Analytic integrability inside a family of degenerate centers

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Abstract

In this paper we study the analytic integrability around the origin inside a family of degenerate centers or perturbations of them. For this family analytic integrability does not imply formal orbital equivalence to a Hamiltonian system. It is shown how difficult is the integrability problem even inside this simple family of degenerate centers or perturbations of them.

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1 Introduction

This work centers in determining the existence of analytic first integrals in a neighborhood of a degenerate center singular point which indeed is a center or which is a perturbation of a degenerate center. It is well–known that a system has a center at a singular point only if it is monodromic and it has either linear part of center type, i.e. with imaginary eigenvalues (nondegenerate point), or nilpotent linear part (nilpotent point) or null linear part (degenerate point). Any nondegenerate center has always a local analytic first integral in a neighborhood of its singular point, see [11, 17, 21, 22]. However, there are nilpotent and degenerate centers without a local analytic first integral, see [12, 13, 14, 15, 16, 19, 20, 22, 23] and references therein. There are methods to detect nondegenerate and nilpotent centers of a given family of polynomial vector field, see [16, 19, 20]. However there is no method to detect centers for a general degenerate singular point.

Any nilpotent center has a local analytic first integral if, and only if, it is analytically equivalent to the Hamiltonian system $\dot{x} = y$, $\dot{y} = -x^{2k-1}$ where $k > 1$, see for instance [3]. The integrability problem has been studied for a few families of degenerate singular points.

In [4] the analytic integrability problem for degenerate systems of the form

$$\dot{x} = y^3 + 3\mu x^2 y + o(|x, y|^3), \qquad \dot{y} = -x^3 - 3\mu x y^2 + o(|x, y|^3), \qquad \mu \in \mathbb{R}, \tag{1.1}$$

was analyzed and the following result was established.

Theorem 1.1 *System* (1.1) *is analytically integrable if, and only if, it is formally equivalent to* $\dot{x} = y^3 + 3\mu x^2 y$, $\dot{y} = -x^3 - 3\mu x y^2$.

In [5] the analytic integrability problem for degenerate systems of the form

$$\dot{x} = y^3 + 2ax^3 y + \cdots, \qquad \dot{y} = -x^5 - 3ax^2 y^2 + \cdots, \qquad a \in \mathbb{R}, \tag{1.2}$$

was also studied where here the dots mean terms of higher order than the first component in the quasi-homogeneous expansion (see definition of quasi-homogeneous expansion below). The next result was established in [5].

Theorem 1.2 *System* (1.2) *is analytically integrable if, and only if, it is formally equivalent to* $\dot{x} = y^3 + 2ax^3y - 2\beta_9 x^4 y$, $\dot{y} = -x^5 - 3ax^2y^2 + 4\beta_9 x^3 y^2$, *where* β_9 *is written in the parameters of the first quasi-homogeneous components of system* (1.2).

The integrability problem for these two previous families can be solved using the following result given in [3].

Theorem 1.3 *Let us assume that the lowest-degree quasi-homogeneous term of the degenerate system is* $\mathbf{F}_r = \mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$, *where* h *has only simple factors. Then, the quoted system is formally integrable if and only if it is formally conjugated, via dissipative transformations, to a divergence-free system.*

First we give some comments and results about the integrability problem for a vector field $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, in function of its first quasi-homogeneous component $\mathbf{F}_r$. As we have said Theorem 1.3 solves the integrability problem in the case that $\mathbf{F}_r = \mathbf{X}_h$ where all the irreducible factors of h over $\mathbb{C}[x, y]$ are simple. The case $\operatorname{div}(\mathbf{F}_r) \neq 0$ with $\mathbf{F}_r$ reducible or $\mathbf{F}_r = \mathbf{X}_h$ where h has multiple factors is not solve and it is an open problem.

