ZDM – Mathematics Education (2024) 56:1123–1135
https://doi.org/10.1007/s11858-024-01605-8

ORIGINAL PAPER

The MTSK model as a tool for designing tasks for teacher education

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Accepted: 11 June 2024 / Published online: 28 June 2024

Abstract
This paper studies the knowledge developed by a group of sixty-two pre-service primary teachers during a training session on the definition of a polygon and the mathematical practice of defining. We explore the knowledge developed by pre-service teachers when they carry out a series of training tasks oriented around the practice of defining with primary pupils, within the theoretical framework of the Mathematics Teachers' Specialised Knowledge model, and using videos of an authentic lesson as prompt. Data were collected by means of video recordings of the implementation of the training tasks, in conjunction with the pre-service teachers' own written observations as they watched the recorded lesson. A content analysis was carried out on the collected data using the Mathematics Teachers' Specialised Knowledge model. The results highlight the way in which knowledge of the mathematical practice of defining is constructed, along with the mathematical objects involved in this practice, and pedagogical aspects of defining with young pupils. They also illustrate how a sequence of tasks oriented around a model of teachers' knowledge can contribute to the development of this knowledge in initial training programmes, and more especially provide orientation about the training of pre-service teachers in the mathematical practice of defining.

Keywords Tasks · Teacher education · Teachers' knowledge · Defining · Polygon

1 Introduction

It is widely recognised that pre-service mathematics teachers face a double discontinuity: first, the transition from school to university in terms of the kind of mathematics studied; and second, the return to the classroom in the role of teacher (Klein, 1945). From this discontinuity there emerges the need to distinguish between school mathematics, with an emphasis on intuition and contextual relationships, and academic mathematics, which foregrounds logical reasoning and systematicity (Dreher et al., 2018). Mathematical practices such as demonstrating and defining are important elements of the primary syllabus (e.g. Melhuish et al., 2020; El-Mouhayar, 2019; Kobiela, 2012), and as such merit attention from the research community (Kobiela & Lehrer, 2015). Of particular interest in this respect are teachers' knowledge of the practice of defining and the focus on the construction of such knowledge (Kobiela, 2012). Although it is widely accepted that teachers should have a good mathematical basis in defining (Leikin & Winnicki-Landman, 2001), there are few studies into the pedagogical content knowledge associated with the process of defining (Kobiela et al., 2023).

There is, then, much to be gained from the use of models of professional knowledge which demonstrate a profound grasp of mathematical practices such as defining, especially those illustrating the combination of mathematical knowledge and pedagogical content knowledge (Delgado-Rebolledo & Zakaryan, 2020). This study starts from the supposition that it is possible to structure the content of teacher education programmes through use of such a model of professional knowledge by means of generating tasks in which the focus is placed on the development of different types of knowledge.

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This studies analyses the development of the knowledge of mathematical definitions displayed by a group of pre-service primary teachers (PPTs) as they carry out two specially designed tasks. The tasks, oriented within the Mathematics Teachers' Specialised Knowledge model (MTSK, Carrillo et al., 2018), take the analysis of classroom practice as their starting point, and foreground the PPTs' development in terms of both their knowledge of how mathematical definitions are constructed in relation to the concept of polygon (recognised as elements of mathematical knowledge in the MTSK model), and their pedagogical content knowledge of the practice of defining. The study aims to contribute to filling the perceived gap in research into incorporating knowledge of the mathematical practice of defining into teacher education programmes. As such, it aims to answer the following question: What knowledge (in terms of the practice of defining, the concept of polygon, and pedagogical content knowledge) do a group of pre-service teachers develop in the course of carrying out a sequence of tasks designed within the MTSK model and focusing on constructing definitions in a primary school context?

