Analytical integrability of perturbations of quadratic systems

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February 19, 2024

AMS Mathematics Subject Classification: 34C20, 34C14

Abstract

We consider analytic perturbations of quadratic homogeneous differential systems having an isolated singularity at the origin. We characterize the systems with an analytic first integral at the origin. We apply the results to two families of degenerate vector fields.

1 Introduction

Given the homogeneous polynomial planar differential system

$$(\dot{x}, \dot{y})^T = \mathbf{F}_n(x, y) = (P_n(x, y), Q_n(x, y))^T,$$

with P_n and Q_n homogeneous polynomials of degree n and whose origin is an isolated singular point, it is interesting to know whether $\dot{\mathbf{x}} = \mathbf{F}_n$+h.o.t., analytic perturbations of $\mathbf{F}_n$, has an analytic first integral at the origin.

For $n = 1$, *i.e.* when the linear part $\mathbf{F}_1$ of the vector field is non-zero, if λ_1, λ_2 are the eigenvaluesxof $D(\mathbf{F}_1)(\mathbf{0})$, it has the following cases (assuming that the origin is an isolated singular point of $\mathbf{F}_1$): if $\lambda_1\lambda_2 \neq 0$, the origin is either a *saddle*, or *node* or a *non-degenerate monodromic singular point* (with imaginary eigenvalues).

The nodes are not analytically integrable. A non-degenerate monodromic point is analytically integrable if, and only if, the system is orbitally equivalent to $(-y, x)^T$, and a resonant saddle has an analytic first integral around the singular point if, and only if, the system is orbitally equivalent to $(px, -qy)^T$ with $p, q \in \mathbb{N}$ [16, 20]. The most studied systems whose origin is a resonant saddle are the Lotka-Volterra systems, see [11–13, 15, 17, 18] and references therein. Therefore, the analytical integrability problem for the vector fields whose origin is an isolated singular point with non-zero linear part is solved theoretically in the following sense:

Theorem 1.1 ([16, 20]) *The analytic vector field $\mathbf{F} = \mathbf{F}_1 + h.o.t.$, with $\mathbf{F}_1$ analytic integrable, is analytically integrable if, and only if, $\mathbf{F}$ and $\mathbf{F}_1$ are orbitally equivalent.*

For $n = 3$, Algaba *et al.* [5] have proved that Theorem 1.1 is true for the perturbations of cubic Kolmogorov systems. However, in general, Theorem 1.1 is not satisfied for

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vector fields with zero linear part (the origin is a *degenerate singular point*). For example, the vector field $\mathbf{F} = \mathbf{F}_4 + \mathbf{F}_5$ with

$$\mathbf{F}_4 = \begin{pmatrix} 4xy^3 - x^4 \\ 4x^3y - y^4 \end{pmatrix}, \quad \mathbf{F}_5 = \begin{pmatrix} -3x^3y^2 \\ 3x^2y^3 \end{pmatrix},$$

it is a Hamiltonian vector field whose Hamiltonian function is a polynomial and therefore is analytically integrable and Algaba *et al.* [5, Theorem 3.20] prove that is non-orbitally equivalent to its leader term $\mathbf{F}_4$.

For vector fields whose origin is a degenerate singular point, the problem remains open. We know only some partial results. The analytical integrability problem when the first quasi-homogeneous component is conservative whose Hamiltonian function h has only simple factors is solved in [2]. A particular case with h having multiple factors is studied in [1].

Recently, Algaba *et al.* [3] have solved the analytic integrability problem around a nilpotent singularity of a planar vector field under generic conditions. Later, Algaba *et al.* [4] solved the remaining case completing the analytic integrability problem for such singularity. In both cases, the vector field has an analytic local first integral if, and only if, it is orbitally quasi-homogenizable (orbitally equivalent to its quasi-homogeneous leader term).

In this work, we focus on the study of the analytical integrability of the vector fields

$$\mathbf{F} = \mathbf{F}_2 + \cdots, \quad \mathbf{F}_2(x, y) = (P_2(x, y), Q_2(x, y))^T$$

with P_2 and Q_2 homogeneous polynomials of degree two (vector fields with zero linear part) and the origin is an isolated singular point of $\dot{\mathbf{x}} = \mathbf{F}_2(\mathbf{x})$.

As far as we know, several authors have provided some normal forms (under conjugation and equivalence) of vector fields whose leader homogeneous term is quadratic, for example, see [7–10, 21].

First, we give an unique orbital normal form under equivalence of the vector fields whose lowest-degree term is quadratic and polynomially integrable (Theorem 2.9). The analysis of this normal form makes it possible to establish whether the systems considered have an analytic first integral at the origin. Concretely, we prove that, under the condition of polynomial integrability of $\mathbf{F}_2$, the vector field $\mathbf{F}$ is analytically integrable if, and only if, it is orbitally equivalent to its lowest-degree component (Theorem 2.11). Thus, Theorem 1.1 is satisfied for $n = 2$. As consequence, we provide an expression of a first integral, if it exists (Theorem 2.12). We characterize its analytical integrability through the existence of a Lie symmetry (Theorem 2.13). It is known that the existence of an analytic inverse integrating factor non-zero at the origin is a sufficient condition of integrability. Here, we prove that the existence of an inverse integrating factor zero at the origin and whose lowest-degree term has a specific expression is also a necessary and sufficient condition of analytical integrability (Theorem 2.14).

Finally, in Section 3, we provide the analytically integrable Volterra systems of the family

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(-x + 3y) \\ y(3x - y) \end{pmatrix} + \begin{pmatrix} xP_2(x, y) \\ yQ_2(x, y) \end{pmatrix}$$

with P_2, Q_2 homogeneous polynomials of degree two. A first integral of these systems is of the form $xy(x - y)^2 + \text{h.o.t.}$.

We also study the analytical integrability of the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2xy \\ 3x^2 + y^2 \end{pmatrix} + \begin{pmatrix} xP_2(x, y) \\ Q_3(x, y) \end{pmatrix}$$

with P_2, Q_3 homogeneous polynomials of degree two and three, respectively. In this case, the line $x = 0$ is the unique line invariant of the family. The primitive first integral of the leader term is $x(x^2 + y^2)$.

1.1 Invariant curves and first integrals of vector fields

We deal with a vector field $\mathbf{F} = (P, Q)^T$ with P, Q analytic functions at the origin and $P(\mathbf{0}) = Q(\mathbf{0}) = 0$. Throughout the paper, we denote the differential operator associated to the vector field $\mathbf{F}$ by F, that is, $F := P\partial_x + Q\partial_y$. We recall the concept of invariant curve and its associated cofactor.

Definition 1.2 *A function $C \in \mathbf{C}[[x, y]]$ (algebra of formal power series in x, y over $\mathbf{C}$), with $C(\mathbf{0}) = 0$, is an invariant curve at the origin of the vector field $\mathbf{F}$, if there exists $K \in \mathbf{C}[[x, y]]$, cofactor of C, such that $F(C) = KC$.*
Moreover, if $K \equiv 0$, the vector field $\mathbf{F}$ is formally integrable and C is a first integral of $\mathbf{F}$. If $K = div(\mathbf{F})$ (divergence of $\mathbf{F}$), C is an inverse integrating factor of $\mathbf{F}$.

It is worth pointing out that for analytic vector fields, by [19, Theorem A], the existence of a formal first integral is equivalent to the existence of an analytic first integral. For this reason, when we use Taylor expansions of functions and vector fields, we do not consider questions of convergence.

We denote by $\mathcal{P}_k$ the vector space of all homogeneous scalar polynomials of degree k and by $\mathcal{H}_k$ the vector space of polynomial homogeneous vector fields of degree k.

