Comparison of the guaranteed state estimator and the zonotopic Kalman filter for linear time-variant systems

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***Abstract*—This manuscript compares two well-known formulations of set-membership estimators for linear time-variant plants based on zonotopes. The guaranteed state estimator is based on computing sequential intersections between the predicted set and the each of strip of states consistent with the different components of the output. The zonotopic Kalman filter proposes a formulation inspired on the Kalman filter. Both formulations seek to minimize the uncertainty of the estimation, measured through the *F-radius*. This letter proves that both formulations are equivalent when the noises affecting the different components of the output vector are not correlated. On the other hand, when there are some cross-relations between the noise components, the zonotopic Kalman filter is able to produce more accurate estimations. Those facts are considered together with the computational requirement of both formulations, leading to a recommendation on the observer choice. The theoretical analysis is complemented with a numerical example.**

***Index Terms*—Linear Time-Variant, Set-membership observer, Zonotopes, Guaranteed state estimator, Zonotopic Kalman filter.**

I. Introduction

State estimation of dynamical plants allows to reconstruct the complete state vector from partial information measured from the plant. It is an essential task when monitoring a plant or, as a part of a control loop. One particular formulation of state estimator is called set-membership observer and its objective is to find a set containing the state vector. It dates back to the end of the 60's, with the work of F. Schweppe [1], in which an ellipsoidal bounding of the state was proposed.

During the years, different mathematical descriptions of the sets have been used: intervals in [2], ellipsoids in [3], parallelotopes in [4], zonotopes in [5], constrained zonotopes in [6], or the more general polytopes in [7]. Without going deep into the details, the more complex or general is the formulation of the set, the more accurate the estimation results are, but the computational cost to find the sets increases.

This is, perhaps, the reason why the use of zonotopes for state estimation has become very common. Zonotopes, which are centrally symmetric convex sets found by an affine transformation of the unit hypercube, allows very simple manipulations such as linear transformations or Minkowskii sums, improving the estimation performance with respect to interval and ellipsoidal approaches (see [8], [9]). Perhaps, the main limitation of zonotopes, as pointed out in [10], is that they are not closed under intersection. Constrained zonotopes or general polytopes can remove this limitation.

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In the literature, one can find two well-known formulations of zonotopic observers, from which many others have been derived: the so-called guaranteed state estimator in [11], and the Zonotopic Kalman Filter (ZKF) in [12]. The former proposes to intersect the set containing the predicted state with the strip of states consistent with the measured output, and hence, is sometimes named strip-based observer (see [13]). For m-dimensional outputs, this intersection is sequentially made, component by component, until the filtered set is found. The authors in [11] present two methods to compute the set that contains the intersection: one of them intends to minimize the size of the generator vectors of the set, and the other one pursues the minimization of the volume of the set. The ZKF takes its name because its resemblance with the Kalman filter. The filtered set, in this case, exploits the information of the output vector in a unique step. The observer gain (similar in concept to the Kalman filter gain) is found by minimizing the *F-radius* of the resulting filtered zonotope, which happens to be similar to the trace of the covariance matrix in the Kalman filter. In [13], this kind of observers were called set-propagation observers.

These formulations, and other set-membership observers, have been extensively compared via numerical experiments (see [13] or [14]). Quite recently, some authors have deepened in the equivalencies between them from a theoretical point of view. This is the case of [15], in which the guaranteed state estimator in [11] was shown to be equivalent to the ZKF in [12] under three conditions: the estimator [11] pursues the minimization of the size of the generator vectors, LTI plant, and scalar output.

In [16], the equivalence of the ZKF to another set-membership observer was shown for time-invariant descriptor systems. This other set-membership observer was not based on a sequential intersection of strips, as the one in [11], and the optimization method was exactly the same one of the ZKF. Those two facts were exploited in the proof of equivalence.

