Analytically integrable centers of perturbations of cubic homogeneous systems

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Abstract

We consider the analytically integrable perturbations of cubic homogeneous differential systems whose origin is an isolated singularity. We prove that are orbitally equivalent to the cubic vector field associated. We also characterize the analytically integrable centers. We apply the results to two families of degenerate vector fields.

1 Introduction

Given the polynomial planar differential system $\dot{\mathbf{x}} = \mathbf{F}_n(\mathbf{x}) = (P_n, Q_n)^T$, with $P_n, Q_n \in \mathcal{P}_n$ (vector space of homogeneous polynomials of degree n) and whose origin is an isolated singular point, it is interesting to characterize the systems $\dot{\mathbf{x}} = \mathbf{F}_n + \text{h.o.t.}$, analytic perturbations of the vector field $\mathbf{F}_n$, which have an analytic first integral at the origin.

We say that an analytic vector field is *homogenizable* if it is orbitally equivalent to a homogeneous polynomial vector field, *i.e.* the system $\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}_n(\mathbf{x}) + \text{h.o.t.}$ by means of a near-identity change of variable $\mathbf{x} = \phi(\mathbf{y})$ and a formal reparameterization of the time $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \eta(\mathbf{x}),\ \eta(\mathbf{0}) = 1$, it is transformed into $\mathbf{y}' = \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = \mathbf{F}_n(\mathbf{y})$.

We ask the following question: For what values of n, assuming that $\mathbf{F}_n$ has a polynomial first integral, does analytic integrability of $\dot{\mathbf{x}} = \mathbf{F}_n(\mathbf{x}) + \text{h.o.t.}$ and homogenizable coincide?

For $n = 1$, linear part $\mathbf{F}_1$ of the vector field is non-zero, one has the following cases depending on the eigenvalues λ_1, λ_2 of $D(\mathbf{F}_1)(\mathbf{0})$ (assuming that the origin is an isolated singular point of $\mathbf{F}_1$): if $\lambda_1\lambda_2 \neq 0$, the origin is either a *saddle*, a *node* or a *non-degenerate monodromic singular point* (with imaginary eigenvalues).

The nodes are not analytically integrable. A non-degenerate monodromic point is analytically integrable if, and only if, the system is orbitally equivalent to $(-y, x)^T$, see [21], and a resonant saddle has an analytic first integral around the singular point if, and only if, the system is orbitally equivalent to $(px, -qy)^T$ with $p, q \in \mathbb{N}$, see [17].

The most studied systems whose origin is a resonant saddle are the Lotka-Volterra systems, see [11–13, 16, 18, 19] and references therein.

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Therefore, for $n = 1, \mathbf{F}_n + $h.o.t. is analytically integrable at the origin if, and only if, it is homogenizable (in this case, *linearizable*).
Ferčec & Giné [14, 15] use the blow-up method to compute necessary conditions of analytic integrability.

For $n = 2$, i.e. $\mathbf{F}_1$ is zero and $\mathbf{F}_2$ non-zero, by Algaba *et al.* [6], the quadratic vector fields, polynomially integrable and whose origin is an isolated singular point are

$$\mathbf{F}_{2,a} = (x(-qx + (q+r)y), y((p+r)x - py))^T, \ p, q, r \in \mathbb{N},$$

and

$$\mathbf{F}_{2,b} = (-2qxy, (p+2q)x^2 + py^2)^T, \ p, q \in \mathbb{N}.$$

Algaba *et al.* [6] have proved that the analytically integrable perturbations of these vector fields are homogenizable.

For $n = 3$, $\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{0}$ and $\mathbf{F}_3$ non-zero, we know the following results. On the one hand, the perturbations of cubic Kolmogorov systems, $(xP_2, yQ_2)^T$, whose origin is analytically integrable, are homogenizable, Algaba *et al.* [7].

On the other hand, Algaba *et al.* [1] study the perturbations of cubic systems $(-y(x^2+y^2), x(x^2+y^2))^T$. The origin is an isolated singular point in $\mathbb{R}^2$ but the vector field is reducible (the origin is a not isolated singular point in $\mathbf{C}^2$). In this class, there are analytically integrable vector fields which are not homogenizable. For example, the vector field $(-y(x^2+y^2), x(x^2+y^2) + 5x^4)^T$ is a Hamiltonian vector field whose Hamilton function is $\frac{1}{4}(x^2+y^2)^2 + x^5$, therefore it is polynomially integrable. Moreover, from [1, Theorem 3.7], it is an orbitally equivalent normal form, consequently it is not homogenizable.

In this paper, we study the analytic integrability of the analytic perturbations of

$$(-d^2(p+q)y^3 - (p+d^2q)x^2y, (p+q)x^3 + (d^2p+q)xy^2)^T,$$

with $p, q \in \mathbb{N}$ and $d \neq 0, \ d \neq \pm 1$. The origin is a monodromic isolated singularity, *i.e.* it is a focus or a center because of the analyticity. So, the analytic integrability is a sufficient condition of center. Here, we prove that the origin is analytically integrable if, and only if, it is homogenizable, Theorem 2.4.

Summarizing, by [1, 7], Theorem 2.3 and Theorem 2.4, we provide the following result, for generic case $n = 3$.

Theorem 1.1 *Consider the analytic vector field* $\mathbf{F} = \mathbf{F}_3 + h.o.t.$ *and assume that the origin of* $\mathbf{F}_3$ *is an isolated singular point and* $\mathbf{F}_3$ *is irreducible.*
If $\mathbf{F}_3$ *is not polynomially integrable, then* $\mathbf{F}$ *is not formally integrable.*
Otherwise, assume that $\mathbf{F}_3$ *is polynomially integrable. Then* $\mathbf{F}$ *is analytically integrable if, and only if,* $\mathbf{F}$ *is homogenizable, i.e.* $\mathbf{F}$ *and* $\mathbf{F}_3$ *are orbitally equivalent.*

However, in general, we note that such a result for $n \geq 4$ is not satisfied. Theorem 3.20 of [7] provides a Hamiltonian vector field (perturbation of $\mathbf{F}_n$, homogeneous vector field of degree $n \geq 4$) whose Hamiltonian function is polynomial, therefore is analytically integrable, and is non-orbitally equivalent to its leading term $\mathbf{F}_n$.

For vector fields whose origin of the lowest degree term is a non-isolated singular point, the problem remains open. We know only some partial results. Recently, it is proved that a nilpotent singular point of an analytic vector field (whose lowest-degree quasi-homogeneous term is integrable) is analytically integrable if, and only if, the vector field is orbital equivalent to its lowest-degree term [3, 4].

We also give a result that allows to characterize the analytic integrability through the existence of a formal inverse integrating factor, Theorem 2.5. This result is used in Section 3 for obtaining sufficient conditions of analytic integrability of two families of polynomial systems whose origin is a center, Theorems 3.6 and 3.7.

We note that the centers obtained are reversible. However, there are non-analytically integrable reversible center of the families considered.

2 Main results

From [2, Prop.2.7], every $\mathbf{F}_n \in \mathcal{H}_n$ (vector space of polynomial homogeneous vector fields of degree n) can be uniquely written as $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D}$ where $\mathbf{D} = (x, y)^T \in \mathcal{H}_1$ (dissipative homogeneous vector field) and $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to the polynomial h)with $h := \frac{1}{n+1}(\mathbf{D} \wedge \mathbf{F}_n) \in \mathcal{P}_{n+1}$ (wedge product of both vector fields) and $\mu := \frac{1}{n+1}\mathrm{div}(\mathbf{F}_n) \in \mathcal{P}_{n-1}$.

