A class of non-integrable systems admitting an inverse integrating factor

A. Algaba, N. Fuentes, C. García, M. Reyes

Department of Mathematics
Research Center of Theoretical Physics and Mathematics FIMAT.
Huelva University, 21071. Huelva, Spain

Abstract

We study the existence of an inverse integrating factor for a class of systems, in general non-integrable, whose lowest-degree quasi-homogeneous term is a Hamiltonian system and its Hamiltonian function only has simple factors over $\mathbb{C}[x, y]$.

1. Introduction and statement of the main results

We consider an autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T, \tag{1}$$

whose origin is an equilibrium point and $P, Q \in \mathbb{C}[[x, y]]$ (algebra of the power series in x and y with coefficient in $\mathbb{C}$) defined in a neighborhood of the origin $U \subset \mathbb{C}^2$.

A function f (or a curve $f(x, y) = 0$) with $f \in \mathbb{C}[[x, y]]$ non-null, is an invariant function (or an invariant curve) of system (1) on U, if there is $K \in \mathbb{C}[[x, y]]$ such that $L_{\mathbf{F}} f = Kf$, being $L_{\mathbf{F}} f := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q$. A function K is named the cofactor of the invariant curve $f = 0$.

A non-null function $V \in \mathbb{C}[[x, y]]$ is an inverse integrating factor of system (1) on U if $V = 0$ is an invariant curve of system (1) whose cofactor is the divergence of the vector field, i.e. $L_{\mathbf{F}} V = \mathrm{div}(\mathbf{F})V$, being $\mathrm{div}(\mathbf{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

In this work, our aim is to provide conditions on the system in order to study the existence of an inverse integrating factor.

It is known that if system (1) has a formal inverse integrating factor, which is non-zero at origin, then the system (1) is formally integrable at origin. Therefore, if system (1) is not formally integrable at origin and it has an inverse integrating factor V, then $V(\mathbf{0}) = 0$. For more details about the relation between the integrability and the inverse integrating factor see [4, 5].

The presence of an inverse integrating factor is also related to the existence of a limit cycle and its hyperbolicity and cyclicity, see [6, 13].

Preprint submitted to Elsevier *May 19, 2015*

Given $\mathbf{t} = (t_1, t_2)$ non-null with t_1 and t_2 non-negative integer numbers without common factors, we denote by $\mathcal{P}_k^{\mathbf{t}}$ the vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k, i.e.

$$\mathcal{P}_k^{\mathbf{t}} = \{f \in \mathbb{C}[x,y] : f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)\},$$

and by

$$\mathcal{Q}_k^{\mathbf{t}} = \{\mathbf{F} = (P,Q)^T : P \in \mathcal{P}_{k+t_1}^{\mathbf{t}},\ Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}\}$$

the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k. Any vector field is expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F}$ becomes

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots,$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$. Such expansions are expressed by $\mathbf{F} = \mathbf{F}_r +$ q-h.h.o.t.

If we select the type $\mathbf{t} = (1,1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees. We cite some properties, see [2, 4].

- $\mathbf{D}_0 := (t_1x, t_2y)^T \in \mathcal{Q}_0^{\mathbf{t}}$,
- if $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $|\mathbf{t}| = t_1 + t_2$, then $\mathbf{X}_h := (-\partial h/\partial y, \partial h/\partial x)^T \in \mathcal{Q}_r^{\mathbf{t}}$,
- every $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ can split as

$$\mathbf{F}_k = \mathbf{X}_h + \mu\mathbf{D}_0 \tag{2}$$

with $h = (\mathbf{D}_0 \wedge \mathbf{F}_k)/(k + |\mathbf{t}|)$ and $\mu = \mathrm{div}(\mathbf{F}_k)/(k + |\mathbf{t}|)$. This sum is known as *the conservative-dissipative splitting of a quasi-homogeneous vector field.*

Given a type $\mathbf{t}$ and $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, we consider the systems which are formally orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mu = \mu_r +$ q-h.h.o.t. and $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$. The following result characterizes them.

Theorem 1. *Given $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, a system $\dot{\mathbf{x}} = \mathbf{X}_h +$ q-h.h.o.t. is formally orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mu = \sum_{j>r}\mu_j$, $\mu_j \in \mathcal{P}_j^{\mathbf{t}}$, if and only if it has an invariant curve $f = 0$ of the form $f = h +$ q-h.h.o.t. with f a function conjugate to h (i.e. there exists a formal diffeomorphism Φ such that $h = f \circ \Phi$).*

In this paper, we deal with a class of systems of the form

$$\dot{\mathbf{x}} = \mathbf{X}_h + \text{q-h.h.o.t.}, \tag{3}$$

where h is a quasi-homogeneous polynomial, whose factorization on $\mathbb{C}[x,y]$ only has simple factors. We note that this condition on h is generic.

We cite the results obtained in the paper. We provide a formal orbital equivalent normal form of system (3), i.e. an expression of this system after a change of state variables and a re-parameterization of the time, Theorem 10. This normal form is of the form $\dot{\mathbf{x}} = \mathbf{X}_{h+g} + \mu\mathbf{D}_0$, with g a polynomial (not quasi-homogeneous, in general) and $\mu = \sum_{j>r}\mu_j$, $\mu_j \in \mathrm{Cor}(\ell_j)$, being $\mathrm{Cor}(\ell_j)$ a complementary subspace to the range of the linear operator

$$
\begin{aligned}
\ell_j \quad : \quad & \mathcal{P}^{t}_{j-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_{j} \\
& \mu_{j-r} \longrightarrow \ell_j(\mu_{j-r}) := L_{\mathbf{F}_r}\mu_{j-r}.
\end{aligned} \tag{4}
$$

For $\mathbf{F}_r = \mathbf{X}_h$, the operator ℓ_j is defined in terms of the Poisson bracket, namely, $\ell_j(\mu_{j-r}) = \{h, \mu_{j-r}\}$.

We focus in the systems with $g \equiv 0$, i.e. the formally orbital equivalent systems to

$$
\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0, \text{ with } \mu = \sum_{j>r} \mu_j, \ \mu_j \in \mathrm{Cor}(\ell_j) \tag{5}
$$

where $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ only has simple factors in its factorization on $\mathbb{C}[x,y]$. They are a wide class of systems. For instance, the systems with linear part non-null are included in this class, among others.

From Algaba et al. [2], the systems formally orbital equivalent to systems (5) are integrable if and only if $\mu \equiv 0$ and, in such a case, they have a first integral of the form $h +$ q-h.h.o.t. Consequently, for $\mu \not\equiv 0$, the systems orbitally equivalent to systems (5) do not have any first integral.

Here we derive conditions preventing the existence of an inverse integrating factor for system (5).

The following theorem is our main result.

Theorem 2. *System (5) has an inverse integrating factor if, and only if, it is formally orbital equivalent either to* $\dot{\mathbf{x}} = \mathbf{X}_h$ *(integrable system) or to* $\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+s(r+|\mathbf{t}|)}\mathbf{D}_0$ *with* s *a natural number and* $\mu_{r+s(r+|\mathbf{t}|)} \in Cor(\ell_{r+s(r+|\mathbf{t}|)})$ *(non-integrable system).*

Remark 1. *There are two main reasons for imposing that the factorization of* h *only should have simple factors on* $\mathbb{C}[x,y]$*:*
On the one hand, it may disappear the cyclicity of the co-ranges whether the condition breaks, see Remark 3, and consequently it is not possible to provide a normal form up any order of the system (5), which is used to prove the necessity of Theorem 2.
Moreover, such condition is necessary so that the inverse integrating factor exists and it is a function of h*, Proposition 19 of [4], which is used to prove the condition sufficient of our theorem.*

As an application, we find the systems of two families which have an inverse integrating factor.

Family 1. We consider the family of systems

$$
\begin{aligned}
(\dot{x}, \dot{y})^T = (& y^2 + \sigma x^2 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
& -2\sigma xy + b_{30}x^3 + b_{12}xy^2 + b_{03}y^3)^T
\end{aligned} \tag{6}
$$

with $\sigma = \pm 1$ and a_{21}, a_{12}, a_{03}, b_{30}, b_{12}, b_{03} real numbers. These systems consist in a quadratic Hamiltonian system, whose Hamiltonian function is $-\frac{y}{3}(y^2 + 3\sigma x^2) \in \mathcal{P}_3^{(1,1)}$, perturbed by cubic terms.

Theorem 3. *System (6) has an inverse integrating factor if and only if it satisfies one of the following conditions:*

1. $a_{12} + 3b_{03} = b_{12} + a_{21} = 0$, *(Hamiltonian system),*
2. $a_{12} = b_{03} = 0,\ b_{12} + a_{21} \neq 0$, *(Reversible system).*

Moreover, all of them are formally integrable except for case 2 with

$$a_{12} = b_{03} = 0, \quad b_{12} + a_{21} \neq 0,\ b_{30}(b_{30} - \sigma(b_{12} - 2a_{21})) \neq 0.$$

Family 2. We consider the family $\dot{\mathbf{x}} = \mathbf{X}_h + \mathbf{F}_4$ with $h = -\frac{1}{3}y(y^2 + 3ax^2y + 3\sigma x^4) \in \mathcal{P}_6^{(1,2)}$, $\sigma = \pm 1$, and $\mathbf{F}_4$ any quasi-homogeneous vector field of type $\mathbf{t} = (1,2)$ of degree 4:

$$(\dot{x}, \dot{y})^T = \mathbf{X}_h + \mathbf{X}_{c_{70}x^7 + c_{51}x^5y + c_{32}x^3y^2 + c_{13}xy^3} + (d_{40}x^4 + d_{21}x^2y + d_{02}y^2)(x, 2y)^T. \tag{7}$$

We note that the polynomial h has only simple factors whenever $3a^2 \neq 4\sigma$.