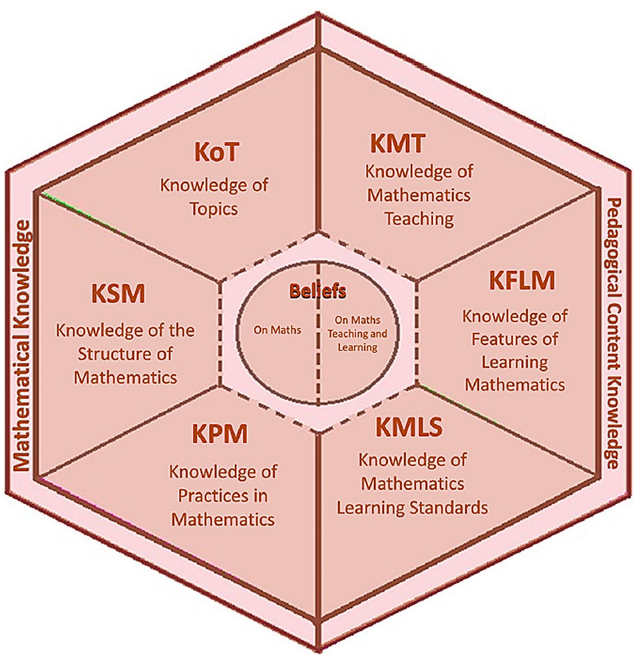

Fig. 1 The mathematics teacher's specialised knowledge model (Carrillo et al., 2018). The subdomains above are designed to be comprehensive. The domain of Mathematical Knowledge includes: (i) knowledge of the concepts and procedures to be taught, their properties and foundations, and contexts outside mathematics associated with a given topic, as well as registers of representation (all of which comprise the subdomain Knowledge of Topics – KoT); (ii) knowledge of relationships between concepts, that is, the connections between elements pertaining to different topics (a subdomain referred to as Knowledge of the Structure of Mathematics – KSM); and (iii) syntactic knowledge (Schwab, 1978), comprising knowledge of mathematical practices such as defining and demonstrating (the subdomain Knowledge of Practices in Mathematics – KPM). For its part, following Shulman (1986) and later contributions, Pedagogical Content Knowledge is divided into: (iv) knowledge of elements which help to guide mathematics teaching, such as teaching resources, teaching theories, and tasks for teaching mathematics (Knowledge of Mathematics Teaching – KMT). (v) knowledge of how learners learn mathematics, both personal beliefs as well as established theories of learning, including knowledge of areas of difficulty and strengths, and learners' emotional responses to learning mathematics (Knowledge of the Features of Learning Mathematics – KFLM). (vi) knowledge of the educational frameworks (such as syllabuses) which are instrumental in determining what and how the students should learn (Knowledge of Mathematics Learning Standards – KMLS)

2 Mathematics teachers' knowledge in teacher education

Initial teacher education tasks should enable PPTs to bridge the gap between the mathematics they have learned and the mathematics they will be teaching (Zaslavsky & Leikin, 2004). Some tasks aim to help PPTs to extend their understanding of mathematical practices so as to incorporate pedagogically oriented mathematical practices (Wasserman et al., 2022).

The use of video-recorded lessons in teacher education can boost the PPTs' capacity for self-reflection, and lay sound theoretical foundations (Vondrová, 2019). For the purposes of reflection, it is useful to have available some kind of analytical framework in order to maintain the focus on the essential aspects of the unfolding lesson and to avoid value judgements (Karsenty & Arcavi, 2017).

In our case, the framework we used for the systematic analysis of practice was the Mathematics Teachers' Specialised Knowledge model (MTSK, Carrillo et al., 2018). The model, based on the work of Shulman (1986) and Ball et al. (2008), aims to overcome the implicit limitations of these models (Silverman & Thompson, 2008) with regard to the delineation of knowledge domains. It does so first by realigning the notion of specialisation in terms of an intrinsic conceptualisation (Scheiner et al., 2019) in contrast to the extrinsic conceptualisation maintained by the other models. Secondly, the model takes an interpretative perspective, with a clear focus on understanding the knowledge used or developed at any point, rather than a focus on evaluating that knowledge (Carrillo et al., 2018). Regarding its analytical use, the MTSK model preserves the distinction between the two principal domains of content knowledge and pedagogical content knowledge. The MTSK model (Fig. 1) includes an important third domain, which brings together the teacher's beliefs about mathematics and the teaching and learning of mathematics (Carrillo et al., 2018).

This paper focuses on two subdomains of Mathematical Knowledge (KoT and KPM), and two from Pedagogical

Content Knowledge (KFLM and KMT) applied to the mathematical practice of defining.

3 Mathematics teachers' specialised knowledge of the mathematical practice of defining

The MTSK model provides a set of subdomains and categories which enables aspects of mathematics teachers' knowledge to be differentiated in detail. Nevertheless, application of the model to any particular topic requires a series of indicators that interface with the corresponding elements of knowledge. Below we set out the indicators for the topics and subdomains of interest to this study.

In the case of the specialised knowledge associated with the practice of defining, a key role is clearly played by the mathematical knowledge involved in this practice (KPM). The syntactic elements of this knowledge include notions of what constitutes the mathematical practice of defining, how a mathematical definition should be constructed, the purposes to which a definition can be put, and what form the definition should take.

Whether a definition is constructed on the basis of properties, or by drawing on previous definitions (descriptive and constructive definitions respectively, de Villiers, 1998), its validity depends on determining the set of necessary (or critical) and sufficient attributes to define the concept. Critical attributes are "those that need to be present in an instance to serve as an example of the defined concept" (Haj-Yahya et al., 2023, p. 609). Nevertheless, defining consists of more than solely proposing definitions. Kobiela and Lehrer (2015) note eight different aspects involved in the process: (a) proposing a definition; (b) constructing or evaluating examples and non-examples; (c) describing properties or relations; (d) constructing explanations or defining arguments; (e) reviewing definitions; (f) establishing and reasoning about systematic relations; (g) asking questions oriented towards defining; and (h) negotiating criteria for judging the adequacy or acceptability of definitions. The definition of a concept can be seen as the set of criteria by which instances can be adjudged to be examples or non-examples of the concept (Haj-Yahya et al., 2023). Examples, non-examples and counterexamples help to construct and validate definitions, and an understanding of this crucial role forms part of the teacher's syntactic knowledge of the practice of defining.

With respect to the characteristics required by a good definition, Winicki-Landman and Leikin (2000) provide a valuable summary: defining is giving a name; for defining the new concept, only previously defined concepts may be used; a definition establishes necessary and sufficient conditions for the concept; the set of conditions should be minimal; and, a definition is arbitrary. There is no consensus on minimality (Zaslavsky & Shir, 2005).

Knowledge of the different aspects involved in defining includes an understanding of its relationship to other mathematical practices, in particular, in the light of the above, to classifying. Defining is, of course, carried out in relation to a particular mathematical concept, in our case, a polygon. Among the elements included within the teacher's KoT in this respect are knowledge of a range of possible definitions, properties and critical attributes, along with examples and non-examples (Carrillo et al., 2018). The involvement of the pupils in generating, reviewing and evaluating definitions challenges their initial conceptual image of the mathematical object by encouraging them to consider new examples and non-examples (Zandieh & Rasmussen, 2010) with analytical rigour so as to identify their critical attributes. Further, the process of defining a polygon leads to the involvement of other related concepts such as side, angle and plane figure.

