Local bifurcation of limit cycles and integrability of a class of nilpotent systems of differential equations.

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Version: April 21, 2009

We study the analytic system of differential equations in the plane which can be written, in a suitable coordinates system, as

$$(\dot{x},\dot{y})^T = \sum_{i=0}^{\infty} \mathbf{F}_{q-p+2is},$$

where $p, q \in \mathbb{N}, p \leq q,\ \ s = (n+1)p - q > 0,\ n \in \mathbb{N}$ and $\mathbf{F}_i = (P_i, Q_i)^T$ are quasi-homogeneous vector fields of type $\mathbf{t} = (p, q)$ and degree i, with $\mathbf{F}_{q-p} = (y, 0)^T$ and $Q_{q-p+2s}(1, 0) < 0$. The origin of this system is a nilpotent and monodromic isolated singular point. We show the Taylor expansion of the return map near the origin for this system, which allow us to generate small amplitude limit cycles bifurcating from the critical point. Also, as an application of the theoretical procedure, we characterize the centers and we generate limit cycles of small amplitude from the origin of several families. Finally, we give a new family integrable analytically which includes the centers of the systems studied.

1. INTRODUCTION.

Probably the most basic and fundamental tool for studying the stability and bifurcations of periodic orbits is the Poincaré map or first return map, defined by Poincaré in 1881, cf. [15].

There are two classes of differential systems whose orbits near an isolated singular point can turn around it (monodromic point): the non-degenerated systems whose linear part has imaginary eigenvalues and the degenerated systems (systems whose matrix of the linear part at the origin is nilpotent or identically null).

When the system has an isolated singular point with imaginary eigenvalues, rather than work directly with the return map, it follows the classical method of searching a Liapunov function V of the form $V(x, y) =$

$x^2 + y^2 + \mathcal{O}(|x,y|^2)$ defined in a neighborhood of the origin. It is known, see [14], that the function V can be constructed such that its rate of change along trajectories be of the form $\dot{V}(x,y) = \eta_2(x^2+y^2) + \eta_4(x^2+y^2)^2 + \ldots$ where η_j are polynomials in the coefficients of the system. We call η_{2k} the k-th focal value and $v_k = \eta_{2k}$ assuming $\eta_{2j} = 0,\ j \leq k-1$, the k-th Liapunov constant. It has that a focus is stable or unstable according to whether the first non-zero focal value is negative or positive. Furthermore, if $\eta_2 = \eta_4 = \ldots = \eta_{2k} = 0$ but $\eta_{2k+2} \neq 0$ (focus of order k), no more than k limit cycles can bifurcate from the origin under perturbations of the system, see [14]. And the critical point is a center (a neighborhood of the origin belongs to a continuous of periodic solutions) if all the focal values are zero; in such case, there exists a local analytic first integral defined at the origin. This technique can not be used for degenerated systems, since, in general, its centers do not have analytic first integral at the origin.

In this work, we are interested in calculating the multiplicity of a focus, to generate small amplitude limit cycles bifurcating from the origin, and to obtain the centers of some families of nilpotent systems. An analytic system of differential equations in the plane having a nilpotent singular point, in some suitable coordinates, can be written as

$$\dot{x} = y + P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are analytic functions without constant nor linear terms defined in a certain neighborhood of the origin.

The center problem for the system (1) was solved theoretically by Moussu [13] and by Sadovskii [16]. Strozyna and Zoladek [17] characterize the centers which have a local analytic first integral at the origin. Giacomini et al. [10, 11] prove that the analytic nilpotent systems with a center can be expressed as limit of systems non-degenerated with a center, and consequently the Poincaré-Liapunov method can be used to find the nilpotent centers.

There are only a few families of polynomial differential systems (1) whose centers are known. The centers of the system (1), where $P(x,y) = P_{2n+1}(x,y)$ and $Q(x,y) = Q_{2n+1}(x,y)$ are homogeneous polynomials of degree $2n+1$, have been characterized for $n = 1, 2$ and 3, see [5] and reference therein. Chavarriga et al. [7] study the centers of (1), with $n = 1$, having an analytic first integral. They also prove that the nilpotent systems time-reversible under the change of variables $(x,y,t) \to (x,-y,-t)$ have an analytic first integral. Sadovskii [16] finds the centers of the family (1) for $P(x,y) = P_2(x,y)$ and $Q(x,y) = Q_3(x,y)$.

Gasull and Torregrosa [8], using the so-called Cherka's method, which consists in doing a change of variables that transforms (1) into a Liénard differential equation, calculate the centers of several families of nilpotent systems. In [9], the authors deal with systems of the form (1) with $P(x,y) = P_{n+1}(x,y) + \ldots$ and $Q(x,y) = -x^{2n-1} + Q_{n+1}(x,y) + Q_{n+2}(x,y) + \ldots$ where, in this case, the vector fields (P_k, Q_k) are quasi-homogeneous vector fields of type $(1,n)$ and degree k. In [2, 3], Álvarez & Gasull apply the normal form theory to study the center problem for monodromic planar nilpotent singularities and calculate the first two generalized Liapunov constants of (1) and they solve the stability problem of several polynomial families.

In this paper, we consider the analytic system of differential equations in the plane whose origin is a nilpotent singular point

$$\dot{x} = y + \sum_{i=1}^{\infty} P_{q-p+2is}(x,y), \qquad \dot{y} = \sum_{i=1}^{\infty} Q_{q-p+2is}(x,y), \tag{2}$$

where $p, q, n \in \mathbb{N}$, $p \leq q$, $s = (n+1)p - q > 0$, and $\mathbf{F}_i = (P_i, Q_i)^T$ is a quasi-homogeneous vector field of type (p,q) and degree i with $Q_{(2n+1)p-q}(1,0) < 0$ (necessary condition of monodromy). That is, according to the degree, $\mathbf{F}_{q-p} = (y,0)^T$ is the quasi-homogeneous component of minor degree, the second one is $\mathbf{F}_{(2n+1)p-q}$ which, among others, has the term $(0, -x^{2n+1})^T$, and the edges of its Newton's polygon of the remaining components are parallel to the edge associated to $\mathbf{F}_{2(n+1)p-q}$. This class includes, among others, the nilpotent systems which are invariant to the change of variables $(x,y) \to (-x,-y)$. In particular, it includes the family

$$\dot{x} = y + X_{2n+1}(x,y), \qquad \dot{y} = Y_{2n+1}(x,y)$$

where X_{2n+1} and Y_{2n+1} are homogeneous polynomials of degree $2n+1$ with $Y_{2n+1}(1,0) < 0$ (case $p = q = 1, P_{2n}(x,y) = X_{2n+1}(x,y),\ Q_{2n}(x,y) = Y_{2n+1}(x,y),\ P_{2i} = Q_{2i} = 0,\ i > n$ in (2)).