In what follows, h is called conservative part of $\mathbf{F}_n$ and μ dissipative part of $\mathbf{F}_n$.

2.1 Necessary condition of analytic integrability

Now we study the integrability problem for the cubic homogeneous polynomial vector fields.

Proposition 2.2 *We consider $\mathbf{F}_3 = (P_3, Q_3)^T$ with P_3 and Q_3 cubic homogeneous polynomials. If $\mathbf{F}_3$ is polynomially integrable, then there exist a linear change of variables and a linear re-parameterization of the time such that $\mathbf{F}_3$ is transformed into one of the following vector fields:*
(i)

$$\mathbf{F}_{3,i} = (0, x^3)^T, \tag{2.1}$$

($\mathbf{F}_{3,i}$ is a reducible polynomial vector field with one invariant line) and x is a polynomial first integral of $\mathbf{F}_{3,i}$.
(ii)

$$\mathbf{F}_{3,ii} = (-y(x^2 + y^2), x(x^2 + y^2))^T, \tag{2.2}$$

($\mathbf{F}_{3,ii}$ is a reducible polynomial vector field with two complex invariant lines) and $(x^2 + y^2)^p$ is a polynomial first integral of $\mathbf{F}_{3,ii}$.
(iii)

$$\mathbf{F}_{3,iii} = (-qx^3, px^2y)^T, \tag{2.3}$$

($\mathbf{F}_{3,iii}$ is a reducible polynomial vector field with two invariant lines) with x^py^q is a polynomial first integral of $\mathbf{F}_{3,iii}$.
(iv)

$$\mathbf{F}_{3,iv} = (-qx^2y, pxy^2)^T, \tag{2.4}$$

($\mathbf{F}_{3,iv}$ is a reducible polynomial vector field with two invariant lines) with x^py^q is a polynomial first integral of $\mathbf{F}_{3,iv}$.
(v)

$$\mathbf{F}_{3,v} = (x^2(-qx + (q + r)y), xy((p + r)x - py))^T, \tag{2.5}$$

($\mathbf{F}_{3,v}$ is a reducible polynomial vector field with three invariant lines) with $x^py^q(x-y)^r$ is a polynomial first integral of $\mathbf{F}_{3,v}$.
(vi)

$$\mathbf{F}_{3,vi} = (-2qx^2y, (p + 2q)x^3 + pxy^2)^T, \tag{2.6}$$

($\mathbf{F}_{3,vi}$ is a reducible polynomial vector field with one real invariant line and two complex invariant lines) with $x^p(x^2 + y^2)^q$ is a polynomial first integral of $\mathbf{F}_{3,vi}$.
(vii)

$$\begin{aligned}\mathbf{F}_{3,vii} = (&x(-qx^2 + [c(q + s) + r + q]xy - c(q + r + s)y^2),\\ &y((p + r + s)x^2 - [c(p + r) + s + p]xy + pcy^2))^T,\end{aligned} \tag{2.7}$$

with $c \in [-1,1), c \neq 0$ *(*$\mathbf{F}_{3,vii}$ *is an irreducible polynomial vector field with four real invariant lines) and* $x^p y^q (x-y)^r (x-cy)^s$ *is a polynomial first integral of* $\mathbf{F}_{3,vii}$.
(viii)

$$\mathbf{F}_{3,viii} = (x(-qx^2 - (q+2r)y^2), y((p+2r)x^2 + py^2))^T, \tag{2.8}$$

($\mathbf{F}_{3,viii}$ *is an irreducible polynomial vector field with two real invariant lines and two complex invariant lines) with* $x^p y^q (x^2+y^2)^r$ *is a polynomial first integral of* $\mathbf{F}_{3,viii}$.
(ix)

$$\mathbf{F}_{3,ix} = (-d^2(p+q)y^3 - (p+d^2q)x^2y, (p+q)x^3 + (d^2p+q)xy^2)^T, \tag{2.9}$$

with $p, q \in \mathbb{N}$ *and* $d \neq 0,\ d \neq \pm 1$, *(*$\mathbf{F}_{3,ix}$ *is an irreducible polynomial vector field with four complex invariant lines) and* $(x^2+y^2]^p(x^2+d^2y^2)^q$ *is a polynomial first integral of* $\mathbf{F}_{3,ix}$.

Proof. Let $I = I_M + \text{h.o.t.}$ be a polynomial first integral of $\mathbf{F}_3$. Equation $F_3(I) = 0$ for degree $M+2$ is $F_3(I_M) = 0$, i.e. I_M is a first integral of $\mathbf{F}_3$.
We seek the vector fields $\mathbf{F}_3$ satisfying the condition $F_3(I_M) = 0$. By [2, Prop. 2.7], $\mathbf{F}_3$ is $\mathbf{F}_3 = \mathbf{X}_h + \mu\mathbf{D}$ with $h \in \mathcal{P}_4$ and $\mu \in \mathcal{P}_2$.
If the polynomial h, conservative part of $\mathbf{F}_3$, is identically zero, it has that $\mathbf{F}_3 = \mu\mathbf{D}_0$ and it is non-formally integrable.
Thus, we assume that h is non-zero. By [9, Theorem 3.1], if $\tilde{\mathbf{F}}_3$ is integrable, a first integral is a factorization on $\mathbf{C}[x,y]$ of the factors of h.

According the factors of h, after a linear change of variables and a linear reparameterization of the time, $\mathbf{F}_3$ is transformed into $\tilde{\mathbf{F}}_3 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}$ with $\tilde{h} \in \mathcal{P}_4$ and $\tilde{\mu} \in \mathcal{P}_2$, where $\tilde{h}$ has one of the following expressions and $\tilde{\mu} = Ax^2 + Bxy + Cy^2$:
• $\tilde{h} = x^4$ (it has only one real factor of order 4). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M = x^p$. By imposing $\tilde{F}_3(\tilde{I}_M) = 0$, we arrive to $\tilde{\mu} = 0$. So, $\mathbf{F}_3$ is an Hamiltonian vector field.
By performing the reparameterization of the time $4t = \tau$, $\tilde{\mathbf{F}}_3$ turns on (2.1).
• $\tilde{h} = (x^2+y^2)^2$ (it has one complex factor of order 2). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M = (x^2+y^2)^p$. By imposing $\tilde{F}_3(\tilde{I}_M) = 0$, we arrive to $\tilde{\mu} = 0$. Again, $\mathbf{F}_3$ is an Hamiltonian vector field.
By performing the reparameterization of the time $4t = \tau$, $\tilde{\mathbf{F}}_3$ turns on (2.2).
• $\tilde{h} = x^3y$ (it has one real factor of order 3 and one real simple factor). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M = x^py^q$. By imposing $\tilde{F}_3(\tilde{I}_M) = 0$, we arrive to $\tilde{\mu} = \frac{p-3q}{p+q}x^2$. By performing the reparameterization of the time $4t = (p+q)\tau$, $\tilde{\mathbf{F}}_3$ turns on (2.3).
• $\tilde{h} = x^2y^2$ (it has two real factors of order 2). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M = x^py^q$. By imposing $\tilde{F}_3(\tilde{I}_M) = 0$, we arrive to $\tilde{\mu} = \frac{2(p-q)}{p+q}xy$. By performing the reparameterization of the time $4t = (p+q)\tau$, $\tilde{\mathbf{F}}_3$ turns on (2.4).
• $\tilde{h} = x^2y(x-y)$ (it has one real factor of order 2 and two real simple factors). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M = x^py^q(x-y)^r$. By imposing $\tilde{F}_3(\tilde{I}_M) = 0$, we arrive to $\tilde{\mu} = \frac{p-3q+r}{p+q+r}x^2 - \frac{2(p-q-r)}{p+q+r}xy$. By performing the reparameterization of the time $4t = (p+q+r)\tau$, $\tilde{\mathbf{F}}_3$ turns on (2.5).
• $\tilde{h} = x^2(x^2+y^2)$ (it has one real factor of order 2 and one complex simple factor). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M = x^p(x^2+y^2)^q$. By imposing $\tilde{F}_3(\tilde{I}_M) = 0$, we arrive to $\tilde{\mu} = \frac{2(p-2q)}{(p+2q)}xy$. By performing the reparameterization of the time $4t = (p+2q)\tau$, $\tilde{\mathbf{F}}_3$ turns on (2.6).
• $h = xy(x-y)(x-cy)$ (it has four real simple factors). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M = x^py^q(x-y)^r(x-cy)^s$. By imposing $\tilde{F}_3(\tilde{I}_M) = 0$, we arrive to