Theories of how mathematical content is learnt, and more particularly how students learn to define, along with the strengths and difficulties associated with doing so, fall within the subdomain of KFLM. Many studies have shown that students have difficulties regarding geometrical definitions and the roles these play in the structure of geometry (such as accepting as correct equivalent definitions, recognising as incorrect incomplete definitions, and perceiving the logic relationships between critical attributes of a concept, Haj-Yahya et al., 2023). These difficulties often arise from the students' concept image, a conceptual representation which includes all the examples, attributes and processes the learner associates with the concept in question (Vinner and Hershkowitz, 1980). Awareness of the relevance of the concept image in learning about a geometric concept, and knowledge of students' difficulties are constituents of the teacher's KFLM.

Finally, theories of teaching mathematical content, specifically definitions, and how to use definitions in classroom settings, are elements of KMT. Within the personal image of a mathematical concept certain examples typically have more weight than others and are thus considered as representative of the concept. These examples tend to be those to which the learner has had greatest exposure, often due to an over-reliance on their use in teaching situations (Hershkowitz, 1990). Awareness of the consequences of such an over-reliance on prototypical examples pertains to theories of teaching. With regard to teaching strategies, the study found that the PPTs favoured developing pupils' ability to define rather than providing them with definitions (de Villiers, 1998). Kobiela and Lehrer (2015) also found examples of teacher interventions aimed at developing students' capacity to define at the same time that they developed the underlying mathematical concepts: (i) asking students to participate in various aspects of practice; (ii) asking questions that serve to expand the mathematical system; (iii) modelling participation in aspects of practice; (iv) proposing (discordant) examples that generate disagreement; and (v) explicitly stating expectations of and purposes for participating in the practice. In a subsequent paper, Kobiela et al. (2023) review three types of activities to involve students in defining: (a) definitional sorting tasks, in which students classify examples and non-examples into categories, (b) evaluating tasks, in which students assess alternative definitions and non-definitions, and (c) authoring tasks, in which students write their own definitions. The authors make a distinction between counterexamples and non-examples, in that the former are non-examples deployed for the purposes of clarifying a definition.

4 Methodology

4.1 Teacher education tasks design and implementation

We carried out two tasks with the PPTs aimed at developing their specialised knowledge of polygons. The sessions took place during a course on teaching and learning geometry delivered in the final year of the degree in Primary Teacher Education, and were designed and delivered by a team of teacher educators at the University of Huelva which included the authors. The PPTs had not received any previous training in polygons.

The first task consisted of two video clips from a recording of a 5th grade lesson and had two phases. In the first phase, the PPTs watched the first clip and were invited to comment freely on what they had seen. They then filled in an observation sheet (see Table 1) devised around the MTSK framework, after which they used their notes as the basis for a whole group discussion. The aspects which the PPTs were invited to comment on corresponded to: Knowledge of the Features of Learning Mathematics, such as knowledge of the strengths and difficulties related to learning about polygons (aspect 1), Knowledge of Topics, such as knowledge of the critical attributes, definitions, and examples of polygons (2 and 5), Knowledge of Practices in Mathematics, specifically in this case the practice of defining (2), Knowledge of Mathematics Teaching, such as knowledge of activities, resources and examples for teaching polygons (3, 4 and 5), and Knowledge of Mathematics Learning Standards (6).

In the second phase, they watched the second clip and analysed it individually (again using the observation sheet). This was likewise followed by a group discussion.

Table 1 Observation sheet (authors' own elaboration)

Aspects to consider	What I observed	What I interpret from this
1. Pupils' thinking strategies and ideas. Intuitive ideas		
2. Subject content studied and placing of emphasis		
3. Type of activities		
4. Resources: potential and limitations of use		
5. Examples used. Representations of content		
6. Fit with syllabus		

Fig. 2 Final grouping of the geometric shapes in the primary lesson

In the first clip (8 min in length), the teacher asked pupils to come to the front of the class, take a cardboard plane shape at random from a bag, and decide which group to place it in. The pupils were told to sort the shapes into groups, but without any further instructions. The first shape out of the bag was a rectangle. The second shape was a right triangle, which was placed in the same group (on the premise that two right triangles can form a rectangle). The next shape, a sector of a circle, was unanimously deemed to belong to a different group (on the grounds that one of the edges was curved). This initial classification set the pattern for the final groupings (see Fig. 2). In the second clip, which followed on from the first (13 min in length), the teacher led the class in a joint construction of a definition of a polygon. This particular video clip was chosen for its potential in highlighting specialised knowledge regarding the practice of defining, and teaching and learning to define (Knowledge of Practices in Mathematics, Knowledge of Mathematics Teaching and Knowledge of Features of Learning Mathematics).

In the second task, the PPTs evaluated six definitions of a polygon given by the teacher educator (Chart 1) in terms of their precision, and suggested examples of shapes which

A polygon is ...