Throughout the paper, we write $\mathbf{D} = (x, y)^T \in \mathcal{Q}_1$ (dissipative homogeneous vector field) and $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to the polynomial h).
The following splitting of a homogeneous vector field plays a main role in our study.

Proposition 1.3 *[2, Prop. 2.7] Every $\mathbf{F}_n \in \mathcal{Q}_n$ can be uniquely written as $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D}$ with $h := \frac{1}{n+1}(\mathbf{D} \wedge \mathbf{F}_n) \in \mathcal{P}_{n+1}$ (product wedge of both vector fields) and $\mu := \frac{1}{n+1} div(\mathbf{F}_n) \in \mathcal{P}_{n-1}$ (divergence of $\mathbf{F}_n$).*

We give the first homogeneous term of the Taylor expansion of a formal invariant curve at the origin of a formal vector field.

Proposition 1.4 *Consider $\mathbf{F} = \sum_{j\geq n} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{H}_j$ with $\mathbf{F}_n \not\equiv \mathbf{0}$. Let C a formal invariant cuve of $\mathbf{F}$ with cofactor K. Then, $C = \sum_{j\geq s} C_j$, $C_j \in \mathcal{P}_j$ and $K = \sum_{j\geq n} K_j$, $K_j \in \mathcal{P}_j$, where C_s is a polynomial invariant curve at the origin of the polynomial vector field $\mathbf{F}_n$ with cofactor K_n.*

Proof. It is enough to consider the lowest-degree homogeneous term of the equation $F(C) - KC = 0$. ■

As a consequence, we have the following necessary condition of integrability.

Corollary 1.5 *Consider $\mathbf{F} = \mathbf{F}_n + h.o.t.$ with $\mathbf{F}_n \in \mathcal{P}_n$. If $\mathbf{F}$ is formally integrable at the origin, then $\mathbf{F}_n$ is polynomially integrable.*

The following two results give an expression of the invariant curves at the origin of a homogeneous vector field.

Proposition 1.6 *Every homogeneous polynomial invariant curve at the origin of a homogeneous vector field $\mathbf{F}_n$ is given by $g_1^{n_1} g_2^{n_2} \ldots g_m^{n_m}$ each g_j being a polynomial invariant curve at the origin of $\mathbf{F}_n$.*
Moreover, its cofactor is $n_1K_1 + \cdots + n_mK_m$, where K_j is the cofactor of g_j.

Proof. We suppose that $g = g_1u$, (g_1 an irreducible homogeneous polynomial and u a suitable homogeneous polynomial), is an invariant curve at the origin of $\mathbf{F}_n$ with K_n cofactor of g. We have that $F_n(g_1u) = g_1F_n(u) + uF_n(g_1) = K_ng_1u$, that is, $g_1(uK_n - F_n(u)) = uF_n(g_1)$. From the irreducibility of g_1, it has two situations: either g_1 is an irreducible invariant curve at the origin of $\mathbf{F}_n$, in such a case, u is also an invariant curve at the origin of $\mathbf{F}_n$ and we repeat the process for u. Or $u = g_1v$ with v a homogeneous polynomial, i.e. $g = g_1^2v$. We now have that $F_n(g_1^2v) = g_1^2F_n(v) + 2vg_1F_n(g_1) = K_ng_1^2v$. Thus, $g_1(vK_r - F_r(v)) = 2vF_r(g_1)$. Reasoning in a

similar way, the proof is completed.
The proof of the second part is straightforward. ■

Proposition 1.7 *Consider* $\mathbf{F}_n \in \mathcal{H}_n$*. Any factor of* $h := \frac{1}{n+1}(\mathbf{D} \wedge \mathbf{F}_n) \in \mathcal{P}_{n+1}$ *is an invariant curve at the origin of* $\mathbf{F}_n$*. Conversely, any homogeneous polynomial invariant curve at the origin of* $\mathbf{F}_n$ *is a factor of* h*.*
Moreover, if I *is a polynomial first integral of* $\mathbf{F}_n$*, then* $I = g_1^{n_1} g_2^{n_2} \cdots g_m^{n_m}$ *where* $g_1, \ldots, g_m$ *are all irreducible factors of* h *and* $n_i > 0$*.*

Proof. From Proposition 1.3, we know that $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D}$ with $h = \frac{1}{n+1}(\mathbf{D} \wedge \mathbf{F}_n)$ and $\mu = \frac{1}{n+1}\mathrm{div}(\mathbf{F}_n)$.
We prove that any factor of h is an invariant curve of $\mathbf{F}_n$.

If h is irreducible, the result follows since it is an invariant curve of $\mathbf{F}_n$. Otherwise, let $f \in \mathcal{P}_s$ a factor of h, that is, $h = fg$ with g a suitable homogeneous polynomial and $F_n(f) = X_{fg}(f) + \mu D(f) = f X_g(f) + s\mu f = (X_g(f) + s\mu)f$, where $X_{(\cdot)}$ and D are the operators associated to the vector fields $\mathbf{X}_{(\cdot)}$ and $\mathbf{D}$. Therefore, f is an invariant curve at the origin of $\mathbf{F}_n$.

We see that any polynomial invariant curve of $\mathbf{F}_n$ is also a factor of h. Indeed, if $f \in \mathcal{P}_s$ is an irreducible invariant curve at the origin of $\mathbf{F}_n$ with cofactor K_n then $K_n f = F_n(f) = X_h(f) + \mu D(f) = X_h(f) + s\mu f$. Thus, $X_h(f) = (K_n - s\mu)f$ and f is an invariant curve at the origin of $\mathbf{X}_h$. So, f divides h.
Last on, if I is a first integral of $\mathbf{F}_n$, it is an invariant curve at the origin of $\mathbf{F}_n$, from Proposition 1.6, a factorization of I is formed by the irreducible factors of h. On the other hand, any first integral satisfying $I(\mathbf{0}) = 0$, is zero on every invariant curve. So, $n_i > 0$. ■

1.2 Necessary condition of analytical integrability for perturbations of quadratic systems

Now we proceed with the study of the integrability of vector fields whose first homogeneous component is a homogeneous quadratic vector field. The following result provides the expression of the lowest-degree component in the case of polynomial integrability of this class of vector fields.

Proposition 1.8 (Necessary condition of analytical integrability) *Let* $\mathbf{F}_2$ *be a homogeneous quadratic vector field. If* $\mathbf{F} = \mathbf{F}_2 + h.o.t.$ *is formally integrable, then there exist a linear change of variables and a re-parameterization linear of the time such that* $\mathbf{F}$ *is transformed into one of the following vector fields:*
(a) $\mathbf{F} = \mathbf{F}_{2,a} + h.o.t.$*, with*

$$\mathbf{F}_{2,a} = (x(-qx + (q+r)y), y((p+r)x - py))^T, \ p, q, r \in \mathbb{N}, \tag{1.1}$$

($\mathbf{F}_{2,a}$ *has three invariant real lines),* $gcd(p,q,r) = 1$ *and* $I_M = x^p y^q (x-y)^r$ *is a polynomial first integral of degree* $M = p + q + r$*.*
(b) $\mathbf{F} = \mathbf{F}_{2,b} + h.o.t.$*, with*

$$\mathbf{F}_{2,b} = (-2qxy, (p+2q)x^2 + py^2)^T, \tag{1.2}$$

($\mathbf{F}_{2,b}$ *has one invariant real line only),* $gcd(p,q) = 1$ *and* $I_M = x^p(x^2+y^2)^q$ *is a polynomial first integral of degree* $M = p + 2q$*.*
(c) $\mathbf{F} = \mathbf{F}_{2,c} + h.o.t.$*, with*

$$\mathbf{F}_{2,c} = (-qxy, py^2)^T, \tag{1.3}$$

($\mathbf{F}_{2,c}$ *has two invariant real line and one line of singular points),* $gcd(p,q) = 1$ *and* $I_M = x^p y^q$ *is a polynomial first integral of degree* $M = p + q$*.*
(d) $\mathbf{F} = \mathbf{F}_{2,d} + h.o.t.$*, with*

$$\mathbf{F}_{2,d} = (0, x^2)^T, \tag{1.4}$$

($\mathbf{F}_{2,d}$ *has one line of singular points) and* $I_1 = x$ *is a polynomial first integral.*