The main result of the paper is Theorem 2.1 which gives the Taylor expansion of the return map of the system (2). This result allows us to solve theoretically the center problem for the systems (2) (Corollary 2.1) and to generate limit cycles bifurcating from the origin of the system (Corollary 2.2). Finally, as an application, we characterize the centers of several families of (2) and we give the number of small amplitude limit cycles bifurcating from the origin. Applying Proposition 3.1, we prove that the systems obtained have a center at the origin. In particular, we conclude that all have a local analytic first integral at the origin and they can be written in the form

$$\dot{x} = y + v_y K(v, y^2) + y\Psi(v, y^2), \qquad \dot{y} = -v_x K(v, y^2)$$

where v, K, Ψ are analytic functions defined in a neighborhood of O with $\Psi(O) = 0$. Also, we give the cyclicity of a weak focus of these families and we find a system with eight limit cycles.

2. POINCARÉ MAP NEAR THE ORIGIN.

Recall that a function f of two variables is quasi-homogeneous of type $\mathbf{t} = (p,q)$ and degree k if $f(\varepsilon^p x, \varepsilon^q y) = \varepsilon^k f(x,y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (F_1, F_2)$ is said quasi-homogeneous of type $\mathbf{t}$ and degree k if $F_1 \in \mathcal{P}_{k+p}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+q}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k.

We consider the analytic system of differential equations

$$(\dot{x}, \dot{y})^T = \sum_{i=0}^{\infty} \mathbf{F}_{q-p+2is}, \tag{3}$$

where $p, q \in \mathbb{N}, p \leq q$ and without common factors, $s = (n+1)p - q > 0$, $n \in \mathbb{N}$ and $\mathbf{F}_i = (P_i, Q_i)^T$ are quasi-homogeneous vector fields of type $\mathbf{t} = (p, q)$ and degree i, with $\mathbf{F}_{q-p} = (y, 0)^T$ and $Q_{q-p+2s}(1,0) < 0$ (without loss of generality, we can assume $Q_{q-p+2s}(1,0) = -1$). In this system, this last condition implies that the origin is a monodromic point, see Andreev [4].

Note that if p or q is even, then the origin is a center of (3). Indeed, we assume, for instance, p is even then q will be odd (since p and q have no common factors), in that case $P_{q-p+2is}(x, -y) = -P_{q-p+2is}(x, y)$ and $Q_{q-p+2is}(x, -y) = Q_{q-p+2is}(x, y)$ since $q+2is$ is odd and $2q-p+2is$ is even. The system (3) is time reversible, i.e. has symmetrical phase portrait with regard to a straight line passing through the origin ($y = 0$, in this case), changing time direction. So, O is a center, since it is monodromic.
In what follows, we assume that p and q are odd.
We already introduce the generalized polar coordinates. Given any natural number $n \in \mathbb{N}$, it defines the *generalized trigonometric functions,* $x(\theta) = Cs(\theta),\ y(\theta) = Sn(\theta)$, as the unique solution of the Cauchy problem

$$\frac{dx}{d\theta} = -y, \qquad \frac{dy}{d\theta} = x^{2n+1},$$

with $x(0) = 1,\ y(0) = 0$.
These functions are T-periodic with

$$T := 2\sqrt{\tfrac{\pi}{n+1}}\frac{\Gamma(\frac{1}{2n+2})}{\Gamma(\frac{n+2}{2n+2})},$$

and they verify the equality $Cs^{2n+2}(\theta) + (n+1)Sn^2(\theta) = 1$, see [12] for a proof.
We can introduce the *generalized polar coordinates*, r and θ of the real plane $(x, y) \in R^2$, as

$$x = rCs(\theta), \qquad y = r^{n+1}Sn(\theta). \tag{4}$$

Furthermore, the following equalities hold

$$\dot{r} = \frac{x^{2n+1}\dot{x} + y\dot{y}}{r^{2n+1}}, \qquad \dot{\theta} = \frac{x\dot{y} - (n+1)y\dot{x}}{r^{n+2}}.$$

The return map of (3) is analytic, see [12]. Now we provide a expression of the Taylor expansion of this return map.

THEOREM 2.1. *Let system (3) with p and q odd. The return map of system (3) has the form*

$$P(u) = u - \frac{1}{2(n+1)} \sum_{l=1}^{\infty} (u^{n+(2l-1)s+1} f_l \int_0^T Cs^{3n-s+2ls+2}(\theta) d\theta)(1 + \mathcal{O}(u)) \ .$$

where $f_l \in \mathbf{R}$, $l \geq 1$, are polynomials in the coefficients of the right-hand sides of (3) (we will call f_k the focus quantities *of the singular point O of the system (3)).*

Proof. By making the change (4), after omitting a common factor r^n, the system (3) takes the form

$$\dot{r} = rf(r,\theta), \quad \dot{\theta} = -1 + rg(r,\theta) \tag{5}$$

with f and g analytic functions and $f(0,\theta) = g(0,\theta) = 0$, for all θ.
We now define the variable u verifying

$$u^{2(n+1)} \quad = \quad W\left(rCs(\theta), r^{n+1}Sn(\theta)\right) \tag{6}$$

being W a $\mathcal{C}^{\infty}$-function such that its derivative along the trajectories of the system (3) has the form

$$\dot{W} = x^{3n-s+2} \sum_{l=1}^{\infty} f_l x^{2ls} + \tau(x,y)$$

where f_l, $l \geq 1$, are polynomials in the coefficients of the right-hand sides of (3) and τ is a flat function at the origin, see [1].
Let suppose $f_l = 0$ for $l = 1, \cdots, m-1$ and $f_m \neq 0$. The expression (6) is valid for $r > 0$ and for all θ. Furthermore, as