This manuscript contributes with an holistic comparison between the two original observer structures in [11] and [12]:

- A proof of equivalence is proposed under the condition of uncorrelated measurement noise, extending the results in [15] for time-variant plants and multidimensional outputs.
- The choice of the order reduction technique chosen is proven to have no effect in the proof of equivalence.
- The ZKF is shown to obtain better estimation performance when the components of the noise are correlated.
- The guaranteed state estimator is shown to require less computational effort to obtain the sequential intersections with the strips than the unique filtering step in ZKF.

The letter includes a numerical example to illustrate the comparison and highlight theses points.

II. Preliminaries

Notation. The concatenation of a sequence of matrices A_i, for $i = 1, \ldots, n$, of appropriate dimensions, is defined with the operator $\underset{i=1}{\overset{n}{\text{cat}}}\{A_i\} = [A_1\ A_2\ \cdots\ A_n]$. The concatenation of arbitrary matrices $A, B, \ldots, C$ is defined with the operator $[A, B, \ldots, C]$. Sequence $A_1, A_2, \ldots, A_n$ of bounded matrices A_i is denoted as $[A_i]$.

Definition 1. A zonotope $\mathcal{X}$, denoted with calligraphic, capital letters, is a centrally symmetric, convex set determined by its center $c \in \mathbb{R}^n$, and by a matrix $H \in \mathbb{R}^{n \times q}$: $\mathcal{X} = \langle c, H \rangle = \{c + \sum_{i=1}^{q} \varsigma_i h_i : \forall i,\ |\varsigma_i| \leq 1\}$, where $h_i \in \mathbb{R}^n$ (columns of H) are called *generator vectors*.

The *order* of a zonotope is given by the number of generator vectors [11]. The *F-radius* of zonotope $\mathcal{X}$ is the Frobenius norm of its generator matrix H, this is, $||\mathcal{X}||_F \equiv \sqrt{tr(H^T H)}$. Let $\mathcal{X} = \langle c_x, H_x \rangle$ and $\mathcal{Y} = \langle c_y, H_y \rangle$ be two zonotopes, and let R be a matrix of appropriate dimensions. A linear transformation of a zonotope is given by $R\mathcal{X} = \langle Rc_x, RH_x \rangle$; the Minkowski sum of two zonotopes is obtained as $\mathcal{X} \oplus \mathcal{Y} = \langle c_x + c_y, [H_x, H_y] \rangle$. Given a matrix A and any vectors such that $x \in \mathcal{X}$ and $w \in \mathcal{W}$, it holds that $y := Ax + w \in A\mathcal{X} \oplus \mathcal{W}$.

III. Plant description

Consider a discrete-time linear time-variant perturbed plant:

$$x(k+1) = A(k)x(k) + B(k)w(k), \tag{1}$$

$$y(k) = C(k)x(k) + v(k), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the output vector, $w \in \mathbb{R}^{n_w}$ represents process noise or disturbances, and $v \in \mathbb{R}^m$ are measurement noises. Matrices $A(k), B(k), C(k)$ are known matrices of appropriate dimensions.

Assumption 1. Process and measurement noises are bounded signals, such that $w(k) \in \mathcal{W}(k) = \langle 0, Q(k) \rangle$, $v(k) \in \mathcal{V}(k) = \langle 0, R(k) \rangle$, being $Q(k)$ and $R(k)$ known matrices of adequate dimensions.

Assumption 2. Given any two sequences of bounded matrices $[A(k)]$ and $[C(k)]$, $k \geq 0$, a bounded sequence of gains $[L(k)]$ and a bounded sequence of semi-positive definite matrices $[W(k)]$ exist such that $x(k+1) = (A(k) - L(k)C(k))x(k)$ is α-stable, that is, stable with decay rate α, with respect to $[A(k)]$ and $[C(k)]$, with $\alpha < 1$, and with $W(k)$ as sequence of Lyapunov matrices.

Assumption 1 requires the noises to belong to some known bounded sets described as zero-centered zonotopes. It is a quite standard assumption in set-membership estimation. Please notice that this assumption is more general than assuming independent bounds of each component of the noises, as in [15], [16], which is equivalent to assume $Q(k)$ and/or $R(k)$ diagonal. This consideration is important later.