However a necessary condition in order that $\mathbf{F}$ be integrable is that $\mathbf{F}_r$ be also integrable. The integrability problem for $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$ with $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, $\mu \neq 0$ is solved in [6], see Theorem 4.12. This theorem shows the necessity of certain resonances in the parameters of the vector field in order that $\mathbf{F}_r$ be integrable. Moreover such resonances determine the exponents of the irreducible factors of h that appear in the first integral.

Lemma 1.4 *Let* $\mathbf{F}_r = (P, Q) \in \mathcal{Q}_r^{\mathbf{t}}$ *irreducible. If* $I \in \mathcal{P}_i^{\mathbf{t}}$ *is a first integral of* $\mathbf{F}_r$, *then* $i \geq r + |\mathbf{t}|$ *and exists* $f \in \mathcal{P}_{i-r-|\mathbf{t}|}^{\mathbf{t}}$ *such that* $f\,\mathbf{F}_r = \mathbf{X}_I$.

Proof. The reasoning is taken from the proof of [6, Theorem 3.1]. We have $\nabla I \cdot \mathbf{F}_r = 0$. As the components of $\mathbf{F}_r$ have not common factors, we have that exist $f \in \mathcal{P}_{i-r-|\mathbf{t}|}^{\mathbf{t}}$ such that $\nabla I = f(-Q, P)$. Thus $\mathbf{X}_I = f(P, Q)^T = f\,\mathbf{F}_r$. ■

On the other hand if $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$ is integrable and I is a first integral, from the previous Lemma we deduce that exists a quasi-homogeneous function f such that $I = \frac{1}{i} \mathbf{D}_0 \wedge \mathbf{X}_I = \frac{1}{i} f\,\mathbf{D}_0 \wedge \mathbf{F}_r = \frac{r+|\mathbf{t}|}{i} f\,h$. Hence the integrability problem of a quasi-homogeneous vector field with not null divergence is equivalent to the integrability problem of a quasi-homogeneous Hamiltonian vector field where its Hamilton function has multiple factors. It is logical to think that the resonances appearing in a problem are also determinants in the other.

We recall that if h has not simple factors Theorem 1.3 can not be applied. This is the case of the family we are going to study in this work which have h with multiple factors. In fact we consider degenerate systems of the form

$$\dot{x} = -y(x^2 + y^2) + \cdots, \quad \dot{y} = x(x^2 + y^2) + \cdots, \tag{1.3}$$

where here the dots mean terms of higher order than the first component in the homogeneous expansion and which corresponds to $\mathbf{X}_h + \cdots$ whose $h = (x^2 + y^2)^2/4$. This type of systems was studied in [25] where it was proved that there exist centers inside this family without an analytic first integral.

We will see that there exist a blow-up and a scale of time that transforms system (1.3) into a nondegenerate system of the form

$$\dot{u} = -u + \cdots, \quad \dot{v} = v + \cdots. \tag{1.4}$$

Using this transformation the center problem of system (1.3) reduce to the center problem for a nondegenerate system for which there are methods as we have mentioned. However the integrability problem of the systems of the form (1.3) is open. It is clear that if system (1.3) is analytic integrable then system (1.4) is also analytic integrable but the converse is not true. Hence analytic integrability of system (1.3) is not solved and in this work we will see that even for such simple family it is a difficult problem. We solve the analytic integrability problem for the generic case and only a degenerate case remains open. The paper also shows the difficulty in obtaining the complete solution for any degenerate differential system.

2 A family of perturbed degenerate centers

The objective of this work is to study the formal integrability around the origin of a system of the form

$$(\dot{x},\dot{y})^T = (x^2+y^2)(-y,x)^T + q-h.h.o.t. \tag{2.5}$$

First we highlight that in [18] it is proved that if a vector field is formal integrable around an isolated singular point then it is analytic integrable. Hence, in what follows the study of the formal integrability is equivalent to the study of the analytic integrability.