Theorem 4. *The system (7) with $3a^2 \neq 4\sigma$, has an inverse integrating factor if and only if it satisfies one of the following conditions:*

1. $d_{40} = d_{21} = d_{02} = 0$,
2. $-2ad_{40} + \sigma d_{21} = \sigma d_{02} - d_{40} = 3\sigma c_{13} - c_{51} = ac_{51} - \sigma c_{32} = 0$ *and* $d_{40} \neq 0$,
3. $-2ad_{40} + \sigma d_{21} = \sigma d_{02} - d_{40} = 3c_{13} - \sigma ac_{32} + (a^2 - \sigma)c_{51} = 0$,
   $(a^2 - 6\sigma)(ac_{51} - \sigma c_{32}) + 14(a^2 - \sigma)c_{70} = 0$,
   $2(a^2 - \sigma)(5c_{51} + 2d_{40}) + 5a(\sigma c_{32} - ac_{51}) = 0$,
   $-4(a^2 - \sigma)^2 d_{40}^2 - 5a\sigma(\sigma c_{32} - ac_{51})(2a^2 - 3\sigma)(a^2 - \sigma)d_{40} + 50(\sigma c_{32} - ac_{51})^2(a^6 - 3a^4\sigma - 18a^2 + 27\sigma) = 0$,
   and $d_{40} \neq 0$.

Moreover, in all cases, system (7) is analytically integrable.

This paper is organized as follows: next section provides a normal form of the systems (3), by using transformations in the state variables and also reparameterizations of the time depending on the state variables (orbital equivalence). We also calculate a reduced normal form orbitally equivalent to the systems (5). Section 3 contains the proofs of Theorems 1-4.

2. Orbitally equivalent normal form.

We recall some notions and results related to the quasi-homogeneous normal forms of systems (1).

We describe the sets $\mathcal{P}_k^{\mathbf{t}}$ of quasi-homogeneous polynomials. For the homogeneous case, any nth degree homogeneous polynomial is written as a linear combination of $n+1$ monomials and therefore the dimension of such spaces is $n+1$. For the quasi-homogeneous case, the situation becomes more involved. Often they are trivial spaces and other times its dimension is not immediately obtained.

The following result provides bases for these spaces.

Lemma 5. *Fixed* $\mathbf{t} = (t_1, t_2)$*, it holds:*

1. $\mathcal{P}_0^{\mathbf{t}} = span\{1\}$.
2. $\mathcal{P}_k^{\mathbf{t}} = \{0\}$, *if* $k \notin \mathcal{I}^{\mathbf{t}}$.
3. $\mathcal{P}_k^{\mathbf{t}} = span\{x^{k_1+t_2(k_3-j)}y^{k_2+t_1 j} : j = 0, \ldots, k_3\}$, *if* $k \in \mathcal{I}^{\mathbf{t}} \setminus \{0\}$,

being $\mathcal{I}^{\mathbf{t}} = \{k = k_1 t_1 + k_2 t_2 + k_3 t_1 t_2 \in \mathbb{N} : k_1, k_2, k_3 \in \mathbb{N}, k_1 < t_2, k_2 < t_1\}$.

Remark 2. *We note that if* $k > t_1 t_2 - |\mathbf{t}|$*, then* $k \in \mathcal{I}^{\mathbf{t}}$*, i.e.* $\mathcal{P}_k^{\mathbf{t}}$ *is a non-trivial space. Table 2 shows the sets* $\mathbb{N} \setminus \mathcal{I}^{\mathbf{t}}$ *for* $t_2 \leq 5$.

$\mathbb{N} \setminus \mathcal{I}^{(1,t_2)} = \emptyset$	
$\mathbb{N} \setminus \mathcal{I}^{(2,3)} = \{1\}$	$\mathbb{N} \setminus \mathcal{I}^{(2,5)} = \{1,3\}$
$\mathbb{N} \setminus \mathcal{I}^{(3,4)} = \{1,2,5\}$	$\mathbb{N} \setminus \mathcal{I}^{(3,5)} = \{1,2,4,7\}$
$\mathbb{N} \setminus \mathcal{I}^{(4,5)} = \{1,2,3,6,7,11\}$	

Table 1: Sets $\mathbb{N} \setminus \mathcal{I}^{\mathbf{t}}$ for $t_2 \leq 5$.

We provide the following properties, which are obtained easily.

Lemma 6. *It holds:*

1. *if* $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}}, \mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ *then* $[\mathbf{F}_i, \mathbf{F}_j] := D\mathbf{F}_i \mathbf{F}_j - D\mathbf{F}_j \mathbf{F}_i \in \mathcal{Q}_{i+j}^{\mathbf{t}}$,
2. *if* $p_i \in \mathcal{P}_i^{\mathbf{t}}$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$*, then* $L_{\mathbf{F}_j} p_i \in \mathcal{P}_{i+j}^{\mathbf{t}}$,
3. $L_{\mathbf{D}_0} p_i = i p_i$*, for any* $p_i \in \mathcal{P}_i^{\mathbf{t}}$*, (Euler's Theorem),*
4. $[p\mathbf{F}, \mathbf{G}] = (L_{\mathbf{G}} p)\mathbf{F} + p[\mathbf{F}, \mathbf{G}]$*, with* p *a smooth scalar function and* $\mathbf{F}$ *and* $\mathbf{G}$ *smooth vector fields.*

We now begin the development of the method, which will allow us to obtain a reduced normal form of system (1).

Definition 1. For every $k \geq 1$, we define the linear operator

$$\begin{aligned} \mathcal{L}_{r+k} \quad : \quad & \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \tau_k) \rightarrow \mathcal{L}_{r+k}(\mathbf{P}_k, \tau_k) = [\mathbf{P}_k, \mathbf{F}_r] - \tau_k \mathbf{F}_r. \end{aligned} \tag{8}$$

The key in the problem of providing a normal form of the system (1) is to analyze the effect of a near-identity transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$ and a reparameterization of the time by $\frac{dt}{dT} = 1 + \tau_k(\mathbf{x})$, where $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ and $\tau_k \in \mathcal{P}_k^{\mathbf{t}}$, with $k \geq 1$.

The quasi-homogeneous terms of the transformed system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$ agree with the original ones up to degree $r+k-1$ and for the degree $r+k$ gets

$$\begin{aligned}\mathbf{G}_{r+k} &= \mathbf{F}_{r+k} - (D\mathbf{P}_k\mathbf{F}_r - D\mathbf{F}_r\mathbf{P}_k) + \tau_k\mathbf{F}_r = \mathbf{F}_{r+k} - [\mathbf{P}_k, \mathbf{F}_r] + \tau_k\mathbf{F}_r \\ &= \mathbf{F}_{r+k} - \mathcal{L}_{r+k}(\mathbf{P}_k, \tau_k).\end{aligned}$$

Thus, $\mathcal{L}_{r+k}$ is actually the homological operator under orbital equivalence.

Following the ideas of the conventional normal form theory, it is enough to choose $(\mathbf{P}_k, \tau_k) \in \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}}$ adequately in order to simplify the $(r+k)$-degree quasi-homogeneous term of system (1), by annihilating the part belonging to the range of the linear operator $\mathcal{L}_{r+k}$. In other words, we can achieve that $\mathbf{F}_{r+k}$ belongs to a complementary subspace to the range of $\mathcal{L}_{r+k}$. When this has been done, we say that the corresponding term has been reduced to normal form under orbital equivalence. So, by means of a sequence of time-reparameterizations and near-identity transformations (by performing the procedure for $k=1$, then for $k=2$ and so on) system (1) can be formally reduced to normal form under orbital equivalence, i.e. the system can be transformed into

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}) = \mathbf{G}_r(\mathbf{y}) + \mathbf{G}_{r+1}(\mathbf{y}) + \cdots = \sum_{k\geq 0} \mathbf{G}_{r+k}(\mathbf{y}), \tag{9}$$

with $\mathbf{G}_r \not\equiv \mathbf{0}$ and $\mathbf{G}_{r+k} \in \mathrm{Cor}(\mathcal{L}_{r+k}) \subseteq \mathcal{Q}_{r+k}^{\mathbf{t}}$ where $\mathrm{Cor}(\mathcal{L}_{r+k})$ is any complementary subspace to the range of the homological operator $\mathcal{L}_{r+k}$. We claim that such space is not unique, in general.

The rest of this section is dedicated to giving an expression of $\mathrm{Cor}(\mathcal{L}_{r+k})$, or equivalently, to finding a normal form of the system.

We first prove the following result.