Assumption 2 introduces a robust detectability condition, required to ensure convergence of the ZKF in [12]. Please note that, in the case of a time-invariant plant, Assumption 2 demands detectability of pair (A, C), which is standard.

IV. Set-membership zonotopic observers

A. Zonotopic observers

A zonotopic observer's goal is to find filtered and predicted sets,

- Filtered set: $\hat{\mathcal{X}}(k|k) = \langle c(k|k), H(k|k) \rangle$,
- Predicted set: $\hat{\mathcal{X}}(k+1|k) = \langle c(k+1|k), H(k+1|k) \rangle$,

described with zonotopes as defined in Section II, intended to contain the actual state of the plant.

Problem 1. Consider a plant described by (1) with $x(k_0) \in \hat{\mathcal{X}}(k_0|k_0-1)$, and outputs (2). The zonotopic observer must:

a) Find sets, by means of zonotopes, in which the actual state is continuously contained. In particular, filtered and predicted sets must be found such that $x(k) \in \hat{\mathcal{X}}(k|k)$ and $x(k+1) \in \hat{\mathcal{X}}(k+1|k)$, $\forall k \geq k_0$.

b) Ensure that the estimation uncertainty, measured through the *F-radius* of the filtered zonotopes, is bounded. In particular, $\exists \beta$ finite such that $||\hat{\mathcal{X}}(k|k)||_F < \beta$, $\forall k \geq k_0$.

c) Minimize the estimation uncertainty, measured through the *F-radius* of the filtered zonotopes.

B. Zonotopic Kalman Filter

The ZKF presented [12] is formulated as follows.

Filtering step:

$$c(k|k) = c(k|k-1) + L(k)\left(y(k) - C(k)c(k|k-1)\right) \tag{3}$$

$$H(k|k) = \left[\left(I - L(k)C(k)\right)H(k|k-1),\ \ -L(k)R(k)\right] \tag{4}$$

$$L(k) = P(k|k-1)C(k)^\top \times \left(C(k)P(k|k-1)C(k)^\top + R(k)R(k)^\top\right)^{-1} \tag{5}$$

Prediction step:

$$c(k+1|k) = A(k)c(k|k) \tag{6}$$

$$H(k+1|k) = [A(k)H(k|k),\ \ B(k)Q(k)] \tag{7}$$

Matrix $P(k|k-1) \equiv H(k|k-1)H(k|k-1)^\top$ represents the covariation of the predicted zonotope, as defined in [12]. It plays a similar role than the covariance matrix in the formulation of the Kalman filter.

Theorem 1. [12] Under Assumptions 1 and 2, the Zonotopic Kalman Filter (3)-(7) solves Problem 1.

To prevent for an unbounded growth of the order of the zonotopes, an order reduction step must be introduced, usually between filtering and prediction steps. There are many methods to reduce the order (the interested reader is forwarded to [12], [17]). Since this operation is not important for the results in this manuscript, no more mention will be given to it.

C. Guaranteed state estimator

The guaranteed state estimator presented in [11] proposed an algorithm to compute a set, represented by a zonotope, that contains the states consistent with the output and noise.

The filtered set is obtained by a sequential intersection between the previous predicted set with m strips $\mathcal{S}_i(k)$ consistent with each component of the measured output. The i-th component of the output is defined by:

$$y_i(k) = C_i(k)x(k) + v_i(k),$$

where $C_i(k)$ is the i-th row of matrix $C(k)$, and $v_i(k)$ is the i-th component of the measurement noise. This measurement noise is a scalar that satisfies $|v_i(k)| \leq r_i(k)$, being $r_i(k) > 0$ the bound, which can be found from the zonotope $\mathcal{V}(k)$ as explained in [11].

Property 1. When $R(k)$ is diagonal for some k, $r_i(k) = |R_i(k)|$, being $R_i(k)$ the i-th element of the diagonal of $R(k)$. Otherwise, $r_i(k) = \sum_j |R_{i,j}(k)|$, being $R_{i,j}(k)$ the element (i, j) of the matrix.