- *a)* *... a flat shape with an edge.*
- *b)* *... a continuous line with angles between its sections.*
- *c)* *... the plane area delimited by a broken line in which I can distinguish what is inside from what is outside.*
- *d)* *... a set of vertices and sides with an area.*
- *e)* *... the plane area delimited by a closed polygonal line which has the same number of vertices sides and angles.*
- f) *... an area delimited by a closed polygonal line, such that, given any two points on the line, the line segment joining them is always inside the area.*

Chart 1 Definitions of a polygon provided for Task 2

met the definition, and others that did not. The PPTs then gave their opinion as to whether these definitions were equivalent to a definition they had previously constructed, and discussed what conditions were lacking, or which were superfluous. We hoped the task would help the PPTs to develop their knowledge of the concept of a polygon and the practice of defining (Knowledge of Topics and Knowledge of Practices in Mathematics).

The combination of the two tasks aimed to situate the construction of specialised knowledge in an authentic classroom context which would promote PCK (in Task 1), and which would afterwards stimulate deeper reflections on the mathematical and epistemological elements relating to defining polygons (in Task 2).

4.2 Data collection and analysis

Data collection was carried out by video recording the group discussions of 4 groups of PPTs (251 PPTs in total), although for reasons of space we will consider the data corresponding to just one of the groups (62 PPTs). In addition to the video recordings of the group discussion in both tasks, further data was provided by the observation sheets completed by the 62 PPTs during the implementation of the first task.

Prior to the analysis of this data, we established categories of analysis for the identification of the knowledge that we anticipated the PPTs would demonstrate in the course of tasks 1 and 2. First we established a frame of reference provided by the MTSK model, which we refined with a literature review of the practice of defining (as summarised in the last paragraph of Sect. 3). This theoretical framework enabled us to pinpoint potential elements of learning which might arise from the lesson which had been selected. Each of the authors independently reviewed the two video clips of the lesson with a view to predicting the kinds of knowledge that the PPTs might be expected to display (Task 1), and we then met to pool ideas. It was at this point that we decided to add a second task aimed at further delving into the construction of knowledge concerning the practice of defining (Task 2). Subsequent analysis of the two tasks in terms of the previously established framework led us to target specific areas of knowledge for development (Table 2), as explained below.

We hoped that by analysing the construction of the definition of a polygon by the primary pupils, the PPTs would identify the critical attributes of the concept of a polygon, and that these would be further questioned when they came to review the definitions provided in Task 2. By retaining the focus on the lesson, allied to the demands of Task 2 to provide examples and non-examples of a polygon, we hoped that the PPTs would develop their knowledge of such examples (and non-examples), and the role these played in the practice of defining. The questioning of both the attributes and the examples of a polygon were intended to contribute to the development of their understanding of possible definitions of the concept. We considered that watching the recording of the primary lesson would foreground the fact that classifying and defining are deeply interrelated, illustrating how classifying can enable a definition to be attempted, and underlining the possibilities of tackling definitions at primary level (as opposed to giving the definition directly). It was expected that the analysis of the definition given in the lesson, and those provided in Task 2, would challenge the PPTs' knowledge of the characteristics of a definition and how to define key elements constituting a polygon (e.g. side and angle). We also believed that analysis of the examples of a polygon given by the teacher and the impact these had on the children's learning would drive home awareness of the importance of using varied examples, the consequences of not going beyond prototypical examples, and the role of the concept image in learning about a geometrical concept. Focusing attention on the children's contributions during the lesson would prompt the PPTs to consider their preconceptions about the difficulties involved in this kind of activity. The defining activity in the lesson and the critical assessment of definitions in Task 2 brought to the fore the mathematical aspects involved in the practice of defining, alongside the teacher's management of the activities and interactions for teaching this practice. The full list of anticipated knowledge is provided in Table 2.

Table 2 Areas of knowledge targeted by the task (authors' own elaboration)

		Task 1	Task 2
KoT	Critical attributes of the concept of a polygon	X	X
	Definitions of a polygon	X	X
	Examples and non-examples of polygons	X	X
	Definitions of concepts involved in the practice of defining a polygon: side, angle, plane figure		X
KPM	Relationships between classifying and defining (defining enables classification; conversely, the characteristics of the definition can be extracted from the classification)	X	
	Aspects involved in the mathematical practice of defining: proposing a definition; constructing and evaluating examples and non-examples; describing properties and relations; constructing explanations and defining arguments; reviewing definitions; establishing and reasoning about systematic relations; asking questions oriented towards defining; and negotiating criteria for judging the adequacy or acceptability of definitions	X	X
	Characteristics of a mathematical definition: defining is giving a name; only previously defined concepts may be used; necessary and sufficient conditions; minimal set of conditions; a definition is arbitrary	X	X
	Role of examples, non-examples and counterexamples in the construction of a definition	X	X
KMT	Use of prototypical examples (overuse restricts the pupils' concept image)	X	
	The potential of defining as a class activity (as opposed to giving an established definition)	X	
	Activities for involving pupils in defining: sorting, evaluating & authoring tasks	X	X
	Types of intervention promoting student development in the practice of defining: encouraging student participation in defining; asking questions aimed at expanding the mathematical system; modelling participation in defining; proposing (discordant) examples that generate disagreement; and explicitly stating expectations of, and purposes for, participating in defining	X	X
KFLM	Relevance of the concept image in learning a geometric concept	X	
	Primary pupils' difficulties with the mathematical practice of defining	X	

Once the recordings were transcribed, the PPTs' contributions, both spoken during the whole group sessions and written on their individual observation sheets, were subjected to a content analysis (Krippendorff, 2018). The purpose was to identify evidence of expressions which might indicate elements of knowledge corresponding to the targeted knowledge categories (Table 2). The content analysis was carried out in the first instance by two of the authors working independently, and then jointly discussed so as to produce a draft analysis, which was then revised by the other two authors until a final consensus was arrived at. The degree of agreement in this process of parallel analysis was very high, thus corroborating the researchers' triangulation (Flick, 2018). The following section provides a number of verbatim examples of the preservice teachers' contributions, and their assignation to the corresponding element of knowledge identified by the MTSK model.