Proof. Let $I = I_M + \text{h.o.t.}$ a formal first integral of $\mathbf{F}$. Equation $F(I) = 0$ for degree $M + 1$ is $F_2(I_M) = 0$, i.e. $\mathbf{F}_2$ is polynomially integrable and I_M is a first integral of $\mathbf{F}_2$.
We seek the vector fields $\mathbf{F}_2$ satisfying the condition $F_2(I_M) = 0$ (necessary condition of analytical integrability). By Proposition 1.3, $\mathbf{F}_2$ is $\mathbf{F}_2 = \mathbf{X}_h + \mu\mathbf{D}$ with $h \in \mathcal{P}_3$ and $\mu \in \mathcal{P}_1$.
If the polynomial h is identically zero, it has that $\mathbf{F}_2 = \mu\mathbf{D}$ and it is non-formally integrable. Otherwise, h has always a linear factor, a_1x+b_1y with a_1 and b_1 constants, since h is a cubic homogeneous polynomial. We can assume that $h = xp_2$ with p_2 a homogeneous polynomial of degree two, since if $a_1 = 0$ we do the change $(x, y) \to (y, x)$. Otherwise, we do $(x, y) \to (a_1x + b_1y, y)$. We distinguish the following cases according to the factors of the polynomial p_2:
• Assume $h = ax(x + by)(x + cy)$ with $b.c \neq 0$ and $b \neq c$, with a, b, c constants.
The linear change of variables $(x, y) \to ((c - b)x, c(x + by))$ and the linear re-parameterization of the time $at = bc(c - b)\tau$, transform $\mathbf{F}_2$ into $\tilde{\mathbf{F}}_2 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}$ with $\tilde{h} = xy(y - x)$. We write the linear polynomial $\tilde{\mu} = Ax + By$. From Proposition 1.7, if it existed a first integral of $\tilde{\mathbf{F}}_2$, it would have the expression $I_M = x^{mp}y^{mq}(x - y)^{mr}$ with $\gcd(p, q, r) = 1$. By imposing $\tilde{F}_2(I_M) = 0$, we arrive to

$$A = \frac{1}{M}(p - 2q + r), B = \frac{1}{M}(q - 2p + r),$$

with $M = p + q + r$. By performing the reparameterization of the time $3t = M\tau$, $\tilde{\mathbf{F}}_2$ turns on (1.1).
• Assume $h = ax((x + by)^2 + x^2)$ with $b \neq 0$.
The linear change $(x, y) \to (x, x + by)$ and the scaling $at = \tau$, transform $\mathbf{F}_2$ into $\tilde{\mathbf{F}}_2 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}$ with $\tilde{h} = x(x^2 + y^2)$. We write $\tilde{\mu} = Ax + By$. Now, by Proposition 1.7, the expression of a first integral of $\tilde{\mathbf{F}}_2$ would be $I_M = x^{mp}(x^2 + y^2)^{mq}$ with $\gcd(p, q) = 1$. By imposing $\tilde{F}_2(I_M) = 0$, we arrive to

$$A = 0, B = \frac{2}{M}(p - q),$$

with $M = p + 2q$. By performing the reparameterization of the time $3t = M\tau$, $\tilde{\mathbf{F}}_2$ turns on (1.2).
• Assume $h = ax(x + by)^2$ with $a, b \neq 0$.
The linear change of variables $(x, y) \to (x, x + by)$ and the scaled of the variable time $at = \tau$, transform $\mathbf{F}_2$ into $\tilde{\mathbf{F}}_2 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}$ with $\tilde{h} = xy^2$. We write $\tilde{\mu} = Ax + By$. From Proposition 1.7, if it existed a first integral of $\tilde{\mathbf{F}}_2$, it would have the expression $I_M = x^{mp}y^{mq}$ with $\gcd(p, q) = 1$. By imposing $\tilde{F}_2(I_M) = 0$, we arrive to

$$A = 0, B = \frac{1}{M}(2p - q),$$

with $M = p+q$. By performing the reparameterization of the time $3t = M\tau$, $\tilde{\mathbf{F}}_2$ turns on (1.3).
• Assume $h = ax^3$ with $a \neq 0$.
The scaled of the variable time $at = \tau$, transforms $\mathbf{F}_2$ into $\tilde{\mathbf{F}}_2 = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}$ with $\tilde{h} = x^3$. We write $\tilde{\mu} = Ax + By$. From Proposition 1.7, if it existed a first integral of $\tilde{\mathbf{F}}_2$, it would have the expression $I_M = x^p$. By imposing $\tilde{F}_2(I_M) = 0$, we arrive to $A = 0$ and $B = 0$. By performing the reparameterization of the time $3t = M\tau$, $\tilde{\mathbf{F}}_2$ turns on (1.4) where $I_1 = x$ is a primitive first integral. ■

2 Main results

Our purpose is to characterize the analytically integrable perturbations of a quadratic system having an isolated singularity at the origin.

First, we are going to provide a suitable expression of the orbital normal form of the vector fields considered.

We do not consider questions of convergence in the normal forms because, as we mention before, the formal integrability is equivalent to the analytical integrability for the vector fields analyzed, see [19, Theorem A].

We will restrict our attention to the vector fields whose first homogeneous component is quadratic, integrable and the origin is an isolated singular point. So, from Proposition 1.8, we only will consider the vector fields $\mathbf{F} = \mathbf{F}_2 + \text{h.o.t.}$ with $\mathbf{F}_2 = \mathbf{F}_{2,a}$ or $\mathbf{F}_2 = \mathbf{F}_{2,b}$ given by (1.1) or (1.2).

Next result provides a orbital normal form of the perturbations of the vector fields considered. To prove this, we have used the technical results given in Appendix 4.

Theorem 2.9 *The vector field $\mathbf{F} = \mathbf{F}_2 + h.o.t.$ with $\mathbf{F}_2 = \mathbf{F}_{2,a}$ (or $\mathbf{F}_{2,b}$) is orbitally equivalent to $\mathbf{F}_2 + \sum_{j\geq 2} \eta_j \mathbf{D}$, with $\eta_j \in \mathrm{Cor}(\ell_j)$, a complementary subspace to Range(ℓ_j), where ℓ_j is the Lie operator of $\mathbf{F}_2$, i.e.*

$$\begin{aligned} \ell_j \quad : \quad & \mathcal{P}_{j-1} \longrightarrow \mathcal{P}_j^{\mathbf{t}} \\ & \eta_{j-1} \longrightarrow F_2(\eta_{j-1}). \end{aligned}$$

Proof. From [5, Theorem A.32], if for all $j \geq 2$, $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{j+2}\right) = \{0\}$, then $\mathbf{F}$ is orbitally equivalent to $\mathbf{G} = \mathbf{F}_2 + \sum_{j\geq 2} \mathbf{G}_{j+1}$, with $\mathbf{G}_{j+1} = \mathbf{X}_{g_{j+2}} + \eta_j \mathbf{D} \in \mathcal{H}_{j+1}$, where $g_{j+2} \in \mathrm{Cor}\left(\ell^{\mathrm{c}}_{j+2}\right)$, $\eta_j \in \mathrm{Cor}\left(\ell_j\right)$ and ℓ^{c}_{j+2} (Lie operator of $\mathbf{F}_2$ moved),

$$\begin{aligned} \ell^{\mathrm{c}}_{j+2} \quad : \quad & \Delta_{j+1} \longrightarrow \Delta_{j+2} \\ & g_{j+1} \longrightarrow \mathrm{Proy}_{\Delta_{j+2}}(F_2 - \tfrac{3}{j+2}\mu D)(g_{j+1}), \end{aligned}$$

with $j \geq 2$ and Δ_{j+2}, the subspaces such that $\mathcal{P}_{j+2} = \Delta_{j+2} \bigoplus h\mathcal{P}_{j-1}$ (such subspaces must be considered as fixed).