$$A = \frac{1}{M}(r+s-3q+p), B = \frac{2}{M}(qc+r-pc-rc-p-s+q+sc), C = \frac{c}{M}(3p-q-r-s),$$

being $M=p+q+r+s$. By performing the reparameterization of the time $4t=M\tau$, $\tilde{\mathbf{F}}_3$ turns on (2.7).
• $\tilde{h}=xy(x^2+y^2)$ (it has two real simple factors and one complex simple factor). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M=x^p y^q (x^2+y^2)^r$. By imposing $\tilde{F}_3(\tilde{I}_M)=0$, we arrive to $\tilde{\mu}=\frac{(p-3q+2r)}{(p+q+2r)}x^2+\frac{(3p-q-2r)}{(p+q+2r)}y^2$. By performing the reparameterization of the time $4t=(p+q+2r)\tau$, $\tilde{\mathbf{F}}_3$ turns on (2.8).

• $\tilde{h}=(x^2+y^2)(x^2+d^2y^2)$ (it has two complex simple factors). If it existed a first integral of $\tilde{\mathbf{F}}_3$, it would have the expression $\tilde{I}_M=(x^2+y^2)^p(x^2+d^2y^2)^q$. By imposing $\tilde{F}_3(\tilde{I}_M)=0$, we arrive to $\tilde{\mu}=\frac{2(d^2-1)(p-q)}{(p+q)}xy$. By performing the reparameterization of the time $4t=(p+q)\tau$, $\tilde{\mathbf{F}}_3$ turns on (2.9). ∎

Theorem 2.3 (Necessary condition of analytic integrability) *We consider the analytic vector field* $\mathbf{F}=\mathbf{F}_3+h.o.t.$ *with* $\mathbf{F}_3\in\mathcal{H}_3$. *If* $\mathbf{F}$ *is analytically integrable, then there exist a linear change of variables and a linear re-parameterization of the time such that* $\mathbf{F}_3$ *is transformed into one of the vector fields (2.1)-(2.9).*

Proof. We assume that $I=I_M+$ h.o.t. is an analytic first integral of $\mathbf{F}$. Equation $F(I)=0$ for degree $M+2$ is $F_3(I_M)=0$, i.e. $\mathbf{F}_3$ is polynomially integrable and I_M is a first integral of $\mathbf{F}_3$. Applying Proposition 2.2, the result follows. ∎

It is easy to check that the necessary condition provided is not a sufficient condition of integrability. In Algaba *et al.* [7], we find some vector fields not analytically integrable vector fields that are a perturbation of the vector field $\mathbf{F}_{3,vii}$ with $(p,q,r,s)=(1,1,1,1)$ and $c=-1$.

2.2 Characterization of analytic integrability

Our purpose is to characterize the analytically integrable perturbations of a cubic homogeneous vector field $\mathbf{F}_3$ whose origin of $\mathbf{F}_3$ is an isolated singularity and $\mathbf{F}_3$ is irreducible.
From Theorem 2.3, these vector fields, after a linear change of variables and linear re-parameterization of the time, can be $\mathbf{F}_{3,vii}+$ h.o.t., $\mathbf{F}_{3,viii}+$ h.o.t. or $\mathbf{F}_{3,ix}+$ h.o.t. only.
[7, Theorem 3.16] proves that the analytically integrable perturbations of $\mathbf{F}_{3,vii}$ and $\mathbf{F}_{3,viii}$ are homogenizable. Therefore, to prove Theorem 1.1 remains only to prove that the analytically integrable perturbations of $\mathbf{F}_{3,ix}$ are also homogenizable. In this paper, we give such result.

Theorem 2.4 *Consider* $\mathbf{F}=\mathbf{F}_{3,ix}+h.o.t.$. *The vector field* $\mathbf{F}$ *is analytically integrable at the origin if, and only if, it is orbitally equivalent to* $\mathbf{F}_{3,ix}$.
Moreover, in such a case, the origin is a center and $\mathbf{F}$ *has an analytic first integral of the form* $I=(x^2+y^2+h.o.t.)^p(x^2+d^2y^2+h.o.t.)^q$.

Proof of Theorem 2.4 is in Section 4.
Remark. For higher degeneracies $\mathbf{F}=\mathbf{F}_n+$ h.o.t. with $n\geq 4$, there are non-homogenizable analytically integrable vector fields, [7, Theorem 3.20]. Also, for $n=5$, we find non-homogenizable analytically integrable centers. For example, the polynomial vector field $\mathbf{F}=(-y^5+3y^2x^4, x^5-4x^3y^3)^T$ has the first integral $x^6+y^6-6y^3x^4$ and its origin is a center since it is an integrable monodromic singular point.
However, it is not orbitally equivalent to $\mathbf{F}_5$. Indeed, from [8, Theorem 1.3], if $\mathbf{F}$ would be orbitally equivalent to $\mathbf{F}_5$, there would be a formal vector field $\mathbf{G}=(x+u_{20}x^2+u_{11}xy+u_{02}y^2+\text{h.o.t.}, y+v_{20}x^2+v_{11}xy+v_{02}y^2+\text{h.o.t.})^T$ and a formal scalar function $\nu=4+d_{10}x+d_{01}y+$ h.o.t. such that $[\mathbf{F}_5+\mathbf{F}_6,\mathbf{G}]=\nu(\mathbf{F}_5+\mathbf{F}_6)$, i.e.,

$$(D\mathbf{F}_6)\mathbf{G}-(D\mathbf{G})\mathbf{F}_6-\nu\mathbf{F}_6+(D\mathbf{F}_5)\mathbf{G}-(D\mathbf{G})\mathbf{F}_5-\nu\mathbf{F}_5=0. \qquad (2.10)$$

The coefficient of x^4y^2 of the first component of (2.10) is 3. So, equation (2.10) is not satisfied and $\mathbf{F}$ is not orbitally equivalent to $\mathbf{F}_5$.

We solve the analytical integrability problem through the existence of a formal inverse integrating factor.

Theorem 2.5 *The vector field* $\mathbf{F} = \mathbf{F}_{3,ix} + h.o.t.$ *is analytically integrable if, and only if, it has a formal inverse integrating factor of the form* $V = (x^2+y^2)(x^2+d^2y^2)+h.o.t.$.

Proof of Theorem 2.5 is in Section 4.

3 Applications

We study the analytic integrability problem for the following family with non-hamiltonian first homogeneous component

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5x^2y - 8y^3 \\ 4x^3 + 7xy^2 \end{pmatrix} + \begin{pmatrix} a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{04}y^4 \\ b_{31}x^3y + b_{13}xy^3 + b_{04}y^4 \end{pmatrix} \tag{3.11}$$

with $a_{ij}, b_{ij} \in \mathbb{R}$.