5 Findings

5.1 Knowledge of mathematical practices

As will be seen below, the analysis of the training tasks provides evidence that the PPTs display knowledge of the association between classifying and defining, the characteristics of a mathematical definition, and the role of examples, non-examples and counterexamples in the process.

5.1.1 The relationship between classifying and defining

The analysis and discussion of the video clips (Task 1) enables the PPTs to discover that it is possible to classify flat shapes before knowing the definition of the concept of a polygon. In the discussion of the first clip, whilst the majority of the PPTs recognise that the lesson involved an activity based on "classifying" or "comparing", one PPT, Alberto, suggests that the definition can be constructed from the classification (Episode 1), as he had already mentioned on his observation sheet (Fig. 3). The identification of the class activity with classifying, on the supposition that the pupils are already familiar with the definition, suggests a recognition that knowing how to define enables classification. The addition of the inverse relationship, that classifying enables

What I observed	What I interpret from this
The main content is the mathematical definition of a polygon. The activity focuses on classifying different plane figures.	*Through this activity, students will be able to construct the definition of a polygon.*

Fig. 3 Excerpt from Alberto's observation sheet, video clip 1

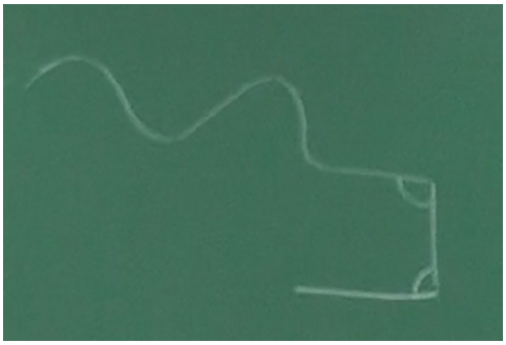

Fig. 4 An unclosed continuous line with angles

definition, means extending their knowledge of the *relationship between classifying and defining*.

Episode 1. Discussion of the primary pupils' activity (video clip 1).

> Alberto: *The pupils are learning to identify flat shapes from what is a polygon* […] *so they see that they all have straight sides and they have to have more than three sides in order to be a polygon, so the classification is polygons and plane shapes that aren't polygons.*
> Educator: *But they haven't done a definition of a polygon, have they?*
> Alberto: *Not directly, but the activity could be directed towards that.*

5.1.2 Characteristics of a mathematical definition

In general, as Episode 1 illustrates, the PPTs recognise that a definition serves to differentiate what something is from what it is not, and that it consists in identifying critical attributes, which for some of the PPTs is associated with identifying patterns and regularities. Alberto notes the attributes that all polygons should have – the critical attributes, such as having straight sides – and that these attributes allow polygons to be distinguished from other plane shapes which are not polygons. In this way, the PPTs demonstrate awareness that *defining is giving a name*. This recognition seems to be reinforced by the analysis of the video clip, as can be seen in the PPTs' written observations.

The need to identify critical attributes is associated with the knowledge that *a definition establishes the necessary characteristics for the concept in question*. In addition, when they analyse the definitions given in Task 2, the PPTs connect the accuracy of the definitions to there being a set of sufficient characteristics (Episode 2). The shortcomings they note are due to incompleteness or insufficiency, meaning that conditions are lacking. Hence, across the two tasks, the PPTs recognise *the need for a definition to establish the necessary and sufficient conditions* of a concept.

Episode 2. Discussing the definition "a polygon is a continuous line with angles between its sections" (task 2).

> Jazmine: *It is a bit more precise than the previous one but it is still lacking something. It has vertices, it has angles, but it is open. We can draw a continuous line which has angles but which isn't closed* (She draws the line in Fig. 4 on the blackboard).

Some of the PPTs question whether there is a single correct definition of a polygon, which raises the notion of *arbitrariness*. After watching the video clips, several PPTs observe that the primary pupils construct "their own definition". This awareness of the arbitrary nature of a mathematical definition, can be seen more clearly in Task 2. In the course of this task, the PPTs discuss the definitions of a polygon that they themselves have constructed, and accept that various different ones can be valid, possibly on the premise that they can be equivalent, thus reinforcing the notion that a definition need not be unique.

Some of the PPTs recognise the lack of minimalism in the definition constructed during the lesson ("all the shapes have straight sides, they have angles, they have vertices, they don't have any curves, and are plane shapes, and all the sides must be joined at both ends"). In this respect, they would seem to recognise that minimalism is a desirable mathematical attribute, but at the same time accept that pedagogical concerns means it can be relaxed (Fig. 5).

It can be noted that in their exchanges on the question of minimalism, the PPTs couch their knowledge of the practice of defining in terms relating to aspects of Pedagogical Content Knowledge. Thus, their focus during the analysis of the video clips remains on a school-based practice of defining throughout, centring above all on what it means for the pupils. However, when in Task 2 the PPTs give their own definitions and discuss their precision, they dispense with pedagogical premises and instead consider what necessary characteristics might be missing, that is to say, the inadequacy of the set of given characteristics (Episode 2 above). On the other hand, the definitions of a polygon given by the PPTs themselves in Task 2 (before the analysis of the given

There are redundancies in the definition. The pupils need this redundancy in order to strengthen [their grasp of] the definition.