Therefore, in order to prove our result it is enough to check that $\mathbf{F}_{2.a}$ and $\mathbf{F}_{2,b}$ satisfy the hypothesis of [5, Theorem A.32] and $\mathrm{Cor}\left(\ell^{\mathrm{c}}_{j+2}\right) = \{0\}$. Indeed, we consider $\mathbf{F} = \mathbf{F}_{2,a} + \text{h.o.t.}$ (similar arguments apply to the case $\mathbf{F}_{2,b}$).The Hamiltonian of the conservative part of vector field $\mathbf{F}_{2,a}$ is $h = \frac{p+q+r}{3} xy(x-y)$, which has only simple factors. On the other hand, the vector field $\mathbf{F}_{2,a} - \frac{3}{j+2}\mu$ with $\mu = \frac{1}{3}((-2q+p+r)x + (q+r-2p)y)$, is irreducible. Therefore, by applying Lemma 4.19, it has that $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{j+2}\right) = \{0\}$.
Second part follows since both Δ_{j+1} and Δ_{j+2} have the same dimension and as $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{j+2}\right) = \{0\}$, it has that $\mathrm{Cor}\left(\ell^{\mathrm{c}}_{j+2}\right)$ is also a trivial subspace.
So, $g_{j+2} = 0$ for all j, and we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{F}_2 + \sum_{j\geq 2} \eta_j \mathbf{D}$ with $\eta_j \in \mathrm{Cor}\left(\ell_j\right)$. ■

Next statement establishes a cyclicity relation between the co-ranges of the operators ℓ_k associated to $\mathbf{F}_{2,a}$ and $\mathbf{F}_{2,b}$.

Proposition 2.10 *Consider $\mathbf{F}_2 = \mathbf{F}_{2,a}$ ($\mathbf{F}_2 = \mathbf{F}_{2,b}$, resp.). For $k \geq 2$, it is always possible to choose $\mathrm{Cor}(\ell_{k+M})$, a complementary subspace to $\mathrm{Range}(\ell_{k+M})$, such that $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$ with $I_M = x^p y^q (x-y)^r$ ($I_M = x^p(x^2+y^2)^q$, resp.).*

Proof. First, we consider $\mathbf{F}_2 = \mathbf{F}_{2,a}$. We see that both subspaces have the same dimension. Indeed, $Ker(\ell_k) = \langle I_M^l \rangle$ if $k-1 = lM$. Otherwise, $Ker(\ell_k) = \{0\}$. Thus, $\dim(\mathrm{Cor}(\ell_k)) = 2$ if $k = lM$ and $\dim(\mathrm{Cor}(\ell_k)) = 1$, otherwise; i.e. $\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{k+M}))$.

The proof is completed by showing that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M\mathrm{Cor}(\ell_k)\cap\mathrm{Range}(\ell_{k+M}) = \{0\}$ by *reductio ad absurdum*. Let $p_k \in \mathrm{Cor}\,(\ell_k)\setminus\{0\}$ such that $p_kI_M \in \mathrm{Range}\,(\ell_{k+M})$, then there exists $p_{k+M-1} \in \mathcal{P}_{k+M-1} \setminus \{0\}$ such that $\ell_{k+M}(p_{k+M-1}) = p_kI_M$, that is, $\ell_{k+M}(p_{k+M-1})$ is multiple of I_M. As $\frac{p(k+M-1)}{j} > \frac{pM}{j} > M$, $j = 1,\ldots,p-1$; $\frac{q(k+M-1)}{j} > M$, $j = 1,\ldots,q-1$; $\frac{r(k+M-1)}{j} > M$, $j = 1,\ldots,r-1$, by applying Lemma 4.21 we have that $p_{k+M-1} \in \langle x^p\rangle \cap \langle y^q\rangle \cap \langle (x-y)^r\rangle$, thus $p_{k+M-1} = p_{k-1}I_M$ with $p_{k-1} \in \mathcal{P}_{k-1} \setminus \{0\}$ and consequently

$$p_kI_M \;=\; F_{2,a}(p_{k+M-1}) = F_2(p_{k-1}I_M) = I_MF_{2,a}(p_{k-1}).$$

Hence $p_k = F_{2,a}(p_{k-1})$, that is, $p_k \in \mathrm{Range}\,(\ell_k)\cap\mathrm{Cor}\,(\ell_k)$ which gives a contradiction. For $\mathbf{F}_2 = \mathbf{F}_{2,b}$, reasoning as before and applying Lemma 4.22, the result follows. ■

We give the main result of our study. As particular case, this result also solves the analytical integrability problem for vector fields which are perturbations of generic quadratic Lotka-Volterra vector fields. It also gives the expression of a first integral.

Theorem 2.11 *Consider* $\mathbf{F} = \mathbf{F}_{2,a} + h.o.t.$ *(*$\mathbf{F} = \mathbf{F}_{2,b} + h.o.t.$*, resp.). The vector field* $\mathbf{F}$ *is analytically integrable at the origin if, and only if, it is orbitally equivalent to* $\mathbf{F}_{2,a}$ *(*$\mathbf{F}_{2,b}$*, resp.).*
Moreover, in such a case, $\mathbf{F}$ *has an analytic first integral of the form* $I = I_M + h.o.t.$ *where* $I_M = x^py^q(x-y)^r$ *is a primitive first integral of* $\mathbf{F}_{2,a}$ *(* $I_M = x^p(x^2+y^2)^q$ *a primitive first integral of* $\mathbf{F}_{2,b}$*, resp.).*

Proof. We do the proof for the case $\mathbf{F}_2 = \mathbf{F}_{2,a}$ only. The other case is similar.
We see the sufficiency. The polynomial $I_M = x^py^q(x-y)^r$ is a first integral of $\mathbf{F}_{2,a}$ which is transformed into a formal first integral $I = I_M +$ h.o.t. of $\mathbf{F}$ and from [19, Theorem A], $\mathbf{F}$ is analytically integrable.
We see the necessity of the condition. Applying Theorem 2.9, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_{2,a} + \sum_{j\geq 2}\eta_j\mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$.
Let note that $\mathbf{F}$ has an analytic first integral equivalents to $\mathbf{G}$ has a formal first integral. Assume that $\mathbf{G}$ is formally integrable and not all the η_j are zero. Let N be defined by $N = \min\{j > 1 : \eta_j \not\equiv 0\}$. A formal first integral of $\mathbf{G}$ is of the form $I = I_M^l + \sum_{j>Ml} I_j$ with $I_j \in \mathcal{P}_j$. Imposing the integrability condition we have

$$\begin{aligned} 0 &= (G(I))_{N+Ml} = (\eta_ND)(I_M^l) + F_2(I_{Ml+N-1}) \\ &= Ml\eta_NI_M^l + \ell_{Ml+N}\left(I_{Ml+N-1}\right). \end{aligned}$$

But this equation is incompatible since, by Proposition 2.10, $Ml\eta_NI_M^l \in \mathrm{Cor}\,(\ell_{Ml+N})$ and $\ell_{Ml+N}\left(I_{Ml+N-1}\right) = -Ml\eta_NI_M^l \in \mathrm{Range}\,(\ell_{Ml+N})$, which is a contradiction. Consequently, $\mathbf{G} = \mathbf{F}_{2,a}$, i.e. $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_{2,a}$.