The vector field associated is $\mathbf{F} = \mathbf{F}_3 + \mathbf{F}_4$ being $\mathbf{F}_3 = \mathbf{F}_{3,ix}$ for $(p,q) = (3,1)$. The origin of this family is a monodromic singular point.

Leading component of the vector field can be expressed as $\mathbf{F}_3 = \mathbf{X}_h + \mu\mathbf{D}$ with $h = (x^2+y^2)(x^2+2y^2)$ and $\mu = xy$. From Proposition 2.2, the vector field $\mathbf{F}_3$ is polynomially integrable and a primitive first integral is $I_8 = (x^2+y^2)^3(x^2+2y^2)$.

The following result solves the analytic integrability problem of the family (3.11).

Theorem 3.6 *System (3.11) is analytically integrable if, and only if, one of the following conditions holds:*
(i) $b_{04} = a_{31} = 3a_{04} + 2b_{13} - 2a_{22} = 7b_{31} - 4b_{13} = 14a_{40} - 7a_{04} - 8b_{13} = 0$,
(ii) $b_{04} = a_{31} = 3b_{31} + 12b_{13} + 4a_{22} = 8a_{04} - 3b_{31} + 12b_{13} = 8a_{40} + b_{31} = 0$,
(iii) $b_{04} = a_{31} = 11b_{31} - 9b_{13} - 2a_{22} = 2a_{04} - 33b_{31} + 22b_{13} = 8a_{40} + 3b_{31} = 0$,
(iv) $b_{04} = a_{31} = 10b_{13} - 11a_{22} = 11b_{31} - 4b_{13} = 88a_{04} + 35b_{13} = 11a_{40} - 6b_{13} = 0$,
(v) $b_{04} = a_{31} = 67b_{13} - 26a_{22} = 13b_{31} - 8b_{13} = 104a_{04} - 209b_{13} = 13a_{40} - 12b_{13} = 0$.

Proof. To prove the necessity, we have computed successively the coefficients of the expression of $F(I)$ up to order 16, with $I = (x^2+y^2)^3(x^2+2y^2)+$h.o.t., and imposing their vanishing. Equation $F(I) = 0$ up to order 16 provides 16 compatibility's conditions (two for each degree of expression of $F(I)$, corresponding to the degrees from 11 to 18) but we do not write them here. Their vanishing leads to the systems (3.11) for cases *(i)*–*(v)*.

We prove the sufficiency for each case.
(i) In this case, system (3.11) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5x^2y - 8y^3 \\ 4x^3 + 7xy^2 \end{pmatrix} + \begin{pmatrix} (\frac{1}{2}a_{04} + \frac{4}{7}b_{13})x^4 + (\frac{3}{2}a_{04} + b_{13})x^2y^2 + a_{04}y^4 \\ \frac{4}{7}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = x^2+y^2$, $C_2 = x^2+2y^2$ and $C_3 = 7+b_{13}y$. The polynomial $V = C_1C_2C_3$ is a polynomial inverse integrating factor starting with h. By Theorem 2.5, the vector field is analytically integrable.

(ii) In this case, system (3.11) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5x^2y - 8y^3 \\ 4x^3 + 7xy^2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{8}b_{31}x^4 - (\frac{3}{4}b_{31} + \frac{1}{2}b_{13})x^2y^2 + (\frac{3}{8}b_{31} - \frac{3}{2}b_{13})y^4 \\ b_{31}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = x^2 + y^2$ and $C_2 = 12x^2 + 24y^2 + 3b_{31}x^2y + (4b_{13} - b_{31})y^3$. The polynomial $I = C_1^3C_2$ is a first integral and $V = C_1C_2$

is a polynomial inverse integrating factor.

(iii) In this case, system (3.11) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5x^2y-8y^3 \\ 4x^3+7xy^2 \end{pmatrix} + \begin{pmatrix} -\frac{3}{8}b_{31}x^4+(\frac{11}{2}b_{31}-\frac{9}{2}b_{13})x^2y^2+(\frac{33}{2}b_{31}-11b_{13})y^4 \\ b_{31}x^3y+b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 4x^2+4y^2+b_{31}x^2y+(4b_{13}-6b_{31})y^3$ and $C_2 = x^2+2y^2$ and $I = C_1^3C_2$ is a polynomial first integral. Moreover, $V = C_1C_2$ is a polynomial inverse integrating factor starting with h.

(iv) In this case, system (3.11) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5x^2y-8y^3 \\ 4x^3+7xy^2 \end{pmatrix} + \begin{pmatrix} \frac{6}{11}b_{13}x^4+\frac{10}{11}b_{13}x^2y^2-\frac{35}{88}b_{13}y^4 \\ \frac{4}{11}b_{13}x^3y+b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 44x^2+44y^2+4b_{13}x^2y+5b_{13}y^3$, $C_2 = 5324x^2+10648y^2-108b_{13}^3x^2y^3-135b_{13}^3y^5-704b_{13}^2x^4-2156b_{13}^2x^2y^2-1298b_{13}^2y^4+1452b_{13}x^2y+3509b_{13}y^3$ and $C_3 = 14641+16b_{13}^4x^4-8b_{13}^4x^2y^2-35b_{13}^4y^4-528b_{13}^3x^2y-528b_{13}^3y^3-1936b_{13}^2x^2-605b_{13}^2y^2+7986b_{13}y$. It has the analytic first integral $I = C_1^3C_2C_3^{-3}$ and $V = C_1C_2C_3^{-1/2}$ is an inverse integrating factor starting with h.

(v) In this case, system (3.11) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5x^2y-8y^3 \\ 4x^3+7xy^2 \end{pmatrix} + \begin{pmatrix} +\frac{12}{13}b_{13}x^4+\frac{67}{26}b_{13}x^2y^2+\frac{209}{104}b_{13}y^4 \\ \frac{8}{13}b_{13}x^3y+b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 1352x^2+1352y^2-27b_{13}^2y^4+208b_{13}x^2y+26b_{13}y^3$ and $C_2 = 1352x^2+2704y^2+64b_{13}^2x^4+304b_{13}^2x^2y^2+361b_{13}^2y^4+624b_{13}x^2y+1274b_{13}y^3$. The function $V = C_1^{1/2}C_2^{5/6}$ is an inverse integrating factor.
We are going to prove that the existence of V implies that the vector field is linearizable. Indeed, the change of variables and the re-parameterization of the time that transforms $\mathbf{F}$ into its normal form $\tilde{\mathbf{F}} = \mathbf{F}_3 + \sum_{j>3}\mu_j\mathbf{D}$, it transforms V into $\tilde{V} = (x^2+y^2+\cdots)^{1/2}(x^2+2y^2+\cdots)^{5/6}$, an inverse integrating factor of $\tilde{\mathbf{F}}$, i.e. $\tilde{F}(\tilde{V}) = \tilde{V}\mathrm{div}(\tilde{\mathbf{F}})$.

On the other hand, the factors of $\tilde{V}$ are invariant curves starting with x^2+y^2 and x^2+2y^2. Moreover, x^2+y^2 and x^2+2y^2 are also invariant curves of $\tilde{\mathbf{F}}$ and applying [5, Theorem 11], we claim that are unique. Therefore, $\tilde{V}$ also has the expression $\tilde{V} = (x^2+y^2)^{1/2}(x^2+2y^2)^{5/6}u$ with u a unit function.