$$u^{2(n+1)} \quad = \quad W\left(rCs(\theta), r^{n+1}Sn(\theta)\right) = r^{2(n+1)} + \mathcal{O}(r^{2n+3}, \theta) \tag{7}$$

from inverse function theorem, it has that $r = u + \mathcal{O}(u^2, \theta)$.
Next, we express (3) in the new coordinates system (u, θ)

$$\begin{aligned} \dot{u} &= \frac{1}{2(n+1)u^{2n+1}} \dot{W} && (8) \\ &= \frac{f_m}{2(n+1)} Cs^{3n-s+2+2ms}(\theta) u^{2ms+n-s+1}(1 + \mathcal{O}(u,\theta)), && (9) \\ \dot{\theta} &= -1 + \mathcal{O}(u,\theta) && (10) \end{aligned}$$

whose differential equation associated can be written as

$$\frac{du}{d\theta} \quad = \quad -\frac{f_m}{2(n+1)} Cs^{3n-s+2+2ms}(\theta) u^{2ms+n-s+1}(1 + \mathcal{O}(u,\theta)). \tag{11}$$

We write the solution of (11) starting at $u = u_0$ when $\theta = 0$ as

$$u(\theta, u_0) = \sum_{i=1}^{\infty} a_i(\theta) u_0^i + \tau(\theta, u_0), \tag{12}$$

with $a_1(0) = 1,\ a_i(0) = 0$ for $i \geq 2,\ \tau(0, u_0) = 0$ and τ flat at $u_0 = 0$. Hence the Poincaré return map from the section $\{(u, \theta) = (u_0, 0),\ u_0 > 0\}$ to itself is given by the series

$$P(u_0) \quad = \quad a_1(T)u_0 + a_2(T)u_0^2 + ... \tag{13}$$

By replacing (12) in the equation differential (11) we obtain

$$a_1(\theta) \equiv 1,\ a_i(\theta) \equiv 0,\ \text{for } i = 2, \cdots, n + (2m-1)s$$

and

$$a_{n+(2m-1)s+1}(T) = -\frac{f_m}{2(n+1)} \int_0^T Cs^{3n-s+2+2ms}(\theta) d\theta. \qquad \blacksquare$$

Remark. The $\mathcal{C}^\infty$-function W above mentioned in the proof of Theorem 2.1 is not a Liapunov function, since it is not a defined positive function in a neighborhood of the origin. Therefore, it cannot be used for finding limit cycles which bifurcate from the origin.

As a consequence, the only significative constant f_l is the first one different from zero. It does that the return map differ from the identity map, and it determines the stability of the origin. Also, let notice that the origin is a center if and only if $P(u) \equiv u$. So, we have the following results: the first one characterizes the centers of the system (3) and the second result is related to the number of small amplitude limit cycles which can bifurcate from the origin.

COROLLARY 2.1. *The origin of (3) with p and q odd is a center if and only if $f_l = 0$, for all $l \geq 1$.*

Proof. As p and q are odd and $s = (n+1)p - q$, it implies that s is even (odd) if and only if n is even (odd). Thereby, $3n - s + 2 + 2ms$ is even. So, $\int_0^T Cs^{3n-s+2+2ms}(\theta)d\theta$ is a positive value. The result is followed as a consequence. $\blacksquare$

If we want to find the systems with a center of a polynomial family $\mathbf{X}(\lambda),\ \lambda \in \mathbb{R}^m$, of systems (3) with p and q odd, we calculate recursively the sets on $\mathbb{R}^m$:

$$\Omega_1 = \{\lambda \in \mathbb{R}^m,\ f_1(\lambda) = 0\},\ \Omega_k = \{\lambda \in \Omega_{k-1},\ f_k(\lambda) = 0\},\ \text{for } k \geq 2.$$

By Hilbert Basis Theorem, we know that there is a M such that

$$\Omega_1 \supset \Omega_2 \supset \cdots \Omega_M \supset \Omega_{M+1} = \Omega_{M+2} = \cdots .$$

So, the systems $\mathbf{X}(\lambda^*)$ with $\lambda^* \in \Omega_{M+1}$ have a center at the origin. Also, in such a case, it is said that M is the order of the family $\mathbf{X}(\lambda)$.

The focus quantities of system (3) can also be used to prove the existence of a certain number of small amplitude limit cycles bifurcating from the nilpotent critical point of a family of systems (3). Next result is used in order to study the degenerate Andronov-Hopf bifurcation, i.e. we analyze the existence of limit cycles which can bifurcate from the origin of $\mathbf{X}(\lambda)$ under variations of the parameters λ.

COROLLARY 2.2. *Let* $\mathbf{X}(\lambda)$ *be a family of systems (3) with* p *and* q *odd, depending on some parameters* $\lambda \in \mathbb{R}^m$. *We assume that* $\lambda^* \in \Omega_r \setminus \Omega_{r+1}$, *(i.e.* O *is a weak focus of order* r *of* $\mathbf{X}(\lambda^*)$*). If there exists* $\bar{\lambda}$ *enough close to* λ^* *such that* $f_1(\bar{\lambda})$, $f_2(\bar{\lambda})$, ..., $f_{r-1}(\bar{\lambda})$, $f_r(\bar{\lambda})$ *alternate sign and*

$$0 < |f_1(\bar{\lambda})| << |f_2(\bar{\lambda})| << ... << |f_{r-1}(\bar{\lambda})| << |f_r(\bar{\lambda})| << 1$$

then system $\mathbf{X}(\bar{\lambda})$ *has exactly* r *limit cycles in a neighborhood of the origin.*

Proof. By Theorem 2.1, the Taylor expansion of the Poincaré return map of $\mathbf{X}(\bar{\lambda})$ has the form

$$P(u) = u - w_1 f_1(\bar{\lambda})(1 + uh_1(u))u^{j_1} - w_2 f_2(\bar{\lambda})(1 + uh_2(u))u^{j_2} - ...$$

where $w_m = \frac{1}{2(n+1)} \int_0^T Cs^{3n-s+2+2ms}(\theta)d\theta > 0$, $j_m = n + (2m-1)s + 1$ and h_m are analytic functions at the origin.
Each small limit cycle around the origin corresponds to each positive fixed point of the Poincaré return map of $\mathbf{X}(\bar{\lambda})$, i.e. positive zeros of the function

$$F(u) = u - P(u) = \sum_{m=1}^{r+1} (1 + uh_m(u)) w_m f_m(\bar{\lambda}) u^{j_m} + \mathcal{O}(j_{r+1} + 1).$$