We now see the second part. A first integral of $\mathbf{F}_{2,a}$ is I_M. So, a first integral of $\mathbf{F}$ is $I_M +$ h.o.t. since $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_{2,a}$. ■

A consequence of Theorem 2.11 is the following result. The important point to note here is the form of the first integral.

Theorem 2.12 *Consider* $\mathbf{F} = \mathbf{F}_{2,a} + h.o.t.$ *(*$\mathbf{F} = \mathbf{F}_{2,b} + h.o.t.$*, resp.). The vector field* $\mathbf{F}$ *is analytically integrable if, and only if,* $\mathbf{F}$ *has a first integral of the form* $I = (x+h.o.t.)^p(y+h.o.t.)^q(x-y+h.o.t.)^r$ *(*$I = (x+h.o.t.)^p(x^2+y^2+h.o.t.)^q$*, resp.).*

Proof. We prove the necessary condition. Consider $\mathbf{F} = \mathbf{F}_{2,a} +$ h.o.t.. From Theorem 2.11 there exist a change of variables and a reparemeterization of the time such that $\mathbf{F}_{2,a}$ is transformed into $\mathbf{F}$. These changes transform the first integral $I_M = x^py^q(y-x)^r$ of $\mathbf{F}_{2,a}$ into a first integral of $\mathbf{F}$, which has the expression $I = (x+\text{h.o.t.})^p(y+\text{h.o.t.})^q(x-y+\text{h.o.t.})^r$.
For $\mathbf{F} = \mathbf{F}_{2,b} +$ h.o.t. the proof is analogous.

Reciprocally, if $\mathbf{F}$ has a formal first integral, from [19, Theorem A], $\mathbf{F}$ has an analytic first integral. ■

The following theorem characterizes the analytical integrability of a vector field whose first homogeneous component is quadratic through the existence of a Lie symmetry.

Theorem 2.13 *Consider $\mathbf{F} = \mathbf{F}_2 + h.o.t.$ with $\mathbf{F}_2 = \mathbf{F}_{2,a}$ or $\mathbf{F}_2 = \mathbf{F}_{2,b}$. Then $\mathbf{F}$ is analytically integrable if, and only if, there exist a formal vector field $\mathbf{G} = \mathbf{D} + h.o.t.$ and a formal scalar function ν, $\nu(\mathbf{0}) = 1$ such that $[\mathbf{F}, \mathbf{G}] = \nu\mathbf{F}$, i.e. $\mathbf{F}$ has a Lie symmetry.*

The proof of Theorem 2.13 follows from Theorem 2.11 and applying [6, Theorem 1.3].

We solve the analytical integrability problem through the existence of a formal inverse integrating factor.

Theorem 2.14 *Consider $\mathbf{F} = \mathbf{F}_{2,a} + h.o.t.$ ($\mathbf{F} = \mathbf{F}_{2,b} + h.o.t.$, resp.). Then $\mathbf{F}$ is analytically integrable if, and only if, it has a formal inverse integrating factor of the form $V = xy(x-y) + h.o.t.$ ($V = x(x^2+y^2) + h.o.t.$, resp.).*

Proof. We do the proof for the case $\mathbf{F}_2 = \mathbf{F}_{2,a}$ only. The other case is analogous.

We prove that the condition is necessary. We assume that $\mathbf{F}$ is analytically integrable. From Theorem 2.11, it is orbitally equivalent to $\mathbf{F}_{2,a} = \mathbf{X}_h + \mu\mathbf{D}$ with $h = \frac{p+q+r}{3}xy(x-y)$ and $\mu = \frac{1}{3}((-2q+p+r)x + (q+r-2p)y)$, which has the inverse integrating factor h. Undoing the change, $\mathbf{F}$ has a formal inverse integrating $V = h +$ h.o.t..

Now we see the sufficiency of the condition. Let $V = h +$ h.o.t. a formal inverse integrating factor of $\mathbf{F}$. From Theorem 2.9, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_{2,a} + \sum_{j\geq 2}\eta_j\mathbf{D}$ with $\eta_j \in \mathrm{Cor}(\ell_j)$. Therefore, $\mathbf{F}$ has a formal inverse integrating factor if, and only if, $\mathbf{G}$ has it too. Moreover, the formal inverse integrating factor W of $\mathbf{G}$ is also of the form $W = h +$ h.o.t.. Also, the unique invariant curves at the origin of $\mathbf{G}$ are $x, y, x-y$. So, we have that $W = hu$ with u a formal function and $u(\mathbf{0}) = 1$. Equation $G(W) - W\mathrm{div}(\mathbf{G}) = 0$ is

$$0 = uG(h) + hG(u) - hu\mathrm{div}(\mathbf{G}).$$

As $G(h) = 3h\mu + \sum_{j>2} 3h\eta_j$ and $\mathrm{div}(\mathbf{G}) = 3\mu + \sum_{j>2}(j+2)\eta_j$, we have that

$$0 = h(G(u) - u\sum_{j>2}(j-1)\eta_j). \tag{2.5}$$

We see that $\eta_j = 0$ for all j. Indeed, otherwise, let $j_0 = \min\{j \in \mathbb{N} : \eta_{j+1} \not\equiv 0\}$. As $\eta_{j_0-k+1} = 0$ for $1 \leq k \leq j_0 - 1$, and expanding $u = 1 + \sum_{i\geq 1} u_i$, equation (2.5) to degree $j_0 + 4$ is $G_2(u_{j_0}) = j_0\eta_{j_0+1}$, i.e. $\eta_{j_0+1} \in \mathrm{Cor}(\ell_{j_0+1})$ and $\eta_{j_0+1} \in \mathrm{Range}(\ell_{j_0+1})$. We arrives to $\eta_{j_0+1} = 0$. ■

3 Applications

Consider the following six-parameter family

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(3y-x) \\ y(3x-y) \end{pmatrix} + \begin{pmatrix} x(a_{20}x^2 + a_{11}xy + a_{02}y^2) \\ y(b_{20}x^2 + b_{11}xy + b_{02}y^2) \end{pmatrix}. \tag{3.6}$$

This family is a perturbation of a Lotka-Volterra system and the edges $x = 0$ and $y = 0$ are invariant curves. The first homogeneous component of the vector field is

$\mathbf{F}_2 = (x(-x+3y),\ y(3x-y))^T$, i.e. system (1.1) with $p = q = 1$, $r = 2$. From Proposition 1.3, we have that $\mathbf{F}_2 = \mathbf{X}_h + \mu\mathbf{D}$ with $h = \frac{4}{3}xy(x-y)$ and $\mu = \frac{1}{3}(x+y)$. The vector field $\mathbf{F}_2$ is polynomially integrable and a primitive first integral is $I_4 = xy(x-y)^2$.

The following result solves the analytical integrability problem for this family.

Theorem 3.15 *System (3.6) is analytically integrable if, and only if, one of the following conditions holds:*
(i) $b_{11} + 5b_{02} = b_{20} + 2b_{02} = a_{11} + 3b_{02} = a_{20} - b_{02} = a_{02} = 0$,
(ii) $b_{11} + 3b_{02} = a_{02} + 2b_{02} = a_{11} + 5b_{02} = a_{20} - b_{02} = b_{20} = 0$,
(iii) $2a_{11} + 2a_{02} - 3b_{20} - 3b_{11} - 5b_{02} = a_{02}b_{20} + a_{02}b_{11} + 3a_{02}b_{02} + 2b_{20}b_{02} = 2a_{20} + b_{20} + b_{11} + 3b_{02} = 0$,
(iv) $a_{02} + 5b_{02} = a_{11} + b_{11} = 5a_{20} + b_{20}$,
(v) $a_{11} + b_{11} = a_{20} + b_{02} = a_{02} = b_{20}$,
(vi) $b_{20} - b_{02} = a_{02} + b_{02} = a_{11} + b_{11} = a_{20} + b_{02}$.

