell_{k-1}\right) \bigoplus \operatorname{Cor}\left(\ell_{k-1}\right)$, we obtain the following scheme for $\mathcal{L}_{n+k-1}$:

$\mathcal{C}_k$	$\mathcal{D}_k$	$\mathcal{F}_k$	$\operatorname{Range}\left(\ell_{k-1}\right)$	$\operatorname{Cor}\left(\ell_{k-1}\right)$	
$\mathbf{X}_{\ell^{\mathrm{c}}_{n+k}(g)}$	0	0	0	0	$\mathcal{C}_{n+k-1}$
$\bullet$	$\ell_{n+k-2}\left(\mu_{k-1}\right)\mathbf{D}$	0	0	0	$\mathcal{D}_{n+k-1}$
$\bullet$	$\bullet$	$\ell_{k-1}\left(\lambda_{k-n}\right)\mathbf{F}_n$	$-\ell_{k-1}\left(\rho_{k-n}\right)\mathbf{F}_n$	0	$\operatorname{Range}\left(\ell_{k-1}\right)\mathbf{F}_n$
		0	0	$-\nu_{k-1}\mathbf{F}_n$	$\operatorname{Cor}\left(\ell_{k-1}\right)\mathbf{F}_n$

From hypothesis $\operatorname{Ker}\left(\ell^{\mathrm{c}}_{n+k}\right) = \{0\}$, we can deduce that the upper left block diagonal of the above matrix has maximum range. The result follows from the structure of the above matrix. ■

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