The strip of states consistent with $y_i(k)$ is:

$$\mathcal{S}_i(k) = \{x(k) \in \mathbb{R}^n : |C_i(k)x(k) - y_i(k)| \leq r_i(k)\}. \quad (8)$$

Each of the sequential filtering steps intends to find a filtered set $\hat{\mathcal{X}}(k|k)|_i$ such that:

$$\hat{\mathcal{X}}(k|k)|_{i-1} \cap \mathcal{S}_i(k) \subseteq \hat{\mathcal{X}}(k|k)|_i = \langle c(k|k)|_i, H(k|k)|_i \rangle, \quad (9)$$

being $\hat{\mathcal{X}}(k|k)|_0 = \hat{\mathcal{X}}(k|k-1)$. In [11] the authors propose two methods to find this set. One of them pursues to minimize the size of the generator vectors of that zonotope (which is shown to be equivalent to the minimization of the Frobenius norm of the generator matrix), while the other one intends to minimize the volume of the zonotope. By solving an adequate optimization problem, the next formulation is obtained:

Sequential filtering steps $i = 1, \ldots, m$**:**

$$c(k|k)|_i = c(k|k)|_{i-1} + \lambda_i(k) \left(y_i(k) - C_i(k)c(k|k)|_{i-1} \right) \quad (10)$$

$$H(k|k)|_i = \left[\left(I - \lambda_i(k)C_i(k) \right) H(k|k)|_{i-1}, \;\; -\lambda_i(k)r_i(k) \right] \quad (11)$$

$$\lambda_i(k) : \min_{\lambda_i(k)} ||H(k|k)_i||_F \text{ or } \min_{\lambda_i(k)} \text{vol}(\hat{\mathcal{X}}(k|k)_i) \quad (12)$$

Final filtered set: after m iterations

$$c(k|k) = c(k|k)|_m \quad (13)$$

$$H(k|k) = H(k|k)|_m \quad (14)$$

Prediction step:

$$c(k+1|k) = A(k)c(k|k) \quad (15)$$

$$H(k+1|k) = [A(k)H(k|k), \;\; B(k)Q(k)] \quad (16)$$

Even though the minimization of the volume produces a more accurate filtered set, the computation of the gains $\lambda_i(k)$ requires to solve a complex optimization problem for each sequential step i. On the other hand, the gain that minimizes the Frobenius norm of the generator matrix or, equivalently, the *F-radius* of the filtered set, can be found analytically:

$$\lambda_i(k) = \frac{H(k|k)|_{i-1}H(k|k)|_{i-1}^\top C_i}{C_i(k)H(k|k)|_{i-1}H(k|k)|_{i-1}^\top C_i(k)^\top + r_i(k)^2}. \quad (17)$$

Theorem 2. [11] Under Assumption 1, the guaranteed state estimator (3)-(7) solves Problem 1a and 1c.

Please realize one important difference between both formulations. By considering the extra Assumption 2, the ZKF ensures the boundedness of the *F-radius* of the filtered sets or, in other words, the size of the generator vectors.

V. Comparison between the two observers

The comparison is focused in three aspects: the equivalence of both formulations when $R(k)$ is diagonal; the computational requirements; and the case of $R(k)$ being non-diagonal.

Please note that both observer structures handle the process noise $w(k)$ (and $Q(k)$) in the prediction step in the same way (see (7)-(16)), with a linear transformation of the filtered set. Then, no particular analysis concerning $Q(k)$ is required.

A. Equivalence of both formulations for diagonal $R(k)$

Next theorem states that, for diagonal $R(k)$ with non-zero diagonal terms, both formulations yield the same filtered and predicted sets and, as a consequence, the guaranteed state estimator inherits the boundedness property from the ZKF.

Theorem 3. For the discrete-time linear time-variant plant (1)-(2) under Assumption 1 with $R(k)$ diagonal with non-zero diagonal terms, the Zonotopic Kalman Filter (3)-(7) and the guaranteed state estimator (10)-(16) with gain (17) produce exactly the same filtered and predicted sets. As a consequence, under Assumption 2, the guaranteed state estimator (10)-(16) solves Problem 1 completely.