Proof. In order to prove the necessary condition, we have computed the first coefficients of the normal form given in Theorem 2.9. By Theorem 2.11, the vanishing of the coefficients leads us to the integrability of (3.6). In this case, it has been necessary to obtain the coefficients of the normal form up order 7,

$$(\dot{x}, \dot{y})^T = \mathbf{F}_2 + (\alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x I_4 + \beta_5 y I_4 + \alpha_6 x^2 I_4)(x, y)^T.$$

The coefficients $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ and β_5 have been obtained following the procedure given in the proof of [5, Theorem A.32]. They are polynomials too long, so we do not given them here. Moreover, the irreducible decomposition of the variety of the ideal generated by the vanishing of the coefficients has been obtained using the computer algebra system SINGULAR [14]. This decomposition leads us to the systems (3.6) cases *(i)*-*(vi)*.

We prove the sufficiency. System (3.6) case *(i)* has an analytic first integral $xy(x - y - b_{02}x^2 + \frac{1}{3}b_{02}xy)^2(1 - b_{02}x - b_{02}y)^{-3}$.
System (3.6) case *(ii)* is transformed into system (3.6) case *(i)* by using the involution $(x, y) \leftrightarrow (y, x)$.
System (3.6) case *(iii)* has an inverse integrating factor $xy(x - y)(2 + b_{20}x + b_{11}x + 3b_{02}x - 2b_{02}y)$ whose first component is h. Applying Theorem 2.14, the vector field is analytically integrable.
System (3.6) case *(iv)* has a polynomial first integral $xy(3x - 3y - 3a_{20}x^2 + b_{11}xy + 3b_{02}y^2)^2$.
System (3.6) case *(v)* has an analytic first integral

$$xy(3x - 3y + (b_{02} + b_{11})xy)^2(2 + 2b_{02}x - 2b_{02}y + (b_{02}b_{11} + b_{02}^2)xy)^{-3}.$$

System (3.6) case *(vi)*, for $b_{11} = -2b_{02}$, has an inverse integrating factor $xy(x-y)(1 + b_{02}x - b_{02}y)$. Otherwise, we don't have found the expression of an inverse integrating factor starting by h. In this case, we center on proving its existence in order to apply Theorem 2.14.
Consider $V = xyC(x, y)$ with C the invariant curve given by Lemma 3.16. Then $F(V) = xyF(C) + xCF(y) + yCF(x) = V(K^{(1)} + K^{(2)} + K^{(3)})$ with $K^{(1)}, K^{(2)}$ and $K^{(3)}$ the cofactors of x, y and C, respectively, $K^{(1)} = -x + 3y - b_{02}x^2 - b_{11}xy - b_{02}y^2$, $K^{(2)} = 3x - y + b_{02}x^2 + b_{11}xy + b_{02}y^2$ and $K^{(3)} = -(x + y)(1 + 2b_{02}x - 2b_{02}y)$ and as $K^{(1)} + K^{(2)} + K^{(3)} = x + y - 2b_{02}x^2 + 2b_{02}y^2 = \mathrm{div}(\mathbf{F})$, then V is an inverse integrating factor of $\mathbf{F}$. This completes the proof. ∎

Lemma 3.16 *System (3.6) case (vi) has an invariant curve $C = x - y + h.o.t.$ with cofactor $K = -(x + y)(1 + 2b_{02}x - 2b_{02}y)$.*

Proof. System (3.6) case *(vi)* is $\dot{\mathbf{x}} = \mathbf{F}_2 + \mathbf{F}_3$ with $\mathbf{F}_2 = (x(-x+3y),\ y(3x-y))^T$ and $\mathbf{F}_3 = (x(-b_{02}x^2 - b_{11}xy - b_{02}y^2),\ y(b_{02}x^2 + b_{11}xy + b_{02}y^2))^T$.
We claim that there exists a formal invariant curve at the origin of $\mathbf{F}$ of the form $C = \sum_{j\geq 1} C_j$ with

$$C_{2j-1} = A_{2j-1}x^{j-1}y^{j-1}(x-y), \quad C_{2j} = x^{j-1}y^{j-1}(A_{2j}x^2 + B_{2j}xy - A_{2j}y^2), \quad (3.7)$$

for any $j \geq 1$, with cofactor $K_1 + K_2$ where $K_1 = -x-y$ and $K_2 = -2b_{02}x^2 + 2b_{02}y^2$. We are going to verify that C satisfies $F(C) - KC = 0$ degree to degree.
For the degree 2, $F_2(C_1) - K_1C_1 = 0$ leads to $C_1 = x - y$, and for the degree 3, $F_2(C_2) + F_3(C_1) - C_2K_1 - C_1K_2 = 0$ yields $C_2 = b_{02}x^2 + \frac{1}{3}(b_{11} - 4b_{02})xy + b_{02}y^2$. Thus, C_1 and C_2 have the form given by (3.7). Assume that (3.7) is true for $2j_0 - 1$ and $2j_0$ and we prove that (3.7) also holds for $2j_0 + 1$ and $2j_0 + 2$.
Equation $F(C) - KC = 0$ for degree $2j_0 + 2$ is

$$F_2(C_{2j_0+1}) - C_{2j_0+1}K_1 = -F_3(C_{2j_0}) + C_{2j_0}K_2 = 2x^{j_0}y^{j_0}(x-y)(x+y)(A_{2j_0}b_{11} - B_{2j_0}b_{02}).$$

A solution of this equation is $C_{2j_0+1} = (A_{2j_0}b_{11} - B_{2j_0}b_{02})x^{j_0-1}y^{j_0-1}(x-y)$, i.e. C_{2j_0+1} is of the form given by (3.7) with $A_{2j_0+1} = A_{2j_0}b_{11} - B_{2j_0}b_{02}$.
Analogously, equation $F(C) - KC = 0$ for degree $2j_0 + 3$ is

$$F_2(C_{2j_0+2}) - C_{2j_0+2}K_1 = -A_{2j_0+1}x^{j_0}y^{j_0}(x+y)(b_{02}x^2 - (b_{11} + 4b_{02})xy + b_{02}y^2).$$

A solution of this equation is

$$C_{2j_0+2} = A_{2j_0+1}x^{j_0}y^{j_0}\left(-\frac{b_{02}}{2j_0-1}x^2 + (\frac{b_{11}}{2j_0+3} + \frac{4(2j_0+1)b_{02}}{(2j_0-1)(2j_0+3)})xy + \frac{b_{02}}{2j_0-1}y^2\right),$$

i.e. C_{2j_0+2} is of the form given by (3.7). Therefore, the proof is complete. ∎

Next, we study the analytical integrability of the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2xy \\ 3x^2 + y^2 \end{pmatrix} + \begin{pmatrix} a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 \\ b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}. \quad (3.8)$$

The leader component of the vector field is $\mathbf{F}_2 = (-2xy, 3x^2 + y^2)^T$ with $\mathbf{F}_2 = \mathbf{F}_{2,b}$ with $(p,q) = (1,1)$. We note that $\mathbf{F}_2$ is a Hamiltonian vector field whose Hamiltonian function is $h = x(x^2 + y^2)$. Moreover, $x = 0$ is an invariant curve of $\mathbf{F}$.

The following result solves the integrability problem of this family.