Proposition 7. *Given a type* $\mathbf{t}$ *and a non-negative integer* k*, it verifies that*

$$\mathit{Range}(\mathcal{L}_{r+k}(\mathcal{Q}_k^{\mathbf{t}} \times \mathit{Cor}(\ell_k))) = \mathit{Range}(\mathcal{L}_{r+k}(\mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}}))$$

being $\mathit{Cor}(\ell_k)$ *a complementary subspace to the range of the linear operator given in (4).*

Proof. Any function $\tau_k \in \mathcal{P}_k^{\mathbf{t}}$ is expressed as $\tau_k = \nu_k + \tau_k^r$ with $\nu_k \in \mathrm{Cor}(\ell_k)$, being $\mathrm{Cor}(\ell_k)$ a complementary subspace to the range of the linear operator ℓ_k, and $\tau_k^r \in \mathrm{Range}(\ell_k)$, i.e. there exists $\tau_{k-r} \in \mathcal{P}_{k-r}^{\mathbf{t}}$ such that $\ell_k(\tau_{k-r}) = \tau_k^r = L_{\mathbf{F}_r}\tau_{k-r}$. So, from Lemma 6 (item 4),

$$\begin{aligned}\mathcal{L}_{r+k}(\mathbf{P}_k, \tau_k) &= [\mathbf{P}_k, \mathbf{F}_r] - \nu_k\mathbf{F}_r - (L_{\mathbf{F}_r}\tau_{k-r})\mathbf{F}_r \\ &= [\mathbf{P}_k, \mathbf{F}_r] - \nu_k\mathbf{F}_r - [\tau_{k-r}\mathbf{F}_r, \mathbf{F}_r] + \tau_{k-r}[\mathbf{F}_r, \mathbf{F}_r] \\ &= [\mathbf{P}_k - \tau_{k-r}\mathbf{F}_r, \mathbf{F}_r] - \nu_k\mathbf{F}_r.\end{aligned}$$

Consequently, both operator $\mathcal{L}_{r+k}$ restricted to $\mathcal{Q}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k)$ and operator $\mathcal{L}_{r+k}$ have the same range. □

In this way, by keeping the notation, we consider $\mathcal{L}_{r+k}$ restricted to $\mathcal{Q}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k)$ as the homological operator under orbital equivalence.

We introduce the following subspaces of $\mathcal{Q}_k^{\mathbf{t}}$, which are used for the study of the homological operator $\mathcal{L}_{r+k}$:

$$\mathcal{D}_k^{\mathbf{t}} := \{\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}} : \mathbf{P}_k = \mu_k \mathbf{D}_0,\ \mu_k \in \mathcal{P}_k^{\mathbf{t}}\}, \quad \mathcal{C}_k^{\mathbf{t}} := \{\mathbf{X}_g \in \mathcal{Q}_k^{\mathbf{t}} :\ g \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}\}.$$

By the uniqueness of conservative-dissipative splitting (2) of a quasi-homogeneous vector field, $\mathcal{Q}_k^{\mathbf{t}}$ can be represented uniquely as $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{C}_k^{\mathbf{t}} \oplus \mathcal{D}_k^{\mathbf{t}}$, for any $k \in \mathbb{N}$.

We give some properties related to the action of the Lie bracket on both subspaces. These properties have a straightforward proof that is omitted.

Lemma 8. *Given $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $g \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$ and $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$, it holds:*

1. $[\mathbf{X}_h, \mathbf{X}_g] = \mathbf{X}_f \in \mathcal{C}_{r+k}^{\mathbf{t}}$, *being* $f = -L_{\mathbf{X}_h} g$.
2. $(r+k+|\mathbf{t}|)\mu_k \mathbf{X}_h = (r+|\mathbf{t}|)\mathbf{X}_{\mu_k h} + (L_{\mathbf{X}_h}\mu_k)\mathbf{D}_0 \in \mathcal{C}_{r+k}^{\mathbf{t}} \oplus \mathcal{D}_{r+k}^{\mathbf{t}}$.
3. $[\mathbf{X}_h, \mu_k \mathbf{D}_0] = \frac{r(r+|\mathbf{t}|)}{r+k+|\mathbf{t}|}\mathbf{X}_{\mu_k h} - \frac{k+|\mathbf{t}|}{r+k+|\mathbf{t}|}(L_{\mathbf{X}_h}\mu_k)\mathbf{D}_0 \in \mathcal{C}_{r+k}^{\mathbf{t}} \oplus \mathcal{D}_{r+k}^{\mathbf{t}}$.

For system (3), from Lemma 8, we re-define the homological operator under orbital equivalence given in (8) as

$$\mathcal{L}_{r+k} : \mathcal{C}_k^{\mathbf{t}} \times \mathcal{D}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k) \longrightarrow \mathcal{C}_{r+k}^{\mathbf{t}} \times \mathcal{D}_{r+k}^{\mathbf{t}}$$

with

$$\begin{aligned}\mathcal{L}_{r+k}(\mathbf{X}_g, \mu_k \mathbf{D}_0, \nu_k) \quad = \quad & \Big(-\tfrac{r(r+|\mathbf{t}|)}{r+k+|\mathbf{t}|}\mathbf{X}_{\mu_k h} - \tfrac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|}\mathbf{X}_{\nu_k h} + \mathbf{X}_{\ell_{r+k+|\mathbf{t}|}(g)},\\ & \tfrac{k+|\mathbf{t}|}{r+k+|\mathbf{t}|}\ell_{r+k}(\mu_k)\mathbf{D}_0 - \tfrac{1}{r+k+|\mathbf{t}|}\ell_{r+k}(\nu_k)\mathbf{D}_0\Big).\end{aligned}$$

Hence, by taking appropriate bases, we get a triangular-block matrix for the above linear operator:

	$\mathbf{X}_g \in \mathcal{C}_k^{\mathbf{t}}$	$\mu_k \mathbf{D}_0 \in \mathcal{D}_k^{\mathbf{t}}$	$\nu_k \in \mathrm{Cor}(\ell_k)$
$\mathcal{C}_{r+k}^{\mathbf{t}}$	$\mathbf{X}_{\ell_{r+k+\lvert\mathbf{t}\rvert}(g)}$	$-\frac{r(r+\lvert\mathbf{t}\rvert)}{r+k+\lvert\mathbf{t}\rvert}\mathbf{X}_{\mu_k h}$	$-\frac{r+\lvert\mathbf{t}\rvert}{r+k+\lvert\mathbf{t}\rvert}\mathbf{X}_{\nu_k h}$
$\mathcal{D}_{r+k}^{\mathbf{t}}$	0	$\frac{k+\lvert\mathbf{t}\rvert}{r+k+\lvert\mathbf{t}\rvert}\ell_{r+k}(\mu_k)\mathbf{D}_0$	$-\frac{1}{r+k+\lvert\mathbf{t}\rvert}\ell_{r+k}(\nu_k)\mathbf{D}_0$

From the expression of the matrix, it deduces that a complementary subspace to the range of the operator $\mathcal{L}_{r+k}$ is expressed in terms of a complementary subspace to the range of the operator ℓ_{r+k}, which acts on scalar functions, instead of vector fields. So, $\mathrm{Cor}(\mathcal{L}_{r+k}) = \mathbf{X}_{S_{k+r+|\mathbf{t}|}} \oplus \mathrm{Cor}(\ell_{r+k})\mathbf{D}_0$, being $S_{k+r+|\mathbf{t}|}$ a subspace verifying $\mathrm{Cor}(\ell_{k+r+|\mathbf{t}|}) = S_{k+r+|\mathbf{t}|} \oplus h\mathrm{Cor}(\ell_k)$.

Next proposition, given in [2], establishes that it is enough to know a finite number of complementary spaces to the range of the operators ℓ_k to compute all of them.
For sake of completeness, we include its proof.

Proposition 9. *Let $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ be, whose factorization on $\mathbb{C}[x,y]$ only has simple factors. If $\mathcal{P}_{k-r}^{\mathbf{t}} \neq \{0\}$ (i.e. $k-r \in \mathcal{I}^{\mathbf{t}}$) with $k > r$, then a complementary subspace to Range$(\ell_{r+k+|\mathbf{t}|})$ is Cor$(\ell_{k+r+|\mathbf{t}|}) = hCor(\ell_k)$.*

PROOF. It is straightforward to prove $\dim(\mathrm{Cor}(\ell_{k+r+|\mathbf{t}|})) = \dim(h\mathrm{Cor}(\ell_k))$. For more details see [2].
To complete the proof, it is enough to show that $h\mu_k \notin \mathrm{Range}(\ell_{k+r+|\mathbf{t}|})$ for each $\mu_k \in \mathrm{Cor}(\ell_k)$.
We assume on the contrary, that is, there exists $\mu_k \in \mathrm{Cor}(\ell_k)$ such that $h\mu_k \in \mathrm{Range}(\ell_{k+r+|\mathbf{t}|})$. Then, there exists $\nu_{k+|\mathbf{t}|} \in \mathcal{P}^{\mathbf{t}}_{k+|\mathbf{t}|}$ such that

$$h\mu_k = \ell_{k+r+|\mathbf{t}|}(\nu_{k+|\mathbf{t}|}) = \nabla\nu_{k+|\mathbf{t}|} \cdot \mathbf{X}_h = -\nabla h \cdot \mathbf{X}_{\nu_{k+|\mathbf{t}|}}.$$

Therefore, $h = 0$ is an invariant curve of $\mathbf{X}_{\nu_{k+|\mathbf{t}|}}$. This implies that all irreducible factors of h are also irreducible factors of $\nu_{r+|\mathbf{t}|}$. As the factorization of h only has simple factors, it has that $\nu_{k+|\mathbf{t}|} = \nu_{k-r}h$. Also, it can be easily derived, by using that $\nabla h \cdot \mathbf{X}_h = 0$, that $\ell_{k+r+|\mathbf{t}|}(\nu_{k-r}h) = h\ell_k(\nu_{k-r})$.
Hence, $h\mu_k = h\ell_k(\nu_{k-r})$ and then $\mu_k \in \mathrm{Range}(\ell_k)$ which is a contradiction. $\square$

Remark 3. *If h has multiple factors, Proposition 9 is not true, in general. For example, for $h = y^2 \in \mathcal{P}^{(1,1)}_2$, it gets $Cor(\ell_j) =< x^j >,\ j \geq 1$.*

The next result provides a formal normal form of (3) when h only has simple factors in its factorization on $\mathbb{C}[x, y]$.

