Kowalevskaya exponents and normal forms of planar dynamical systems. Application to integrability problem

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Abstract

In this work dynamical properties of the solutions of differential systems in the plane are provided using certain constants related with the Kowalevskaya exponents. An orbital normal form that allows to study the existence of analytic and non-analytic first integrals is given. We apply the results to study the algebraic and analytic integrability around several singularities.

1 Introduction

The first attempt to solve a differential equation either explicitly or by series expansions was made by Euler, Newton and Leibniz and expanded by Lagrange, Poisson, Hamilton and Liouville. The main idea was that the solutions can always be represented by a combination of known functions or by series expansions. The notion of *integrability* was introduced to describe property that the solutions can be obtained explicitly or implicitly from the constants of motion. Two main different approaches have been developed from the Kowalevskaya' studies and the Poincaré works. The different theories of integrability arise essentially from these works and some of them have relation with the form of the constants of motion or first integrals admissible in each theory. Hence, the definition of *integrable system* depends of the integrability theory used and there is not a unified definition. The universal definition of integrability should be obtained when the connections between different integrability theories can be established. One of the main theories to study integrability is *singularity analysis*. This theory studies the local behavior of the dependent variables in function of the independent variable (usually the time) [21]. As in generical is not possible to solve any differential equation or differential system, then these functions (solutions) are not explicitly known. Nevertheless, a local analysis can be done around singularities of solutions. The singularities of a solution are the locations in time where the solution diverges. In fact, singularity analysis is indeed an extension of Frobenius' method used for studying linear differential equations with time-dependent coefficients.

The singularity analysis and its relation with the integrability problem has been studied by many authors, see for instance the seminal work of Painlevé [32]. The solutions of a differential system can exhibit different types of behavior. Single-valued solutions appear when all solutions can be represented as Laurent series in the time parameter, i.e. the solutions are locally meromorphic around the singularities. The lack of meromorphic solutions can be used to prove the non-integrability of a differential system in this theory. The *Painlevé test* is an algorithm (see [1]) used to detect integrable systems, see for instance [22, 35]. However, more general singularities for the solutions can appear. In this case we have to consider Puiseux series and also it is possible to find a consistent expansion involving logarithmic terms near a singularity. These series containing logarithmic terms were called Ψ-series or Dulac expansions, see [16, 22, 24, 25] and references therein.

In this paper, we study the connections between singularity analysis and Kowalevskaya exponents with the normal form for differential systems on the plane. We apply this study to analytical and algebraic over $\mathbb{C}[[x,y]]$ integrability problem of such systems, where $\mathbb{C}[[x,y]]$ is the algebra of the formal power series in x,y with coefficients in $\mathbb{C}$, that is, given a system $(\dot{x},\dot{y})^T = \mathbf{F}(x,y) = (P(x,y),Q(x,y))^T$ with P and Q analytic in a neighborhood of the origin $\Omega \subset \mathbb{C}^2$, we analize when there exists a function $I : \Omega \to \mathbb{C}$, analytic or algebraic over $\mathbb{C}[[x,y]]$, satisfying $F(I) := \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q \equiv 0$ in Ω.

The singularity analysis is given in sections 2 and 3 where the main results of the paper are proved. In section 4 we apply our results to some families of differential systems.

First, we recall that a function of two variables f is a quasi-homogeneous (weight homogeneous) function of degree $k \in \mathbb{N}$ with respect to type (or weight) $\mathbf{t} = (t_1,t_2) \in \mathbb{N}^2$ if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)$. We denote by $\mathcal{P}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomials in x,y of degree k. A two-dimensional vector field $\mathbf{F} = (P,Q)^T$ is quasi-homogeneous of degree $k \in \mathbb{Z}$ with respect to the type $\mathbf{t}$ if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of quasi-homogeneous polynomials vector fields of degree k will be denoted by $\mathcal{Q}_k^{\mathbf{t}}$. So, fixed a type $\mathbf{t}$, every analytic vector field $\mathbf{F}$ can be expanded as a sum of quasi-homogeneous polynomial terms $\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots, r \in \mathbb{Z}$, and $\mathbf{F}_r = (P_{r+t_1}, Q_{r+t_2})^T$ is called leading term of $\mathbf{F}$ respect to the type $\mathbf{t}$. Thus, for any arbitrarily fixed $\mathbf{t}$, every analytic vector field $\mathbf{F}$ can be considered as a perturbation of its leading term $\mathbf{F}_r$.

We see an example. Under the degree, the first quasi-homogeneous polynomials of type $(3,4)$ are:

$$\begin{aligned}
&\mathcal{P}_3^{(3,4)} = \{x\}, &&\mathcal{P}_4^{(3,4)} = \{y\}, &&\mathcal{P}_6^{(3,4)} = \{x^2\},\\
&\mathcal{P}_7^{(3,4)} = \{xy\}, &&\mathcal{P}_8^{(3,4)} = \{y^2\}, &&\mathcal{P}_9^{(3,4)} = \{x^3\},\\
&\mathcal{P}_{10}^{(3,4)} = \{x^2y\}, &&\mathcal{P}_{11}^{(3,4)} = \{xy^2\}, &&\mathcal{P}_{12}^{(3,4)} = \{y^3,\ x^4\}
\end{aligned}$$

and $\mathcal{P}_j^{(3,4)} = \{0\},\ j = 1,2,5$. Consider the vector field $\mathbf{F} = (\sum_{j\geq 1} P_j(x,y), \sum_{j\geq 1} Q_j(x,y))^T$

with $P_j(x,y)=\sum_{m+n\geq 1} a_{mn}x^m y^n$ and $Q_j(x,y)=\sum_{m+n\geq 1} b_{mn}x^m y^n$. The quasi-homogeneous expansion of $\mathbf{F}$ with respect to $\mathbf{t}=(3,4)$ is

$$\begin{aligned}\mathbf{F} \;=\; & \overbrace{\begin{pmatrix}0\\ b_{10}x\end{pmatrix}}^{\mathbf{F}_{-1}}+\overbrace{\begin{pmatrix}a_{10}x\\ b_{01}y\end{pmatrix}}^{\mathbf{F}_{0}}+\overbrace{\begin{pmatrix}a_{01}y\\ 0\end{pmatrix}}^{\mathbf{F}_{1}}+\overbrace{\begin{pmatrix}0\\ b_{20}x^2\end{pmatrix}}^{\mathbf{F}_{2}}+\overbrace{\begin{pmatrix}a_{20}x^2\\ b_{11}xy\end{pmatrix}}^{\mathbf{F}_{3}}+\overbrace{\begin{pmatrix}a_{11}xy\\ b_{02}y^2\end{pmatrix}}^{\mathbf{F}_{4}}\\ & +\overbrace{\begin{pmatrix}a_{02}y^2\\ b_{30}x^3\end{pmatrix}}^{\mathbf{F}_{5}}+\overbrace{\begin{pmatrix}a_{30}x^3\\ b_{21}x^2y\end{pmatrix}}^{\mathbf{F}_{6}}+\overbrace{\begin{pmatrix}a_{21}x^2y\\ b_{12}xy^2\end{pmatrix}}^{\mathbf{F}_{7}}+\overbrace{\begin{pmatrix}a_{12}xy^2\\ b_{03}y^3+b_{40}x^4\end{pmatrix}}^{\mathbf{F}_{8}}+\text{q-h.h.o.t..}\end{aligned}$$

We note that for $\mathbf{t}=(1,1)$, we have Taylor expansion of a vector field.

Summarizing, the quasi-homogeneous expansion of a vector field is a reordering of the non-linear terms that is established by the weight $\mathbf{t}=(t_1,t_2)$. In this sense, Newton diagram of the vector field provides information about the nonlinear terms that are essential to study the dynamics of the vector field. Thus, the suitable weights are the associated types to the edges of its Newton diagram, see [10, 11, 13].

Throughout this paper, it is denoted by $\mathbf{X}_f$, the Hamiltonian vector field associated to the $\mathcal{C}^1$ function f, i.e. $\mathbf{X}_f=(-\frac{\partial f}{\partial y},\frac{\partial f}{\partial x})^T$ and by $\mathbf{D}_0^{\mathbf{t}}$, the dissipative quasi-homogeneous vector field of degree 0, $\mathbf{D}_0^{\mathbf{t}}(x,y)=(t_1x,t_2y)^T$ (Eulerian vector field).

Algaba *et al.* [6] prove that any quasi-homogeneous polynomial vector field can be univocally expressed as sum of two terms, one hamiltonian and another dissipative, $\mathbf{F}_r=\mathbf{X}_{h_{r+|\mathbf{t}|}}+\mu_r\mathbf{D}_0^{\mathbf{t}}$ where $h_{r+|\mathbf{t}|}=\frac{1}{r+|\mathbf{t}|}(t_1xQ_{r+t_2}-t_2yP_{r+t_1})\in\mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ (the wedge product of $\mathbf{D}_0^{\mathbf{t}}$ and $\mathbf{F}_r$) and $\mu_r=\frac{1}{r+|\mathbf{t}|}\text{div}(\mathbf{F}_r)\in\mathcal{P}_r^{\mathbf{t}}$ (the divergence of $\mathbf{F}_r$) being $|\mathbf{t}|=t_1+t_2$.

The polynomial $h_{r+|\mathbf{t}|}$ provides information about the dynamics of the solutions of the quasi-homogeneous system $\dot{\mathbf{x}}=\mathbf{F}_r(\mathbf{x})$, see [6, 7].

Proposition 1.1 *The following statements hold:*
(i) If $h_{r+|\mathbf{t}|}\equiv 0$, then for any quasi-homogeneous polynomial f, the curve $f(x,y)=0$ is an invariant curve of $\mathbf{F}_r$ and $\mathbf{F}_r$ is not analytically integrable.
(ii) If $h_{r+|\mathbf{t}|}\not\equiv 0$, then the irreducible invariant curves of $\mathbf{F}_r$ are $f(x,y)=0$ with f an irreducible quasi-homogeneous factor of $h_{r+|\mathbf{t}|}$.

The classical Kowalevskaya exponents arose from the study of the existence of particular solutions of the form $(x(t),y(t))=(c_1t^{-t_1},c_2t^{-t_2})$ of the nonlinear system $\dot{\mathbf{x}}=\mathbf{F}_r(\mathbf{x})$ $(r>0)$ where the coefficients $\mathbf{c}=(c_1,c_2)\in\mathbb{C}^2\setminus\{(0,0)\}$ are given by the vectorial equation

$$\mathbf{F}_r(\mathbf{c})+\tfrac{1}{r}\mathbf{D}_0^{\mathbf{t}}(\mathbf{c})=\mathbf{0}.$$

For a given type $\mathbf{t}$, there may exist different $\mathbf{c}$ called system *balances* [18, 19, 20, 21].

Now, for each balance $\mathbf{c}$, we compute the differential of $\mathbf{F}_r+\frac{1}{r}\mathbf{D}_0^{\mathbf{t}}$ evaluated at $\mathbf{c}$, that is $K(\mathbf{c})=D\left(\mathbf{F}_r+\frac{1}{r}\mathbf{D}_0^{\mathbf{t}}\right)(\mathbf{c})$. The eigenvalues of $K(\mathbf{c})$ are called *Kowalevskaya exponents* of the balance $\mathbf{c}$, see Kowalevskaya [26]. It can be shown that there always exists a Kowalevskaya exponent equals to -1 [37, 38].