By writing $1 + uh_m(u) = (1 + uh_1(u))(1 + \bar{h}_m(u))$, where $\bar{h}_m$ are analytic functions at the origin, it has $F(u) = (1 + uh_1(u))u^{j_1} F_0(u)$ where

$$F_0(u) = w_1 f_1(\bar{\lambda}) + \sum_{m=2}^{r+1} w_m f_m(\bar{\lambda})(1 + u\bar{h}_m(u))u^{j_m - j_1} + \mathcal{O}(j_{r+1} - j_1 + 1).$$

We must look for positive zeros of F_0.
By differentiating, we have

$$F_0'(u) = \sum_{m=2}^{r+1} (1 + u\hat{g}_m(u))(j_m - j_1) w_m f_m(\bar{\lambda}) u^{j_m - j_1 - 1} + \mathcal{O}(j_{r+1} - j_1)$$

where

$$(j_m - j_1)(1 + u\hat{g}_m(u)) = (j_m - j_1)(1 + u\bar{h}_m(u)) + u(\bar{h}_m(u) + u\bar{g}'_m(u)).$$

By writing $1 + u\hat{g}_m(u) = (1 + u\hat{g}_1(u))(1 + \tilde{h}_m(u))$, where $\tilde{h}_m$ are analytic functions at the origin, F'_0 has the form $F'_0(u) = (1 + u\tilde{h}_m(u))u^{j_2 - j_1 - 1}F_1$ where

$$\begin{aligned} F_1(u) = \;\; & (j_2 - j_1)w_2 f_2(\bar{\lambda}) + \textstyle\sum_{m=3}^{r+1}(j_m - j_1)w_m f_m(\bar{\lambda})(1 + u\tilde{h}_m(u))u^{j_m - j_2} \\ & + \mathcal{O}(j_{r+1} - j_2 + 1). \end{aligned}$$

Now, the number of positive zeros of F_0 cannot exceed the number of positive zeros of F_1 by more than unity. By continuing this process a further step we obtain a function F_2 such that the number of positive zeros of F_1 cannot exceed the number of positive zeros of F_1 by more than unity. So, the number of positive zeros of F_0 cannot exceed the number of positive zeros of F_2 by more than two.
This process stops at the r^{th} step when we obtain a function F_r of the form

$$F_r(u) = (j_{r+1} - j_1)w_{r+1}f_{r+1}(\bar{\lambda}) + \mathcal{O}(1),$$

which does not have zeros in a neighborhood of origin, since $f_{r+1}(\bar{\lambda})$ is close to $f_{r+1}(\lambda^*) \neq 0$, by continuity. Therefore, F cannot have more than r positive zeros. Moreover, as $f_1(\bar{\lambda})$, $f_2(\bar{\lambda})$, ..., $f_{r-1}(\bar{\lambda})$ and $f_r(\bar{\lambda})$ alternate sign and satisfy $0 < |f_1(\bar{\lambda})| << |f_2(\bar{\lambda})| << ... << |f_{r-1}(\bar{\lambda})| << |f_r(\bar{\lambda})|$, we can assure the existence of r limit cycles of small amplitude bifurcating of the origin of $\mathbf{X}(\lambda^*)$. ■

3. NILPOTENT CENTERS AND CYCLICITY OF A WEAK FOCUS OF SEVERAL FAMILIES OF POLYNOMIAL SYSTEMS

In this section, by applying Corollary 2.1 and 2.2, we characterize the centers and the order of a weak focus of three families of systems (3).
First, on the one hand, we give the following result which we will use in order to prove the analytic integrability of the centers of several families of (3). We recall that if the system (3) is monodromic, the existence of an analytic first integral is a sufficient condition so that the origin be a center.

Proposition 3.1. *The nilpotent systems*

$$\dot{x} = y + v_y K(v, y^2) + y\Psi(v, y^2), \qquad \dot{y} = -v_x K(v, y^2), \tag{14}$$

where v, K, Ψ are analytic functions defined in a neighborhood of the origin with $\Psi(O) = 0$, are integrable analytically in a neighborhood of the origin.

Proof. Doing the change of variables $u = y^2$, $v = v(x, y)$, by redefining the variable time by $d\tau = yv_x(1 + \Psi(x, y))dt$ and by denoting $\frac{d}{d\tau} ='$, the system (14) becomes

$$u' = -\frac{2K(u, v)}{1 + \psi(u, v)}, \qquad v' = 1. \tag{15}$$

From Cauchy-Arnold's Theorem (see Bruno [6], page 98), the system (15) has got an analytic first integral $H(u, v) = cte$ defined in a neighborhood of O. Undoing the change of variable, (14) has a first integral of (14), $\tilde{H}(x, y) = H(y^2, v(x, y)) = cte$, which is analytic in a neighborhood N, since it is a composition of analytic functions. Also, if we denote $\mathbf{X}$ the vector field associated to (14), it has $\nabla\tilde{H} \cdot \mathbf{X} = 0$, for all $(x, y) \in N \setminus \{\nabla v \cdot \mathbf{X} \neq 0\}$. So, by continuity, $\nabla\tilde{H} \cdot \mathbf{X} = 0$, for all $(x, y) \in N$, that is $\tilde{H}$ is a local analytic first integral of (14). ■

Remarks.

- For $v(x, y) = x$ the systems (14) turn out

$$\dot{x} = y + y\bar{\Psi}(x, y^2), \qquad \dot{y} = \bar{K}(x, y^2), \tag{16}$$

that is, the family of nilpotent systems time reversible under the change of variables $(x, y, t) \to (x, -y, -t)$.
The analytic integrability of the nilpotent systems (16) is one of the main results of Chavarriga et al. [7].