Theorem 3.17 *System (3.8) is analytically integrable if, and only if, one of the following conditions holds:*
(i) $b_{12} = b_{30} = b_{21} + 3b_{03} = a_{12} + b_{03} = a_{30} - 3b_{03} = 0$,
(ii) $b_{12} = b_{30} = a_{21} = b_{21} - 6b_{03} = 2a_{12} - b_{03} = 2a_{30} - 3b_{03} = 0$,
(iii) $a_{12} + 3b_{03} = a_{21} + b_{12} = 3a_{30} + b_{21} = 0$,
(iv) $b_{21} - 3b_{03} = a_{21} + b_{30} - b_{12} = a_{30} - a_{12} - 2b_{03} = a_{12}b_{30} - a_{12}b_{12} + b_{30}b_{03} + b_{12}b_{03} = 0$,
(v) $a_{12} = a_{30} = b_{21} - 9b_{03} = b_{30} + 3b_{12} = a_{21}b_{12} - 2b_{12}^2 - 6b_{03}^2 = 0$.

Proof. We assume that $\mathbf{F}_2$ is non-zero; otherwise, the vector field is polynomially integrable.

To prove the necessary condition, it has obtained the expression of $F(I)$ with $I = x(x^2 + y^2) + \text{h.o.t.}$ up to order 9. Its vanishing arrives to systems (3.8) cases (i)–(v).

We prove the sufficiency. System (3.8) case (i) has an inverse integrating factor

$$V = (1 - \tfrac{1}{4}a_{21}x - \tfrac{1}{4}\lambda x + b_{03}y)^{1+\frac{a_{21}}{\lambda}}(1 - \tfrac{1}{4}a_{21}x + \tfrac{1}{4}\lambda x + b_{03}y)^{1-\frac{a_{21}}{\lambda}},$$

with $\lambda = a_{21}^2 + 48b_{03}^2 > 0$, it which is non-zero at origin. Thus,

$$-\int P/V\,dy + \int \left(Q/V + \frac{\partial}{\partial x}\int P/V\,dy\right)dx$$

is a formal first integral defined in a neighborhood of the origin and, by Mattei and Moussu [19], there exists an analytic first integral at the origin. Therefore, the vector field is analytically integrable.
In case (*ii*), system (3.8) has an analytic first integral

$$x(x^2+y^2+b_{03}x^2y+\tfrac{3}{10}b_{03}^2x^4)(1+b_{03}y+\tfrac{3}{4}b_{03}^2x^2)^{-5/2}.$$

System (3.8) case (*iii*) is a Hamiltonian system whose Hamiltonian function is

$$x(x^2+y^2)-a_{30}x^3y-\tfrac{1}{2}a_{21}x^2y^2-\tfrac{1}{3}a_{12}xy^3.$$

In case (*iv*), system (3.8) for $b_{30}\neq b_{12}$ has the analytic first integral

$$x(x^2+y^2+b_{03}x^2y+\tfrac{3}{10}b_{03}^2x^4)(1+b_{03}y+\tfrac{3}{4}b_{03}^2x^2)^{-5/2}.$$

Otherwise, $b_{30}=b_{12}$ and $b_{03}=0$. System (3.8) has an inverse integrating factor $x(x^2+y^2)$ whose first component is h. Applying Theorem 2.14, the vector field is analytically integrable.
If $b_{12}\neq 0$, system (3.8) case (*v*) has an inverse integrating factor

$$(1-xb_{12}+b_{03}y)^3(-b_{12}+xb_{12}^2+3xb_{03}^2)^{6b_{03}^2/(b_{12}^2+3b_{03}^2)},$$

it which is non-zero at origin. Reasoning as before, the vector field is analytically integrable. Otherwise, $b_{12}=0$, then $b_{03}=0$. In such a case,

$$-\frac{2(xy^2a_{21}^3-6a_{21}^2x^2+12a_{21}x+24)}{a_{21}^3(a_{21}x-2)}+\frac{\log(a_{21}x-2)^{48}}{a_{21}^3}$$

is an analytic first integral at the origin. This completes the proof. ∎

4 Appendix

We give the following technical lemmas.

Lemma 4.18 *Consider* $\mathbf{F}_n\in\mathcal{H}_n$ *irreducible (its component are coprime) and* $f\in\mathbb{C}[x,y]$ *an irreducible invariant curve at the origin of* $\mathbf{F}_n$. *If* $F_n(p_k)\in\langle f\rangle$ *with* $p_k\in\mathcal{P}_k$, *then* $p_k\in\langle f\rangle$.

Proof. If $F_n(p_k)=0$ then p_k is a first integral of $\dot{\mathbf{x}}=\mathbf{F}_n$. A first integral of $\mathbf{F}_n$ vanishes on any invariant curve of it, i.e. $p_k(\mathbf{x})=0$ when $f(\mathbf{x})=0$. Therefore, by Hilbert‘s Nullstellensatz $p_k\in rad\langle f\rangle$. Since $\langle f\rangle$ is a prime ideal, then $\langle f\rangle=rad\langle f\rangle$, in consequence $p\in\langle f\rangle$.

If $F_n(p_k)\neq 0$, let $\nu\in\mathbb{C}[x,y]\setminus\{0\}$ such that $f\nu=F_n(p_k)$. Consider $\gamma(t)$, real or complex, a solution curve of $\dot{\mathbf{x}}=\mathbf{F}_n(\mathbf{x})$ which is a parametrization of $f(\mathbf{x})=0$. We assume that $\lim_{t\to-\infty}\gamma(t)=\mathbf{0}$, (the other case $\lim_{t\to+\infty}\gamma(t)=\mathbf{0}$ is proved in a similar way). Taking into account that $p_k(\mathbf{0})=0$ then

$$\begin{aligned} p_k(\gamma(t)) &= p_k(\gamma(t))-p_k(\mathbf{0})=\int_{-\infty}^{t}\tfrac{dp_k(\gamma(s))ds}{ds}=\int_{-\infty}^{t}\nabla_{\mathbf{x}}p_k\cdot\mathbf{F}_n(\gamma(s))ds \\ &= \int_{-\infty}^{t}F_n(p_k)(\gamma(s))ds=\int_{-\infty}^{t}f(\gamma(s))\nu(\gamma(s))ds=0. \end{aligned}$$

Recalling that $f(\mathbf{x})=0$ is the union of orbits, we have that $p_k(\mathbf{x})=0$ when $f(\mathbf{x})=0$. Therefore, by Hilbert‘s Nullstellensatz $p_k\in rad\langle f\rangle$. Since $\langle f\rangle$ is a prime ideal, then $\langle f\rangle=rad\langle f\rangle$, in consequence $p_k\in\langle f\rangle$. ∎

Remark. The hypothesis of the irreducibility of the invariant curve at the origin is fundamental. For instance, if we consider $\mathbf{F}_2:=(-2x^2,-3x^2-2xy+3y^2)^T\in\mathcal{H}_2$ irreducible and the invariant curve $(y-x)^2$, for $p_3=x^2(y-x)$ we have that $F_2(p_3)=3x^2(y-x)^2\in\langle(y-x)^2\rangle$ and nevertheless $p_3\notin\langle(y-x)^2\rangle$.

Lemma 4.19 *Consider* $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D}$ *with* h *having simple factors. Assume that* $\mathbf{F}_n - \frac{(n+1)}{n+k-1}\mu\mathbf{D}$ *is irreducible, i.e. it has coprime components. Then,* $\mathrm{Ker}\left(\ell^{c}_{k+n-1}\right) = \{0\}$.

Proof. Let $p_k \in \mathrm{Ker}\left(\ell^{c}_{k+n-1}\right) \subset \Delta_k$, i.e. $(F_n - \frac{(n+1)}{n+k-1}\mu D)(p_k) \in \langle h \rangle$, (we note that h is the Hamiltonian part of both $\mathbf{F}_n$ and $\mathbf{F}_n - \frac{(n+1)}{n+k-1}\mu\mathbf{D}$. So, $(F_n - \frac{(n+1)}{n+k-1}\mu D)(p_k) \in \langle f_i \rangle$, for all f_i invariant curve of $\mathbf{F}_n - \frac{(n+1)}{n+k-1}\mu\mathbf{D}$.