From Proposition 1.1, the irreducible invariant curves of $\mathbf{F}_r$ are $f(x,y)=0$ with f an irreducible factor of $h_{r+|\mathbf{t}|}$. A parameterization of $f(x,y)=0$ is given by $(x(t),y(t))=(c_1t^{-t_1},c_2t^{-t_2})$. Thus, such exponent is associated with a irreducible factor f of $h_{r+|\mathbf{t}|}$, which we denote by $\rho(f)$.

Several authors have studied the integrability of quasi-homogeneous systems and its relation with both planar and n-dimensional systems, see, Algaba *et al.* [5], Cairó & Llibre [12], Furta [18], Giné [19], Goriely [20], Llibre & Zhang [29] and Tsygvintsev [37], among others.

The ratio between the dissipative part (μ_r) and the Hamiltonian part ($h_{r+|\mathbf{t}|}$) of $\mathbf{F}_r$ (assuming that $h_{r+|\mathbf{t}|}\not\equiv 0$) is given by $\frac{\mu_r(x,y)}{h_{r+|\mathbf{t}|}(x,y)}=x^{t_2-1}y^{t_1-1}\eta(x^{t_2},y^{t_1})$ being η a homogeneous rational function, see [5]. Replacing (x^{t_1},y^{t_2}) by (x,y), we have that

$$\eta(x,y):=\frac{\mu_r(x^{1/t_2},y^{1/t_1})}{x^{1-1/t_2}y^{1-1/t_1}h_{r+|\mathbf{t}|}(x^{1/t_2},y^{1/t_1})}.$$

This function plays a main role in our study.
Moreover,
if $x\in\mathcal{P}^{\mathbf{t}}_{t_1}$ is a simple factor of $h_{r+|\mathbf{t}|}$, then $\lambda=\infty$ is a pole of $\eta(1,y)$,
if $y\in\mathcal{P}^{\mathbf{t}}_{t_2}$ is a simple factor of $h_{r+|\mathbf{t}|}$, then $\lambda=0$ is a pole of $\eta(1,y)$,
if $y^{t_1}-ax^{t_2}\in\mathcal{P}^{\mathbf{t}}_{t_1t_2}$ with $a\in\mathbb{C}\setminus\{0\}$, is a simple factor of $h_{r+|\mathbf{t}|}$, then $\lambda=a$ is a pole of $\eta(1,y)$.
So, for each f irreducible simple factor of $h_{r+|\mathbf{t}|}$, we can define the complex number

$$r(f):=\frac{1}{r+|\mathbf{t}|}\left(1-\frac{t_1t_2}{\deg_{\mathbf{t}}(f)}\mathrm{Res}[\eta(1,y),\lambda]\right), \tag{1.1}$$

($\lambda=\infty$, if $f=x$; $\lambda=0$, if $f=y$; $\lambda=a$, if $f=y^{t_1}-ax^{t_2}$) where $\mathrm{Res}[\eta(1,y),\lambda]$ denotes the residue of the function $\eta(1,y)$ associated to λ and $\deg_{\mathbf{t}}(f)$ denotes the degree of the quasi-homogeneous polynomial with respect to the type $\mathbf{t}$; that is, $\deg_{\mathbf{t}}(f)$ is t_1, t_2 or t_1t_2, respectively.

There exists a link between the constants $r(f)$ and the Kowalevskaya exponents $\rho(f)$ of the system $\dot{\mathbf{x}}=\mathbf{F}_r(\mathbf{x})$, with f an irreducible factor of $h_{r+|\mathbf{t}|}$. The proof of the following result is given in [5, 7].

Proposition 1.2 *The following statements hold:*
(i) If f is a simple factor of $h_{r+|\mathbf{t}|}$ and $f(x,y)=0$ is a curve of singular points of $\mathbf{F}_r$, we have that $r(f)=0$. Otherwise, $r(f)$ is not zero.

(ii) Let $\rho(f)$ be the Kowalevskaya exponent different from -1 associated with the factor f of $h_{r+|\mathbf{t}|}$ with $r>0$. If f is a multiple factor of $h_{r+|\mathbf{t}|}$, then $\rho(f)=0$. Otherwise, $r(f)=\frac{1}{r\rho(f)}$.

(iii) $\mathbf{F}_r$ is rationally (or polynomially) integrable if and only if div($\mathbf{F}_r$) $\equiv 0$, or the two following properties hold:
(iii.1) $h_{r+|\mathbf{t}|}$ has only simple factors on $\mathbb{C}[x,y]$ and $h_{r+|\mathbf{t}|}$ has at least two factors.
(iii.2) $r(f)\in\mathbb{Q}$ (or $r(f)\in\mathbb{Q}^+$), for any f factor of $h_{r+|\mathbf{t}|}$.

(iv) If $\mathbf{F}_r$ is polynomially integrable then it has a quasi-homogeneous polynomial first integral whose degree is $M=lcm(\{denominator(r(f_i))\}_{i=1}^m)$ being f_i the irreducible factors

of $h_{r+|\mathbf{t}|}$, $i = 1, \ldots, m$. *Moreover,* $I_M = f_1^{n_1} \cdots f_m^{n_m} \in \mathcal{P}^{\mathbf{t}}_M$ *with* $n_i = r(f_i)M$ *is the primitive quasi-homogeneous first integral. (Note that* I_M *is the lowest-degree polynomial first integral since gcd*$(n_1, \ldots, n_m) = 1$*).*

(v) *If* f *is a factor of* $h_{r+|\mathbf{t}|}$ *such that* $r(f) \notin \mathbb{Q} \cap [-1, 0)$, *then there exists an unique analytic invariant curve of* $\mathbf{F}_r + \cdots$ *starting with* f.

We deal with analytic vector fields $\mathbf{F} = \mathbf{F}_r + \cdots$ with $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0^{\mathbf{t}}$ an irreducible quasi-homogeneous vector field with non-zero $h_{r+|\mathbf{t}|}$ having only simple factors on $\mathbb{C}[x, y]$. The aim of this article is to provide dynamical properties of the solutions of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ through the constants defined in (1.1). For that, first we give an appropriated orbital normal form. Its expression allows to provide dynamic properties of the system. More specifically, we give necessary and sufficient conditions of existence of non-analytic and analytic first integrals. The main results of this work are Theorem 2.11 and Theorem 3.19 which characterize the algebraic and analytic first integrals, respectively. As an application, we study the algebraic integrability over $\mathbb{C}[[x, y]]$ of the resonant linear node and of some differential systems whose origin is a degenerate singular point.

2 Orbitally equivalent normal forms

We recall the concept of orbital equivalence of vector fields. Two vector fields $\mathbf{F}$ and $\mathbf{G}$ are orbitally equivalent if there exist a near-identity change of variable and a reparemeterization of the time-variable such that one is transformed into the other, i.e. there exist a diffeomorphism ϕ and a formal scalar function $\nu : \mathbb{C}^2 \to \mathbb{C}$ with $\nu(\mathbf{0}) = 1$, such that $\mathbf{G} = \phi_*(\nu \mathbf{F})$, where we denote by ϕ_* the push-forward defined by the diffeomorphism ϕ, $\phi_* \mathbf{F}(\mathbf{y}) := (D\phi(\mathbf{y}))^{-1} \mathbf{F}(\phi(\mathbf{y}))$, see [30].

We provide an orbitally equivalent normal form of $\mathbf{F}$.

Theorem 2.3 *Consider the analytic vector field* $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}^{\mathbf{t}}_j$, $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}^{\mathbf{t}}_r$ *with* $h_{r+|\mathbf{t}|} \not\equiv 0$ *having only simple factors on* $\mathbb{C}[x, y]$. *Also, assume that for any* f *irreducible factor of* $h_{r+|\mathbf{t}|}$ *it verifies that* $r(f) \neq -\frac{1}{k}$ *(or* $\rho(f) \neq -\frac{k}{r}$ *for* $r > 0$*) for all* $k \in \mathbb{N}$. *Then,* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G} = \mathbf{F}_r + \sum_{j=1}^{r-1} \mathbf{X}_{g_{r+j+|\mathbf{t}|}} + \sum_{\substack{j=r+1 \\ \mathcal{P}^{\mathbf{t}}_{j-r} = \{0\}}}^{r+t_1t_2-|\mathbf{t}|} \mathbf{X}_{g_{r+j+|\mathbf{t}|}} + \sum_{j \geq 1} \mu_{r+j} \mathbf{D}_0^{\mathbf{t}} \tag{2.2}$$

with $g_{r+j+|\mathbf{t}|} \in \operatorname{Cor}\left(\ell^{c}_{r+j+|\mathbf{t}|}\right)$ *(a complementary subspace to Range of* $\ell^{c}_{r+j+|\mathbf{t}|}$*) and* $\mu_{r+j} \in \operatorname{Cor}(\ell_{r+j})$ *(a complementary subspace to Range of* ℓ_{r+j}*),* $j \geq 1$, *being* ℓ_{r+j} *and* $\ell^{c}_{r+j+|\mathbf{t}|}$, *the following linear operators:*

$$\begin{aligned} \ell_{r+j} \quad &: \quad \mathcal{P}^{\mathbf{t}}_j \longrightarrow \mathcal{P}^{\mathbf{t}}_{r+j} && (2.3) \\ & \quad\quad \ell_{r+j}(p_j) = F_r(p_j) := \nabla p_j \cdot \mathbf{F}_r, \\ \ell^{c}_{r+j+|\mathbf{t}|} \quad &: \quad \Delta_{j+|\mathbf{t}|} \longrightarrow \Delta_{r+j+|\mathbf{t}|} && (2.4) \\ & \quad\quad \ell^{c}_{r+j+|\mathbf{t}|}\left(\delta_{j+|\mathbf{t}|}\right) = \operatorname{Proj}_{\Delta_{r+j+|\mathbf{t}|}}\left(F_r - \tfrac{r+|\mathbf{t}|}{r+j+|\mathbf{t}|} \mu_r D_0^{\mathbf{t}}\right)\left(\delta_{j+|\mathbf{t}|}\right) \end{aligned}$$

where $\Delta_{j+|\mathbf{t}|}$ *is a complementary subspace to* $h_{r+|\mathbf{t}|}\mathcal{P}^{\mathbf{t}}_{j-r}$.

We note that, in general, given a subspace, a complementary subspace is not unique. Therefore, the complementary subspaces mentioned in Theorem 2.3 are not unique and hence any vector field has infinitely many orbitally equivalent normal forms. Here, our objective is providing a suitable normal form that allows to solve the integrability problem.

To prove the above theorem, we will need a series of technical results which are already provide in previous works and other results that will be proved here. The following orbital normal form has been provided by Algaba *et al.* [4].

Proposition 2.4 *Let* $\mathbf{F} = \mathbf{F}_r + \sum_{j\geq 1}\mathbf{F}_{r+j}$ *with* $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r\mathbf{D}^{\mathbf{t}}_0 \in \mathcal{Q}^{\mathbf{t}}_r$ *and* $\mathbf{F}_{r+j} \in \mathcal{Q}^{\mathbf{t}}_{r+j}$. *If* $\mathrm{Ker}\left(\ell^c_{r+j+|\mathbf{t}|}\right) = \{0\}$ *for any* j, *then* $\mathbf{F}$ *is orbitally equivalent to* $\mathbf{G} = \mathbf{F}_r + \sum_{j>0}\mathbf{G}_{r+j}$ *being* $\mathbf{G}_{r+j} = \mathbf{X}_{\delta_{r+j+|\mathbf{t}|}} + \eta_{r+j}\mathbf{D}^{\mathbf{t}}_0$, *with* $\eta_{r+j} \in \mathrm{Cor}\,(\ell_{r+j})$ *and* $\delta_{r+j+|\mathbf{t}|} \in \mathrm{Cor}\left(\ell^c_{r+j+|\mathbf{t}|}\right)$.