Theorem 10. *Let $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ be, whose factorization on $\mathbb{C}[x, y]$ only has simple factors. Then, a formal normal form under orbital equivalence for system $\dot{\mathbf{x}} = \mathbf{X}_h +$ q-h.h.o.t. is*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \sum_{j=1}^{n_0-1} \mathbf{X}_{g_{j+r+|\mathbf{t}|}} + \sum_{j=r+1}^{n_0+r+|\mathbf{t}|-1} \eta_j^{(0)}\mathbf{D}_0 + \sum_{i=1}^{\infty}\sum_{j=0}^{r+|\mathbf{t}|-1} \eta_{j+n_0}^{(i)} h^i \mathbf{D}_0, \tag{10}$$

with $n_0 := 1 + r + max\left\{\mathbb{N}_0 \setminus \mathcal{I}^{\mathbf{t}}\right\}$, $\eta_j^{(i)} \in Cor(\ell_j)$ and $S_{j+r+|\mathbf{t}|}$ a subspace verifying $Cor(\ell_{j+r+|\mathbf{t}|}) = S_{j+r+|\mathbf{t}|} \oplus hCor(\ell_j)$. Moreover, $r + 1 \leq n_0 \leq r + 1 + max\{0, t_1t_2 - |\mathbf{t}|\}$.

PROOF. From Lemma 5 and Remark 2, it claims that n_0 exists and it holds that $r + 1 \leq n_0 \leq max\{0, t_1t_2 - |\mathbf{t}|\} + r + 1$. The co-range of the homological operator $\mathcal{L}_{r+k}$ is determined by the co-range of the linear operator ℓ_{r+k}. From Proposition 9, if $k \geq n_0 + r + |\mathbf{t}|$, then $\mathrm{Cor}(\ell_k) = h\mathrm{Cor}(\ell_{k-r-|\mathbf{t}|})$ and $S_k=\{0\}$. Taking $i = \left\lfloor \frac{k-n_0}{r+|\mathbf{t}|} \right\rfloor$ and $j = k - (r + |\mathbf{t}|)i$, it holds $n_0 \leq j < n_0 + r + |\mathbf{t}|$. Applying, Proposition 9, it gets $\mathrm{Cor}(\ell_k) = h^i\mathrm{Cor}(\ell_j)$. This completes the proof. $\square$

Remark 4. *a) It is necessary only the computation of a certain number of these co-ranges, concretely, from $r + 1$ to $n_0 + r + |\mathbf{t}| - 1$, to obtain the normal form of (3).*
b) The systems analyzed in this paper, systems (5), are the systems which satisfy one of the following conditions:

1. *the sets* S_j, $j = r + |\mathbf{t}| + 1, \ldots, n_0 + r + |\mathbf{t}| - 1$ *are empty-sets, or*
2. *the sets* S_j *are not empty-sets and have the normal form given by (10) with* $g_j = 0$ *for all* $r + |\mathbf{t}| + 1 \leq j \leq n_0 + r + |\mathbf{t}| - 1$, *i.e. the coefficients of* g_j *are nulls. For example, the system* $(\dot{x}, \dot{y})^T = (-y^3, x^5)^T +$ *q-h.h.o.t. is orbitally equivalent to*

$$(\dot{x}, \dot{y})^T = (-y^3, x^5)^T + (\alpha_8 x^4 + \beta_8 xy^2 + \sum_{j \geq 9} \mu_j)(2x, 3y)^T + \alpha_9 \mathbf{X}_{x^4 y^2},$$

with $\mu_j \in Cor(\ell_j)$. *In such a case, it is possible to study the system for the values of the coefficients that verify the condition* $\alpha_9 = 0$.

2.1. Reduced Normal Form under orbital equivalence of system (5) non-integrable.

We consider the systems (5), which are non-integrable; i.e. the systems $\dot{\mathbf{x}} = \mathbf{X}_h + \mu \mathbf{D}_0$, with $\mu = \mu_N + \sum_{j>N} \mu_j$, $\mu_j \in \mathrm{Cor}(\ell_j)$ and $\mu_N \not\equiv 0$.

It is possible to compute a reduced normal form by proceeding as follows. Firstly, it performs a re-parameterization of the time $dt/dT = 1 - \nu_k(\mathbf{x})$ and a coordinate transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$, with $\nu_k \in \mathrm{Cor}(\ell_k)$, $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$, that puts the k-degree quasi-homogeneous terms in normal form as it is explained before. Later, it performs a new re-parameterization of the time $dT/d\tau = 1 - \tilde{\nu}_{r+k-N}(\mathbf{y})$ and a near-identity transformation $\mathbf{y} = \mathbf{z} + \tilde{\mathbf{P}}_{r+k-N}(\mathbf{z})$ with $(\tilde{\mathbf{P}}_{r+k-N}, \tilde{\nu}_{r+k-N}) \in \mathrm{Ker}(\mathcal{L}_{2r+k-N}) \subset \mathcal{Q}_{r+k-N}^{\mathbf{t}} \times \mathrm{Cor}(\ell_{r+k-N})$. This process defines the two-step homological operator:

Definition 2. For every $k \geq 1$, we define the linear operator

$$\begin{aligned}
\mathcal{L}_{r+k}^{(2)} \quad : \quad & \mathcal{Q}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k) \times \mathrm{Ker}(\mathcal{L}_{2r+k-N}) \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\
& (\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{r+k-N}, \tilde{\nu}_{r+k-N})) \to \mathcal{L}_{r+k}^{(2)}(\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{r+k-N}, \tilde{\nu}_{r+k-N})) \\
& = [\mathbf{P}_k, \mathbf{F}_r] - \nu_k \mathbf{F}_r + [\tilde{\mathbf{P}}_{r+k-N}, \mathbf{F}_N] - \tilde{\nu}_{r+k-N} \mathbf{F}_N \\
& = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + [\tilde{\mathbf{P}}_{r+k-N}, \mathbf{F}_N] - \tilde{\nu}_{r+k-N} \mathbf{F}_N. \qquad (11)
\end{aligned}$$

It is clear that the quasi-homogeneous terms of the transformed system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$ remain unaltered up to degree $r + k - 1$ and for the degree $r + k$ get

$$\mathbf{G}_{r+k} = \mathbf{F}_{r+k} - \mathcal{L}_{r+k}^{(2)}(\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{r+k-N}, \tilde{\nu}_{r+k-N})).$$

To obtain the expression of a reduced normal form of system (5) with μ_N non-null, it is necessary to know the bases of $\mathrm{Ker}(\ell_{r+k})$ and $\mathrm{Ker}(\mathcal{L}_{r+k})$ which will allow us to get the expression of a complementary subspace to the range of the operator $\mathcal{L}_{r+k}^{(2)}$.

Lemma 11. *Let* $k = l_1(r + |\mathbf{t}|) + l_2$ *be, with* $0 \leq l_2 < r + |\mathbf{t}|$. *It holds:*

1. $Ker(\ell_{r+k}) = span\{h^{l_1}\}$, *if* $l_2 = 0$,
2. $Ker(\mathcal{L}_{r+k}) = span\{(\mathbf{0}, h^{l_1} \mathbf{D}_0, -r h^{l_1})\}$, *if* $l_2 = 0$,
3. $Ker(\mathcal{L}_{r+k}) = span\{(\mathbf{X}_{h^{l_1+1}}, \mathbf{0}, 0)\}$, *if* $l_2 = r$.

Otherwise, both sets Ker(ℓ_{r+k}) and Ker($\mathcal{L}_{r+k}$), are trivial spaces.

Proof. The basis of $\mathrm{Ker}(\ell_{r+k})$ is obtained in [2].
Now we are going to obtain the second one. Let $(\mathbf{X}_{g_{k+|\mathbf{t}|}}, \mu_k \mathbf{D}_0, \nu_k) \in \mathrm{Ker}(\mathcal{L}_{r+k})$ be. It holds that:

$$\ell_{r+k+|\mathbf{t}|}(g_{k+|\mathbf{t}|}) = \tfrac{r(r+|\mathbf{t}|)}{r+k+|\mathbf{t}|}\mu_k h + \tfrac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|}\nu_k h \tag{12}$$

$$(k+|\mathbf{t}|)\mu_k - \nu_k \in \mathrm{Ker}(\ell_{r+k}). \tag{13}$$

We suppose that $l_2 \neq 0$. As $\mathrm{Ker}(\ell_{r+k}) = \{0\}$, from (13), it deduces that $\nu_k = (k+|\mathbf{t}|)\mu_k$; i.e. $\mu_k \in \mathrm{Cor}(\ell_k)$. So (12) becomes $\ell_{r+k+|\mathbf{t}|}(g_{k+|\mathbf{t}|}) = (r+|\mathbf{t}|)\mu_k h \in \mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}) \cap \mathrm{Range}(\ell_{r+k+|\mathbf{t}|})$, that is, $\mu_k = \nu_k = 0$. Moreover, $g_{k+|\mathbf{t}|} \in \mathrm{Ker}(\ell_{r+k+|\mathbf{t}|})$. Consequently, $g_{k+|\mathbf{t}|} = \alpha h^{l_1+1}$ if $l_2 = r$, and $g_{k+|\mathbf{t}|} = 0$ if $l_2 \neq r$.
Now, we suppose that $l_2 = 0$. It has that $\mathrm{Ker}(\ell_{r+k}) = \mathrm{span}\{h^{l_1}\}$ and $\mathrm{Ker}(\ell_{r+k+|\mathbf{t}|}) = \{0\}$. Hence, by (13), it leads $\nu_k = (k+|\mathbf{t}|)\mu_k + \alpha h^{l_1}$ (it claims that $\mu_k \in \mathrm{Cor}(\ell_k)$ since $\nu_k, h^{l_1} \in \mathrm{Cor}(\ell_k)$) and (12) gets $\ell_{r+k+|\mathbf{t}|}(g_{k+|\mathbf{t}|}) = (r+|\mathbf{t}|)\mu_k h + \frac{k+|\mathbf{t}|}{r+k+|\mathbf{t}|}h^{l_1+1} \in \mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}) \cap \mathrm{Range}(\ell_{r+k+|\mathbf{t}|})$. So, by (12) and (13), $g_{r+k+|\mathbf{t}|} = 0$ and $\mu_k = -\frac{\alpha}{r+k+|\mathbf{t}|}h^{l_1}$, $\nu_k = \frac{\alpha r}{r+k+|\mathbf{t}|}h^{l_1}$. This completes the proof. □