As we have said in [26], the blow up

$$x = u(u^2+v^2)^2, \quad y = v(u^2+v^2)^2, \tag{2.6}$$

and the scaling of time $dT = (u^2+v^2)^5 dt$ transforms system (2.5) in a nondegenerate analytic system

$$(u',v')^T = (-v,u)^T + q-h.h.o.t. \tag{2.7}$$

Note that this transformation is an homeomorphism but not a diffeomorphism. Therefore this transformation preserves the topological properties but not the differentiable. The center problem is completely solved because the existence of a center for system (2.5) is equivalent to the existence of a center for system (2.7). However, it is not the same for the formal integrability. If system (2.5) is formal integrable then system (2.7) is also formal integrable and it has a center at the origin. But the converse is not true. If system(2.7) is $\mathcal{C}^\omega$-integrable with a formal first integral H, undoing the transformation (2.6) we obtain a first integral of system (2.5) which is not always formal. Therefore, the fact that system (2.7) be $\mathcal{C}^\omega$-integrable does not implies that system (2.5) be also $\mathcal{C}^\omega$-integrable.

The following example shows the situation. Consider the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} x^2y^2+y^4 \\ x^3y \end{pmatrix}. \tag{2.8}$$

This system is monodromic, to see that it is enough to apply [8, Theorem 2]. Moreover it is R_x-reversible, hence it has a center at the origin. The associated nondegenerate system (2.7) is $\mathcal{C}^\omega$-integrable. However, using Corollary 5.15, it is possible to show that system (2.8) is not formal integrable, i.e., the integrability problem is not equivalent for system (2.5) that for system (2.7). Therefore, we have to propose another method to study the formal integrability for system (2.5), because with the study of the non-degenerate associated system is not enough.

In the rest of the paper first we give some previous results and we obtain conditions to have formal integrability of system (2.5). Finally we apply our results to two particular families.

3 Preliminary definitions and results

We write a system of differential equations as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{3.9}$$

where $\mathbf{x} = (x,y)^{\mathrm{T}} \in \mathbb{R}^2$ and $\mathbf{F} = (P,Q)^{\mathrm{T}}$ is an analytic planar vector field defined in a neighborhood of the origin $U \subset \mathbb{R}^2$.

We recall the following concepts and definitions. Given $\mathbf{t} = (t_1,t_2)$ non-null with t_1 and t_2 non-negative integer numbers without common factors, a function f of two variables is quasi-homogeneous of type $\mathbf{t}$ and degree k if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (F_1,F_2)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $F_1 \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k.

We write the vector field $\mathbf{F}$ as

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots,$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$, because any vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Such expansions will be expressed as $\mathbf{F} = \mathbf{F}_r + q-h.h.o.t.$

We will denote by $\mathbf{D}_0 = (t_1 x, t_2 y)^T \in \mathcal{Q}_0^{\mathbf{t}}$ (a dissipative quasi-homogeneous vector field) and by $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (the Hamiltonian vector field associated to the polynomial h). If $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, then $\mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$, where $|\mathbf{t}| = t_1 + t_2$. Moreover, it is proved that every $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_k = \mathbf{X}_h + \mu \mathbf{D}_0 \tag{3.10}$$

with $h = (\mathbf{D}_0 \wedge \mathbf{F}_k)/(k + |\mathbf{t}|)$ and $\mu = \operatorname{div}(\mathbf{F}_k)/(k + |\mathbf{t}|)$, where $\mathbf{D}_0 \wedge \mathbf{F}_k \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$ is the wedge product of both vector fields and $\operatorname{div}(\mathbf{F}_k) \in \mathcal{P}_k^{\mathbf{t}}$ is the divergence of $\mathbf{F}_k$, see [3]. This sum is known as the conservative-dissipative splitting of a quasi-homogeneous vector field.

The procedure to obtain a normal form under equivalence for the system (3.9) is described in, for example, [7]. Here we recall some notions.

The key in the problem of obtaining a normal form for the system (3.9) is to analyze the effect of a near-identity transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$ and a reparametrization of the time given by $dt/dT = 1 + \mu_k(\mathbf{x})$, where $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ and $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$, with $k \geq 1$.