We describe the sets $\mathcal{P}^{\mathbf{t}}_j$ of quasi-homogeneous polynomials according the type $\mathbf{t} = (t_1, t_2)$. The following result provides bases for these spaces. Its proof was given in [3].

Lemma 2.5 *Fixed* $\mathbf{t} = (t_1, t_2)$, *we define the set* $\mathcal{I}^{\mathbf{t}} = \{j = k_1t_1 + k_2t_2 + k_3t_1t_2 \in \mathbb{N} : k_1, k_2, k_3 \in \mathbb{N}\cup\{0\}, k_1 < t_2, k_2 < t_1\}$. *The following statements are verified:*

(i) If $j \notin \mathcal{I}^{\mathbf{t}}$, *then* $\mathcal{P}^{\mathbf{t}}_j = \{0\}$. *Otherwise,* $\mathcal{P}^{\mathbf{t}}_j = \mathit{span}\{x^{k_1+t_2(k_3-j)}y^{k_2+t_1 i} : i = 0, \ldots, k_3\}$, *i.e. we have that* $\dim\left(\mathcal{P}^{\mathbf{t}}_j\right) = k_3 + 1$.

(ii) if $j > t_1t_2 - |\mathbf{t}|$, *then* $j \in \mathcal{I}^{\mathbf{t}}$, *i.e.* $\mathcal{P}^{\mathbf{t}}_j$ *is a non-trivial space.*

Lemma 2.6 *Consider* $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r\mathbf{D}^{\mathbf{t}}_0 \in \mathcal{Q}^{\mathbf{t}}_r$, $h_{r+|\mathbf{t}|} \not\equiv 0$ *having only simple factors on* $\mathbb{C}[x, y]$, *i.e.* $h_{r+|\mathbf{t}|}(x, y) = x^{\delta_1}y^{\delta_2}\prod_{i=1}^m(y^{t_1} - \lambda_i x^{t_2})$ *with* $\delta_1, \delta_2 \in \{0, 1\}$, $\lambda_i \in \mathbb{C}\setminus\{0\}$ *and* $m \in \mathbb{N}_0$. *Assume that* $f(x, y) = 0$ *with* $f \in \mathcal{P}^{\mathbf{t}}_N$ *is a quasi-homogeneous invariant curve (irreducible or not) of degree* $N \in \mathbb{N}$ *of* $\mathbf{F}_r$. *Then for any* j *with* $\mathcal{P}_j \neq \{0\}$ *it satisfies that*

$$\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r+N}\right) - \dim\left(\mathcal{P}^{\mathbf{t}}_{j+N}\right) = \dim\left(\mathcal{P}^{\mathbf{t}}_{j+r}\right) - \dim\left(\mathcal{P}^{\mathbf{t}}_j\right).$$

Proof. From the expression of $h_{r+|\mathbf{t}|}$, we have that $r + |\mathbf{t}| = \delta_1t_1 + \delta_2t_2 + mt_1t_2$. We write $j + r = k_1t_1 + k_2t_2 + k_3t_1t_2$ and $N = n_1t_1 + n_2t_2 + n_3t_1t_2$ since the invariant curves of $\mathbf{F}_r$ are factorization of the factors of $h_{r+|\mathbf{t}|}$ and their degree are t_1, t_2 and t_1t_2, respectively. As $\mathcal{P}^{\mathbf{t}}_j \neq \{0\}$, the others sets also are non-trivial. Moreover,

$$\begin{aligned}
\dim\left(\mathcal{P}^{\mathbf{t}}_j\right) &= [\tfrac{k_1+1-\delta_1}{t_2}] + [\tfrac{k_2+1-\delta_2}{t_1}] + k_3 - m + 1,\\
\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r}\right) &= [\tfrac{k_1}{t_2}] + [\tfrac{k_2}{t_1}] + k_3 + 1,\\
\dim\left(\mathcal{P}^{\mathbf{t}}_{j+N}\right) &= [\tfrac{k_1+n_1+1-\delta_1}{t_2}] + [\tfrac{k_2+n_2+1-\delta_2}{t_1}] + k_3 + n_3 - m + 1,\\
\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r+N}\right) &= [\tfrac{k_1+n_1}{t_2}] + [\tfrac{k_2+n_2}{t_1}] + k_3 + n_3 + 1.
\end{aligned}$$

We distinguish the following cases:
If $\delta_1=1,\delta_2=1$, we have that $\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r+N}\right)-\dim\left(\mathcal{P}^{\mathbf{t}}_{j+N}\right)=\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r}\right)-\dim\left(\mathcal{P}^{\mathbf{t}}_{j}\right)=m$. If $\delta_1=0,\delta_2=1$, in such a case $n_1=0$, we have that $\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r+N}\right)-\dim\left(\mathcal{P}^{\mathbf{t}}_{j+N}\right)=\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r}\right)-\dim\left(\mathcal{P}^{\mathbf{t}}_{j}\right)=[\frac{k_1}{t_2}]-[\frac{k_1+1}{t_2}]+m$.
If $\delta_1=1,\delta_2=0$, this case is analogous to the above case.
If $\delta_1=0,\delta_2=0$, in such a case $n_1=0,n_2=0$, we have that $\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r+N}\right)-\dim\left(\mathcal{P}^{\mathbf{t}}_{j+N}\right)=\dim\left(\mathcal{P}^{\mathbf{t}}_{j+r}\right)-\dim\left(\mathcal{P}^{\mathbf{t}}_{j}\right)=[\frac{k_1}{t_2}]+[\frac{k_2}{t_1}]-[\frac{k_1+1}{t_2}]-[\frac{k_2+1}{t_1}]+m$. ■

The following result was proved in [4, Lemma 3.21].

Lemma 2.7 *Let $f(x,y)=0$ be with $f\in\mathcal{P}^{\mathbf{t}}_s$, a quasi-homogeneous irreducible invariant curve of $\mathbf{F}_r\in\mathcal{Q}^{\mathbf{t}}_r$ such that f is not a factor of $\mathbf{F}_r$. If $p\in\mathcal{P}^{\mathbf{t}}_j$ and $\ell_{r+j}(p)\in f\mathcal{P}^{\mathbf{t}}_{r+j-s}$ then $p\in f\mathcal{P}^{\mathbf{t}}_{j-s}$.*

Lemma 2.8 *Consider $\mathbf{F}_r=\mathbf{X}_{h_{r+|\mathbf{t}|}}+\mu_r\mathbf{D}^{\mathbf{t}}_0$, with $h_{r+|\mathbf{t}|}\mu_r\not\equiv 0$ and $h_{r+|\mathbf{t}|}$ having only simple factors, and $f(x,y)=0$ with $f\in\mathcal{P}^{t}_s$, an irreducible invariant curve of $\mathbf{F}_r$. Fixed a real constant C, assume that f is a factor of $\mathbf{G}_r=\mathbf{F}_r+C\mu_r\mathbf{D}^{\mathbf{t}}_0$. Then, $C\neq-1$ and $r(f)=\frac{C}{(r+|\mathbf{t}|)(1+C)}$.*

Proof. Consider $\mathbf{G}_r:=\mathbf{F}_r+C\mu_r\mathbf{D}^{\mathbf{t}}_0=\mathbf{X}_{h_{r+|\mathbf{t}|}}+(1+C)\mu_r\mathbf{D}^{\mathbf{t}}_0$. If $C=-1$, $\mathbf{G}_r=\mathbf{X}_{h_{r+|\mathbf{t}|}}$ and f is not a factor of $\mathbf{G}_r$ since $h_{r+|\mathbf{t}|}$ has only simple factors; therefore $C\neq-1$. So, $r_{\mathbf{G}_r}(f)=\frac{1}{r+|\mathbf{t}|}\left(1-\frac{t_1t_2}{s}(1+C)\mathrm{Res}\left[\eta(1,y),\omega\right]\right)$. As f is a factor of $\mathbf{G}_r$, $f(x,y)=0$ is a curve of singular points and from [7, Theorem 12], we have that $r_{\mathbf{G}_r}(f)=0$. Therefore, $\mathrm{Res}\left[\eta(1,y),\omega\right]=\frac{s}{t_1t_2(1+C)}$. So, $r(f)=\frac{1}{r+|\mathbf{t}|}(1-\frac{1}{1+C})=\frac{C}{(r+|\mathbf{t}|)(1+C)}$. ■

Now we give sufficient conditions so that $\mathrm{Ker}\left(\ell^c_{r+j+|\mathbf{t}|}\right)$ be a trivial set.

Proposition 2.9 *Consider $\mathbf{F}_r=\mathbf{X}_{h_{r+|\mathbf{t}|}}+\mu_r\mathbf{D}^{\mathbf{t}}_0\in\mathcal{Q}^{\mathbf{t}}_r$, with $h_{r+|\mathbf{t}|}$ having only simple factors on $\mathbb{C}[x,y]$ and assume that $r(f)\neq-\frac{1}{j}$ for any $f(x,y)=0$ irreducible invariant curve of $\mathbf{F}_r$ and $j\in\mathbb{N}$. Then,*

(i) $\mathrm{Ker}\left(\ell^c_{r+j+|\mathbf{t}|}\right)=\{0\}$,

(ii) $\mathrm{Cor}\left(\ell^c_{r+j+|\mathbf{t}|}\right)=\{0\}$, *for $j>r+t_1t_2-|\mathbf{t}|$ or $j=r$.*

Proof. We prove the first part. Let $p\in\mathrm{Ker}\left(\ell^c_{r+j+|\mathbf{t}|}\right)$, that is $p\in\Delta_{j+|\mathbf{t}|}\setminus\{0\}$ such that $\ell^c_{r+j+|\mathbf{t}|}(p)\in h_{r+|\mathbf{t}|}\mathcal{P}^{\mathbf{t}}_j$. As $h_{r+|\mathbf{t}|}=\prod_{i=1}^m f_i$, $f_i\in\mathcal{P}^{\mathbf{t}}_{s_i}$ irreducible factor of $h_{r+|\mathbf{t}|}$ then $\ell^c_{r+j+|\mathbf{t}|}(p)\in f_i\mathcal{P}^{\mathbf{t}}_{r+j+|\mathbf{t}|-s_i}$ for any $i=1,\cdots,m$.
We now prove that f_i is not a factor of $\mathbf{G}_r:=\mathbf{F}_r-\frac{r+|\mathbf{t}|}{r+j+|\mathbf{t}|}\mu_r\mathbf{D}^{\mathbf{t}}_0$. Otherwise, from Lemma 2.8 for $C=-\frac{r+|\mathbf{t}|}{r+j+|\mathbf{t}|}$, we have that $r(f_i)=-\frac{1}{j}$, and we arrive to contradiction. Therefore f_i is not a factor of $\mathbf{G}_r$. Applying Lemma 2.7 to the vector field $\mathbf{G}_r$ and taking into account

that both $\mathbf{F}_r$ and $\mathbf{G}_r$ have the same invariant curves, we have that $p \in f_i \mathcal{P}^{\mathbf{t}}_{j+|\mathbf{t}|-s_i}$, for any i. So, $p \in \Delta_{j+|\mathbf{t}|} \cap \langle h_{r+|\mathbf{t}|} \rangle$, *i.e.* $p = 0$.
We see the second part. The operator $\ell^c_{r+j+|\mathbf{t}|} : \Delta_{j+|\mathbf{t}|} \to \Delta_{r+j+|\mathbf{t}|}$ is injective since $\operatorname{Ker}\left(\ell^{\mathbf{c}}_{r+j+|\mathbf{t}|}\right) = \{0\}$. On the other hand,

$$\begin{aligned} \dim\left(\Delta_{r+j+|\mathbf{t}|}\right) &= \dim\left(\mathcal{P}^{\mathbf{t}}_{r+j+|\mathbf{t}|}\right) - \dim\left(\mathcal{P}^{\mathbf{t}}_{j}\right) \\ \dim\left(\Delta_{j+|\mathbf{t}|}\right) &= \dim\left(\mathcal{P}^{\mathbf{t}}_{j+|\mathbf{t}|}\right) - \dim\left(\mathcal{P}^{\mathbf{t}}_{j-r}\right). \end{aligned}$$

So, by Lemma 2.6, replacing j by $j-r$ and N by $r+|\mathbf{t}|$ since $h_{r+|\mathbf{t}|}(x,y) = 0$ is an invariant curve of $\mathbf{F}_r$, if $\mathcal{P}^{\mathbf{t}}_{j-r} \neq \{0\}$, we have that $\Delta_{j+|\mathbf{t}|}$ and $\Delta_{r+j+|\mathbf{t}|}$ have the same dimension, that is the operator $\ell^c_{r+j+|\mathbf{t}|}$ is bijective and therefore $\operatorname{Cor}\left(\ell^c_{r+j+|\mathbf{t}|}\right) = \{0\}$.