We give the following linear operator:

Definition 3. For given $k = N + l_1(r+|\mathbf{t}|) + l_2$, with $0 \leq l_2 < r + |\mathbf{t}|$, we define the linear operator $\ell_k^{(2)} : \mathcal{P}^t_{k-r} \times \mathrm{Ker}(\ell_{r+k-N}) \longrightarrow \mathcal{P}^{\mathbf{t}}_k$ as

$$\begin{aligned} &\ell_k^{(2)}(\mu_{k-r}, \alpha h^{l_1}) := \ell_k(\mu_{k-r}) + \alpha\mu_N h^{l_1}, && \text{if } l_2 = 0,\\ &\ell_k^{(2)}(\mu_{k-r}, 0) := \ell_k(\mu_{k-r}), && \text{if } l_2 \neq 0. \end{aligned}$$

The operator above is well-defined and provides an expression of $\mathrm{Cor}(\mathcal{L}^{(2)}_{r+k})$.

Proposition 12. *$Cor(\mathcal{L}^{(2)}_{r+k}) = Cor(\ell^{(2)}_{r+k})\mathbf{D}_0$, if $r+k > N$.*

Proof. The natural number $r+k-N$ is written as $r+k-N = l_1(r+|\mathbf{t}|) + l_2$ with $0 \leq l_2 < r + |\mathbf{t}|$.
We distinguish two cases:
We suppose that $l_2 \neq 0$. As $\mathrm{Ker}(\ell_{2r+k-N})$ is a trivial space, it deduces that $\ell^{(2)}_{r+k} = \ell_{r+k}$.
Moreover, if $l_2 \neq r$, then $\mathrm{Ker}(\mathcal{L}_{2r+k-N})$ is actually a trivial space and thus $\mathcal{L}^{(2)}_{r+k} = \mathcal{L}_{r+k}$. Otherwise, $l_2 = r$ and $\mathrm{Ker}(\mathcal{L}_{2r+k-N}) = \mathrm{span}\{(\mathbf{X}_{h^{l_1+1}}, \mathbf{0}, 0)\}$. So, by applying Lemma 6 (item 4), it gets

$$\begin{aligned} &\mathcal{L}^{(2)}_{r+k}(\mathbf{X}_g + \eta\mathbf{D}_0, \nu, (\mathbf{X}_{h^{l_1+1}}, \mathbf{0}, 0))\\ &\qquad = \mathcal{L}_{r+k}(\mathbf{X}_g + \eta\mathbf{D}_0, \nu) + \alpha[\mathbf{X}_{h^{l_1+1}}, \mu_N\mathbf{D}_0]\\ &\qquad = \mathcal{L}_{r+k}(\mathbf{X}_g + \eta\mathbf{D}_0, \nu) + \alpha(l_1+1)\mathcal{L}_{r+k}(h^{l_1}\mu_N\mathbf{D}_0, -(k-N)h^{l_1}\mu_N). \end{aligned}$$

Therefore, $\text{Range}(\mathcal{L}^{(2)}_{r+k}) = \text{Range}(\mathcal{L}_{r+k})$. For both cases, $l_2 \neq r$ or $l_2 = r$, it gets

$$\text{Cor}(\mathcal{L}^{(2)}_{r+k}) = \text{Cor}(\mathcal{L}_{r+k}) = \text{Cor}(\ell_{r+k})\mathbf{D}_0 = \text{Cor}(\ell^{(2)}_{r+k})\mathbf{D}_0.$$

Otherwise, we suppose that $l_2 = 0$. In this case, $\text{Ker}(\mathcal{L}_{2r+k-N}) = \{(\mathbf{0}, h^{l_1}\mathbf{D}_0, -rh^{l_1})\}$. Hence,

$$\begin{aligned}\mathcal{L}^{(2)}_{r+k}&(\mathbf{X}_g + \eta\mathbf{D}_0, \nu, (\mathbf{0}, \alpha h^{l_1}\mathbf{D}_0, -\alpha r h^{l_1}))\\ &= \mathcal{L}_{r+k}(\mathbf{X}_g + \eta\mathbf{D}_0, \nu) + \alpha(r+k-N)\mu_N h^{l_1}\mathbf{D}_0,\end{aligned}$$

with $g \in \mathcal{P}_{k-r},\ \eta \in \mathcal{P}_k,\ \nu \in \text{Cor}(\ell_k)$ and α real number. Therefore,

$$\text{Range}(\mathcal{L}^{(2)}_{r+k}) = \text{Range}(\mathcal{L}_{r+k}) + \text{span}\{h^{l_1}\mu_N\mathbf{D}_0\}.$$

Moreover, as the system (5) is a normal form, then

$$\text{Range}(\mathcal{L}_{r+k}) = \mathbf{X}_{\mathcal{P}^{\mathbf{t}}_{r+k+|\mathbf{t}|}} + \text{Range}(\ell_{r+k})\mathbf{D}_0.$$

Thus, it concludes that $\text{Cor}(\mathcal{L}^{(2)}_{r+k}) = \text{Cor}(\ell^{(2)}_{r+k})\mathbf{D}_0$. This completes the proof. □

Proposition 12 yields the next theorem, which determines a reduced normal form of non-integrable systems.

Theorem 13. *Let $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ be, whose factorization on $\mathbb{C}[x, y]$ only has simple factors. A formal normal form under orbital equivalence for system $\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0$, with $\mu = \sum_{j\geq N}\mu_j,\ \mu_j \in Cor(\ell_j),\ j \geq N,\ \mu_N \neq 0$, is*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_N\mathbf{D}_0 + \sum_{j>N}\tilde{\mu}_j\mathbf{D}_0,$$

where $\tilde{\mu}_j \in Cor(\ell^{(2)}_j)$.

3. Proofs of the main results.

The following lemmas are used to prove Theorem 1.

Lemma 14. *Let Φ be a near identity diffeomorphism on $U \subset \mathbb{C}^2$. If $f(\mathbf{x}) = 0$ is an invariant curve of the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, whose cofactor is K, then $f(\Phi(\mathbf{y})) = 0$ is an invariant curve of the system $\dot{\mathbf{y}} = (\Phi_*((1+\alpha)\mathbf{F}))(\mathbf{y})$, whose cofactor is $(1+\alpha\circ\Phi)(K\circ\Phi)$, for any α a $\mathcal{C}^\infty$-class scalar function with $\alpha(\mathbf{0}) = 0$.*

Proof. Indeed, let $\mathbf{G} = \Phi_*((1+\alpha)\mathbf{F})$ be. It has that

$$\begin{aligned}L_{\mathbf{G}}(f\circ\Phi)(\mathbf{y}) &= \nabla f(\Phi(\mathbf{y}))\cdot D^{-1}\Phi(\mathbf{y})(1+\alpha(\Phi(\mathbf{y})))\mathbf{F}(\Phi(\mathbf{y}))\\ &= (1+\alpha(\mathbf{x}))\nabla f(\mathbf{x})D\Phi(\mathbf{y})D^{-1}\Phi(\mathbf{y})\mathbf{F}(\mathbf{x})\\ &= (1+\alpha(\mathbf{x}))\nabla f(\mathbf{x})\cdot\mathbf{F}(\mathbf{x}) = (1+\alpha(\mathbf{x}))K(\mathbf{x})f(\mathbf{x})\\ &= (1+\alpha(\Phi(\mathbf{y})))K(\Phi(\mathbf{y}))f(\Phi(\mathbf{y})),\end{aligned}$$

as desired. □

Lemma 15. *Let $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ be. Every $\mathbf{F}_j \in \mathcal{Q}^t_j$ with $j > r$ is expressed as $\mathbf{F}_j = \mathbf{X}_{g_{j+|\mathbf{t}|}} + \eta_j \mathbf{D}_0 + \lambda_{j-r}\mathbf{X}_h$, with $\lambda_{j-r} \in \mathcal{P}^{\mathbf{t}}_{j-r}$, $\eta_j \in \mathcal{P}^{\mathbf{t}}_j$ and $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$, being $\Delta_{j+|\mathbf{t}|}$ a complementary subspace of $h\mathcal{P}^{\mathbf{t}}_{j-r}$, i.e. $\mathcal{P}^{\mathbf{t}}_{j+|\mathbf{t}|} = \Delta_{j+|\mathbf{t}|} \oplus h\mathcal{P}^{\mathbf{t}}_{j-r}$.*