The quasi-homogeneous terms of the transformed system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$ agree with the original ones up to degree $r + k - 1$ and for the degree $r + k$ it has

$$\mathbf{G}_{r+k} \quad = \quad \mathbf{F}_{r+k} - \mathcal{L}_k(\mathbf{P}_k, \mu_k)$$

where we have introduced the homological operator under formal orbital equivalence:

$$\begin{aligned} \mathcal{L}_k \quad : \quad & \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \mu_k) \to \mathcal{L}_k(\mathbf{P}_k, \mu_k) = [\mathbf{P}_k, \mathbf{F}_r] - \mu_k \mathbf{F}_r. \end{aligned} \tag{3.11}$$

Following the ideas of the conventional normal form theory, it is enough to choose $(\mathbf{P}_k, \mu_k) \in \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}}$ adequately in order to simplify the $(r+k)$-degree quasi-homogeneous term in system (3.9), by annihilating the part belonging to the range of the linear operator $\mathcal{L}_k$. And we say that the corresponding term has been reduced to normal form under orbital equivalence. So, by means of a sequence of time-reparametrizations and near identity transformations system (3.9) can be formally reduced to normal form under orbital equivalence.

To study the homological operator, we define the linear operator

$$\begin{aligned} \ell_{k-r} \quad : \quad & \mathcal{P}_{k-r}^{\mathbf{t}} \longrightarrow \mathcal{P}_k^{\mathbf{t}} \\ & \mu_{k-r} \to L_{\mathbf{F}_r} \mu_{k-r} := \nabla \mu_{k-r} \cdot \mathbf{F}_r, \end{aligned}$$

i.e. the Lie derivative of the lowest degree quasi-homogeneous term of $\mathbf{F}$.

Following the developing made in [7], we have the following lemma adapted to our purposes.

Lemma 3.5 *Let us assume that the lowest-degree quasi-homogeneous term of system* (3.9) *is* $\mathbf{F}_r = \mathbf{X}_H \in \mathcal{Q}_r^{\mathbf{t}}$, *where* $H \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$. *A complementary subspace to the range of* $\mathcal{L}_k$, *with* $k \geq 1$, *can be written as*

$$Cor(\mathcal{L}_k) = \mathbf{X}_{S_{r+|\mathbf{t}|+k}} \oplus Cor(\ell_k)\mathbf{D}_0,$$

being $S_{r+|\mathbf{t}|+k}$ *a subspace verifying* $Cor(\ell_{k+|\mathbf{t}|}) = S_{r+|\mathbf{t}|+k} \oplus \left(HCor(\ell_{k-r}) \cap Cor(\ell_{k+|\mathbf{t}|})\right)$.

Proposition 3.6 *Let us assume that the lowest-degree quasi-homogeneous term of system* (3.9) *is* $\mathbf{F}_r = \mathbf{X}_H \in \mathcal{Q}_r^{\mathbf{t}}$, *with* $H = h^2$. *A complementary subspace of the range of* $L_{\mathbf{F}_r}$ *in* $\mathcal{P}_k^{\mathbf{t}}$ *is*

$$Cor(L_{\mathbf{X}_{h^2}}) \cap \mathcal{P}_k^{\mathbf{t}} = h\left(Cor(L_{\mathbf{X}_h}) \cap \mathcal{P}_{k-\frac{r+|\mathbf{t}|}{2}}^{\mathbf{t}}\right) \oplus \Delta_k,$$

where Δ_k *is such that* $\mathcal{P}_k^{\mathbf{t}} = \Delta_k \oplus < h >$.