- For

$$v = v_{2m},\; K(y^2, v) = v_{2m}^p,\; \Psi(y^2, v) = \sum_{k=0}^{p} a_k v_{2m}^{p-k} y^{2m(k+1)-2}$$

where $p \geq 0$, v_{2m} is a homogeneous polynomial of degree $2m$ and a_k arbitrary constants, the systems (14) come given by

$$\left\{ \begin{array}{l} \dot{x} = y + \frac{\partial v_{2m}}{\partial y} v_{2m}^p + \sum_{k=0}^{p} a_k v_{2m}^{p-k} y^{2m(k+1)-1}, \\ \dot{y} = -\frac{\partial v_{2m}}{\partial x} v_{2m}^p. \end{array} \right. \tag{17}$$

These systems include the nilpotent family integrable analytically given in [5], Lemma 2.

On the other hand, we prove the following results which will use in order to obtain a simpler expression of the focus quantities of (3).

Proposition 3.2. *For all $\mathcal{C}^1$ vector fields $\mathbf{X}$ and $\mathbf{U}$ such that $[\mathbf{X}, \mathbf{U}] = \nu\mathbf{X}$ with ν a $\mathcal{C}^1$-function in a neighborhood of the origin and $V = \mathbf{X} \wedge \mathbf{U} \neq 0$, it has that*

$$(\nu + div(\mathbf{U}))\mathbf{X} = -J\nabla V + div(\mathbf{X})\mathbf{U},$$

where $[\mathbf{X}, \mathbf{U}]$ denotes Lie's bracket of the fields $\mathbf{X}$ and $\mathbf{U}$, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. It is easy to prove that for all $\mathcal{C}^1$ vector fields $\mathbf{X}$ and $\mathbf{U}$, it holds

$$\nabla V \cdot \mathbf{U} = [\mathbf{X}, \mathbf{U}] \wedge \mathbf{U} + V div(\mathbf{U}), \qquad \nabla V \cdot \mathbf{X} = [\mathbf{X}, \mathbf{U}] \wedge \mathbf{X} + V div(\mathbf{X}). \quad (18)$$

In our case, $\nabla V \cdot \mathbf{U} = (\nu + div(\mathbf{U}))V$ and $\nabla V \cdot \mathbf{X} = div(\mathbf{X})V$.
As $V \neq 0$, it has that $J\mathbf{X}$ and $J\mathbf{U}$ are transversals. Therefore, there exist a, b functions of class one in a neighborhood of the origin such that $\nabla V = aJ\mathbf{X} + bJ\mathbf{U}$. Hence, $\nabla V \cdot \mathbf{X} = -bV$, $\nabla V \cdot \mathbf{U} = aV$. Thus,

$$\begin{aligned} V \cdot \nabla V &= (aV)J\mathbf{X} + (bV)J\mathbf{U} = (\nabla V \cdot \mathbf{U})J\mathbf{X} - (\nabla V \cdot \mathbf{X})J\mathbf{U} \\ &= (\nu + div(\mathbf{U}))VJ\mathbf{X} - div(\mathbf{X})VJ\mathbf{U}, \end{aligned}$$

it follows the result easily. ■

We emphasize the following decomposition which is obtained from Proposition 3.2 with $\mathbf{X} = \mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ and $\mathbf{U} = \mathbf{D}_t = (px, qy)^T$ where $\mathbf{t} = (p, q)$, and by taking into account the quasi homogeneous character of $\mathbf{F}_k$ and $[\mathbf{F}_k, \mathbf{D}_t] = k\mathbf{F}_k$.

Corollary 3.1. *(Conservative-dissipative decomposition). For each* $\mathbf{t} = (p, q)$, *given* $\mathbf{F}_k = (P_k, Q_k)^T \in \mathcal{Q}_k^{\mathbf{t}}$, *there exist* $h_k \in \mathcal{P}_{k+p+q}^{\mathbf{t}}$ *and* $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$ *such that*

$$\mathbf{F}_k = \frac{1}{k+p+q}(\mathbf{X}_{h_k} + \mu_k \mathbf{D}_{\mathbf{t}}), \qquad (19)$$

with $h_k = \mathbf{F}_k \wedge \mathbf{D}_{\mathbf{t}}$ *and* $\mu_k = div(\mathbf{F}_k)$, *where* $\mathbf{D}_{\mathbf{t}} := (px, qy)^T$ *and* $\mathbf{X}_{h_k} := (\frac{\partial h_k}{\partial y}(x, y), -\frac{\partial h_k}{\partial x}(x, y))^T$.

We now show several applications of our research. We first consider the 11-parameter nilpotent system

$$\begin{aligned} \dot{x} &= y + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_6xy^4 + a_7y^5, \\ \dot{y} &= -x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + b_5xy^4 + b_6y^5. \end{aligned} \qquad (20)$$

This is a subfamily of (3) given by $(\dot{x}, \dot{y})^T = \mathbf{F}_0 + \mathbf{F}_2 + \mathbf{F}_4$, with $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}}$, $i = 0, 2, 4$, $\mathbf{t} = (1, 1)$, and

$$\mathbf{F}_0 = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \mathbf{F}_2 = \begin{pmatrix} a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \\ -x^3 + b_1x^2y + b_2xy^2 + b_3y^3 \end{pmatrix},$$

$$\mathbf{F}_4 = \begin{pmatrix} a_6xy^4 + a_7y^5 \\ b_5xy^4 + b_6y^5 \end{pmatrix}.$$

The following result characterizes the centers of the systems (20). We note that the focus quantities of (3) have been computed by means of the recursive procedure developed in [1], and they have the form

$$f_1 = \alpha_1 g_1, \quad f_i = \alpha_i g_i + \sum_{j=1}^{i-1} \beta_{i,j} f_j, \ i \geq 2,$$

with α_i positive constants and $\beta_{i,j}$ polynomials in the coefficients of the right-hand sides of (3).

THEOREM 3.1. *The origin of the system (20) is a center if and only if one of the following two series is satisfied:*
i) $3a_1 + b_1 = 2a_2 + b_2 = a_3 + 3b_3 = b_5 = a_6 + 5b_6 = 0$, *(Hamiltonian system).*
ii) $b_1 + 3a_1 = b_3 - a_1(b_2 + 2a_1^2) = a_3 + a_1(b_2 - 2a_2 + 6a_1^2) = b_6 - a_1 b_5 = a_6 + a_1 b_5 = 0$.
Moreover, each one of them has a local analytic first integral.