Proof. Hence, by the conservative-dissipative splitting of a quasi-homogeneous vector field (2), $\mathbf{F}_j = \mathbf{X}_{h_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0$ with $\mu_j \in \mathcal{P}^t_j$ and $h_{j+|\mathbf{t}|} \in \mathcal{P}^t_{j+|\mathbf{t}|}$. As $\mathcal{P}^t_{j+|\mathbf{t}|} = \Delta_{j+|\mathbf{t}|} \oplus h\mathcal{P}^{\mathbf{t}}_{j-r}$, it expresses $h_{j+|\mathbf{t}|} = g_{j+|\mathbf{t}|} + \tilde{\lambda}_{j-r}h$ with $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$ and $\tilde{\lambda}_{j-r} \in \mathcal{P}^t_{j-r}$. Therefore

$$\mathbf{F}_j = \mathbf{X}_{g_{j+|t|}} + \mathbf{X}_{\tilde{\lambda}_{j-r}h} + \mu_j \mathbf{D}_0.$$

On the other hand, by applying Lemma 8, 2), it gets

$$\mathbf{X}_{\tilde{\lambda}_{j-r}h} = \frac{j+|\mathbf{t}|}{r+|\mathbf{t}|}\tilde{\lambda}_{j-r}\mathbf{X}_h - \frac{1}{r+|\mathbf{t}|}\ell_j(\tilde{\lambda}_{j-r})\mathbf{D}_0.$$

Hence,

$$\begin{aligned} \mathbf{F}_j &= \mathbf{X}_{g_{j+|\mathbf{t}|}} + \frac{j+|\mathbf{t}|}{r+|\mathbf{t}|}\tilde{\lambda}_{j-r}\mathbf{X}_h + \left(\mu_j - \frac{1}{r+|\mathbf{t}|}\ell_j(\tilde{\lambda}_{j-r})\right)\mathbf{D}_0 \\ &= \mathbf{X}_{g_{j+|\mathbf{t}|}} + \left(\mu_j - \frac{1}{r+|\mathbf{t}|}\ell_j(\tilde{\lambda}_{j-r})\right)\mathbf{D}_0 + \frac{j+|\mathbf{t}|}{r+|\mathbf{t}|}\tilde{\lambda}_{j-r}\mathbf{X}_h. \end{aligned}$$

The result follows by taking $\eta_j = \mu_j - \frac{1}{r+|\mathbf{t}|}\ell_j(\tilde{\lambda}_{j-r})$ and $\lambda_{j-r} = \frac{j+|\mathbf{t}|}{r+|\mathbf{t}|}\tilde{\lambda}_{j-r}$. □

Lemma 16. *Let $\mathbf{F} = \sum_{j\geq r}\mathbf{F}_j$ be, with $\mathbf{F}_j \in \mathcal{Q}^{\mathbf{t}}_j$ and $\mathbf{F}_r = \mathbf{X}_h$, being $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$. If $h = 0$ is an invariant curve of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, then $\mathbf{F} = (1+\lambda)\mathbf{X}_h + \eta\mathbf{D}_0$, where $\lambda = \sum_{j\geq 1}\lambda_j$, $\lambda_j \in \mathcal{P}^{\mathbf{t}}_j$ and $\eta = \sum_{j>r}\eta_j$, $\eta_j \in \mathcal{P}^{\mathbf{t}}_j$.*

Proof. If h is an invariant curve of $\mathbf{F}$, as h is a quasi-homogeneous polynomial, it deduces that h is an invariant curve of each $\mathbf{F}_j$, that is $L_{\mathbf{F}_j}h := \nabla h \cdot \mathbf{F}_j = K_j h$ with $K_j \in \mathcal{P}^{\mathbf{t}}_j$. By above lemma, $\mathbf{F}_j = \mathbf{X}_{g_{j+|\mathbf{t}|}} + \eta_j\mathbf{D}_0 + \lambda_{j-r}\mathbf{X}_h$, $\lambda_{j-r} \in \mathcal{P}^{\mathbf{t}}_{j-r}$, $\eta_j \in \mathcal{P}^{\mathbf{t}}_j$ and $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$. Thus,

$$K_j h \;=\; \nabla h \cdot \mathbf{F}_j = \nabla h \cdot \left(\mathbf{X}_{g_{j+|\mathbf{t}|}} + \eta_j\mathbf{D}_0 + \lambda_{j-r}\mathbf{X}_h\right) = \nabla h \cdot \mathbf{X}_{g_{j+|\mathbf{t}|}} + (r+|\mathbf{t}|)\eta_j h,$$

i.e. h is an invariant curve of $\mathbf{X}_{g_{j+|\mathbf{t}|}}$. Therefore, $g_{j+|\mathbf{t}|}$ belongs to the ideal generated by h and as $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$ this leads to $g_{j+|\mathbf{t}|} = 0$.

So, for each $j > r$, $\mathbf{F}_j = \eta_j\mathbf{D}_0 + \lambda_j\mathbf{X}_h$ and then $\mathbf{F} = (1+\lambda)\mathbf{X}_h + \eta\mathbf{D}_0$ with $\lambda = \sum_{j>r}\lambda_{j-r}$ and $\eta = \sum_{j>r}\eta_j$. □

Proof of Theorem 1.
We prove the necessity. We assume that there is a $\mu = \sum_{j>r}\mu_j$, $\mu_j \in \mathcal{P}^{\mathbf{t}}_j$ such that $\mathbf{F}$ and $\mathbf{G} := \mathbf{X}_h + \mu\mathbf{D}_0$ are orbitally equivalent. That is, there is a Ψ near-identity diffeomorphism and α scalar function with $\alpha(\mathbf{0}) = 0$ such that $\Psi_*(1+\alpha)(\mathbf{X}_h + \mu\mathbf{D}_0) = \mathbf{F}$. We claim that h is an invariant curve of $\mathbf{X}_h + \mu\mathbf{D}_0$ with cofactor $(r+|\mathbf{t}|)\mu$ since $L_{\mathbf{G}}h = \mu L_{\mathbf{D}_0}h = (r+|\mathbf{t}|)\mu h$.

Hence, by Lemma 14, $f = h(\Psi) = h + $ q-h.h.o.t. is an invariant curve of $\mathbf{F}$, that is, f is actually conjugated to h.

Now, we prove the sufficient condition. We suppose that there exists an invariant curve f of $\mathbf{F}$ such that $f = h + $ q-h.h.o.t. and it is conjugated to h, thus there is a formal diffeomorphism Φ such that $h = f(\Phi)$. From Lemma 14 for $\alpha \equiv 0$, h is an invariant curve of $\tilde{\mathbf{G}} := \Phi_* \mathbf{F}$. Applying Lemma 16, it gets $\tilde{\mathbf{G}} = (1+\lambda)\mathbf{X}_h + \eta \mathbf{D}_0$. Consequently,

$$\mathbf{F} \;=\; \Phi^* \tilde{\mathbf{G}} = \Phi^*((1+\lambda)(\mathbf{X}_h + \frac{\eta}{1+\lambda}\mathbf{D}_0)) = \Phi^*((1+\lambda)(\mathbf{X}_h + \mu \mathbf{D}_0))$$

with $\mu = \frac{\eta}{1+\lambda}$. So, $\mathbf{F}$ is orbitally equivalent to $\mathbf{X}_h + \mu \mathbf{D}_0$. $\square$

The following result gives a necessary condition so that an inverse integrating factor exists.

Proposition 17. *Let the system $\dot{\mathbf{x}} = \mathbf{X}_h + \lambda \mathbf{D}_0$ be, with $h \in \mathcal{P}_{r+|\mathbf{t}|}$ and $\lambda \in \mathcal{P}_{r+s(r+|\mathbf{t}|)}$ being s a natural number. Then, the polynomial h^{s+1} is an inverse integrating factor of the system.*

Proof. It names $V(h) = h^{s+1}$. From Euler's Theorem (i.e. $L_{\mathbf{D}_0} p_i = i p_i$, for any $p_i \in \mathcal{P}_i^{\mathbf{t}}$), it gets

$$L_{\mathbf{X}_h + \lambda \mathbf{D}_0} V = L_{\lambda \mathbf{D}_0} V = V'(h) L_{\mathbf{D}_0} h = (s+1)(r+|\mathbf{t}|)\lambda h^{s+1}$$

and $\operatorname{div}(\mathbf{X}_h + \lambda \mathbf{D}_0) = \operatorname{div}(\lambda \mathbf{D}_0) = (L_{\mathbf{D}_0}\lambda + |\mathbf{t}|\lambda) = (s+1)(r+|\mathbf{t}|)\lambda$, thus, the result follows. $\square$

Next we relate the terms of an inverse integrating factor and we provide its lowest-degree quasi-homogeneous term.