Proof. By the definition of the linear operator, we have $L_{\mathbf{X}_{h^2}}(\mu) = \nabla\mu \cdot \mathbf{X}_{h^2} = 2h\nabla\mu \cdot \mathbf{X}_h$, from where we deduce $Range(L_{\mathbf{X}_{h^2}}) \in h \cdot Range(L_{\mathbf{X}_h})$. On the other hand, $h \cdot Range(L_{\mathbf{X}_h}) \in Range(L_{\mathbf{X}_{h^2}})$ because $h \cdot L_{\mathbf{X}_h}(\mu) = h \cdot \nabla\mu \cdot \mathbf{X}_h = \nabla\frac{\mu}{2} \cdot \mathbf{X}_{h^2} = L_{\mathbf{X}_{h^2}}(\frac{\mu}{2})$. Hence we have the equality $Range(L_{\mathbf{X}_{h^2}}) = h \cdot Range(L_{\mathbf{X}_h})$. From here the result follows. ■

In order to study the formal integrability of system (2.5) we first compute a formal equivalent normal form for such system.

Theorem 3.7 *A formally orbital equivalent normal form for system (2.5) is*

$$(\dot{x}, \dot{y})^T = (x^2+y^2)(-y, x)^T + \sum_{j=3}^{\infty} \mathbf{X}_{a_j x^{j+2} + b_j x^{j+1} y} + (c_j x^j + d_j x^{j-1} y)\mathbf{D}_0 + \sum_{j=2}^{\infty} e_{2j} h^j \mathbf{D}_0. \tag{3.12}$$

Proof. In this case, $r = 2$ and $|\mathbf{t}| = 2$. From Lemma 3.5, $Cor(\mathcal{L}_k) = \mathbf{X}_{S_{4+k}} \oplus Cor(\ell_k)\mathbf{D}_0$ with $k \geq 1$. So we must calculate $Cor(\ell_k)$ and the subspace S_{4+k}.

From proposition 3.6, we have $Cor(\ell_k) = Cor(L_{\mathbf{X}_{h^2}}) \cap \mathcal{P}^{\mathbf{t}}_{k+2} = h\left(Cor(L_{\mathbf{X}_h}) \cap \mathcal{P}^{\mathbf{t}}_k\right) \oplus \Delta_{k+2}$, $k \geq 1$, where we can choose $\Delta_j =< x^j, x^{j-1}y >$. It is easy to proof that $Cor(L_{\mathbf{X}_h}) \cap \mathcal{P}^{\mathbf{t}}_k = 0$ if k is odd and $Cor(L_{\mathbf{X}_h}) \cap \mathcal{P}^{\mathbf{t}}_k =< h^l >$ if $k = 2l$ with $l \geq 1$. So, $Cor(\ell_k) =< x^{k+2}, x^{k+1}y >$ for k odd and $Cor(\ell_k) =< x^{k+2}, x^{k+1}y, h^{l+1} >$ for $k = 2l$, $l \geq 1$.

The subspace S_{4+k} is such that $Cor(\ell_{k+2}) = S_{4+k} \oplus (HCor(\ell_{k-2}) \cap Cor(\ell_{k+2}))$. Taking into account the previous calculations, we can deduce $S_{4+k} =< x^{k+4}, x^{k+3}y >$, with $k \geq 1$.

Therefore, $Cor(\mathcal{L}_k) =< \mathbf{X}_{x^{k+4}}, \mathbf{X}_{x^{k+3}y}, x^{k+2}\mathbf{D}_0, x^{k+1}y\mathbf{D}_0 >$ if k is odd and $Cor(\mathcal{L}_k) =< \mathbf{X}_{x^{k+4}}, \mathbf{X}_{x^{k+3}y}, x^{k+2}\mathbf{D}_0, x^{k+1}y\mathbf{D}_0, h^{\frac{k}{2}+1}\mathbf{D}_0 >$ if k is even, with $k \geq 1$. ■

As a consequence we obtain the following result

Theorem 3.8 *System* (2.5) *is formal integrable if and only if, system* (3.12) *is formal integrable.*

From now on we will study the formal integrability of system (3.12). In the following lemma we apply the blow-up technique to transform system (3.12) into another whose first quasi-homogeneous component be irreducible and thus learn more about necessary conditions of integrability and existence of invariant curves of system (3.12).