The formal function $\tilde{W} := \tilde{V}^6 = HU$ with $H = (x^2+y^2)^3(x^2+2y^2)^5 \in \mathcal{P}_{16}$ and U the unit function given by $u^6 = U$, satisfies the equation

$$\tilde{F}(\tilde{W}) = 6\tilde{W}\mathrm{div}(\tilde{\mathbf{F}}). \tag{3.12}$$

The Eq. (3.12) is

$$0 = \tilde{F}(HU) - 6HU\mathrm{div}(\tilde{\mathbf{F}}) = U\tilde{F}(H) + H\tilde{F}(U) - 6HU\mathrm{div}(\tilde{\mathbf{F}}).$$

As $F_3(H) = 6H\mathrm{div}(\mathbf{F}_3)$, $D(H) = 16H$ and $\mathrm{div}(\mu_j\mathbf{D}) = (j+2)\mu_j$, the Eq. (3.12) becomes $0 = H(\tilde{F}(U) - U\sum_{j>3}(6(j+2)-16)\mu_j)$, i.e. $\tilde{F}(U) = U\sum_{j>3}(6j-4)\mu_j$.

We claim that $\mu_{2+j} = 0$ for all $j \geq 1$ because otherwise we can consider $j_0 = \min\{j \in \mathbb{N} : \mu_{2+j} \neq 0\}$ and Eq. (3.12) for degree $2+j_0$ is $F_3(U_{j_0}) = (6j_0+8)\mu_{2+j_0}$, therefore $F_3(U_{j_0}) \in \mathrm{Range}(\ell_{2+j_0})$ and $\mu_{2+j_0} \in \mathrm{Cor}(\ell_{2+j_0})$, consequently $\mu_{2+j_0} = 0$ which is a contradiction and the claim is proved.

This concludes the proof. ∎

Last on, we consider the following family

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2+4y^2) \\ 3x(x^2+2y^2) \end{pmatrix} + \begin{pmatrix} a_{40}x^4 + a_{22}x^2y^2 + a_{04}y^4 \\ b_{31}x^3y + b_{13}xy^3 + b_{04}y^4 \end{pmatrix} \tag{3.13}$$

with $a_{ij}, b_{ij} \in \mathbb{R}$. We limit our study for $b_{13} \neq 0$.

The vector field associated is $\mathbf{F} = \mathbf{F}_3 + \mathbf{F}_4$ being $\mathbf{F}_3 = \mathbf{F}_{3,ix}$ for $(p,q) = (1,2)$. The origin of this family is a monodromic singular point and is a reversible family, therefore, the origin is a center.

The vector field is given by $\mathbf{F}_3 = \mathbf{X}_h + \mu\mathbf{D}$ with $h = \frac{3}{4}(x^2+y^2)(x^2+4y^2)$ and $\mu = -\frac{3}{2}xy$. From Proposition 2.2, $\mathbf{F}_3$ is polynomially integrable and a primitive first integral is $I_6 = (x^2+y^2)(x^2+4y^2)^2$.

The following result solves the analytic integrability problem of the family (3.13) with $b_{13} \neq 0$.

Theorem 3.7 *System (3.13) with $b_{13} \neq 0$, is analytically integrable if, and only if, one of the following conditions holds:*
(i) $b_{04} = 10a_{40} - 3b_{13} - 2a_{22} = 2b_{31} - b_{13} = 4a_{40} - a_{04} - 2b_{13} = 0$,
(ii) $b_{04} = 28b_{31} + 9b_{13} + 12a_{22} = 3a_{04} - 28b_{31} + 21b_{13} = 6a_{40} + b_{31} = 0$,
(iii) $b_{04} = 4b_{31} - 9b_{13} - 3a_{22} = 3a_{04} - 8b_{31} + 12b_{13} = 3a_{40} + b_{31} = 0$,
(iv) $b_{04} = 11b_{13} + 6a_{22} = 6b_{31} - b_{13} = a_{04} + 6b_{13} = 6a_{40} + b_{13} = 0$,
(v) $b_{04} = b_{13} + 6a_{22} = 3b_{31} - 2b_{13} = 3a_{04} + 2b_{13} = 6a_{40} - b_{13} = 0$,
(vi) $b_{04} = 5b_{13} + 6a_{22} = 3b_{31} - b_{13} = 3a_{04} + 10b_{13} = 12a_{40} - b_{13} = 0$,
(vii) $b_{04} = 3b_{13} - 4a_{22} = a_{04} = 8b_{31} - 3b_{13} = 16a_{40} - 9b_{13} = 0$,
(viii) $b_{04} = 5b_{13} + 2a_{22} = 2b_{31} - b_{13} = 3a_{04} + 10b_{13} = 4a_{40} + b_{13} = 0$,
(ix) $b_{04} = 2b_{13} - a_{22} = 2b_{31} - b_{13} = 3a_{04} - 4b_{13} = 4a_{40} - 3b_{13} = 0$.

Proof. To prove the necessity, we have computed successively the coefficients of the expression of $F(I)$ up to order 17, with $I = (x^2+y^2)(x^2+4y^2)^2 + \text{h.o.t.}$, and imposing their vanishing. Equation $F(I) = 0$ up to order 16 provides six compatibility's conditions, corresponding to the degrees 9, 11, 12, 13, 15 and 17, but we do not write them here. Their vanishing leads to the systems (3.13) for cases *(i)–(ix)*.

We prove the sufficiency for each case. We have found an analytic first integral for each case. Also, we provide an inverse integrating factor starting with $(x^2+y^2)(x^2+4y^2)$ for each case except for *(viii)*.

(i) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2+4y^2) \\ 3x(x^2+2y^2) \end{pmatrix} + \begin{pmatrix} a_{40}x^4 + (5a_{40} - \frac{3}{2}b_{13})x^2y^2 + (4a_{40} - 2b_{13})y^4 \\ \frac{1}{2}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = x^2+y^2$, $C_2 = x^2+4y^2$ and $C_3 = 6+b_{13}y$. It has the analytic first integral $I = C_1C_2^2C_3^{-12a_{40}/b_{13}}$. Moreover, $V = C_1C_2C_3$ is a polynomial inverse integrating factor starting with h.

(ii) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2+4y^2) \\ 3x(x^2+2y^2) \end{pmatrix} + \begin{pmatrix} -\frac{1}{6}b_{31}x^4 - (\frac{7}{3}b_{31} + \frac{3}{4}b_{13})x^2y^2 + (\frac{28}{3}b_{31} - 7b_{13})y^4 \\ b_{31}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 6x^2 + 6y^2 + 2b_{31}x^2y + (3b_{13} - 4b_{31})y^3$ and $C_2 = x^2+4y^2$. The polynomial $I = C_1C_2^2$ is a first integral and $V = C_1C_2$ is a polynomial inverse integrating factor.

(iii) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2+4y^2) \\ 3x(x^2+2y^2) \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}b_{31}x^4 + (\frac{4}{3}b_{31} - 3b_{13})x^2y^2 + (\frac{8}{3}b_{31} - 4b_{13})y^4 \\ b_{31}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = x^2 + y^2$ and $C_2 = 3x^2 + 12y^2 + b_{31}x^2y + (3b_{13} - 2b_{31})y^3$ and $I = C_1C_2^2$ is a polynomial first integral. Moreover, $V = C_1C_2$ is a polynomial inverse integrating factor starting with h.