Moreover, if $i - r > t_1 t_2 - \; |\mathbf{t}|$ by Lemma 2.5 then $\mathcal{P}^{\mathbf{t}}_{i-r} \neq \{0\}$; if $i < r$, $\mathcal{P}^{\mathbf{t}}_{i-r} = \{0\}$; if $i = r$, $\mathcal{P}^{\mathbf{t}}_{i-r} = \operatorname{span}\{1\} \neq \{0\}$. ■

Proof of Theorem 2.3. Applying Proposition 1.2.(ii) and Propositions 2.4 and 2.9, the result follows. ■

2.1 Application to the algebraic integrability problem over $\mathbb{C}[[x,y]]$

A vector field $\mathbf{F} = \mathbf{F}_r + \cdots$ is algebraically integrable over $\mathbb{C}[[x,y]]$ if it has a first integral which is algebraic over $\mathbb{C}[[x,y]]$. In such a case, from [8, 34], the vector field $\mathbf{F}$ has a generalized rational first integral, that is, it has a first integral that is quotient of two formal functions. Moreover, from [5] a necessary condition of algebraic integrability over $\mathbb{C}[[x,y]]$ of $\mathbf{F}$ is the existence of a rational first integral for its leading term $\mathbf{F}_r$.

Next, we give the following auxiliary result.

Lemma 2.10 *If $I_{rat} = \frac{f}{g}$ with $f \in \mathcal{P}^{\mathbf{t}}_{m_0}$ and $g \in \mathcal{P}^{\mathbf{t}}_{n_0}$ is a rational first integral of an irreducible quasi-homogeneous vector field $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}^{\mathbf{t}}_0$ with $h_{r+|\mathbf{t}|} \not\equiv 0$, then $m_0 \neq n_0$.*

Proof. If I_{rat} is a first integral of $\mathbf{F}_r$, it holds $F_r(I_{rat}) = 0$ in $\Omega_g := \{(x,y) \in \mathbb{C}^2, g(x,y) \neq 0\}$ i.e. $P(gf_x - fg_x) + Q(gf_y - fg_y) = 0$ in Ω_g, where f_x, f_y and g_x, g_y are the partial derivatives of f and g and $\mathbf{F}_r = (P,Q)^T$. As the components of $\mathbf{F}_r$ haven't got common factors, we have that there exists $K \in \mathcal{P}^{\mathbf{t}}_{m_0+n_0-r-|\mathbf{t}|}$, such that

$$(g_y f - f_y g) = KP, \qquad -(g_x f - f_x g) = KQ,$$

thus, it follows that $K\mathbf{F}_r = g\mathbf{X}_f - f\mathbf{X}_g = g^2 \mathbf{X}_{\frac{f}{g}}$ in Ω_g ($K \not\equiv 0$ since $P.Q \not\equiv 0$). So,

$$K h_{r+|\mathbf{t}|} = \mathbf{D}^{\mathbf{t}}_0 \wedge (K\mathbf{F}_r) = \mathbf{D}^{\mathbf{t}}_0 \wedge (g^2 \mathbf{X}_{\frac{f}{g}}) = \mathbf{D}^{\mathbf{t}}_0 \wedge (g\mathbf{X}_f - f\mathbf{X}_g),$$

and as $\mathbf{D}^{\mathbf{t}}_0 \wedge \mathbf{X}_f = m_0 f$ and $\mathbf{D}^{\mathbf{t}}_0 \wedge \mathbf{X}_g = n_0 g$, we have that $K h_{r+|\mathbf{t}|} = (m_0 - n_0) f g$ in Ω_g, and this equality can be extend to $\mathbb{C}^2$. Therefore, $m_0 \neq n_0$. ■

We now give a result that characterizes vector fields algebraically integrable over $\mathbb{C}[[x, y]]$ of a wide class of analytic vector fields. Moreover, this result provides an algorithm for calculating vector fields algebraically integrable over $\mathbb{C}[[x, y]]$.

Theorem 2.11 *Let* $\mathbf{F} = \mathbf{F}_r + \cdots$ *such that the origin is an isolated singular point of* $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0^{\mathbf{t}}$ *and* $\mathbf{F}_r$ *is rationally integrable with* $\operatorname{div}(\mathbf{F}_r) \neq 0$ *and* $r(f) \notin \mathbb{Q} \cap [-1, 0)$ *(or* $\frac{1}{r\rho(f)} \notin \mathbb{Q} \cap [-1, 0)$ *for* $r > 0$*) for any* f *irreducible factor of* $h_{r+|\mathbf{t}|}$*. Assume that* $\ell^c_{r+i+|\mathbf{t}|}$ *is a full range for* $i = 1, \ldots, r + t_1 t_2 - |\mathbf{t}|$*. Then* $\mathbf{F}$ *is algebraically integrable over* $\mathbb{C}[[x, y]]$ *if and only if* $\mathbf{F}$ *is orbitally equivalent to* $\mathbf{F}_r$*.*

Proof. The sufficient condition is direct. If $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_r$ and $\mathbf{F}_r$ has a rational first integral, undoing the change that transforms $\mathbf{F}$ into $\mathbf{F}_r$, it arrives to that $\mathbf{F}$ has a generalized rational first integral and by [8, 34], the vector field $\mathbf{F}$ is algebraically integrable over $\mathbb{C}[[x, y]]$.

Let prove the necessary condition. From [6], any first integral of $\mathbf{F}_r$ is a factorization of the irreducible factors of $h_{r+|\mathbf{t}|}$, so $\mathbf{F}_r$ has a rational first integral of the form $I = f_1^{p_1} f_2^{p_2} \cdots f_m^{p_m}$ with f_i irreducible factor of $h_{r+|\mathbf{t}|}$ and p_i integer numbers.
As $r(f_i) \neq -\frac{1}{k}$ for any k natural since $r(f_i) \notin [-1, 0)$, by Theorem 2.3, $\mathbf{F}$ is orbitally equivalent to $\mathbf{G} = \mathbf{F}_r + \sum_{j>r} \mu_j \mathbf{D}_0^{\mathbf{t}}$ with $\mu_j \in \operatorname{Cor}(\ell_j)$. We suppose that there exists j_0 such that $j_0 = \min\{j > r : \mu_j \neq 0\}$. If $\mathbf{F}$ is algebraically integrable over $\mathbb{C}[[x, y]]$, then $\mathbf{G}$ is also algebraically integrable. Moreover, $f_1(x, y) = 0, \ldots, f_m(x, y) = 0$ are also invariant curves of $\mathbf{G}$ since $G(f_i) = (K_i + \sum_{j>r} j\mu_j) f_i$, with K_i cofactor of f_i with respect to $\mathbf{F}_r$. From [7, Therorem 11], there exists a unique (module to product by an unit function) invariant curve of $\mathbf{G}$ by starting f_i since $r(f_i) \notin \mathbb{Q} \cap [-1, 0)$, for any f_i irreducible factor of $h_{r+|\mathbf{t}|}$. Therefore, $f_i(x, y) = 0$ are the irreducible invariant curves of $\mathbf{G}$ and I is the primitive first integral of $\mathbf{G}$. By imposing that $G(I) = 0$, by Euler Theorem we have that $G(I) = (p_1 \deg_{\mathbf{t}}(f_1) + p_2 \deg_{\mathbf{t}}(f_2) + \cdots + p_m \deg_{\mathbf{t}}(f_m)) \mu_{j_0} I + \cdots = 0$. From Lemma 2.10, we have that $p_1 \deg_{\mathbf{t}}(f_1) + p_2 \deg_{\mathbf{t}}(f_2) + \cdots + p_m \deg_{\mathbf{t}}(f_m) \neq 0$ (the degree of I), thus $\mu_{j_0} = 0$.
■

A direct consequence is the following result.

Corollary 2.12 *Consider the analytic system* $(\dot{x}, \dot{y}) = (P_2 + \cdots, Q_2 + \cdots)$ *with* P_2 *and* Q_2 *quadratic homogeneous polynomials,* (P_2, Q_2) *rationally integrable and* $r(f) \notin \mathbb{Q} \cap [-1, 0)$ *(or* $\frac{1}{r\rho(f)} \notin \mathbb{Q} \cap [-1, 0)$ *for* $r > 0$*) for any* f *irreducible factor of* $xQ_2(x, y) - yP_2(x, y)$*. The system is algebraically integrable over* $\mathbb{C}[[x, y]]$ *if and only if it is orbitally equivalent to* $(\dot{x}, \dot{y}) = (P_2, Q_2)$*.*

Proof. The vector field (P_2, Q_2) is a quasi-homogeneous vector field of type $\mathbf{t} = (1, 1)$ and degree $r = 1$. So, $r + t_1 t_2 - |\mathbf{t}| = 0$. Applying Theorem 2.11, the result follows. ■

3 Normal form of vector fields whose leading term is polynomially integrable

We give an orbital normal form of a vector field whose quasi-homogeneous leading term has a polynomial first integral. Notice that this condition is a necessary condition for that a vector field be analytically integrable.