Fig. 5 Excerpt from Luisa's observation sheet, video clip 2

definitions) are not generally speaking minimal, underlining that for them this is not an important condition.

5.1.3 The role of examples, non-examples and counterexamples

The PPTs demonstrate awareness that both examples and non-examples are necessary elements in the process of defining, facilitating the identification of critical attributes (Episode 3). This awareness is first demonstrated (Task 1) when they note the role played by examples, and particularly the lack of non-examples which might help the pupils to identify certain critical attributes, and is related, as we will see below, to the characteristics of the set of examples provided by the teacher.

Episode 3. Discussing non-examples.

> María: *I also think that as they only have the visual representation of polygonal and non-polygonal shapes, and because both are plane, it doesn't occur to the pupils to say that they are plane shapes… later on, for example, they have to draw a shape with one side that isn't closed, so they lack any reference to other shapes to be able to complete the definition.*

In the PPTs' analysis of the given definitions (Task 2) a *counterexample* becomes the main tool for arguing a point (thus involving the PPTs in the deployment of definitional arguments) (Fig. 4 in Episode 2). The example illustrates awareness of the use of counterexamples, although this is not made explicit by the PPTs. This use of both non-examples and counterexamples by the teacher is noted by the PPTs in their observation of the video clips (Episode 4), which they may have taken as a model of the strategy.

Episode 4. Discussing the school teacher's use of counterexamples.

> Ana: *The teacher rebuts what the pupils say, using examples to contradict them*.

Likewise, in Task 2 the PPTs engage in a discussion of *aspects of the mathematical practice of defining with primary pupils*: proposing a definition, constructing or evaluating examples and non-examples, constructing defining arguments, and reviewing definitions. However, as there is no explicit mention by the PPTs that these elements are involved in the practice of defining with primary pupils, it is not possible to say that they demonstrate knowledge pertaining to this category.

5.2 Knowledge of the concept of polygon

Over the course of the two tasks the PPTs demonstrate knowledge of the *critical attributes of the concept of a polygon*, by virtue of their analysis of the examples given in the lesson, their observation of the difficulties encountered by the pupils in constructing a definition, and their analysis of the definitions given in Task 2. This can be seen when they consider the plane shapes the pupils work with in the lesson. To the attribute employed by the pupils to differentiate polygons from non-polygons (straight sides), they add a critical attribute (polygons must have more than three sides – Episode 1). They also note other non-critical attributes, which include regularity, convexity and the number of sides (stating that polygons can be regular or irregular, concave or convex, and have a different number of sides). Prompted by their recognition of the difficulties facing the pupils, the PPTs highlight some of the other necessary attributes of a polygon: "plane shape" and "closed" (Episode 3), and including "an interior" (Episode 5). In the first task we can again see how the demonstration of Pedagogical Content Knowledge, in this instance with respect to examples for teaching the concept of a polygon and concomitant pupil difficulties, fosters the deployment of Mathematical Knowledge.

Episode 5. Interior of a polygon.

> Carmen: *I've put that the difficulties they face were that the pupils didn't know whether a polygon was just the interior or the exterior.*

The discussion about the definitions given in Task 2 leads the PPTs to bring to bear their knowledge of critical attributes at a more advanced level of mathematics than in Task 1. Prompted by an example given by the teacher educator (See Fig. 6), one PPT raises the question of whether a polygon must be comprised of a single connected component (Episode 6). In the ensuing discussion, the teacher educator raises the question of what is the interior and exterior of the shape. This question leads the PPTs to discard as a polygon a shape with more than one connected component.

Episode 6. Elements associated to polygon.

> Melisa: *But this shape can't be cut out of card because however finely you cut it you always have a space in the centre.*
> […]
> E: *In this shape, what is inside the broken line?*
> Melisa: *This broken line might make us think that what is excluded is the inside part of the triangle above, but*

Fig. 6 A complex polygon

when the line comes this way and cuts across the other one we might think instead that the inside part is what is outside this triangle underneath.
[…]
E: *But is it one polygon as a single unit?*
Melisa: *Both are inside, but they're separate* […] *If there are two things inside, it can't be.*

With respect to the deployment of *knowledge regarding definitions of a polygon*, the PPTs, as mentioned above, subscribe to the institutionalized primary level definition. The definition they provide (at the start of Task 2) is almost identical to that constructed in the primary lesson, thus underlining their tacit consideration of it as adequate, and/or their assumption that the task was situated in a classroom context. Although it is clear that the second task presented PPTs with different definitions of a polygon, featuring different sets of necessary and sufficient properties, there is no evidence of the kind of definition they might construct after completing the two tasks.

With regard to the use of examples and non-examples, in the discussion of the video clips the PPTs appear satisfied with those of each type mentioned in the lesson, and offer no further types. Although in their individual written texts analysing the first video clip some of the PPTs consider the circular sector as an example of a polygon, this does not occur in the written analysis of the second clip. This seems to suggest that as the task progresses, the PPTs problematize their concept images of a polygon and refine their *knowledge of example and non-examples*. This development can be seen more clearly in the second task, in which the examples, and especially the non-examples, are considerably more elaborate, in particular the "monster" or "extreme" examples (Fig. 4).