(iv) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2 + 4y^2) \\ 3x(x^2 + 2y^2) \end{pmatrix} + \begin{pmatrix} -\frac{1}{6}b_{13}x^4 - \frac{11}{6}b_{13}x^2y^2 - 6b_{13}y^4 \\ \frac{1}{6}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 46656x^2 + 46656y^2 + b_{13}^4x^4y^2 + 8b_{13}^4x^2y^4 + 16b_{13}^4y^6 + 36b_{13}^3x^4y + 288b_{13}^3x^2y^3 + 576b_{13}^3y^5 + 324b_{13}^2x^4 + 1296b_{13}^2x^2y^2 - 432b_{13}^2y^4 + 15552b_{13}y^3$, $C_2 = x^2 + 4y^2$ and $C_3 = 324 + 2b_{13}^2x^2 + 9b_{13}^2y^2 - 36b_{13}y$. It has the analytic first integral $I = C_1C_2^2C_3^{-3}$ and $V = C_1C_2C_3^{-1/2}$ is an inverse integrating factor starting with h.

(v) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2 + 4y^2) \\ 3x(x^2 + 2y^2) \end{pmatrix} + \begin{pmatrix} +\frac{1}{6}b_{13}x^4 - \frac{1}{6}b_{13}x^2y^2 - \frac{2}{3}b_{13}y^4 \\ \frac{2}{3}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = x^2 + y^2$, $C_2 = \frac{1}{810}b_{13}^2x^4 + \frac{1}{405}b_{13}^2x^2y^2 + \frac{1}{810}b_{13}^2y^4 + \frac{1}{9}b_{13}x^2y + \frac{1}{9}b_{13}y^3 + x^2 + 4y^2$ and $C_3 = 324 + b_{13}^2x^2 + b_{13}^2y^2 + 36b_{13}y$. It has the analytic first integral $I = C_1C_2^2C_3^{-5}$ and the polynomial $V = C_1C_2C_3$ is an inverse integrating factor starting with h.

(vi) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2 + 4y^2) \\ 3x(x^2 + 2y^2) \end{pmatrix} + \begin{pmatrix} \frac{1}{12}b_{13}x^4 - \frac{5}{6}b_{13}x^2y^2 - \frac{10}{3}b_{13}y^4 \\ \frac{1}{3}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 2592x^2 + 2592y^2 - b_{13}^2x^4 - 8b_{13}^2x^2y^2 - 16b_{13}^2y^4 + 144b_{13}x^2y + 576b_{13}y^3$, $C_2 = x^2 + 4y^2$ and $C_3 = 648 - b_{13}^2x^2 - 4b_{13}^2y^2 + 36b_{13}y$. It has the analytic first integral $I = C_1C_2^2C_3^{-4}$. Moreover, $V = C_1C_2C_3$ is a polynomial inverse integrating factor starting with h.

(vii) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2 + 4y^2) \\ 3x(x^2 + 2y^2) \end{pmatrix} + \begin{pmatrix} \frac{9}{16}b_{13}x^4 + \frac{3}{4}b_{13}x^2y^2 \\ \frac{3}{8}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 768x^2 + 768y^2 - 9b_{13}^2x^4 + 288b_{13}x^2y + 256b_{13}y^3$, $C_2 = 3x^2 + 12y^2 + b_{13}y^3$ and $C_3 = 2304 - 3b_{13}^3x^2y + 4b_{13}^3y^3 - 24b_{13}^2x^2 + 144b_{13}^2y^2 + 1152b_{13}y$. The analytic function $I = C_1C_2^2C_3^{-3}$ is an analytic first integral and $V = C_1C_2C_3^{-1/2}$ is an inverse integrating factor starting with h.

(viii) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2 + 4y^2) \\ 3x(x^2 + 2y^2) \end{pmatrix} + \begin{pmatrix} -\frac{1}{4}b_{13}x^4 - \frac{5}{2}b_{13}x^2y^2 - \frac{10}{3}b_{13}y^4 \\ \frac{1}{2}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 18x^2 + 18y^2 + 3b_{13}x^2y + 4b_{13}y^3$ and $C_2 = 18x^2 + 72y^2 + 3b_{13}x^2y + 10b_{13}y^3$. It has the polynomial first integral $I = C_1C_2^2$.

(ix) In this case, system (3.13) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3y(3x^2 + 4y^2) \\ 3x(x^2 + 2y^2) \end{pmatrix} + \begin{pmatrix} \frac{3}{4}b_{13}x^4 + 2b_{13}x^2y^2 + \frac{4}{3}b_{13}y^4 \\ \frac{1}{2}b_{13}x^3y + b_{13}xy^3 \end{pmatrix}.$$

This vector field has the invariant curves $C_1 = 18x^2+18y^2-b_{13}y^3,\ C_2 = 9x^2+36y^2+b_{13}y^3$ and $C_3 = 6+b_{13}y$. It has the analytic first integral $I = C_1C_2^2C_3^{-9}$. Moreover, $V = C_1C_2C_3$ is a polynomial inverse integrating factor starting with h.
This concludes the proof. ■

4 Proofs of Theorems 2.4 and 2.5

Basov [10] has provided an orbital normal forms for perturbations of homogeneous cubic systems. Next result provides also an orbital normal form of the perturbations of the vector field $\mathbf{F}_{3,ix}$. From [2, Prop.2.7], the homogeneous vector field is $\mathbf{F}_{3,ix} = \mathbf{X}_h + \mu\mathbf{D}$, with

$$h := \tfrac{p+q}{4}(x^2+y^2)(x^2+d^2y^2), \quad \mu := \tfrac{p-q}{2}(d^2-1)xy.$$

Proposition 4.8 *The vector field* $\mathbf{F} = \mathbf{F}_{3,ix} + h.o.t.$ *is orbitally equivalent to* $\mathbf{F}_{3,ix} + \sum_{j\geq 3}\eta_j\mathbf{D}$, *with* $\eta_j \in \mathrm{Cor}(\ell_j)$, *a complementary subspace to Range*(ℓ_j), *where* ℓ_j *is the Lie operator of* $\mathbf{F}_{3,ix}$, *i.e.*

$$\begin{aligned}\ell_j \quad : \quad & \mathcal{P}_{j-2} \longrightarrow \mathcal{P}_j^{\mathbf{t}} \\ & \eta_{j-2} \longrightarrow F_{3,ix}(\eta_{j-2}).\end{aligned}$$

Proof. From [7, Theorem A.32], to prove this result it is enough to check that for all $j > 3$, $\mathrm{Ker}\left(\ell_{j+4}^{\mathrm{c}}\right) = \{0\}$, where ℓ_j^{c} is the Lie operator of $\mathbf{F}_{3,ix}$ moved,

$$\begin{aligned}\ell_j^{\mathrm{c}} \quad : \quad & \Delta_{j-2} \longrightarrow \Delta_j \\ & g_{j-2} \longrightarrow \mathrm{Proy}_{\Delta_j}(F_{3,ix} - \tfrac{4}{j}\mu D)(g_{j-2}),\end{aligned}$$

with $\Delta_j,\ j > 2$, the subspaces such that $\mathcal{P}_j = \Delta_j \bigoplus h\mathcal{P}_{j-3}$ (such subspaces must be considered as fixed).