Theorem 3.13 *Consider the vector field* $\mathbf{F} = \sum_{j\geq r} \mathbf{F}_j$, $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}}$, $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}_r^{\mathbf{t}}$ *with* $h_{r+|\mathbf{t}|} \not\equiv 0$ *having only simple factors on* $\mathbb{C}[x,y]$ *and assume that* $I_M \in \mathcal{P}_M^{\mathbf{t}}$ *is the primitive quasi-homogeneous polynomial first integral of degree* M *of* $\mathbf{F}_r$ *(given in Proposition 1.2.(iv)). Then,* $\mathbf{F}$ *is orbitally equivalent to*

$$\begin{aligned} \mathbf{G} \;=\; & \mathbf{F}_r + \sum_{j=1}^{r-1} \mathbf{X}_{g_{r+j+|\mathbf{t}|}} + \sum_{\substack{j=r+1 \\ \mathcal{P}_{j-r}^{\mathbf{t}}=\{0\}}}^{r+t_1t_2-|\mathbf{t}|} \mathbf{X}_{g_{r+j+|\mathbf{t}|}} + \sum_{\substack{j>r \\ \mathcal{P}_{j-r}^{\mathbf{t}}=\{0\}}} \mu_j \mathbf{D}_0^{\mathbf{t}} + \sum_{\substack{j=r+1 \\ \mathcal{P}_{j-r}^{\mathbf{t}}\neq\{0\}}}^{M-1} \mu_j^{(0)} \mathbf{D}_0^{\mathbf{t}} \\ & + \sum_{l>0} \sum_{\substack{j=0 \\ \mathcal{P}_{j-r}^{\mathbf{t}}\neq\{0\}}}^{M-1} \mu_j^{(l)} I_M^l \mathbf{D}_0^{\mathbf{t}} \end{aligned} \tag{3.5}$$

with $g_{r+j+|\mathbf{t}|} \in \mathrm{Cor}\left(\ell_{r+j+|\mathbf{t}|}^c\right)$ *and* $\mu_j, \mu_j^{(l)} \in \mathrm{Cor}\left(\ell_j\right)$, $j \geq 1$.

To prove Theorem 3.13, we will need a series of technical results which have been proved in previous works. The following was proved in [4, Lemma 3.22].

Lemma 3.14 *Consider* $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}_r^{\mathbf{t}}$ *and* $f(x,y) = 0$ *with* $f \in \mathcal{P}_s^{\mathbf{t}}$ *an irreducible quasi-homogeneous invariant curve of* $\mathbf{F}_r$ *with* K_r *its cofactor and assume that* f *is not a factor of* $\mathbf{F}_r + \frac{jK_r}{k-js}\mathbf{D}_0^{\mathbf{t}}$ *for* $j = 0, \cdots, m-1$. *If* $p \in \mathcal{P}_k^{\mathbf{t}}$ *such that* $\ell_{r+k}(p) \in f^m \mathcal{P}_{r+k-ms}^{\mathbf{t}}$, *then* $p \in f^m \mathcal{P}_{k-ms}^{\mathbf{t}}$.

From the previous result we can establish the following ones.

Lemma 3.15 *Consider* $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0^{\mathbf{t}}$, *with* $h_{r+|\mathbf{t}|}\mu_r \neq 0$ *and* $h_{r+|\mathbf{t}|}$ *having only simple factors, and* $f(x,y) = 0$ *with* $f \in \mathcal{P}_s^t$ *an irreducible invariant curve of* $\mathbf{F}_r$ *and* K_r *its cofactor. Fixed a real constant* C, *assume that* f *is a factor of* $\tilde{\mathbf{F}}_r = \mathbf{F}_r + CK_r\mathbf{D}_0^{\mathbf{t}}$. *Then,* $1 + Cs \neq 0$ *and* $r(f) = \frac{C}{1+Cs}$.

Proof. As f is a factor of $\tilde{\mathbf{F}}_r$, we have that there exists a natural number n such that $\tilde{\mathbf{F}}_r = f^n \mathbf{G}_{r-ns}$ and f does not divide to $\mathbf{G}_{r-ns}$. Moreover, $h_{r+|\mathbf{t}|}$ is the Hamiltonian part of both $\tilde{\mathbf{F}}_r$ and $\mathbf{F}_r$ and it has only simple factors, so that $n = 1$. We write $\mathbf{G}_{r-s} = \mathbf{X}_g + \lambda \mathbf{D}_0^{\mathbf{t}}$ with $g \in \mathcal{P}_{r+|\mathbf{t}|-s}^{\mathbf{t}}$ and $\lambda \in \mathcal{P}_{r-s}^{\mathbf{t}}$. Thus, $\tilde{\mathbf{F}}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + (\mu_r + CK_r)\mathbf{D}_0^{\mathbf{t}} = f\mathbf{X}_g + f\lambda\mathbf{D}_0^{\mathbf{t}}$, with

$$\begin{aligned} h_{r+|\mathbf{t}|} &= \tfrac{1}{r+|\mathbf{t}|}(\mathbf{D}_0^{\mathbf{t}} \wedge (f\mathbf{X}_g)) = \tfrac{r+|\mathbf{t}|-s}{r+|\mathbf{t}|} fg, \\ \mu_r + CK_r &= \tfrac{1}{r+|\mathbf{t}|}\mathrm{div}(\tilde{\mathbf{F}}_r) = \tfrac{1}{r+|\mathbf{t}|}\left[\mathrm{div}\left(f\mathbf{X}_g\right) + \mathrm{div}\left(f\lambda\mathbf{D}_0^{\mathbf{t}}\right)\right] = \tfrac{1}{r+|\mathbf{t}|}X_g(f) + f\lambda. \end{aligned}$$

Solving g of the first equation, $g=\frac{r+|\mathbf{t}|}{r+|\mathbf{t}|-s}\frac{h_{r+|\mathbf{t}|}}{f}$, and replacing it in the second expression, we have that

$$f\lambda=\mu_r+CK_r-\tfrac{1}{r+|\mathbf{t}|-s}\frac{X_{h_{r+|\mathbf{t}|}}(f)}{f}.$$

As $f(x,y)=0$ is an invariant curve of $\mathbf{F}_r$ and K_r its cofactor, we have that

$$K_rf=F_r(f)=X_{h_{r+|\mathbf{t}|}}(f)+(\mu_rD_0)(f)=X_{h_{r+|\mathbf{t}|}}(f)+s\mu_rf.$$

Solving K_r and replacing it in the above expression, we have

$$f\lambda \;=\; \mu_r(1+Cs)+\frac{X_{h_{r+|\mathbf{t}|}}(f)}{f}(C-\tfrac{1}{r+|\mathbf{t}|-s}). \tag{3.6}$$

We check that $1+Cs\neq 0$. Otherwise,

$$\tilde{F}_r(f)=F_r(f)+CK_rD_0(f)=K_rf+CK_r(sf)=K_r(1+Cs)f=0.$$

Thus f is a first integral of $f\mathbf{G}_{r-s}$ and therefore f is also a first integral of $\mathbf{G}_{r-s}=\mathbf{X}_g+\lambda\mathbf{D}_0^{\mathbf{t}}$ with $\mathbf{G}_{r-s}$ irreducible. So, $g=cf, c\in\mathbb{R}$ and $\lambda\neq 0$ since $\mu_r\neq 0$. As g has only a factor, $\mathbf{G}_{r-s}$ is not polynomially integrable. This fact yields contradiction.

The factorization of $h_{r+|\mathbf{t}|}$ on $\mathbb{C}[x,y]$ can be written as $h_{r+|\mathbf{t}|}(x,y)=x^{\delta_1}y^{\delta_2}\Pi_{i=1}^m f_i$, with $\delta_1,\delta_2\in\{0,1\}$ and $f_i=y^{t_2}-\lambda_ix^{t_1}\in\mathcal{P}^t_{t_1t_2}$ with $\lambda_i\in\mathbb{C}\setminus\{0\}$, that is the invariant curve is $f(x,y)=0$ with $f=x$, $f=y$ or $f=y^{t_1}-\lambda_ix^{t_2}$. Thus,

$$\begin{aligned}X_{h_{r+|\mathbf{t}|}}(f) &= \left(\delta_1\frac{h_{r+|\mathbf{t}|}}{x}X_x+\delta_2\frac{h_{r+|\mathbf{t}|}}{y}X_y+\sum_{i=1}^m\frac{h_{r+|\mathbf{t}|}}{f_i}X_{f_i}\right)(f)\\ &= h_{r+|\mathbf{t}|}\left[\delta_1\frac{X_x(f)}{x}+\delta_2\frac{X_y(f)}{y}+\sum_{i=1}^m\frac{X_{f_i}(f)}{f_i}\right].\end{aligned}$$

Replacing $X_{h_{r+|\mathbf{t}|}}(f)$ into (3.6) we have that

$$\mu_r+\frac{C(r-s+|\mathbf{t}|)-1}{(1+Cs)(r-s+|\mathbf{t}|)}\frac{h_{r+|\mathbf{t}|}}{f}\left[\delta_1\frac{X_x(f)}{x}+\delta_2\frac{X_y(f)}{y}+\sum_{i=1}^m\frac{X_{f_i}(f)}{f_i}\right]=\frac{f\lambda}{1+Cs}$$

As f is a simple factor of $h_{r+|\mathbf{t}|}$ then f does not divide to $\frac{h_{r+|\mathbf{t}|}}{f}$ and thus there exists a rational function $\tilde{\lambda}$ such that $\tilde{\lambda}=\lambda\frac{f}{h_{r+|\mathbf{t}|}}$ where $f\tilde{\lambda}\,|_{f=0}=0$. To clarify, we denote $A=\frac{C(r-s+|\mathbf{t}|)-1}{(1+Cs)(r-s+|\mathbf{t}|)}$ and, dividing by $\frac{h_{r+|\mathbf{t}|}}{f}$, the above equality becomes

$$f\frac{\mu_r}{h_{r+|\mathbf{t}|}}+A\left[\delta_1\frac{X_x(f)}{x}+\delta_2\frac{X_y(f)}{y}+\sum_{i=1}^m\frac{X_{f_i}(f)}{f_i}\right] \;=\; f\frac{\tilde{\lambda}}{1+Cs}. \tag{3.7}$$

We know that the rational function η, by Algaba *et al.* [5] satisfies

$$\frac{\mu_r(x,y)}{h_{r+|\mathbf{t}|}(x,y)}=x^{t_2-1}y^{t_1-1}\eta(x^{t_2},y^{t_1}).$$

So, taking $X=x^{t_2},Y=y^{t_1}$, by (3.7), we have that

$$\eta(X,Y) \;=\; -\frac{A}{fx^{t_2-1}y^{t_1-1}}\left[\delta_1\frac{X_x(f)}{x}+\delta_2\frac{X_y(f)}{y}+\sum_{i=1}^m\frac{X_{f_i}(f)}{f_i}\right]+\frac{\tilde{\lambda}}{(1+Cs)x^{t_2-1}y^{t_1-1}}. \tag{3.8}$$

Next, we prove that

$$\mathrm{Res}\,[\eta(1,Y),\lambda]=\frac{s}{t_1t_2}(1-C\frac{r+|\mathbf{t}|}{1+Cs}), \tag{3.9}$$

where $s = \deg_{\mathbf{t}}(f_i)$ and λ is the pole of $\eta(1,Y)$ associated to f_i, for $i = 1, \ldots, m+2$. We see each case separately:

- For $f = x$, in this case $\delta_1 = 1$, $k_1 = 0$ and $s = t_1$, equation (3.8) becomes

$$\eta(X,Y) = -\tfrac{1}{Xy^{t_1-1}}A(-\tfrac{\delta_2}{y} - t_1 y^{t_1-1}\textstyle\sum_{i=1}^{m}\tfrac{1}{Y-\lambda_i X}) + \tfrac{x\tilde{\lambda}}{(1+Cs)Xy^{t_1-1}}.$$

We have that

$$\begin{array}{rcl} \mathrm{Res}\,[\eta(1,Y),\infty] & = & -A(\delta_2 + t_1 m) = -A\frac{\delta_2 t_2 + t_1 t_2 m}{t_2} = -A\frac{r-s+|\mathbf{t}|}{t_2} \\ & = & -\frac{C(r-s+|\mathbf{t}|)-1}{(1+Cs)(r-s+|\mathbf{t}|)}\frac{r-s+|\mathbf{t}|}{t_2} = \frac{1}{t_2}\left(1 - C\frac{r+|\mathbf{t}|}{1+Cs}\right). \end{array}$$