Proposition 18. *Let the system $\dot{\mathbf{x}} = \mathbf{X}_h + \mu \mathbf{D}_0$ be, with $\mu = \sum_{j>r} \mu_j$, $\mu_j \in Cor(\ell_j)$ and $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ having only simple factors in its factorization on $\mathbb{C}[x,y]$ and $\mu \not\equiv 0$. We assume that the system has an inverse integrating factor V. Then, it has that*

1. *$V = h^{s+1} + \sum_{j>s+1} b_j h^j$, for a certain s natural number.*
   Moreover, it holds
$$0 = \sum_{l=1}^{k-(s+1)(r+|\mathbf{t}|)} (k - (r+|\mathbf{t}|) - 2l) V_{k-l} \mu_{r+l}, \tag{14}$$
   with V_j the quasi-homogeneous term of degree j of V.
2. *$\mu = \mu_{r+s(r+|\mathbf{t}|)} + $ q-h.h.o.t. with $\mu_{r+s(r+|\mathbf{t}|)} \neq 0$.*

Proof. The fist part is Proposition 19 of [4].
We see the second one. We suppose that $\mu = \mu_N + $ q-h.h.o.t. with $\mu_N \neq 0$. Equation (14), for $k = (s+1)(r+|\mathbf{t}|) + N - r$, is

$$\begin{aligned} 0 &= ((s+1)(r+|\mathbf{t}|) + N - r - r - |\mathbf{t}| - 2(N-r)) V_{(s+1)(r+|\mathbf{t}|)}\, \mu_N \\ &= (s(r+|\mathbf{t}|) - N + r) h^{s+1}\, \mu_N. \end{aligned}$$

It follows that $N = r + s(r + |\mathbf{t}|) > r$. $\square$

Proof of Theorem 2. The sufficiency follows from Proposition 17.
We prove the necessity. If $\mu_j = 0$ for all j, the system is a Hamiltonian system whose first integral is h. In such a case, h is also an inverse integrating factor. Otherwise, let $N = \min\{j,\ \mu_j \neq 0\}$. Hence, by Theorem 13, system (5) is formally orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu_N \mathbf{D}_0 + \sum_{j>N} \tilde{\mu}_j \mathbf{D}_0$, with $\tilde{\mu}_j \in \mathrm{Cor}(\ell_j^{(2)})$. By Proposition 18, if $\mu = \mu_N +$ q-h.h.o.t. with $\mu_N \neq 0$, then $N = r + s(r + |\mathbf{t}|)$ and $V = h^{s+1} + \sum_{j>s} b_j h^j$.
We do the proof in two steps:
Step 1. We see that $V = h^{s+1}$ and $\tilde{\mu}_{r+(j-1)(r+|\mathbf{t}|)} = 0$, for any $j > s + 1$. Indeed, we assume the contrary, i.e. there exists an inverse integrating factor V of the form $V = h^{s+1} + \sum_{j=s+2}^{\infty} b_j h^j$, with some $b_j \neq 0$ and let $j_0 = min\{j > s+1,\ b_j \neq 0\}$.

For $k = (j_0 + s)(r + |\mathbf{t}|)$, there are only two components $V_{k-l} \neq 0$ with $1 \leq l \leq k - (s+1)(r + |\mathbf{t}|)$. In particular, for $l = (j_0 - 1)(r + |\mathbf{t}|)$ we have $V_{k-l} = V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$ and for $l = s(r + |\mathbf{t}|)$, $V_{k-l} = V_{j_0(r+|\mathbf{t}|)} = b_{j_0} h^{j_0}$. Then, the equality (14), for $k = (j_0 + s)(r + |\mathbf{t}|)$, becomes

$$
\begin{aligned}
0 &= ((j_0+s)(r+|\mathbf{t}|) - (r+|\mathbf{t}|) - 2s(r+|\mathbf{t}|))V_{j_0(r+|\mathbf{t}|)}\tilde{\mu}_{r+s(r+|\mathbf{t}|)} \\
&\quad +((j_0+s)(r+|\mathbf{t}|) - (r+|\mathbf{t}|) - 2(j_0-1)(r+|\mathbf{t}|))V_{(s+1)(r+|\mathbf{t}|)}\tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} \\
&= (j_0 - s - 1)(r+|\mathbf{t}|)b_{j_0}h^{j_0}\mu_N - (j_0 - s - 1)(r+|\mathbf{t}|)h^{s+1}\tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} \\
&= (j_0 - s - 1)(r+|\mathbf{t}|)h^{s+1}\left[b_{j_0}h^{j_0-s-1}\mu_N - \tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)}\right].
\end{aligned}
$$

Consequently, $\tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} = b_{j_0} h^{j_0-s-1}\mu_N \in \mathrm{Cor}(\ell^{(2)}_{r+(j_0-1)(r+|\mathbf{t}|)}) \setminus \{0\}$, but also $b_{j_0} h^{j_0-s-1}\mu_N = \ell^{(2)}_{r+(j_0-1)(r+|\mathbf{t}|)}(0, b_{j_0} h^{j_0-s-1})$, which is a contradiction. So, $V = h^{s+1}$ and $\tilde{\mu}_{r+(j-1)(r+|\mathbf{t}|)} = 0$, for all $j > s + 1$.
Step 2. We prove that $\tilde{\mu}_j = 0$, for all $j > N$. We use a contradiction. Let $j_0 = min\{j > N,\ \tilde{\mu}_j \neq 0\}$ be. For $k = j_0 - r + (s+1)(r + |\mathbf{t}|)$, (14) gets

$$
\begin{aligned}
0 &= \sum_{l=1}^{j_0-r} (j_0 - r + s(r+|\mathbf{t}|) - 2l)V_{j_0-r+(s+1)(r+|\mathbf{t}|)-l}\,\mu_{r+l} \\
&= [s(r+|\mathbf{t}|) - (j_0 - r)]V_{(s+1)(r+|\mathbf{t}|)}\,\mu_{j_0},
\end{aligned}
$$

and as $V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$ and $j_0 - r \neq s(r+|\mathbf{t}|)$ (by step 1), we obtain $\mu_{j_0} = 0$, a contradiction. $\square$

The coefficients of the formal normal form (10) of system (3) have been obtained by using the procedure given in Algaba et al. [1]. This method consists in a recursive procedure to compute quasi-homogeneous normal form under orbital equivalence, which uses the Lie triangle.
Proof of Theorem 3. System (6) is a system (3) with $\mathbf{t} = (1,1)$, $h = -\frac{1}{3}y^3 - \sigma x^2 y \in \mathcal{P}_3^{(1,1)}$, i.e. $r = 1$ and h only simple factors on $\mathbb{C}[x,y]$. By

Theorem 10, (in this case $n_0 = 2$) a formal normal form of system (6) is

$$(\dot{x}, \dot{y})^T = \mathbf{X}_h + \mathbf{X}_{g_4} + \sum_{i=0}^{\infty} \sum_{j=0}^{2} \eta_{j+2}^{(i)} h^i (x, y)^T, \tag{15}$$

with $\eta_j^{(i)} \in \mathrm{Cor}(\ell_j),\ g_4 \in S_4$. The co-ranges of $\ell_j,\ j = 2, 3, 4$ are

$$\mathrm{Cor}(\ell_2) = \mathrm{span}\{y^2\}, \quad \mathrm{Cor}(\ell_3) = \mathrm{span}\{h\}, \quad \mathrm{Cor}(\ell_4) = \mathrm{span}\{xh, yh\},$$

i.e. $\mathrm{Cor}(\ell_4) = h\mathrm{Cor}(\ell_1)$, ($S_4 = \{0\}$). Thus, $g_4 \equiv 0$. Therefore, system (15) up to order 4 has the form

$$(\dot{x}, \dot{y})^T = \mathbf{X}_h + (\alpha_2 y^2 + \alpha_3 h + \alpha_4 xh + \beta_4 yh)(x, y)^T, \tag{16}$$

with $\alpha_2, \alpha_3, \alpha_4, \beta_4$ real numbers.

From Theorem 2, if system (6) has an inverse integrating factor then $\mu_j = 0$ for $2 \leq j < 1 + 3s$, for some $s \in \mathbb{N}$. Thus, the coefficients α_2 and α_3 must be null. So, by applying the algorithm given in [1], it obtains that $\alpha_2 = 3b_{03} + a_{12}$. For $\alpha_2 = 0$, it has that $\alpha_3 = b_{03}(b_{12} + a_{21})$.
If $b_{12} + a_{21} = 0$, it is in case 1). In this case, system (6) is a Hamiltonian system; therefore, $f(h)$, being f any scalar function, is an inverse integrating factor of system (6).
Otherwise, $b_{03} = 0$ (it arrives to $a_{12} = 0$) and $b_{12} + a_{21} \neq 0$ (case b)). System (6) is an $\mathbb{R}_x$-reversible system. Performing the change of variables $x^2 = \frac{2\sigma}{b_{30}}(u + v)$, $y = v$ and re-parameterizing the time $dT = 2\sigma x dt$, the system is transformed into

$$\begin{aligned} u' &= \ v + \tfrac{1}{2}(b_{12}\sigma)u^2, \\ v' &= \ u + a_{21}\sigma uv + \tfrac{1}{2}(-b_{12}\sigma + b_{30} + 2a_{21}\sigma)u^2 + \tfrac{1}{2}(b_{30}a_{03})u^3. \end{aligned}$$

The result follows because any perturbation of a weak saddle has an inverse integrating factor, see [4, Theorem 6].

We see the second part. System (6) for case 1) is a Hamiltonian system and it is analytically integrable. We focus in the case 2). In such a case, the coefficient of xh of the normal form is $\alpha_4 = b_{30}(b_{12} + a_{21})(2a_{21}\sigma + b_{30} - b_{12}\sigma)$. At this point, we distinguish two cases:

- Case $b_{30} = 0$. If $b_{12} \neq 0$, system (6) has the inverse integrating factor W^A being $W = 1 - \frac{1}{2}(b_{12}\sigma)y$ and $A = \frac{2}{b_{12}}(a_{21} + b_{12})$. Otherwise, $b_{12} = 0$, $e^{a_{21}\sigma y}$ is an exponential inverse integrating factor. Thus, system is analytically integrable for both cases because the inverse integrating factor is non-null at the origin.
- Case $b_{30} = (b_{12} - 2a_{21})\sigma \neq 0$. Ii is easy to check that it has the complex inverse integrating factor W^A with

$$\begin{aligned} W = 1 - \frac{1}{A}(a_{21} + b_{12})\sigma y + \frac{1}{2A}(2a_{21}^2 + a_{21}b_{12} - b_{12}^2)\sigma x^2 \\ - \frac{1}{2A^2}((A-1)a_{21}^2 + (A-2)a_{21}b_{12} - b_{12}^2), \end{aligned}$$

where A is a solution of the equation

$$(-\sigma a_{03}b_{12}+2\sigma a_{03}a_{21}+a_{21}b_{12})A^2-(a_{21}+b_{12})A+(a_{21}+b_{12})^2=0,$$

whose discriminant is $\Delta=-(a_{21}+b_{12})^2[4\sigma(2a_{21}+b_{12})a_{03}-(a_{21}+b_{12})^2]$.