Proof. Taking $\mathbf{t} = (1,2)$, (that is, the type of the vector field $(y, -x^3)^T$, i.e. the quasi-homogeneous principal part of the system (20)) and applying Corollary 3.1, degree to degree, the system (20) comes given by $(\dot{x}, \dot{y})^T = \mathbf{X}_h + \mu \mathbf{D_t}$, being h the defined positive function

$$\begin{aligned} h(x,y) = {} & \tfrac{1}{4}(2y^2 + x^4) - \tfrac{1}{5}c_1 x^3 y - \tfrac{1}{6}c_2 x^2 y^2 - \tfrac{1}{7}c_3 x y^3 - \tfrac{1}{8}c_4 y^4 - \tfrac{1}{10}c_5 x^2 y^4 \\ & - \tfrac{1}{11}c_6 x y^5 - \tfrac{1}{12}c_7 y^6, \end{aligned} \tag{21}$$

and $\mu(x,y) = \frac{1}{5}d_1 x^2 + \frac{1}{6}d_2 xy + \frac{1}{7}d_3 y^2 + \frac{1}{10}d_5 x y^3 + \frac{1}{11}d_6 y^4$ where

$$\begin{array}{llll} c_1 = b_1 - 2a_1, & d_1 = 3a_1 + b_1, & c_2 = b_2 - 2a_2, & d_2 = 2a_2 + 2b_2, \\ c_3 = b_3 - 2a_3, & d_3 = a_3 + 3b_3, & c_4 = -2a_4, & d_5 = 4c_5 = 4b_5, \\ c_6 = 2a_6, & d_6 = a_6 + 5b_6, & c_7 = -2a_7. & \end{array}$$

The first four focus quantities are

$$\begin{aligned} & g_1 = d_1, \quad g_2 = d_2 c_1 + 5d_3, \\ & g_3 = d_2(25 d_2 c_1 + 175 c_1 c_2 + 42 c_1^3 + 375 c_3), \\ & g_4 = 4 c_5 c_1 + 5 d_6. \end{aligned}$$

In this point, we distinguish two cases depending on the coefficient d_2. If $d_2 \neq 0$, some necessary conditions for the origin to be a center are $g_1 = g_2 = g_3 = g_4 = 0$, i.e. $d_1 = 0$, $d_3 = -\frac{1}{5}d_2 c_1$, $c_3 = -\frac{1}{375}c_1(25d_2 + 175c_2 + 42c_1^2)$, $d_6 = -\frac{4}{5}c_1 c_5$. In such a case, it has $g_5 = d_2(3c_1 c_5 + 5c_6)$. Hence, $c_6 = -\frac{3}{5}c_1 c_5$. Thus, we arrive to the system given by **ii)**,

$$\begin{aligned} \dot{x} = {} & y + a_1 x^3 + a_2 x^2 y - a_1(b_2 - 2a_2 + 6a_1^2) x y^2 + a_4 y^3 - a_1 b_5 x y^4 + a_7 y^5, \\ \dot{y} = {} & -x^3 - 3a_1 x^2 y + b_2 x y^2 + a_1(b_2 + 2a_1^2) y^3 + b_5 x y^4 + a_1 b_5 y^5. \end{aligned} \tag{22}$$

This system belongs to the family (14) given in Proposition 3.1 where

$$\begin{aligned}
&v = \tfrac{1}{2}x^2 + a_1xy + (\tfrac{1}{2}a_2 - a_1^2)y^2,\\
&K(v, y^2) = 2h - (a_2 + b_2)y^2 - b_5y^4,\\
&\Phi(v, y^2) = [a_4 + (a_2 - 2a_1^2)(2a_1^2 + b_2)]y^2 + [a_7 + b_5(a_2 - 2a_1^2)]y^4
\end{aligned}$$

and, therefore, the origin is a center, since the system has a local analytic first integral and O is a monodromic point.
If $d_2 = 0$, from the vanishing of the first four constants above, we have $d_1 = d_3 = 0$ and $d_6 = -\frac{4}{5}c_1c_5$. In this case, the next focus quantities are $g_5 = c_5(175c_1c_2 + 42c_1^3 + 375c_3)$, $g_6 \equiv 0$ and $g_7 = c_5(3c_1c_5 + 5c_6)$. If $c_5 = 0$, the necessary conditions to have a center leads us to the hamiltonian system **i)** whose hamiltonian function is (21). We note that the curves $h(x, y) = cte$ are closed, therefore, it is a center.
And if $c_5 \neq 0$, g_5 and g_7 must be zero; hence,

$$c_3 = -\frac{7}{375}c_1(25c_2 + 6c_1^2),\ c_6 = -\frac{3}{5}c_1c_5.$$

In such a case, the system is of the form (22) with $d_2 = 0$. So, O is a center.
■

THEOREM 3.2. *Under perturbations of the parameters of the system (20), it has:*
a) *if $a_2 + b_2 \neq 0$ or $b_5 \neq 0$, it can bifurcate 0,1,2,3 or 4 limit cycles around the origin.*
b) *if $a_2 + b_2 = b_5 = 0$, it can bifurcate 0,1 or 2 limit cycles around the origin.*

Proof. We fix the constants c_1, d_2, c_2, c_5 above defined and consider the critical values

$$\begin{aligned}
&d_1^* = 0,\ d_3^* = -\tfrac{1}{5}d_2c_1,\ c_3^* = -\tfrac{1}{375}c_1(25d_2 + 175c_2 + 42c_1^2),\\
&d_6^* = -\tfrac{4}{5}c_1c_5,\ c_6^* = -\tfrac{3}{5}c_1c_5.
\end{aligned}$$

Firstly, we assume that $a_2 + b_2 \neq 0$. From the expression of g_1, g_2, g_3, g_4 and g_5, applying Corollary 2.2, it deduces the following one: if $d_1 \neq d_1^*$ there isn't limit cycles around the origin. If d_1 is near zero and $d_3 \neq d_3^*$, then it can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0 < |d_1| << |d_3|$ and also c_3 is different from c_3^*, then there are 2 small amplitude limit cycles. If also $d_3c_3 < 0$, $|d_1| << |d_3|$ and $d_6 \neq d_6^*$, then 3 limit cycles bifurcate of the origin. If we take d_1, d_3, c_3 and d_6 different from d_1^*, d_3^*, c_3^* and d_6^* but near each of them, respectively, and $c_6 \neq c_6^*$ such that it satisfy the hypothesis of Corollary 2.2 with $r = 4$, it has that we can bifurcate 4 limit cycles, at least.