- For $f = y$, ($\delta_2 = 1$, $s = t_2$). The equation (3.8) is

$$\begin{array}{rcl} \eta(X,Y) & = & -\frac{A}{x^{t_2-1}Y}(\frac{\delta_1}{x} - t_2 x^{t_2-1}\sum_{i=1}^{m}\frac{\lambda_i}{Y-\lambda_i X}) + \frac{y\tilde{\lambda}}{(1+Cs)x^{t_2-1}Y} \\ & = & -\frac{A}{Y}(\frac{\delta_1}{X} - t_2\sum_{i=1}^{m}\frac{\lambda_i}{Y-\lambda_i X}) + \frac{y\tilde{\lambda}}{(1+Cs)x^{t_2-1}Y}. \end{array}$$

Therefore,

$$\begin{array}{rcl} \mathrm{Res}\,[\eta(1,Y),0] & = & -A(\delta_1 + t_2 m) = -A\frac{r-s+|\mathbf{t}|}{t_1} \\ & = & -\frac{C(r-s+|\mathbf{t}|)-1}{(1+Cs)(r-s+|\mathbf{t}|)}\frac{r-s+|\mathbf{t}|}{t_1} = \frac{1}{t_1}\left(1 - C\frac{r+|\mathbf{t}|}{1+Cs}\right). \end{array}$$

- For $f = y^{t_1} - \lambda_{i_0}x^{t_2}$, $s = t_1 t_2$. As $X_x(y^{t_1} - \lambda_i x^{t_2}) = t_1 y^{t_1-1}$, $X_y(y^{t_1} - \lambda_i x^{t_2}) = \lambda_i t_2 x^{t_2-1}$ and $X_{y^{t_1}-\lambda_i x^{t_2}}(y^{t_1} - \lambda_{i_0}x^{t_2}) = t_1 t_2 x^{t_2-1}y^{t_1-1}(\lambda_{i_0} - \lambda_i)$, (3.8) becomes

$$\begin{array}{rcl} \eta(X,Y) & = & -\frac{1}{(Y-\lambda_{i_0}X)}\frac{1}{x^{t_2-1}y^{t_1-1}}A\left(\frac{\delta_1 t_1 y^{t_1-1}}{x} + \frac{\delta_2\lambda_{i_0}t_2 x^{t_2-1}}{y}\right. \\ & & \left. + t_1 t_2 x^{t_2-1}y^{t_1-1}\sum_{\substack{i=1\\ i\neq i_0}}^{m}\frac{\lambda_{i_0}-\lambda_i}{Y-\lambda_i X}\right) + \frac{\tilde{\lambda}}{(1+Cs)x^{t_2-1}y^{t_1-1}} \\ & = & -\frac{1}{Y-\lambda_{i_0}X}A\left(\frac{\delta_1 t_1}{X} + \frac{\delta_2\lambda_{i_0}t_2}{Y} + t_1 t_2\sum_{\substack{i=1\\ i\neq i_0}}^{m}\frac{\lambda_{i_0}-\lambda_i}{Y-\lambda_i X}\right) + \frac{\tilde{\lambda}}{(1+Cs)x^{t_2-1}y^{t_1-1}} \end{array}$$

$$\begin{array}{rcl} \mathrm{Res}\,[\eta(1,Y),\lambda_{i_0}] & = & -A(\delta_1 t_1 + \delta_2 t_2 + (m-1)t_1 t_2) = -A(r + |\mathbf{t}| - s) \\ & = & -\frac{C(r-s+|\mathbf{t}|)-1}{(1+Cs)(r-s+|\mathbf{t}|)}(r + |\mathbf{t}| - s) = 1 - C\frac{r+|\mathbf{t}|}{1+Cs}. \end{array}$$

So, we have proved (3.9) and by (1.1), we have that $r(f_i) = \frac{C}{1+Cs}$, for $i = 1, \ldots, m+2$. ■

Proposition 3.16 *Consider* $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r\mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}_r^{\mathbf{t}}$ *irreducible and* $f(x,y) = 0$ *with* $f \in \mathcal{P}_s^{\mathbf{t}}$ *an irreducible invariant curve of* $\mathbf{F}_r$ *and assume that* $r(f) \neq \frac{j}{k}$ *for* $j = 0, \cdots, m-1$. *If* $p \in \mathcal{P}_k^{\mathbf{t}}$, $\ell_{r+k}(p) \in f^m\mathcal{P}_{r+k-ms}^{\mathbf{t}}$ *then* $p \in f^m\mathcal{P}_{k-ms}^{\mathbf{t}}$.

Proof. By Lemma 3.14, it enough to prove that f is not a factor of $\mathbf{G}_r = \mathbf{F}_r + \frac{jK_r}{k-js}\mathbf{D}_0^{\mathbf{t}}$. The vector field $\mathbf{G}_r$ is of the form $\mathbf{F}_r + CK_r\mathbf{D}_0^{\mathbf{t}}$ with $C = \frac{j}{k-js}$. The real number $\frac{C}{1+Cs}$ is $\frac{C}{1+Cs} = \frac{\frac{j}{k-js}}{1+s\frac{j}{k-js}} = \frac{j}{k}$. So, from Lemma 3.15, if $r(f) \neq \frac{j}{k}$ para $j = 0, \cdots, m-1$, we have the result. ■

Proposition 3.17 *Consider* $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}_r^{\mathbf{t}}$*,* $h \not\equiv 0$ *with simple factors on* $\mathbb{C}[x,y]$ *and let* $I_M \in \mathcal{P}_M^{\mathbf{t}}$ *be the primitive first integral of* $\mathbf{F}_r$ *(given in Proposition 1.2.(iv)),* $(I_M = h_{r+|\mathbf{t}|}$ *if* $\mu_r \equiv 0)$*. Given* $p \in \mathcal{P}_k^{\mathbf{t}}$*, if* $\ell_{r+k}(p)$ *is multiple of* I_M *then* p *is zero or a non null multiple of* I_M*.*

Proof. We have that $k \geq M$ otherwise, if $k < M$ and $\ell_{r+k}(p)$ is a multiple of I_M we obtain $\ell_{r+k}(p) = 0$ which implies that p is a first integral of $\mathbf{F}_r$ of degree less than M which gives a contradiction since I_M is a primitive first integral of $\mathbf{F}_r$.

If $h_{r+|\mathbf{t}|} = \prod_{i=1}^m f_i$, $f_i \in \mathcal{P}_{s_i}^{\mathbf{t}}$, the unique irreducible quasi-homogeneous invariant curves of $\mathbf{F}_r$ are $f_i(x,y) = 0$, $i = 1, \ldots, m$ with cofactors K_i, $i = 1, \ldots, m$.

As $n_i = Mr(f_i) \in \mathbb{N}$ for any $i = 1, \ldots, m$, the polynomial $I_M = \prod_{i=1}^m f_i^{n_i} \in \mathcal{P}_M^{\mathbf{t}}$, with $M = \sum_{i=1}^m n_i \deg_{\mathbf{t}}(f_i)$ is a primitive first integral. It is sufficient to prove that if $\ell_{r+k}(p) \in f_i^{n_i} \mathcal{P}_{r+k-n_i s_i}^{\mathbf{t}}$ then $p \in f_i^{n_i} \mathcal{P}_{k-n_i s_i}^{\mathbf{t}}$ for $i = 1, \ldots, m$.

For $j = 0, \cdots, n_i - 1$ and $k \geq M$ we have $r(f_i) = \frac{n_i}{M} > \frac{j}{M} \geq \frac{j}{k}$. Applying Proposition 3.16, $p \in f_i^{n_i} \mathcal{P}_{k-n_i s_i}^{\mathbf{t}}$ for any $i = 1, \ldots, m$ and the result follows. ■

Next statement establishes a cyclicity relation between the co-ranges of the operators ℓ_k when $\mathbf{F}_r$ has a polynomial first integral.

Proposition 3.18 *Assume that* $\mathbf{F}_r$ *is polynomially integrable and let* $I_M \in \mathcal{P}_M^{\mathbf{t}}$ *be the primitive first integral of* $\mathbf{F}_r$ *(given in Proposition 1.2.(iv)). Fixed* $k \geq r$ *with* $\mathcal{P}_{k-r}^{\mathbf{t}} \neq \{0\}$*, we can choose a complementary subspace to* Range(ℓ_{k+M}) *such that* Cor$(\ell_{k+M}) = I_M$Cor(ℓ_k)*.*

Proof. We will prove that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M\mathrm{Cor}(\ell_k) \cap \mathrm{Range}(\ell_{k+M}) = \{0\}$ by reductio ad absurdum. Given $p \in \mathrm{Cor}(\ell_k) \setminus \{0\}$ such that $pI_M \in \mathrm{Range}(\ell_{k+M})$, then there exists $q \in \mathcal{P}_{k+M-r}^{\mathbf{t}} \setminus \{0\}$ (exists because $k \geq r$ and therefore $\mathcal{P}_{k-r}^{\mathbf{t}} \neq \{0\}$), such that $\ell_{k+M}(q) = pI_M$. Applying Proposition 3.17 we have that $q = \tilde{q} I_M$ with $\tilde{q} \in \mathcal{P}_{k-r}^{\mathbf{t}} \setminus \{0\}$ and consequently

$$pI_M \;=\; F_r(q) = F_r(\tilde{q} I_M) = I_M F_r(\tilde{q}).$$

Hence $p = F_r(\tilde{q})$, that is, $p \in \mathrm{Range}(\ell_k) \cap \mathrm{Cor}(\ell_k)$ which gives a contradiction.

We have shown that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$. Therefore it is enough to prove that $\dim(I_M\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{k+M}))$.

Since ℓ_k and ℓ_{k+M} are linear operators, we get

$$\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathcal{P}_k^{\mathbf{t}}) - \dim(\mathcal{P}_{k-r}^{\mathbf{t}}) + \dim(\mathrm{Ker}(\ell_k)). \tag{3.10}$$

$$\dim(\mathrm{Cor}(\ell_{k+M})) = \dim(\mathcal{P}_{k+M}^{\mathbf{t}}) - \dim(\mathcal{P}_{k+M-r}^{\mathbf{t}}) + \dim(\mathrm{Ker}(\ell_{k+M})). \tag{3.11}$$

Applying Lemma 2.6 for $N = M$ and from (3.10), (3.11), we get

$$\dim(\mathrm{Cor}(\ell_{k+M})) = \dim(\mathrm{Cor}(\ell_k)) - \dim(\mathrm{Ker}(\ell_k)) + \dim(\mathrm{Ker}(\ell_{k+M})). \tag{3.12}$$

We note that $Ker(\ell_k) = \text{span}\left\{I_M^l\right\}$, if $k-r = lM$. Otherwise, $Ker(\ell_k) = \text{span}\left\{0\right\}$. Thus, by (3.12), we obtain $\dim(\text{Cor}(\ell_k)) = \dim(\text{Cor}(\ell_{k+M}))$ and this completes the proof. ■

Now we can proof the main theorem of this section.

Proof of Theorem 3.13. Let $f_1, \ldots, f_n$ the simple factors of $h_{r+|\mathbf{t}|}$. If $\mathbf{F}_r$ is a Hamiltonian system, *i.e.* $\mu_r \equiv 0$, then $\eta \equiv 0$ and thus for every $i = 1, \cdots, n$, $r(f_i) = \frac{1}{r+|\mathbf{t}|}$, therefore $r(f_i) \neq -\frac{1}{k}$ for any $k \in \mathbb{N}$. By Theorem 2.3 and Proposition 3.18, we have the result.