The authors in [15] proved that both observer formulations are equivalent for $m = 1$, and for a LTI system, which is direct from equations (5) and (17). Theorem 3 introduces, for the first time, a general proof of the equivalence of both formulations when $m > 1$ for linear time-variant systems.

Proof of Theorem 3. The proof is organised as follows. Firstly, a reformulation of the ZKF is introduced. Later, it is proven that the covariation matrices, centers and generation matrices of the filtered sets of both observers are the same. Then, as a consequence, the proof of Theorem 3 follows.

a) Reformulation of the ZKF: Considering (4), the covariation matrix of the filtered set, this is $P(k|k) \equiv H(k|k)H(k|k)^\top$, satisfies:

$$P(k|k) = L(k)R(k)R(k)^\top L(k)^\top + \\ (I - L(k)C(k))H(k|k-1)H(k|k-1)^\top (I - L(k)C(k))^\top .$$

Defining, in a similar way, the covariation matrix of the predicted set as $P(k|k-1) \equiv H(k|k-1)H(k|k-1)^\top$:

$$\begin{aligned} P(k|k) &= (I - L(k)C(k))P(k|k-1)(I - L(k)C(k))^\top \\ &\quad + L(k)R(k)R(k)^\top L(k)^\top \\ &= P(k|k-1) + L(k)S(k|k-1)L(k)^\top \\ &\quad - L(k)C(k)P(k|k-1) - P(k|k-1)C(k)^\top L(k)^\top, \end{aligned}$$

where $S(k|k-1) = C(k)P(k|k-1)C(k)^\top + R(k)R(k)^\top$. From (5), it is satisfied that $L(k)S(k|k-1)L(k)^\top = P(k|k-1)C(k)^\top S(k|k-1)^{-1}S(k|k-1)L(k)^\top$. Then,

$$\begin{aligned} P(k|k) &= P(k|k-1) - L(k)C(k)P(k|k-1) \\ &= P(k|k-1) \\ &\quad - P(k|k-1)C(k)^\top S(k|k-1)^{-1}C(k)P(k|k-1). \end{aligned}$$

Using Woodbury formula ($(E + FGH)^{-1} \equiv E^{-1} - E^{-1}F(G^{-1} + HE^{-1}F)^{-1}HE^{-1}$), it is obtained:

$$P(k|k)^{-1} = P(k|k-1)^{-1} + C(k)^\top \left(R(k)R(k)^\top\right)^{-1} C(k).$$

The existence of the inverses is ensured by the assumption of $R(k)$ diagonal with non-zeros diagonal terms, and by an adequate initialization of $H(k_0|k_0-1)H(k_0|k_0-1)^\top$ as a positive definite matrix. Hence, considering the diagonal structure of matrix $R(k)$ (see Property 1):

$$P(k|k)^{-1} = P(k|k-1)^{-1} + \sum_{i=1}^{m} \frac{C_i(k)^\top C_i(k)}{R_i(k)^2}. \tag{18}$$

Using the Matrix Inversion Lemma ($A^{-1}B(D-CA^{-1}B)^{-1} = (A-BD^{-1}C)^{-1}BD^{-1}$), the observer gain (5) can be rewritten as:

$$\begin{aligned} L(k) &= \left(P(k|k-1)^{-1} + C(k)^\top \left(R(k)R(k)^\top\right)^{-1} C(k)\right)^{-1} \\ &\quad \times C(k)^\top \left(R(k)R(k)^\top\right)^{-1} \\ &= P(k|k)C(k)^\top \left(R(k)R(k)^\top\right)^{-1}. \end{aligned}$$

Considering again the diagonal structure of matrix $R(k)$:

$$L(k) = \begin{bmatrix} L_1(k) & L_2(k) & \cdots & L_m(k) \end{bmatrix}, \tag{19}$$

where $L_i(k) = \frac{P(k|k)C_i(k)^\top}{R_i(k)^2}$.