Applying Lemma 4.18 to $\mathbf{F}_n - \frac{(n+1)}{n+k-1}\mu\mathbf{D}$, it has that $p_k \in \langle f_i \rangle$, for all f_i and therefore $p_k \in \langle h \rangle$ since h has simple factors and h is a first integral of $\mathbf{F}_n$. Hence $p_k \in \Delta_k$, it has that $p_k = 0$. ■

Lemma 4.20 *Let* $f \in \mathbf{C}[x, y]$ *an irreducible polynomial invariant curve of* $\mathbf{F}_n$, $K \in \mathcal{P}_{n-1}$ *its cofactor and* $k, m \in \mathbb{N}$ *with* $n + k - 1 \geq m$. *Assume that the vector fields of* $\mathcal{H}_n$, $\mathbf{F}_n + \frac{jK}{k-j}\mathbf{D}$, $0 \leq j \leq m-1$, *are irreducible. Then, if* $F_n(p_k) \in \langle f^m \rangle$ *with* $p_k \in \mathcal{P}_k$, *it satisfies that* $p_k \in \langle f^m \rangle$.

Proof. Lemma 4.18 proves the statement for $m = 1$.

We first consider the case $m = 2$. If $F_n(p_k) \in \langle f^2 \rangle$ then $F_n(p_k) \in \langle f \rangle$ and, by Lemma 4.18, we have that there exists $p_{k-1} \in \mathcal{P}_{k-1}$ such that $p_k = f p_{k-1}$. Therefore,

$$\begin{aligned} F_n(p_k) &= F_n(f p_{k-1}) = p_{k-1} F_n(f) + f F_n(p_{k-1}) = p_{k-1} K f + f F_n(p_{k-1}) \\ &= f\left(\frac{K}{k-1} D(p_{k-1}) + F_n(p_{k-1})\right) = f(F_n + \frac{K}{k-1} D)(p_{k-1}) \in \langle f^2 \rangle. \end{aligned}$$

Hence $(F_n + \frac{K}{k-1} D)(p_{k-1}) \in \langle f \rangle$. Applying Lemma 4.18 for the irreducible vector field $\mathbf{F}_n + \frac{K}{k-1}\mathbf{D}$, we have that $p_{k-1} \in \langle f \rangle$ and consequently $p_k \in \langle f^2 \rangle$.

Consider now the case $m = 3$. If $F_n(p_k) \in \langle f^3 \rangle$ then $F_n(p_k) \in \langle f^2 \rangle$ and by the previous paragraph we have that there exists $p_{k-2} \in \mathcal{P}_{k-2}$ such that $p_k = f^2 p_{k-2}$, therefore

$$\begin{aligned} F_n(p_k) &= F_n(f^2 p_{k-2}) = p_{k-2} F_n(f^2) + f^2 F_n(p_{k-2}) = 2 p_{k-2} K f^2 + f^2 F_n(p_{k-2}) \\ &= f^2\left(\frac{2K}{k-2} D(p_{k-2}) + F_n(p_{k-2})\right) = f^2(F_n + \frac{2K}{k-2} D)(p_{k-2}) \in \langle f^3 \rangle. \end{aligned}$$

Hence $(F_n + \frac{2K}{k-2} D)(p_{k-2}) \in \langle f \rangle$ and as $\mathbf{F}_n + \frac{2K}{k-2}\mathbf{D}$ is irreducible, applying Lemma 4.18 we have that $p_{k-2} \in \langle f \rangle$ and consequently $p_k \in \langle f^3 \rangle$. Reasoning by induction we get the result for all m natural number. ■

Lemma 4.21 *Consider the vector field* $\mathbf{F}_{2,a} = (x(-qx + (q+r)y), y((p+r)x - py))^T$ *with* p, q, r *natural numbers. Given* m *a natural number, assume that for every* $k \geq m$, *it satisfies that* $p + q + r \neq p\frac{k}{j}$, $p + q + r \neq q\frac{k}{j}$, $p + q + r \neq r\frac{k}{j}$, $j = 1, \ldots, m-1$. *If* $p_k \in \mathcal{P}_k$ *such that* $F_{2,a}(p_k) \in \langle f_i^m \rangle$, *where* $f_1 = x$, $f_2 = y$, $f_3 = x - y$, *are invariant curves at the origin of* $\mathbf{F}_{2,a}$, *then* $p_k \in \langle f_i^m \rangle$, $i = 1, 2, 3$.

Proof. We prove the case $i = 1$, $(f_1 = x)$, the cases $i = 2, 3$ are analogous.
The cofactor of x is $K_1 = -qx + (q+r)y$. From Lemma 4.20, it is enough to prove that $\mathbf{F}_{2,a} + \frac{jK_1}{k-j}\mathbf{D}$, $0 \leq j \leq m-1$, are irreducible, i.e.

$$\left(-q + \frac{jq}{-k+j}\right) x^2 + \left(q + r - \frac{j(q+r)}{-k+j}\right) yx, \quad \left(p + r + \frac{jq}{-k+j}\right) yx - \left(p + \frac{j(q+r)}{-k+j}\right) y^2$$

are coprime. Analyzing the different factorization of both polynomials, one has that both polynomials are coprime if and only if $p + q + r \neq p\frac{k}{j}$, $j = 1, \ldots, m-1$.

For $f_2 = y$ and $f_3 = x - y$, reasoning in a similar way, it is easy to prove that the conditions are $p + q + r \neq q\frac{k}{j}$ and $p + q + r \neq r\frac{k}{j}$, $j = 1, \ldots, m-1$, respectively. ■

Lemma 4.22 *Consider* $\mathbf{F}_{2,b} = (-2qxy, (p+2q)x^2 + py^2)^T$*, with* p, q *natural numbers. Given* m *a natural number, assume that for every* $k \geq m$*, it satisfies that* $p + 2q \neq p\frac{k}{j}$*,* $p + q \neq q\frac{k}{j}$*,* $j = 1, \ldots, m-1$*. If* $p_k \in \mathcal{P}_k$ *such that* $F_{2,a}(p_k) \in \langle f_i^m \rangle$*, where* $f_1 = x$*,* $f_2 = x^2 + y^2$*, are invariant curves at the origin of* $\mathbf{F}_{2,b}$*, then* $p_k \in \langle f_i^m \rangle$*,* $i = 1, 2$*.*

Proof. We prove the case $i = 1$, $(f_1 = x)$, the cases $i = 2, 3$ are analogous.
The cofactor of x is $K_1 = -2qy$. From Lemma 4.20, it is enough to prove that $\mathbf{F}_{2,b} + \frac{jK_1}{k-j}\mathbf{D}$, $0 \leq j \leq m-1$, are irreducible, i.e.

$$\tfrac{2qk}{-k+j}xy, \quad (p+2q)x^2 + \tfrac{(-pk+jp+2qj)}{-k+j}y^2$$

are coprime. Both polynomials are coprime if and only if $p + 2q \neq p\frac{k}{j}$, $j = 1, \ldots, m-1$.

For $f_2 = x^2 + y^2$, reasoning in a similar way, it is easy to prove that the conditions are $p + q \neq q\frac{k}{j}$, $j = 1, \ldots,\ m-1$. ■

Acknowledgments. This work has been partially supported by Ministerio de Ciencia, Innovación y Universidades, Spain (project PGC2018-096265-B-I00) and by Consejería de Economía, Innovación, Ciencia y Empleo de la Junta de Andalucía, Spain (projects FQM-276, UHU-1260150 and P12-FQM-1658).

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