Otherwise, $\mu_r \not\equiv 0$. As $\mathbf{F}_r$ is polynomially integrable, there are some natural numbers n_i, $i = 1, \cdots, m$ such that $r(f_i) = \frac{n_i}{M}$ where $M = \sum_{i=1}^{m} n_i \deg_{\mathbf{t}}(f_i)$ is the quasi-homogeneous degree of the primitive first integral, thus for each $i = 1, \cdots, m$, we have that $r(f_i) \neq -\frac{1}{k}$ for all $k \in \mathbb{N}$. From Theorem 2.3 and Proposition 3.18, the result follows. ■

3.1 Application to the analytic integrability problem

There are wide families of analytic vector fields whose normal form has the expression $\mathbf{F}_r + \mu \mathbf{D}_0^{\mathbf{t}}$ with $\mu = \sum_{j>r} \mu_j$. For example, the resonant saddle singular point, center-focus singular point, nilpotent singular point. Next, we study the analytic integrability problem for these families. We prove that are analytically integrable if, and only if, they are orbitally equivalent to its leading quasi-homogeneous term. Notice that the polynomial integrability of the leading term of a vector field is a necessary condition for the analytic integrability of the vector field.

Theorem 3.19 *Let* $\mathbf{F} = \mathbf{F}_r + \cdots$ *such that the origin is an isolated singular point of* $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ *and* $\mathbf{F}_r$ *is polynomially integrable with* $\text{div}\,(\mathbf{F}_r) \neq 0$ *and assume that* $\ell^c_{r+i+|\mathbf{t}|}$ *is a full range for* $i = 1, \ldots, r + t_1 t_2 - |\mathbf{t}|$. *Then* $\mathbf{F}$ *is analytically integrable if and only if* $\mathbf{F}$ *is orbitally equivalent to* $\mathbf{F}_r$.

Proof. The sufficient condition is trivial. Let's now test the necessary condition. As $\text{Cor}\left(\ell^c_{r+i+|\mathbf{t}|}\right) = \{0\}$ for $i = 1, \ldots, r + t_1 t_2 - |\mathbf{t}|$, by Theorem 3.13, $\mathbf{F}$ is orbitally equivalent to $\mathbf{G} = \mathbf{F}_r + \sum_{j>r} \mu_j \mathbf{D}_0^{\mathbf{t}}$ with $\mu_j \in \text{Cor}\,(\ell_j)$. If $\mu_j = 0$ for all $j > r$ the result is proved, otherwise let $j_0 = \min\{j > r : \mu_j \neq 0\}$. If $\mathbf{F}$ is integrable, $\mathbf{G}$ is still integrable. Let $I_M \in \mathcal{P}_M^{\mathbf{t}}$ a primitive first integral of $\mathbf{F}_r$. A first integral of $\mathbf{G}$ is $I = I_M + \sum_{j>M} I_j$ with $I_j \in \mathcal{P}_j^{\mathbf{t}}$. The condition of integrability is $G(I) = 0$. If we consider the degree $M + j_0$ of above condition, we obtain

$$0 = \left(\mu_{j_0} D_0\right)(I_M) + F_r(I_{M+j_0-r}) = M \mu_{j_0} I_M + F_r(I_{M+j_0-r})$$

i.e. $\mu_{j_0} I_M \in \text{Range}\,(\ell_{M+j_0}) \setminus \{0\}$ and Proposition 3.18 arrives to contradiction.

The polynomial I_M is a first integral of $\mathbf{F}_r$ which is transformed into a formal first integral $I = I_M + \text{h.o.t.}$ of $\mathbf{F}$ and from [31, Theorem A], there exists an analytic first integral of $\mathbf{F}$. ■

4 Some applications and examples

4.1 Perturbations of a resonant linear node

In this subsection, we study the algebraic integrability over $\mathbb{C}[[x,y]]$ of a resonant linear node.

First, we give a normal form of a vector field whose origin is a resonant node critical point. This result can be proved using Poincaré–Dulac normal form, [2, 14, 17, 33]. Here, we have used our techniques to prove it.

Proposition 4.20 *Consider the analytic vector field* $\mathbf{F} = (px + h.o.t., qy + h.o.t.)^T$ *with* p, q *natural numbers and* $p > q$. *We have that*
(i) if $\frac{p}{q} \notin \mathbb{N}$, *then* $\mathbf{F}$ *is orbitally equivalent to* $(px, qy)^T$.
(ii) if $p = (k+1)q$ *with* $k \geq 1$, *then* $\mathbf{F}$ *is orbitally equivalent to* $((k+1)x + by^{k+1}, y)^T$ *with* b *real number.*

Proof. If we consider Taylor expansion (quasi-homogeneous expansion with respect to the type $\mathbf{t} = (1,1)$), the vector field $\mathbf{F}$ can be written as $\mathbf{F} = \mathbf{F}_0 + \sum_{j>0} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ with $\mathbf{F}_0 = (px, qy)^T$. The leading term of $\mathbf{F}$ has the following conservative-dissipative embledding $\mathbf{F}_0 = \begin{pmatrix} px \\ qy \end{pmatrix} = \frac{1}{2}\mathbf{X}_{(q-p)xy} + \frac{p+q}{2}\begin{pmatrix} x \\ y \end{pmatrix}$. Thus $\eta(x,y) = \frac{p+q}{xy(q-p)}$. The factors of h_2 are simple factors. So,
$\operatorname{Res}\left[\eta(1,y), \infty\right] = -\operatorname{Res}\left[\eta(x,1), 0\right] = -\frac{p+q}{q-p}$ and $r(x) = \frac{1}{2}\left(1 + \frac{p+q}{q-p}\right) = \frac{q}{q-p}$,
$\operatorname{Res}\left[\eta(1,y), 0\right] = \frac{p+q}{q-p}$ and $r(y) = \frac{1}{2}\left(1 - \frac{p+q}{q-p}\right) = -\frac{p}{q-p}$.
We already compute an orbitally equivalent normal form of $\mathbf{F}$. In this case, for $\mathbf{t} = (1,1)$, the sets $\mathcal{P}_j^{\mathbf{t}}$ are not trivial for any j natural. We distinguish the following situations:
(a) If $r(x) \neq -\frac{1}{k}$ and $r(y) \neq -\frac{1}{k}$ for all k natural number, *i.e.* $p \neq (k+1)q$ and $q \neq (k+1)p$, applying Theorem 2.3, $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_0 + \sum_{j>0} \mu_j \mathbf{D}_0^{\mathbf{t}}$ with $\mu_j \in \operatorname{Cor}(\ell_j)$, $j \geq 1$. Moreover, $\operatorname{Cor}(\ell_j) = \{0\}$ for all $j \in \mathbb{N}$ since $\operatorname{Ker}(\ell_j) = \{0\}$. Therefore, $\mathbf{F}_0$ is a orbitally equivalent normal form of $\mathbf{F}$, case (i).
(b) If there is a $k \in \mathbb{N}$ such that $p = (k+1)q$, the leading term of $\mathbf{F}$ is $\mathbf{F}_0 = ((k+1)qx, qy)^T$ and scaling the time, we can assume that $\mathbf{F}_0 = ((k+1)x, y)^T$. We see the expression of linear operator ℓ_{j+2}^c. We denote $\mathbf{G}_0 = \mathbf{F}_0 - \frac{k+2}{j+2}\mathbf{D}_0^{\mathbf{t}}$. If we choose $\Delta_{j+2} = \langle x^{j+2}, y^{j+2} \rangle$, we have that

$$\begin{aligned} \ell_{j+2}^c(x^{j+2}) &= \operatorname{Proj}_{\Delta_{j+2}}\left(G_0(x^{j+2})\right) = \left((k+1)(j+2) - (k+2)\right)x^{j+2}. \\ \ell_{j+2}^c(y^{j+2}) &= \operatorname{Proj}_{\Delta_{j+2}}\left(G_0(y^{j+2})\right) = \left(j+2-(k+2)\right)y^{j+2}. \end{aligned}$$

It is direct to check that $\operatorname{Ker}(\ell_{k+2}^c) = \operatorname{Cor}\left(\ell_{k+2}^c\right) = \langle y^{k+2} \rangle$ and $\operatorname{Ker}(\ell_{j+2}^c) = \operatorname{Cor}\left(\ell_{j+2}^c\right) = \{0\}$ for $j \neq k$. Therefore the vector field $\mathbf{F}$ is orbitally equivalent to $((k+1)x, y)^T + \mathbf{X}_{\beta y^{k+2}}$, that is $((k+1)x + by^{k+1}, y)^T$ with b real number, case (ii). ■

Next, we characterize the resonant node singular points having an algebraic first integral over $\mathbb{C}[[x,y]]$. This problem has been studied in [28] by using blows up techniques and the Poincaré-Dulac normal form.

Theorem 4.21 *The analytic vector field* $\mathbf{F} = (px + h.o.t., qy + h.o.t.)^T$ *with* p, q *natural numbers and* $p > q$*, is algebraically integrable over* $\mathbb{C}[[x, y]]$ *if, and only if, it is orbitally equivalent to* $(px, qy)^T$*. In such a case, a generalized rational first integral is of the form* $I = (y + h.o.t.)^p(x + h.o.t.)^{-q}$*.*

Proof. The sufficient condition is direct since $\mathbf{F}_0 = (px, qy)^T$ has the rational first integral $y^p x^{-q}$ and undoing the change that transforms $\mathbf{F}$ into $\mathbf{F}_0$, the result follows.
We see the necessary condition. We assume that $\mathbf{F}$ has an algebraic first integral. If p/q is not natural, from Proposition 4.20 we have that $\mathbf{F}$ and $\mathbf{F}_0$ are orbitally equivalent.
If $p = (k + 1)q$ with k natural number, by Proposition 4.20, the vector field $\mathbf{F}$ is orbitally equivalent to $\mathbf{G} := ((k + 1)x + by^{k+1}, y)^T$. The invariant curves of $\mathbf{G}$ are $y = 0$ and $C(x, y) = 0$ with $C := x + \sum_{j>1} C_j$, $C_j \in \mathcal{P}_j^{\mathbf{t}}$ and its cofactors are 1 and $(k + 1) +$ h.o.t., respectively.
From [8, 34], if $\mathbf{G}$ has an algebraic first integral, then we have a first integral that is ratio of two coprime formal functions. Moreover, we have two formal invariant curves with the same cofactor. So, if $\mathbf{G}$ is algebraically integrable, then C and y^{k+1} has the same cofactor, *i.e.* $k + 1$ is the cofactor of C. Therefore, it must accomplish that $G(C) = (k + 1)C$. This equation to order $k + 1$ is

$$\nabla C_{k+1} \cdot \begin{pmatrix} (k+1)x \\ y \end{pmatrix} + \nabla x \cdot \begin{pmatrix} by^{k+1} \\ 0 \end{pmatrix} = (k+1)C_{k+1},$$

which arrives to $kx\frac{\partial}{\partial x}C_{k+1} = -by^{k+1}$; that is, $b = 0$ and the result follows. ■

4.2 Perturbations of a class of Kolmogorov systems

Consider the quadratic homogeneous vector fields (a quasi-homogeneous with respect to the type $\mathbf{t} = (1, 1)$ of degree 1)

$$\mathbf{F}_1 = (x(-qx + (q + r)y), y((p + r)x - py))^T, \ p, q, r \in \mathbb{Q}, \tag{4.13}$$

The systems $\dot{\mathbf{x}} = \mathbf{F}_1$ are a class of Kolmogorov systems and have been studied by several authors, see [27, 36]. Applications of Kolmogorov systems can be found particularly in modeling population dynamics in biology and ecology. The systems $\dot{\mathbf{x}} = \mathbf{F}_1(\mathbf{x})$ have three invariant real lines and $x^p y^q (x - y)^r$ is a first integral. The following result characterizes the analytic and algebraic integrability over $\mathbb{C}[[x, y]]$ of any perturbation of $\mathbf{F}_1$. The characterization of the analytic integrability is also proved in [9]. The algebraic integrability of these vector fields is a new result.