We prove that $\mathrm{Ker}\left(\ell_{j+4}^{\mathrm{c}}\right) = \{0\}$, for all $j > 3$. Indeed, let $p_{j+2} \in \mathrm{Ker}\left(\ell_{j+4}^{\mathrm{c}}\right) \subset \Delta_{j+2}$, i.e. $(F_{3,ix} - \frac{4}{j+4}\mu D)(p_{j+2}) \in \langle h\rangle$, (we note that h is the Hamiltonian part of both $\mathbf{F}_{3,ix}$ and $\mathbf{F}_{3,ix} - \frac{4}{j+4}\mu\mathbf{D}$. So, $(\mathbf{F}_{3,ix} - \frac{4}{j+4}\mu\mathbf{D})(p_{j+2}) \in \langle f_i\rangle$, with $f_1 = x^2+y^2,\ f_2 = x^2+d^2y^2$, irreducible invariant curve of $\mathbf{F}_{3,ix} - \frac{4}{j+4}\mu\mathbf{D}$.
Next, we prove that $\mathbf{F}_{3,ix} - \frac{4}{j+4}\mu\mathbf{D}$ is a irreducible vector field. Indeed, its components are

$$-y((d^2qj+2d^2p+2d^2q+pj+2p+2q)x^2+d^2(p+q)(j+4)y^2) = -y(Mx^2+Ny^2),$$

$$x((p+q)(j+4)x^2+(d^2pj+2d^2p+2d^2q+pj+2p+2q)y^2) = x(\tilde{M}x^2+\tilde{N}y^2).$$

Both components have a common factor if $jp = jq = -2(p+q)$. It is a contradiction since j, p, q are natural numbers.

So $\mathbf{F}_{3,ix} - \frac{4}{j+4}\mu\mathbf{D}$ is a irreducible vector field. Applying Lemma 5.10 to $\mathbf{F}_{3,ix} - \frac{4}{j+4}\mu\mathbf{D}$, it has that $p_{j+2} \in \langle f_i\rangle,\ i = 1,2$. Therefore, $p_{j+2} \in \langle h\rangle$ since h has simple factors and h is a first integral of $\mathbf{F}_{3,ix}$. As $p_{j+2} \in \Delta_{j+2}$, it has that $p_{j+2} = 0$. ■

Next statement establishes a cyclicity relation between the co-ranges of the operators ℓ_k.

Proposition 4.9 *Consider* $\mathbf{F}_{3,ix}$ *and* $M = 2(p+q)$, *degree of the primitive polynomial first integral. For* $k \geq 3$, *it is always possible to choose* $\mathrm{Cor}(\ell_{k+M})$, *a complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M\mathrm{Cor}(\ell_k)$ *with* $I_M = (x^2+y^2)^p(x^2+d^2y^2)^q$.

Proof. We see that both subspaces, $\mathrm{Cor}(\ell_{k+M})$ and $I_M\mathrm{Cor}(\ell_k)$, have the same dimension. Indeed, $Ker(\ell_k) = \langle I_M^l \rangle$ if $k-2 = lM$. Otherwise, $Ker(\ell_k) = \{0\}$. Thus, $\dim(\mathrm{Cor}(\ell_k)) = 2$ if $k = lM$ and $\dim(\mathrm{Cor}(\ell_k)) = 1$, otherwise; i.e. $\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{k+M}))$.

The proof is completed by showing that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M\mathrm{Cor}(\ell_k) \cap \mathrm{Range}(\ell_{k+M}) = \{0\}$ by *reductio ad absurdum.* Let $p_k \in \mathrm{Cor}\,(\ell_k) \setminus \{0\}$ such that $p_k I_M \in \mathrm{Range}\,(\ell_{k+M})$, then there exists $p_{k+M-2} \in \mathcal{P}_{k+M-2} \setminus \{0\}$ such that $\ell_{k+M}(p_{k+M-2}) = p_k I_M$, that is, $\ell_{k+M}(p_{k+M-2})$ is multiple of I_M. Consequently, $\ell_{k+M}(p_{k+M-2})$ is multiple of f_1^p and $f_2 q$ with $f_1 = x^2 + y^2$ and $f_2 = x^2 + d^2y^2$.
We prove that p_{k+M-2} is multiple of f_1^p and f_2^q. From Lemma 5.12, it is enough to prove that

$$\tfrac{2j_1}{k+M-2+j_1} \neq \tfrac{p}{q},\ j_1 = 1, \cdots, p-1, \qquad \tfrac{2j_2}{k+M-2+j_2} \neq \tfrac{q}{p},\ j_2 = 1, \cdots, q-1.$$

Solving with respect to j_1 and j_2, it is easy to check that the conditions are satisfied.

Therefore, we have that $p_{k+M-2} \in \langle f_1^p \rangle \cap \langle f_2^q \rangle$, thus $p_{k+M-2} = p_{k-2} I_M$ with $p_{k-2} \in \mathcal{P}_{k-2} \setminus \{0\}$ and consequently

$$p_k I_M \;=\; F_{3,ix}(p_{k+M-2}) = F_{3,ix}(p_{k-2} I_M) = I_M F_{3,ix}(p_{k-2}).$$

Hence $p_k = F_{3,ix}(p_{k-2})$, that is, $p_k \in \mathrm{Range}\,(\ell_k) \cap \mathrm{Cor}\,(\ell_k)$ which gives a contradiction. ■

Proof of Theorem 2.4. We see the sufficiency. The polynomial $I_M = (x^2+y^2)^p(x^2+d^2y^2)^q$ is a first integral of $\mathbf{F}_{3,ix}$ which is transformed into a formal first integral $I = I_M + \text{h.o.t.}$ of $\mathbf{F}$ and from [20, Theorem A], $\mathbf{F}$ is analytically integrable.
We see the necessity of the condition. Applying Proposition 4.8, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_{3,ix} + \sum_{j\geq 3} \eta_j \mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$.
Let note that $\mathbf{F}$ has an analytic first integral equivalents to $\mathbf{G}$ has a formal first integral. Assume that $\mathbf{G}$ is formally integrable and not all the η_j are zero. We denote $N = \min\{j > 2 : \eta_j \not\equiv 0\}$. A formal first integral of $\mathbf{G}$ is of the form $I = I_M^l + \sum_{j>Ml} I_j$ with $I_j \in \mathcal{P}_j$. Imposing the integrability condition we have

$$\begin{aligned} 0 &= (G(I))_{N+Ml} = (\eta_N D)(I_M^l) + F_{3,ix}(I_{Ml+N-2}) \\ &= Ml\eta_N I_M^l + \ell_{Ml+N}\,(I_{Ml+N-2})\,. \end{aligned}$$

But this equation is incompatible since, by Proposition 4.9, $Ml\eta_N I_M^l \in \mathrm{Cor}\,(\ell_{Ml+N})$ and $\ell_{Ml+N}\,(I_{Ml+N-2}) = -Ml\eta_N I_M^l \in \mathrm{Range}\,(\ell_{Ml+N})$, which is a contradiction. Consequently, $\mathbf{G} = \mathbf{F}_{3,ix}$, i.e. $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_{3,ix}$.

We now see the second part. A first integral of $\mathbf{F}_{3,ix}$ is I_M. The change of variables and the reparemeterization of the time that transform $\mathbf{F}_{3,ix}$ into $\mathbf{F}$, transform $I_M +$ h.o.t. (first integral of $\mathbf{F}$) into $I = (x^2+y^2+\text{h.o.t.})^p(x^2+d^2y^2+\text{h.o.t.})^q$. ■

Proof of Theorem 2.5. We prove that the condition is necessary. We assume that $\mathbf{F}$ is analytically integrable. From Theorem 2.4, it is orbitally equivalent to $\mathbf{F}_{3,ix} = \mathbf{X}_h + \mu\mathbf{D}$, it which has the inverse integrating factor h, i.e. $F_{3,ix}(h) = \mathrm{div}(\mathbf{F}_{3,ix})h$. Undoing the change, it has that $\mathbf{F}$ has a formal inverse integrating $V = h + \text{h.o.t.}$.