We note that A is non-null and it can be a complex number. As the field is real, both the real and imaginary parts of W^A are also real inverse integrating factors of system (6) and non-null at the origin.
Notice that if $b_{30}(b_{30}-\sigma(b_{12}-2a_{21}))\neq 0$, the normal form has dissipative terms and from [2, Theorem 3.19], the system is actually not integrable. $\square$

Proof of Theorem 4.
System (7) is a system (3) with $\mathbf{t}=(1,2)$, $r=3$, $h=-\frac{1}{3}y(y^2+3ax^2y+3\sigma x^4)\in\mathcal{P}_6^{(1,2)}$. The polynomial h has only simple factors on $\mathbb{C}[x,y]$ if and only if $3a^2\neq 4\sigma$. The co-ranges of ℓ_j, $j=4,5,6,7,8,9$ are

$$\begin{array}{lll}\mathrm{Cor}(\ell_4)=\mathrm{span}\{x^2y,\ y^2\}, & \mathrm{Cor}(\ell_5)=\mathrm{span}\{xy^2\}, & \mathrm{Cor}(\ell_6)=\mathrm{span}\{x^2y^2,\ h\},\\ \mathrm{Cor}(\ell_7)=\mathrm{span}\{xh\} & \mathrm{Cor}(\ell_8)=\mathrm{span}\{x^2h,\ yh\}, & \mathrm{Cor}(\ell_9)=\mathrm{span}\{xyh,\ x^3h\}.\end{array}$$

So, from Theorem 10, in this case $n_0=4$, system (7) is orbitally equivalent to system (5). A normal form of system (7) up to order 7 is

$$(\dot{x},\dot{y})^T=\mathbf{X}_h+(\alpha_4x^2y+\beta_4y^2+\alpha_5xy^2+\alpha_6x^2y^2+\beta_6h+\alpha_7xh)(x,2y)^T, \quad (17)$$

with $\alpha_4,\beta_4,\alpha_5,\alpha_6,\beta_6$ and α_7 real numbers.

We prove the sufficient condition. From Theorem 2, $\mu_j=0$ for $3\leq j<3+6s$, for some $s\in\mathbb{N}$. In particular, $\alpha_4,\beta_4,\alpha_5,\alpha_6,\beta_6$ and α_7 must be zero. The first coefficients are $\alpha_4=-2ad_{40}+d_{21}\sigma$ and $\beta_4=d_{02}\sigma-d_{40}$. Taking $d_{02}=\sigma d_{40}$ and $d_{21}=2a\sigma d_{40}$, the next coefficient α_5 is $\alpha_5=d_{40}\left(3c_{13}-\sigma ac_{32}+(a^2-\sigma)c_{51}\right)$.

So, if $d_{40}=0$, it is case 1). Otherwise, $3c_{13}-\sigma ac_{32}+(a^2-\sigma)c_{51}=0$ and $d_{40}\neq 0$, taking $c_{13}=-\frac{1}{3}(a^2c_{51}-c_{32}\sigma a-c_{51}\sigma)$, the following coefficients are

$$\begin{aligned}\alpha_6 &= d_{40}(ac_{51}-c_{32}\sigma)[(a^2-6\sigma)(ac_{51}-c_{32}\sigma)+14(a^2-\sigma)c_{70}],\\ \beta_6 &= d_{40}(ac_{51}-c_{32}\sigma)[(a^2-10\sigma)c_{51}-c_{32}\sigma a+14ac_{70}-4d_{40}\sigma].\end{aligned}$$

If $ac_{51}-c_{32}\sigma=0$, it is case 2).
Otherwise, $ac_{51}-c_{32}\sigma\neq 0$ and $(a^2-6\sigma)(ac_{51}-c_{32}\sigma)+14(a^2-\sigma)c_{70}=0$ (observe that, $a^2-\sigma\neq 0$ since, otherwise, $a^2=\sigma$ and, consequently $a^2-6\sigma=0$, which does not hold). Imposing $\alpha_6=\beta_6=0$, it obtains that $c_{70}=\frac{(a^2-6\sigma)(\sigma c_{32}-ac_{51})}{14(a^2-\sigma)}$ and $c_{51}=-\frac{2}{5}d_{40}-\frac{a(\sigma c_{32}-ac_{51})}{2(a^2-\sigma)}$.

Under these hypotheses, the next coefficient α_7 is zero if it holds

$$\begin{aligned}&-4(a^2-\sigma)^2d_{40}^2-5a\sigma(\sigma c_{32}-ac_{51})(2a^2-3\sigma)(a^2-\sigma)d_{40}\\ &\qquad+50(\sigma c_{32}-ac_{51})^2(a^6-3a^4\sigma-18a^2+27\sigma)=0,\end{aligned}$$

i.e., it gets case 3).

We see the necessity. If it imposes condition 1), system (7) is a Hamiltonian system and consequently, the system has an inverse integrating factor.

System (7) for case 2) and $d_0 \neq c_{51}$ has the inverse integrating factor $W = V^A$, with $V = 1 + \sigma(d_{40} - c_{51})x$ and $A = \frac{7d_{40}}{d_{40}-c_{51}}$. Otherwise, $exp(-7\sigma c_{51}\sigma x)$ is an inverse integrating factor.

If condition 3) is satisfied, there is an inverse integrate factor of the form V^A, with

$$\begin{aligned} V &= 1 + \alpha_{10}x + \alpha_{20}x^2 + \alpha_{01}y + \alpha_{30}x^3 + \alpha_{11}xy + \alpha_{40}x^4 + \alpha_{21}x^2y + \alpha_{02}y^2, \\ A &= \frac{15d_{40}(a^2-\sigma)}{5a\sigma(2a^2-3\sigma)(\sigma c_{32} - ac_{51}) + 8(a^2-\sigma)d_{40}}, \end{aligned}$$

(we have omitted the expressions of α_{ij} because they are too long), under the condition $5a\sigma(2a^2-3\sigma)(\sigma c_{32} - ac_{51}) + 8(a^2-\sigma)d_{40} \neq 0$. Otherwise, we find an exponential integrating factor $exp(\frac{5(2a^2-3\sigma)a(\sigma c_{32}-ac_{51})}{128(a^2-\sigma)^2}V)$, where V is

$$V = a(2a^2 - 7\sigma)(\sigma c_{32} - ac_{51})x^2 + 16(a^2-\sigma)x - 8(a^2-\sigma)(\sigma c_{32} - ac_{51})y.$$

□

Acknowledgments. This work has been partially supported by *Ministerio de Ciencia y Tecnología, Plan Nacional I+D+I* co-financed with FEDER funds, in the frame of the project MTM2010-20907-C02-02, and by *Consejería de Educación y Ciencia de la Junta de Andalucía* (FQM-276 and P12-FQM-1658).

References

[1] Algaba, A.; Freire, E.; Gamero, E.; García C. *An algorithm for computing quasi-homogeneous formal normal forms under equivalence.* Acta Applicandae Mathematicae, **80**, (2004), 335-359.

[2] Algaba, A., Gamero, E., García, C. *The integrability problem for a class of planar systems.* Nonlinearity, **22**, (2009), 2, 395-420.

[3] A. Algaba , C. García and M. Reyes. *I*ntegrability of two dimensional quasi-homogeneous polynomial differential systems. Rocky Mountain Journal of Mathematics, **41**, 1, (2011), 1-22.

[4] A. Algaba , C. García and M. Reyes. *Existence of an inverse integrating factor, center problem and integrability of a class of nilpotent systems.* Chaos, Solitions and Fractals, **45**, (2012), 869-878.

[5] J. Chavarriga; H. Giacomini; J. Giné; J. Llibre, *On the integrability of two-dimensional flows.* J. Differential Equations **157**, (1999), 163-182.

[6] A. Enciso and D. Peralta-Salas. *Existence and vanishing set of inverse integrating factor for analytic vector fields.* Bull. London Math. Soc. **41**, (2009), 1112-1124.

[7] I. García, H. Giacomini, M. Grau. *The inverse integrating factor and the Poincaré map,* Trans. Am. Math. Soc., **362**, 7, (2010), 3591-3612.

[8] I. García, H. Giacomini, M. Grau. *Generalized Hopf Bifurcation for planar vector fields via the inverse integrating factor,* J. Dyn. Differ. Equat., (2011), **23**, (2), 251-281.

[9] I. García; M. Grau, *A survey on the inverse integrating factor.* Qual. Theory Dyn. Sist., **9**, (2010), 1-2, 115-166.

[10] I. García; D. Shafer, *Integral invariants and limit sets of planar vector fields.* J. Differential Equations **217**, (2005), 2, 363-376.

[11] H. Giacomini, J. Llibre, M. Viano. *On the nonexistence, existence and uniqueness of limit cycles.* Nonlinearity, **9**, (1996), 501-516.

[12] H. Giacomini, J. Llibre, M. Viano. *On the shape of limit cycles that bifurcate from Hamiltonian centers.* Nonlinear Anal., (2000), **41**, 523-37.

[13] H. Giacomini, J. Llibre, M. Viano. *On the shape of limit cycles that bifurcate from non-Hamiltonian centers.* Nonlinear Anal., (2001), **43**, 837-859.