Theorem 4.22 *Consider an analytic vector field* $\mathbf{F} = \mathbf{F}_1 + h.o.t.$ *and assume that* $p+q+r \neq 0$ *and* $\frac{p}{p+q+r}, \frac{q}{p+q+r}, \frac{r}{p+q+r} \notin [-1, 0)$*,*
(i) if $\mathbf{F}_1$ *is polynomially integrable (i.e.* p, q, r *have same sign), then* $\mathbf{F}$ *is analytically integrable if, and only if,* $\mathbf{F}$ *is orbitally equivalent to* $\mathbf{F}_1$*.*
(ii) if $\mathbf{F}_1$ *is rationally integrable (i.e.* p, q, r *do not have all same sign), then* $\mathbf{F}$ *is algebraically integrable over* $\mathbb{C}[[x, y]]$ *if, and only if,* $\mathbf{F}$ *is orbitally equivalent to* $\mathbf{F}_1$*.*

Proof. It is easy to check that $x^p y^q (x - y)^r$ is a first integral of $\mathbf{F}_1$. It only enough to prove that the vector field $\mathbf{F}_1 := \mathbf{X}_{h_3} + \mu_1(x, y)^T$ with $h_3 = (p + q + r)xy(x - y)$ and

$\mu_1 = (p-2q+r)x + (-2p+q+r)y$ satisfies the hypothesis of Theorems 3.19 and 2.11. On the one hand, as $r + t_1 t_2 - |\mathbf{t}| = 0$, we have that $\mathrm{Cor}\left(\ell_{i+4}^c\right) = \{0\}$ for any i. On the other hand, the rational function η is $\eta(x,y) = \frac{(p-2q+r)x+(-2p+q+r)y}{(p+q+r)xy(x-y)}$. The factors of h_3 are simple factors. We have that

$\mathrm{Res}\,[\eta(1,y),\infty] = -\mathrm{Res}\,[\eta(x,1),0] = \frac{-2p+q+r}{p+q+r}$ and $r(x) = \frac{1}{3}\left(1 - \frac{-2p+q+r}{p+q+r}\right) = \frac{p}{p+q+r}$,

$\mathrm{Res}\,[\eta(1,y),0] = \frac{p-2q+r}{p+q+r}$ and $r(y) = \frac{1}{3}\left(1 - \frac{p-2q+r}{p+q+r}\right) = \frac{q}{p+q+r}$,

$\mathrm{Res}\,[\eta(1,y),1] = \frac{p+q-2r}{p+q+r}$ and $r(x-y) = \frac{1}{3}\left(1 - \frac{p+q-2r}{p+q+r}\right) = \frac{r}{p+q+r}$.

Thus, by hypothesis, $r(f) \notin [-1,0)$. From Theorem 3.19 and 2.11, we have (i) and (ii), respectively. ■

4.3 Perturbations of a resonant linear saddle

Here we study the analytic and algebraic integrability over $\mathbb{C}[[x,y]]$ of a resonant linear saddle. First, we give a normal form of a vector field whose origin is a resonant saddle critical point (this result can also be proved using Poincaré-Dulac normal form, [2, 14, 17, 23, 33]).

Proposition 4.23 *The analytic vector field* $\mathbf{F}(x,y) = (-px + h.o.t., qy + h.o.t.)^T$ *with* p, q *natural numbers and* $p \neq q$, *is orbitally equivalent to* $(-px, qy)^T + \sum_{l>0} \alpha_l (x^q y^p)^l (x,y)^T$ *with* α_l *real numbers.*

Proof. The leading term of $\mathbf{F}$ with respect to the type $\mathbf{t} = (1,1)$ is $\mathbf{F}_0 = (-px, qy)^T = \frac{1}{2}\mathbf{X}_{(q+p)xy} + \frac{q-p}{2}\mathbf{D}_0^{\mathbf{t}}$ that is polynomially integrable and $I_{p+q} = x^q y^p$ is a primitive first integral. As $\mathcal{P}_j^{\mathbf{t}} \neq \{0\}$ for all $j \in \mathbb{N}$, applying Theorem 3.19, we have that $\mathbf{F}$ is orbitally equivalent to

$$\begin{pmatrix} -px \\ qy \end{pmatrix} + \sum_{j=1}^{p+q-1} \mu_j^{(0)} \mathbf{D}_0^{\mathbf{t}} + \sum_{l>0} \sum_{j=0}^{p+q-1} \mu_j^{(l)} I_M^l \mathbf{D}_0^{\mathbf{t}}$$

with $\mu_j^{(l)} \in \mathrm{Cor}\,(\ell_j)$, $j \geq 1$. Moreover, for j not multiple of $(p+q)$, $\mathrm{Ker}\,(\ell_j) = \mathrm{Cor}\,(\ell_j) = \{0\}$ and, otherwise $j = l(p+q)$ we have that $\mathrm{Ker}\,(\ell_j) = \mathrm{Cor}\,(\ell_j) = (x^q y^p)^l$. So, from Theorem 3.5, $\mathbf{F}$ is orbitally equivalent to $(-px, qy)^T + \sum_{l>0} \alpha_l (x^q y^p)^l (x,y)^T$. ■

Next, we characterizes the resonant saddle singular points with an analytic and algebraic first integral over $\mathbb{C}[[x,y]]$. The algebraic integrability for these systems is also studied in [28].

Theorem 4.24 *The analytic vector field* $\mathbf{F} = (-px + h.o.t., qy + h.o.t.)^T$ *with* p, q *natural numbers and* $p \neq q$, *is analytically integrable if, and only if, it is orbitally equivalent to* $(-px, qy)^T$.
Moreover, if $\mathbf{F}$ *is algebraically integrable over* $\mathbb{C}[[x,y]]$ *then it is analytically integrable.*

Proof. The polynomial $I_{p+q} = (x^q y^p)$ is a first integral of $(-px, qy)^T$ and the operator ℓ_{j+2}^c has full range. From Proposition 4.23 and Theorem 3.19, $\mathbf{F}$ is analytically integrable

if and only if $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_0(x,y) = (-px, qy)^T$.
Let prove the second part. we assume the contrary, *i.e.* $\mathbf{F}$ is algebraically integrable over $\mathbb{C}[[x,y]]$ and it is not analytically integrable. So, there exists a first integral $I(x,y) = \frac{f(x,y)}{g(x,y)}$ with $f, g \in \mathbb{C}[[x,y]]$ and $g(0,0) = 0$. Then $f(x,y) - \lambda g(x,y) = 0$ with $\lambda \in \mathbb{R}$ is an invariant curve. So, we arrive to contradiction since $\mathbf{F}$ has only two formal invariant curves. ■

4.4 Perturbations of a nilpotent vector field

Consider the quasi-homogeneous system $(\dot{x}, \dot{y}) = (-2y + dx^2, -4x^3 + 2dxy) \in \mathcal{Q}_1^{\mathbf{t}}$ with $\mathbf{t} = (1,2)$. This system has the invariant curves $y - x^2 = 0$ and $y + x^2 = 0$ and we assume that the origin is an isolated singular point (that is, $d \neq \pm 2$). Note that $(y+x^2)^{2+d}(y-x^2)^{2-d}$ is a first integral of the system.

The analytic and algebraic integrability over $\mathbb{C}[[x,y]]$ of the analytic perturbations of this system have been studied in [4, 8]. Here, we apply our approaches. First, we give a normal form orbitally equivalent. Notice that in this case the Poincaré–Dulac normal form is not applicable.

Proposition 4.25 *Consider the vector field* $\mathbf{F} = (-2y + dx^2 + \cdots, -4x^3 + 2dxy + \cdots)^T$ *with* $d \neq \pm 2$ *such that* $-\frac{8}{d+2}$ *and* $\frac{8}{d-2}$ *are not natural numbers. The vector field* $\mathbf{F}$ *is orbitally equivalent to* $(-2y + dx^2, -4x^3 + 2dxy)^T + \sum_{j>1} \mu_j (x, 2y)^T$ *with* $\mu_j \in \operatorname{Cor}(\ell_j)$, *a complementary subspace to Range of the linear operator given by* (2.3).

Proof. If we choose the type $\mathbf{t} = (1,2)$, the vector field is $\mathbf{F} = \mathbf{F}_1 + \sum_{j>1} \mathbf{F}_j$ with $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_1 = \mathbf{X}_{(y-x^2)(y+x^2)} + dx\mathbf{D}_0^{\mathbf{t}}$.

The rational function associated to $\mathbf{F}_1$ is $\eta(x,y) = \frac{dx}{(y-x)(y+x)}$. Thus,

$\operatorname{Res}[\eta(1,y), -1] = -\frac{d}{2}$ and $r(y+x^2) = \frac{1}{4}\left(1 + \frac{2}{2}\frac{d}{2}\right) = \frac{2+d}{8}$,

$\operatorname{Res}[\eta(1,y), 1] = \frac{d}{2}$ and then $r(y-x^2) = \frac{1}{4}\left(1 - \frac{2}{2}\frac{d}{2}\right) = \frac{2-d}{8}$.

Taking into account that $\mathcal{P}_{j-r}^{\mathbf{t}} \neq \{0\}$ for all $j \in \mathbb{N}$ and also $r(y-x^2)$ and $r(y+x^2)$ are different from $-\frac{1}{k}$, applying Theorem 2.3, we have the result. ■

This result allows us to characterize the analytic and algebraic integrability over $\mathbb{C}[[x,y]]$ of the above vector field. As we have mentioned before, the polynomial (rational) integrability of the leading term of an analytic vector field is a necessary condition of analytic (algebraic) integrability.

Theorem 4.26 *Consider the vector field* $\mathbf{F} = (-2y + dx^2 + \cdots, -4x^3 + 2dxy + \cdots)^T$ *with* d *a rational number such that* $d \neq \pm 2$ *and* $-\frac{8}{d+2}$ *and* $\frac{8}{d-2}$ *are not natural numbers. We have that:*
(i) $\mathbf{F}$ *is analytically integrable if, and only if,* $d^2 - 4 > 0$ *and* $\mathbf{F}$ *is orbitally equivalent to* $(-2y + dx^2, -4x^3 + 2dxy)^T$.
(ii) $\mathbf{F}$ *is algebraically integrable over* $\mathbb{C}[[x,y]]$ *if, and only if,* $d^2 - 4 < 0$ *and* $\mathbf{F}$ *is orbitally equivalent to* $(-2y + dx^2, -4x^3 + 2dxy)^T$.

Proof. It is enough to check that if $d^2 - 4 > 0$ the vector field $\mathbf{F}_1$ has a polynomial first integral, and if $d^2 - 4 < 0$ the vector field $\mathbf{F}_1$ has a rational first integral. The result follows from Proposition 4.25 and Theorems 3.19 and 2.11. ■

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