Both the definition constructed during the lesson and the PPTs' discussion of the definitions provided in Task 2 lead them to engage with *concepts associated with a polygon*, and so add to their knowledge of the definitions of these concepts. To this end, they consider the concepts of a side (whether forming part of a straight line is a critical attribute, as prompted by the affirmation that "a side has to be straight" during the construction of the definition in the lesson), the interior of a shape (Episode 6) and angles (Episode 7). The example below provides a brief illustration of how the PPTs engage with this knowledge.

Episode 7. Concept of angle.

Ángela: (referring to Fig. 6 on the board) *But to make an angle can the shape be open?*

It seems that for some PPTs, the notion of angle is associated with polygons, prompting the question of whether non-polygonal plane shapes can have angles. This represents an extension of the notion of angle in terms of analysis of its basic characterisation. To an extent, the discussion of what constitutes a polygon undermines the PPTs' definition of side, angle and interior.

5.3 Pedagogical content knowledge of the concept of a polygon and the mathematical practice of defining

In this section we note how the PPTs are challenged to consider their knowledge regarding constructing a definition as a class activity rather than being given a pre-packaged definition, the importance of examples in learning about polygons and the possibility that unsuitable examples might result in a restrictive concept image of the concept, and the most likely difficulties primary pupils will encounter in the practice of defining.

5.3.1 Constructing a definition versus applying a given definition

One of the issues which seems to have most impact on the PPTs is that the lesson revolves around constructing a definition of a polygon. In this respect, the session challenges the PPTs' awareness of defining as a class activity, and the potential inherent in this approach, as opposed to simply giving pupils an established definition. In the discussion of the first video clip several PPTs seem not to appreciate the difference between asking pupils to come up with a definition and providing one ready-made, and fail to see why so much time is spent on the classifying activity (Episode 8).

The teacher works with the students on the classification of plane geometric figures. Polygons and their types, identifying their characteristics. Emphasis is placed on the immediate classification of polygons belonging to the same group and to the opposite group. Polygonal and non-polygonal figures are distinguished in terms of their sides.	*The contents that are worked on are the classification of plane geometric figures, along with the creation of the definition of a polygon in a group. The teacher does not give the definition as he gives more importance to the construction of the knowledge itself. The students name the characteristics that will allow them to identify them.*
Observation notes on video clip 1	Observation notes on video clip 2

Fig. 7 Excerpt from the observation sheet by Isabel, video clips 1 and 2

What I observed	What I interpret from this
Regular and irregular polygons, concave and convex figures are used. [The materials] *are manipulable. But you can never give all the examples, there can be problems with generalizing.*	*The teacher wants to have as many examples as possible, so as to get a variety of characteristics.*

Fig. 8 Excerpt from the observation sheet by Miguel, video clip 1

Episode 8. Suitability of the activity for 5th grade (video clip 1)

> Miguel: *For me, this is something I'd do two years before. And the activity strikes me as completely beginner, because that first big classification task is the most basic thing to do.*

Although according to their notes for the first video clip, the majority regard the activity as a way to consolidate or review the concept, the PPTs' attitude appears to shift by the end of Task 1, and some voice their reconsideration of the complexity inherent in the activity and in defining (Episode 9). The shift seems generalised among the PPTs, as in their notes on the observation sheets for the second clip, the majority identify the aim of the activity as gradually building up "the definition or concept of a polygon", showing awareness of the practice of defining at primary level (e.g., Fig. 7).

Episode 9. Suitability of the activity for 5th grade (video clip 2).

Sergio: *To me it does seem to be appropriate because eliciting a definition really is something appropriate for fifth year primary. It's not something that can be easily done in previous years.*

5.3.2 Prototypical examples and concept image

A large proportion of the KMT displayed by the PPTs occurs in reference to the examples that the teacher gives and the materials provided for the pupils (cardboard shapes). In this respect, they show awareness of the importance of the set of examples selected to aid comprehension of a concept, and how poorly chosen examples might act to restrict comprehension (*overuse of prototypical examples*) (Fig. 8). Underlying this reflection is a recognition of the significance of examples in constructing a mental image of a concept. In particular, the PPTs recognise the importance of taking care not to provide examples which add an unnecessary condition, such as position (Episode 10).

Episode 10. Advantage of the materials.

> Sandra: *You can see other kinds of shapes, which if you see in a book you can't move them or you can't handle them.*

5.3.3 Primary students' difficulties in the mathematical practice of defining

Watching the video clips leads the PPTs to recognise that the practice of defining at primary level is more complex than they had anticipated, as it encompasses identifying all the characteristics that enable pupils to differentiate what something is from what it is not (Episode 11).

Episode 11. The complexity of defining at primary level.

> Jazmine: *I've seen that supposedly they should have done this last year, but they had a lot of difficulty finalising what a polygon is and all its characteristics.*
> Rosa: *What they find difficult is listing all the characteristics which include certain shapes and exclude other shapes.*

The findings presented in the previous subsections (5.3.1, 5.3.2 and 5.3.3.) correspond to data collected from Task 1.

In Task 2 there are no reflections that explicitly demonstrate the use of PCK, although the teacher educator's management, as well as the task itself, could serve as models for the kind of activities which involve the pupils in the mathematical practice of defining, and demonstrate how the PPTs might intervene as teachers in order to encourage this. The task includes two of the activities which Kobiela et al. (2023) recommend, one in which the PPTs assess alternative definitions and non-definitions, and one in which they write their own definitions (evaluating and authoring tasks, respectively). Nevertheless, there is no evidence that the PPTs demonstrate explicit recognition of these activities and types of interaction, and we found just one such explicit mention, which referred to how the teacher suggested (discordant) examples that generate disagreement (Episode 4).