Now, we assume that $a_2+b_2=0$ and $b_5 \neq 0$. The first constants g_i different from zero are

$$g_1=d_1,\ g_2=d_3,\ g_4=4c_1c_5+5d_6,\ g_5=c_5(175c_1c_2+42c_1^3+375c_3),$$
$$g_7=c_5(3c_1c_5+5c_6).$$

Thus, if $d_1 \neq 0$ there isn't limit cycles around the origin. If d_1 is close to zero and $d_3 \neq 0$, can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0<|d_1|<<|d_3|$ and also d_6 is different from d_6^*, then there are 2 limit cycles of small amplitude. If we now take d_6 such that d_3 and d_6 alternate sign and $|d_3|<<|d_6|$ and also c_3 is different from $c_3^*=-\frac{7}{375}c_1(25c_2+6c_1^2)$ then there exist 3 limit cycles around the origin. If we also choose c_3 near c_3^* such that $d_6c_3<0$ and $|d_6|<<|c_3|$ and take $c_6 \neq c_6^*$, then there exist, at least, 4 limit cycles around the origin.
Last on, if $a_2+b_2=b_5=0$, that is $d_2=c_5=0$, the g_i different from zero are

$$g_1=d_1,\ g_2=d_3,\ g_4=d_6.$$

Therefore, if $d_1 \neq 0$ there isn't limit cycles near the origin. If d_1 is close zero and $d_3 \neq 0$, it can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0<|d_1|<<|d_3|$ and also d_6 is different from zero, then 2 small amplitude limit cycles bifurcate from the origin. ■

We now consider the 9-parameter subfamily of (3) given by

$$(\dot{x},\dot{y})^T=\mathbf{F}_2+\mathbf{F}_4+\mathbf{F}_6, \tag{23}$$

with $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}},\ i=2,4,6,\ \mathbf{t}=(1,3)$, and

$$\mathbf{F}_2=\begin{pmatrix} y \\ 0 \end{pmatrix},\quad \mathbf{F}_4=\begin{pmatrix} a_1x^5+a_2x^2y \\ -x^7+b_1x^4y+b_2xy^2 \end{pmatrix},$$
$$\mathbf{F}_6=\begin{pmatrix} a_3x^7+a_4x^4y+a_5xy^2 \\ b_6x^9+b_3x^6y+b_4x^3y^2+b_5y^3 \end{pmatrix}$$

with $b_6=0$ and $b_2=-a_2$.
We already get a lower bound for its cyclicity.

THEOREM 3.3. *Under perturbations of the parameters of the system (23), it has:*
a) *if* $2a_4+b_4=0$, *it can bifurcate 0,1 or 2 limit cycles from the origin.*
b) *if* $2a_4+b_4 \neq 0$ *and* $(-5a_2+2(b_1-4a_1)^2)(b_1-4a_1)=0$, *it can bifurcate 0,1,2,3 or 4 limit cycles around the origin.*
c) *if* $2a_4+b_4 \neq 0$ *and* $(-5a_2+2(b_1-4a_1)^2)(b_1-4a_1) \neq 0$, *it can bifurcate 0,1,2,3,4,5,6,7 or 8 limit cycles around the origin.*

Proof. In this case, we apply Corollary 3.1, degree to degree, with $\mathbf{t}=(1,4)$, (that is, the type of the vector field $(y,-x^7)^T$, i.e. the quasi-homogeneous principal part of the system (23)). The system (23) comes given by $(\dot{x}, \dot{y})^T = \mathbf{X}_h + \mu \mathbf{D_t}$, being h the defined positive function

$$h(x,y) = \frac{1}{8}(x^8+4y^2) - \frac{1}{9}c_1x^5y - \frac{1}{10}c_2x^2y^2 - \frac{1}{11}c_3x^7y - \frac{1}{12}c_4x^4y^2 - \frac{1}{13}c_5xy^3,$$

and $\mu(x,y) = \frac{1}{9}d_1x^4 + \frac{1}{11}d_3x^6 + \frac{1}{12}d_4x^3y + \frac{1}{13}d_5y^2$ where

$$\begin{array}{llll} c_1 = b_1 - 4a_1, & d_1 = 5a_1 + b_1, & c_2 = -5a_2, & \\ c_3 = b_3 - 4a_3, & d_3 = 7a_3 + b_3, & c_4 = b_4 - 4a_4, & d_4 = 4a_4 + 2b_4, \\ c_5 = b_5 - 4a_5 & d_5 = a_5 + 3b_5. & & \end{array}$$

First, we assume that $2a_4 + b_4 = 0$, that is $d_4 = 0$. The only g_i different from zero are $g_1 = d_1,\ g_2 = d_3,\ g_3 = d_5$. Therefore, if $d_1 \neq 0$ there is a neighborhood of the origin where the system (23) does not have any limit cycle around the origin. If d_1 is close to zero and $d_3 \neq 0$, it can exist, at least, 1 limit cycle. If $d_1d_3 < 0$ with $0 < |d_1| << |d_3|$ and also $d_5 \neq 0$, then there are 2 limit cycles of small amplitude.
We now assume that $2a_4 + b_4 \neq 0$ and $(-5a_2 + 2(b_1 - 4a_1)^2)(b_1 - 4a_1) = 0$, i.e. $d_4 \neq 0$ and $(c_2 + 2c_1^2)c_1 = 0$. In this case,

$$\begin{array}{l} g_1 = d_1, \quad g_2 = d_3, \quad g_3 = d_5 + \frac{12}{13}c_1d_4, \quad g_4 = d_4c_3, \\ g_5 = d_4(c_5 + 4c_1(c_4 + \frac{1}{13}d_4)). \end{array}$$

and the remain are zero.
So, if $d_1 \neq 0$ there isn't limit cycles around the origin. If d_1 is close to zero and $d_3 \neq 0$, can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0 < |d_1| << |d_3|$ and also d_5 different from $d_5^* = -\frac{12}{13}c_1d_4$, then there are 2 small amplitude limit cycles. If we now take d_5 such that d_3 and d_5 alternate sign and $|d_3| << |d_5|$ and also $c_3 \neq 0$, then there exist 3 limit cycles around the origin. If we also choose c_3 close to 0 such that $d_5c_3 < 0$ and $|d_5| << |c_3|$ and take $c_5 \neq -4c_1(c_4 + \frac{1}{13}d_4)$ then there exist, at least, 4 limit cycles bifurcating from the origin.
Lastly, if $2a_4 + b_4 \neq 0$ and $(-5a_2 + 2(b_1 - 4a_1)^2)(b_1 - 4a_1) \neq 0$, by denoting $q = d_4c_1(c_2 + 2c_1^2) \neq 0$, the first nine constants $g_i,\ i = 1, ..., 9$ of (23) are