Finally, the center and generator matrix of the filtered set (3)-(4) can be reformulated with some manipulations:

$$c(k|k) = c(k|k-1) + \sum_{i=1}^{m} L_i(k)(y_i(k) - C_i(k)c(k|k-1)), \tag{20}$$

$$\begin{aligned} H(k|k) = \Bigg[&\left(I - \sum_{i=1}^{m} L_i(k)C_i(k)\right) H(k|k-1), \\ &\operatorname*{cat}_{i=1}^{m} \{-L_i(k)R_i(k)\} \Bigg]. \end{aligned} \tag{21}$$

b) Equivalence of covariation matrices: Although the concept of covariation was not used in [11], we are going to use the same nomenclature in the guaranteed state estimator. Using again Woodbury formula and similar manipulations, and considering (11), the evolution of the covariation $P(k|k)|_i \equiv H(k|k)|_i H(k|k)|_i^\top$ for the sequential filtering steps satisfies:

$$P(k|k)|_i^{-1} = P(k|k)|_{i-1}^{-1} + \frac{C_i(k)^\top C_i(k)}{r_i(k)^2},$$

for $i = 1, \ldots, m$. Please realize that, according to Property 1, $r_i(k)$ matches the diagonal element of $R(k)$ denoted as $R_i(k)$. Iterating this equations m times:

$$P(k|k)|_m^{-1} = P(k|k)|_0^{-1} + \sum_{i=1}^{m} \frac{C_i(k)^\top C_i(k)}{r_i(k)^2}.$$

Since $P(k|k)|_0 = P(k|k-1)$ and $P(k|k)|_m = P(k|k)$, then:

$$P(k|k)^{-1} = P(k|k-1)^{-1} + \sum_{i=1}^{m} \frac{C_i(k)^\top C_i(k)}{r_i(k)^2},$$

which matches equation (18) of the ZKF.

c) Equivalence of centers: From equation (10), and knowing that $c(k|k)|_0 = c(k|k-1)$, the centers for the first sequential steps can be found:

$$\begin{aligned} c(k|k)|_1 &= c(k|k)|_0 + \lambda_1(k)\left(y_1(k) - C_1(k)c(k|k)|_0\right) \\ &= c(k|k-1) + \lambda_1(k)e_1(k), \\ c(k|k)|_2 &= c(k|k)|_1 + \lambda_2(k)\left(y_2(k) - C_2(k)c(k|k)|_1\right) \\ &= c(k|k-1) + \left(I - \lambda_2(k)C_2(k)\right)\lambda_1(k)e_1(k) \\ &\quad + \lambda_2(k)e_2(k), \end{aligned}$$

where $e_i(k) \equiv (y_i(k) - C_i(k)c(k|k-1))$. The sequential iteration of m reveals a structure:

$$\begin{aligned} c(k|k) &= c(k|k)|_m \\ &= c(k|k-1) + \lambda_m(k)e_m(k) \\ &\quad + \sum_{i=1}^{m-1} \left\{ \prod_{j=i+1}^{m} (I - \lambda_j(k)C_j(k)) \right\} \lambda_i(k)e_i(k) \end{aligned}$$

It is proven next that the previous equation for the center of the filtered set is equivalent to (20) found for the ZKF.

Using similar manipulations as before, one can find that:

$$P(k|k)|_{i+1} = (I - \lambda_{i+1}(k)C_{i+1}(k))P(k|k)|_i.$$

Then,

$$\left\{\prod_{j=i+1}^{m}(I-\lambda_j(k)C_j(k))\right\}\lambda_i(k)=$$
$$\left\{\prod_{j=i+2}^{m}(I-\lambda_j(k)C_j(k))\right\}(I-\lambda_{i+1}(k)C_{i+1}(k))\lambda_i(k).$$