6 Discussion and conclusions

Models of teachers' professional knowledge have been used largely as a means of evaluating this knowledge (e, g, Ball et al., 2008). This paper brings to the fore the possibility of a complementary approach involving the planning, implementation and evaluation of teacher education tasks with prospective primary teachers. In this respect, the MTSK model constitutes a powerful tool for researchers and teacher educators in structuring tasks for teacher education (Carrillo et al., 2020), taking the conceptualisation of training plans based on the model as pre-service.

We have shown how, in the course of carrying out a task framed by the MTSK model, the PPTs explored their specialised knowledge. They broadened their mathematical knowledge of polygons and the mathematical practice of defining, and also developed their knowledge of the teaching and learning of these two elements. They recognised that while definition enables shapes to be classified, classification conversely participates in the process of defining, thus developing a deeper understanding of the complementary relationship between the two procedures. In the course of these sessions, the PPTs collectively constructed knowledge of the critical attributes of a polygon, and at the same time developed their awareness that certain attributes are not essential for the definition itself, but rather enable subsequent groupings among the different shapes considered as polygons. This was a key step to identifying the necessary and sufficient characteristics for making a definition. These results concur with previous studies about PPT' perceptions of the characteristics of a mathematical definition (Linchevski et al., 1992), with minimalism turning out to be the one given least weight by the PPTs in our study. This allocation of diminished importance results from their reasoning on the basis of their PCK. In addition, they showed awareness that definitions are arbitrary (Haj Yahya et al., 2019), that they can be constructed and hence revised (Kobiela et al., 2023), and that different definitions can be compared and their equivalence discussed. Likewise, the PPTs developed their recognition of the role of examples and non-examples in the process of constructing a definition in the classroom, and even the role of counterexamples as a contrastive device.

The second task also allowed the PPTs to "unpack" (Ma, 1998) concepts associated with polygons, deepening in their knowledge of the topic, epistemologically and mathematically, as they progressed from the first to the second task. This sequence enabled the tasks to be situated in a realistic context. This in turn allowed the PPTs to observe the three activities that are instrumental in involving students in the practice of defining (Kobiela et al., 2023). At the same time, the eight aspects of definitional practice listed by Kobiela and Lehrer (2015) could be seen to be brought into play.

When the PPTs watched the video, they focused mostly on pedagogical issues. This is consistent with Kobiela et al. (2023), albeit with an interesting difference, namely that in our study the PPTs also considered how the pupils responded to the content. This could be a reflection of the distinct research designs. In our study, the PPTs' attention was particularly drawn to the pupils' thinking, and above all their difficulties in learning, leading them to take on the role of teacher when faced with an authentic lesson sample, as opposed to other perspectives which favour the role of student (Kobiela et al., 2023). This potential of the use of video has been noted in previous studies (Mitchell & Marin, 2015). When the PPTs are discussing these difficulties they consider the mathematical content encompassed by the concept of polygon. In other words, development of Knowledge of Topics basically occurs through use of Knowledge of Features of Learning Mathematics. Although this subdomain is developed, greater emphasis is given to questions concerning Knowledge of Mathematics Teaching, in that the discussion raises ideas about how to approach this content area, and about the potential of the material to overcome the tendency of textbooks to illustrate concepts with prototypical pictures by underlining the importance of a suitable set of examples for teaching the concept. With respect to Knowledge of Mathematics Teaching, the activity is noteworthy for its impact on distinguishing between the mathematical practice of defining and giving a classroom definition, a distinction which could be instrumental in bringing about a change in the way definitions are approached in education, something which has been advocated for decades (de Villiers, 1998). We believe the impact was prompted by viewing this practice in an authentic primary classroom. Further, the aspects for consideration provided on the observation sheet, drawn from MTSK subdomains, helped to guide the PPTs towards what seemed like a discovery. At the same time, the choice of an appropriate range of examples (productively controversial) is essential in order to challenge established mental images of the concept (Kobiela & Lehrer, 2015). In Task 2, the examples of polygons, and especially non-polygons, increase in complexity, as the PPTs consider a richer range of images, which go much further than those shown in the primary lesson.

This study has also enabled us to identify some limitations. Although analysis of the PPTs' contributions allowed certain elements of knowledge to be identified, the system of data collection was unable to identify elements pertaining to other categories of knowledge. Despite this shortcoming, we think it is likely that the teacher educator's interventions during the sessions led the PPTs to reflect on the issues raised in a meaningful way, and in future studies we intend to expand the role of teacher educator (Appova & Taylor, 2019).

The scope of the research presented here, focused on the notion of polygon, can we believe be broadened. Our approach is appropriate, for example, to studying the knowledge underpinning the practice of defining other mathematical objects, or to other mathematical practices associated with polygons, or indeed to other mathematical practices in general. Given the scarcity of research of this type (Kobiela et al., 2023), more studies are needed which, along the lines of the work presented in this paper, tackle the issues involved in constructing knowledge about teaching and learning how to define (Kobiela, 2012), along with other mathematical practices (Delgado-Rebolledo, & Zakaryan, 2020).

Acknowledgements Project PID2021-122180OB-I00 (Government of Spain). Project ProyExcel_00297 (Andalusian Government). COIDESO research centre (University of Huelva). DESYM (HUM-168). MTSK network, sponsorized by the AUIP.

Funding Funding for open access publishing: Universidad de Huelva/CBUA

Declarations

Conflict of interest The authors declare no competing interests.

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