$$\begin{array}{l} g_1 = d_1, \quad g_2 = d_3, \quad g_3 = d_5 + \frac{12}{13}c_1d_4, \\ g_4 = d_4\left[c_3 + 2c_1(c_2 + 2c_1^2)\right], \\ g_5 = d_4\left[c_5 + 4c_1(c_4 + \frac{1}{13}d_4) - \frac{100}{3}c_1^3(c_2 + 2c_1^2)\right], \\ g_6 = -q\left[c_4 + \frac{1}{2}d_4 - \frac{62}{3}c_1^2(c_2 + 2c_1^2)\right], \\ g_7 = q\left[d_4 - \frac{24}{5}c_2^2 - \frac{748}{15}c_2c_1^2 - \frac{1408}{15}c_1^4\right], \\ g_8 = -q\left[1548c_2 - (4681 + 5\sqrt{54049})c_1^2\right]\left[1548c_2 - (4681 - 5\sqrt{54049})c_1^2\right], \\ g_9 = -q(381374c_2 + 859813c_1^2). \end{array}$$

We can choose $d_1, d_3, d_5, c_3, c_5, c_4, d_4$ and c_2 adequately such that $g_i g_{i+1}$ is negative, g_9 different from zero and

$$0 < |g_1| << |g_2| << |g_3| << |g_4| << |g_5| << |g_6| << |g_7| << |g_8|.$$

Applying Corollary 2.2, for $r = 1, 2, 3, 4, 5, 6, 7$ and 8, there are regions of the parameters where the system has 0,1,2,3,4,5,6,7 or until 8 limit cycles around the origin. ∎

The centers of (23) have been characterized in [1]. These one are included in the class of systems given by Proposition 3.1.

Last on, we study the systems

$$\dot{x} = y + \sum_{i=1}^{[\frac{2n+2}{5}]} a_{i,n} x^{2n+2-5i} y^{3i-1}, \quad \dot{y} = -x^{2n+1} + \sum_{i=1}^{[\frac{2n+1}{5}]} b_{i,n} x^{2n+1-5i} y^{3i}. \tag{24}$$

with $1 \leq n \leq 9$. ($[x]$ means integer part of x). These systems are of the family (3) given by $(\dot{x}, \dot{y})^T = \mathbf{F}_2 + \mathbf{F}_{6n-2}$, with $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}}$, $i = 2, 6n-2$, $\mathbf{t} = (3, 5)$.
Analogously to the above application, we obtain the ciclicity of the origin of (24).

Theorem 3.4. *The system (24) with n equal to 1,2,3 or 4, does not have limit cycles of small amplitude surrounding the origin. There are systems inside this family for $n = 5$ or $n = 6$ with 0 or 1 limit cycle around the origin. For $n = 7$ or $n = 8$ there exist systems inside this family with 0,1,2 or 3 limit cycles around the origin. There exist systems (24) with $n = 9$ which have 0, 1 or 2 limit cycles around the origin.*

Proof. Taking $\mathbf{t} = (1, n+1)$ and by applying Corollary 3.1, degree to degree, the system (24) takes the form

$$(\dot{x}, \dot{y})^T = -\tfrac{1}{2n+2} \mathbf{X}_{x^{2n+2}+(n+1)y^2} +$$
$$\textstyle\sum_{i=1}^{[\frac{2n+1}{5}]} \frac{1}{2n+2+(3n-2)i} \left(d_{i,n} x^{2n+1-5i} y^{3i-1} \mathbf{D}_{(1,n+1)} + c_{i,n} \mathbf{X}_{x^{2n+2-5i} y^{3i}} \right)$$

where the new coefficients that appear are

$$c_{i,n} = (n+1) a_{i,n} - b_{i,n}, \qquad d_{i,n} = (2n+2-5i) a_{i,n} + 3i b_{i,n}$$

The expressions of the focus quantities different from zero are:

$$\begin{array}{ll}
n=2, & g_1 = d_{1,2}, \\
n=3, & g_1 = d_{1,3}, \\
n=4, & g_1 = d_{1,4}, \\
n=5, & g_1 = d_{1,5}, g_2 = c_{1,5}d_{2,5} \\
n=6, & g_1 = d_{1,6},\ g_2 = c_{1,6}d_{2,6}, \\
n=7, & g_1 = d_{1,7},\ g_2 = c_{1,7}d_{2,7} + 35d_{3,7}, \\
 & g_3 = d_{2,7}(33075c_{3,7} + 1314c_{1,7}^3 + (2660d_{2,7} + 7665c_{2,7})c_{1,7}), \\
 & g_4 = -c_{1,7}^5 d_{2,7}, \\
n=8, & g_1 = d_{1,8},\ g_2 = 3c_{1,8}d_{2,8} + 40d_{3,8}, \\
 & g_3 = d_{2,8}(49600c_{3,8} + 2387c_{1,8}^3 + (3960d_{2,8} + 13440c_{2,8})c_{1,8}), \\
 & g_4 = -c_{1,8}^5 d_{2,8}, \\
n=9, & g_1 = d_{1,9},\ g_2 = c_{1,9}d_{2,9} + 9d_{3,9}, \\
 & g_3 = d_{2,9}(5103c_{3,9} + (266c_{1,9}^2 + 405d_{2,9} + 1539c_{2,9})c_{1,9}).
\end{array}$$

Choosing adequately the constants $c_{i,n}$ and $d_{i,n}$ and by applying Corollary 2.2 it follows the result. ■

ACKNOWLEDGMENTS

This work has been partially supported by *Ministerio de Ciencia y Tecnología, Plan Nacional I+D+I* co-financed with FEDER funds, in the frame of the project MTM2007-64193 and by *Consejería de Educación y Ciencia de la Junta de Andalucía* (FQM-276 and EXC/2008)

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