Considering again that (17) and the Matrix Inversion Lemma lead to $\lambda_i(k)=\frac{P(k|k)|_i C_i(k)^\top}{r_i(k)^2}$, previous equation can be rewritten as:

$$\left\{\prod_{j=i+1}^{m}(I-\lambda_j(k)C_j(k))\right\}\lambda_i(k)=$$
$$\left\{\prod_{j=i+2}^{m}(I-\lambda_j(k)C_j(k))\right\}\frac{P(k|k)|_{i+1}C_i(k)^\top}{r_i(k)^2}.$$

Continuing the chain until the first element of the product:

$$\left\{\prod_{j=i+1}^{m}(I-\lambda_j(k)C_j(k))\right\}\lambda_i(k)=\frac{P(k|k)|_m C_i(k)^\top}{r_i(k)^2},$$

which is equal to $L_i(k)$. So, finally, it is obtained that the center of the sequential filtered set evolves as:

$$c(k|k)=c(k|k-1)+\sum_{i=1}^{m}L_i(k)(y_i(k)-C_i(k)c(k|k-1)),$$

which matches equation (20).

d) Equivalence of generator matrices: With similar mathematical manipulations as before, the first elements of the sequence of generator matrices can be found from (11), revealing a structure that can be written as:

$$H(k|k)|_m=\Bigg[\prod_{i=1}^{m}(I-\lambda_i(k)C_i(k))\,H(k|k-1),$$
$$\operatorname*{cat}_{i=1}^{m-1}\left\{-\left\{\prod_{j=i+1}^{m}(I-\lambda_j(k)C_j(k))\right\}\lambda_i(k)r_i(k)\right\},$$
$$-\lambda_m(k)R_m(k)\Bigg],$$

which, according to the relation between $\lambda_i(k)$ and $L_i(k)$ found before, can be rewritten as:

$$H(k|k)|_m=\Bigg[\prod_{i=1}^{m}(I-\lambda_i(k)C_i(k))\,H(k|k-1),$$
$$\operatorname*{cat}_{i=1}^{m}\{-L_i(k)r_i(k)\}\Bigg].$$

The first productory can be reformulated, after some mathematical manipulations, as:

$$\prod_{i=1}^{m}(I-\lambda_i(k)C_i(k))=I-\lambda_m(k)C_m(k)$$
$$-\sum_{i=1}^{m-1}\left\{\prod_{j=i+1}^{m}(I-\lambda_j(k)C_j(k))\right\}\lambda_i(k)C_i(k).$$

Then, it is proven that:

$$\prod_{i=1}^{m}(I-\lambda_i(k)C_i(k))=I-\sum_{i=1}^{m}L_i(k)C_i(k).$$

Therefore, the generator matrix for the filtered set of the guaranteed state estimator matches that of the ZKF in (21).

e) Proof of Theorem 3: The proof is now immediate realizing that the filtered set for the both observers are exactly the same (even when the observer gains are different), and the prediction step is the same in both observers. Then, under Assumptions 1 and 2, the guaranteed state estimator also solves Problem 1 completely.

The inclusion of an order reduction operation does not affect the proof, as long as both observers perform the same reduction technique, and no reduction is made in between the sequential filtering steps. □

Remark. The case of $R(k)$ with zero diagonal terms requires a perfectly accurate sensor, which is unreal in practical applications. Furthermore, adding an infinitely small $\delta>0$ to the zero diagonal terms will generate the same estimation sets, and the proof of equivalence would still hold.

B. Computational requirements

From a computational point of view, both structures only require standard matrix manipulations (inverse, transpose, multiplication and addition). However, for a large-scale plant, with a large number of sensors, the computation of $L(k)$ in (5) demands the inversion of a $m\times m$ matrix. This operation can be computationally hard in the ZKF if m grows. In contrast, the guaranteed state estimator needs to compute m scalar divisions in (17), which is simpler in terms of computation and memory requirements.

C. The general case of non-diagonal $R(k)$

When matrix $R(k)$ is non-diagonal there exist cross relations between the different components of the noise. In other words, the fact that some component takes some value, might affect to the bound of the other components.

These cross relations are not exploited in the guaranteed state estimator as the value of $r_i(k)$ computed with Property 1 considers the worst case for each sequential filtering step. On the other hand, the ZKF is formulated in a unique step and, hence, it is able the exploit this information to further reduce the final filtered set.