Now we will see the sufficiency of the condition. Let $V = h + \text{h.o.t.}$ a formal inverse integrating factor of $\mathbf{F}$. By Proposition 4.8, $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_{3,ix} + \sum_{j\geq 3} \eta_j \mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$. Therefore, $\mathbf{F}$ has a formal inverse integrating factor if, and only if, $\mathbf{G}$ has it too. Moreover, the formal inverse integrating factor W of $\mathbf{G}$ is also of the form $W = h + \text{h.o.t.}$. On the other hand, the unique invariant curves of $\mathbf{G}$ are x^2+y^2 and $x^2+d^2y^2$. So, we get $W = hu$ being u formal and $u(\mathbf{0}) = 1$. Equation $G(W) - W\mathrm{div}(\mathbf{G}) = 0$ is

$$0 = uG(h) + hG(u) - hu\mathrm{div}(\mathbf{G}).$$

As $G(h) = 4h\mu + \sum_{j>3} 4h\eta_j$ and $\mathrm{div}(\mathbf{G}) = 4\mu + \sum_{j>3}(j+2)\eta_j$, it has that

$$0 = h(G(u) - u\sum_{j>3}(j-2)\eta_j).$$

Expanding $u = 1 + \sum_{i\geq 1} u_i$, the above equation to degree $i+2$ becomes

$$0 = F_{3,ix}(u_i) - i\eta_{i+2} + \sum_{k=1}^{i}(2k-i)\eta_{i-k+2}u_k. \quad (4.14)$$

We see that $\eta_j = 0$ for all j. Indeed, otherwise, let $j_0 = \min\{j \in \mathbb{N} : \eta_{j+2} \not\equiv 0\}$. Expression (4.14) to degree $j_0 + 2$ is

$$F_{3,ix}(u_{j_0}) = j_0\eta_{j_0+2} - \sum_{k=1}^{j_0}(2k-j_0)\eta_{j_0-k+2}u_k.$$

As $\eta_{j_0-k+2} = 0$ for $1 \leq k \leq j_0 - 1$, we get $F_{3,ix}(u_{j_0}) = j_0\eta_{j_0+2}$, i.e. $\eta_{j_0+2} \in \mathrm{Cor}\,(\ell_{j_0+2})$ and $\eta_{j_0+2} \in \mathrm{Range}\,(\ell_{j_0+2})$. We conclude that $\eta_{j_0+2} = 0$. ■

5 Appendix

The following results will be used to compute a normal form of the vector fields considered.

The following technical lemma is a direct consequence of Hilbert's Nussstellnsatz.

Lemma 5.10 *Consider $\mathbf{F}_n \in \mathcal{H}_n$ irreducible (its components are coprime) and $f \in \mathbb{C}[x,y]$ an irreducible invariant curve at the origin of $\mathbf{F}_n$. If $F_n(p_k) \in \langle f\rangle$ with $p_k \in \mathcal{P}_k$, then $p_k \in \langle f\rangle$.*

Lemma 5.11 *Let $f \in \mathbf{C}[x,y]$ an irreducible polynomial invariant curve of $\mathbf{F}_n$, $K \in \mathcal{P}_{n-1}$ its cofactor and $k,m \in \mathbb{N}$ with $n+k-1 \geq m$. Assume that the vector fields of $\mathcal{H}_n$, $\mathbf{F}_n + \frac{jK}{k-j}\mathbf{D}$, $0 \leq j \leq m-1$, are irreducible. Then, if $F_n(p_k) \in \langle f^m\rangle$ with $p_k \in \mathcal{P}_k$, it satisfies that $p_k \in \langle f^m\rangle$.*

Proof. Lemma 5.10 proves the statement for $m = 1$.

We first consider the case $m = 2$. If $F_n(p_k) \in \langle f^2\rangle$ then $F_n(p_k) \in \langle f\rangle$ and, by Lemma 5.10, we have that there exists $p_{k-1} \in \mathcal{P}_{k-1}$ such that $p_k = fp_{k-1}$. Therefore,

$$\begin{aligned} F_n(p_k) &= F_n(fp_{k-1}) = p_{k-1}F_n(f) + fF_n(p_{k-1}) = p_{k-1}Kf + fF_n(p_{k-1}) \\ &= f\left(\tfrac{K}{k-1}D(p_{k-1}) + F_n(p_{k-1})\right) = f(F_n + \tfrac{K}{k-1}D)(p_{k-1}) \in \langle f^2\rangle. \end{aligned}$$

Hence $(F_n + \frac{K}{k-1}D)(p_{k-1}) \in \langle f\rangle$. Applying Lemma 5.10 for the irreducible vector field $\mathbf{F}_n + \frac{K}{k-1}\mathbf{D}$, we have that $p_{k-1} \in \langle f\rangle$ and consequently $p_k \in \langle f^2\rangle$.

Consider now the case $m = 3$. If $F_n(p_k) \in \langle f^3\rangle$ then $F_n(p_k) \in \langle f^2\rangle$ and by the previous paragraph we have that there exists $p_{k-2} \in \mathcal{P}_{k-2}$ such that $p_k = f^2p_{k-2}$, therefore

$$\begin{aligned} F_n(p_k) &= F_n(f^2p_{k-2}) = p_{k-2}F_n(f^2) + f^2F_n(p_{k-2}) = 2p_{k-2}Kf^2 + f^2F_n(p_{k-2}) \\ &= f^2\left(\tfrac{2K}{k-2}D(p_{k-2}) + F_n(p_{k-2})\right) = f^2(F_n + \tfrac{2K}{k-2}D)(p_{k-2}) \in \langle f^3\rangle. \end{aligned}$$

Hence $(F_n + \frac{2K}{k-2}D)(p_{k-2}) \in \langle f\rangle$ and as $\mathbf{F}_n + \frac{2K}{k-2}\mathbf{D}$ is irreducible, applying Lemma 5.10 we have that $p_{k-2} \in \langle f\rangle$ and consequently $p_k \in \langle f^3\rangle$. Reasoning by induction we get the result for all m natural number. ■

Lemma 5.12 *The following statements are satisfied:*
(i) *The vector field* $\mathbf{F}_{3,ix} + \alpha K_1 \mathbf{D}$ *with* $K_1 := -2q(d^2-1)xy$ *(cofactor of* $f_1 = x^2+y^2$*, invariant curve at the origin of* $\mathbf{F}_{3,ix}$*) and* $\alpha \in \mathbb{Q}^+$*, is irreducible if and only if* $2\alpha \neq \frac{p}{q}$*.*
(ii) *The vector field* $\mathbf{F}_{3,ix} + \alpha K_2 \mathbf{D}$ *with* $K_2 := 2p(d^2-1)xy$ *(cofactor of* $f_2 = x^2+d^2y^2$*, invariant curve at the origin of* $\mathbf{F}_{3,ix}$*) and* $\alpha \in \mathbb{Q}^+$*, is irreducible if and only if* $2\alpha \neq \frac{q}{p}$*.*

Proof. We prove item **(i)**, being the case **(ii)** analogous. The components of $\mathbf{F}_{3,ix} + \alpha K_1 \mathbf{D}$ are of the form

$$\begin{array}{l} -y(a_{20}x^2 + a_{02}) := -y\left((2\alpha d^2 q + d^2 q - 2\alpha q + p)x^2 + d^2(p+q)y^2\right), \\ x(b_{20}x^2 + b_{02}) := x\left((p+q)x^2 + (-2\alpha d^2 q + d^2 p + 2\alpha q + q)y^2\right). \end{array}$$

Both polynomials are coprime if and only if $a_{20}b_{02} - a_{02}b_{20} \neq 0$, i.e.

$$q(d^2-1)^2(2\alpha+1)(2\alpha q - p) \neq 0,$$

that is, $2\alpha \neq \frac{p}{q}$. ∎

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Data availability statement

The authors declare that data sharing is not applicable to this article as no datasets were generated or analysed during the current study.