VI. Numerical example

This example illustrates the equivalence between both formulations when $R(k)$ is diagonal, and the improvements obtained with the ZKF when $R(k)$ is non-diagonal.

To do so, a set of random LTI discrete-time systems of the form of (1)-(2) with $A, B \in \mathbb{R}^{4\times 4}, C \in \mathbb{R}^{3\times 4}$ is generated using `drss` function in MATLAB. The initial condition $x(k_0)$ is randomly chosen within the unit hypercube centered in zero. Process noise is modeled with $Q = 0.02I$, being I an identity matrix of appropriate dimensions. Concerning $R(k)$, two situations are considered: $R_1(k) = R_1 = 0.02I$, and $R_2(k) = R_2 = 0.02\mathbf{1}$, being $\mathbf{1}$ a matrix of ones of appropriate dimensions. Random process and measurement noises are generated for each experiment meeting Assumption 1, but the same random signals are applied to both observers.

Regarding the estimators, the initial zonotope is chosen as $\hat{\mathcal{X}}(k_0|k_0-1) = \langle 0, I\rangle$, so $x(k_0) \in \hat{\mathcal{X}}(k_0|k_0-1)$. Finally, two different order reduction techniques are compared. The first one is based on a Principal Component Analysis (see Algorithm 2 in [17]), while the second one, based in an ordering of the generator vectors by their norms, was proposed in [11]. The PCA-based method generates a set with the same order than the dimension of the system. The second method generates a set with maximum order of q. The order reduction step is applied right after the filtered set is found.

In order to present a numerical comparison of the performance of the observers, we use the next index (defined in [18]) for a set of N_s simulation of N_i instants:

$$I = \frac{1}{N_s}\sum_{j=1}^{N_s}\left(\frac{1}{N_i}\sum_{k=1}^{N_i}||\hat{\mathcal{X}}(k|k)||_F\right).$$

This index measures the average value of the *F-radius* of the sets computed for all time instants and simulations.

The results are included in Table I for different values of N_i, N_s. As expected, the value of the index obtained by both methods is exactly the same for R_1 provided that the same order reduction technique is used. On the other hand, the ZKF obtains more accurate estimation for R_2 by considering the cross-relations between the noise components.

TABLE I
VALUE OF I FOR THE TWO OBSERVERS (GSE AND ZKF) AND DIFFERENT ORDER REDUCTION TECHNIQUES

$N_s = 100, N_i = 100$	GSE R_1	ZKF R_1	GSE R_2	ZKF R_2
[17] PCA	0.0601	0.0601	0.0599	0.0545
[11] $q = 100$	0.0504	0.0504	0.0495	0.0451
[11] $q = 10$	0.1024	0.1024	0.0941	0.0903
$N_s = 10, N_i = 100$	GSE R_1	ZKF R_1	GSE R_2	ZKF R_2
[17] PCA	0.0554	0.0554	0.0683	0.0546
[11] $q = 100$	0.0486	0.0486	0.0534	0.0440
[11] $q = 10$	0.0764	0.0764	0.1094	0.0969
$N_s = 10, N_i = 10$	GSE R_1	ZKF R_1	GSE R_2	ZKF R_2
[17] PCA	0.1391	0.1391	0.1585	0.1526
[11] $q = 100$	0.1391	0.1391	0.1585	0.1526
[11] $q = 10$	0.1622	0.1622	0.1983	0.1896

VII. Conclusions

The paper has compared two well-known formulations of set-membership observers for linear time-variant systems: the Zonotopic Kalman Filter and the guaranteed state estimator. Based on the conducted analysis, the guaranteed state estimator is recommended when matrix $R(k)$ is diagonal for all k, as the estimated filtered and predicted sets are exactly the same, but the computational effort is lower. For general non-diagonal $R(k)$, there exists a trade-off between computational effort and filtering performance that needs to be considered.

Future work will explore if the equivalence still holds for particular nonlinear systems, or when the optimization metric is other than the *F-radius*.

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