Lemma 3.9 *The blow up $x = u$, $y = uv$, the scaling of time $dT = u^2dt$ and then the translation $v = x_2 - i\sigma$, where $\sigma = \pm 1$ with $i = \sqrt{-1}$ transforms system (3.12) into*

$$\begin{aligned} u' &= 2ux_2 + 3i\sigma u x_2^2 - u x_2^3 - \sum_{j\geq 3} b_j u^{j-1} + \sum_{j\geq 3}(c_j - i\sigma d_j)u^{j-1} + \sum_{j\geq 3} d_j u^{j-1}x_2 \\ &\quad + \sum_{j\geq 2} e_{2j} x_2^j (x_2 - 2i\sigma)^j u^{2j-1}, \qquad (3.13) \\ x_2' &= -4x_2^2 - 4i\sigma x_2^3 + x_2^4 + \sum_{j\geq 3}(j+2)(a_j - i\sigma b_j)u^{j-2} + \sum_{j\geq 3}(j+2)b_j u^{j-2}x_2. \end{aligned}$$

In order to study the integrability conditions for the system (3.12), we studied the different Newton diagrams in terms of the parameters of system (3.13), and we make some definitions in terms of the coefficients of the normal form which we will be useful later.

The Newton diagram associated to system (3.13), (see [9, 10]), has always a non compact edge leaving from the inner vertex $V_0 = (1, 2)$ associated to the vector $(2ux_2, -4x_2^2)^T$, see Figure 1.

We define the following non-negative integers

$$\begin{aligned} m &:= \min\left\{j \in \mathbb{N}, j \geq 3 : a_j^2 + b_j^2 \neq 0\right\}, \\ l &:= \min\left\{j \in \mathbb{N}, j \geq 3 : c_j^2 + d_j^2 \neq 0\right\}, \qquad (3.14) \\ k &:= \min\left\{j \in \mathbb{N}, j \geq 2 : e_{2j} \neq 0\right\}, \end{aligned}$$

where $\min(\emptyset) := +\infty$.

In the case $m < 2(l-1)$ with $m < +\infty$ the Newton diagram has a exterior vertex $V_2 = (m-1, 0)$ associated to the vector $(0, (m+2)(a_m - i\sigma b_m)u^{m-2})^T$.

The case $2(l-1) < m < +\infty$ has also the inner vertex $V_1 = (l-1, 1)$ associated to the vector field $((c_l - i\sigma d_l)u^{l-1}, 0)^T$ and two compact edges.

In the case $m = 2(l-1) < +\infty$, the vector field $((c_l - i\sigma d_l)u^{l-1}, 0)^T$ is not associated to any vertex, its points in the Newton diagram are located on the unique compact edge of the Newton diagram.

If $m = +\infty$, we will see by Theorem 4.13 that system (2.5) is integrable if $l = k = +\infty$. In this case the system is $\mathcal{C}^\infty$ orbitally equivalent to system $(x^2+y^2)(-y, x)^T$. Hence the system has a first integral of the form $I = (x^2 + y^2) + \cdots$.

If $m < +\infty$, with $l = k = +\infty$, system (2.5) is formally integrable because system (3.12) is Hamiltonian of the form $(\dot{x}, \dot{y})^T = (x^2 + y^2)(-y, x)^T + \sum_{j=3}^{\infty} \mathbf{X}_{a_j x^{j+2} + b_j x^{j+1} y} = \mathbf{X}_h + \sum_{j=3}^{\infty} \mathbf{X}_{a_j x^{j+2} + b_j x^{j+1} y}$ where $h = (x^2 + y^2)^2/4$. Hence system (2.5) has a first integral of the form $I = (x^2 + y^2)^2 + \cdots$.

Consequently, from now on we assume $l < +\infty$ o $k < +\infty$; i.e., we assume that the vector field is not formally equivalent to a Hamiltonian one.

We remark the new scenario in which we are. Theorem 1.3 does not characterize the integrability of these vector fields, see below Lemma 6.19.

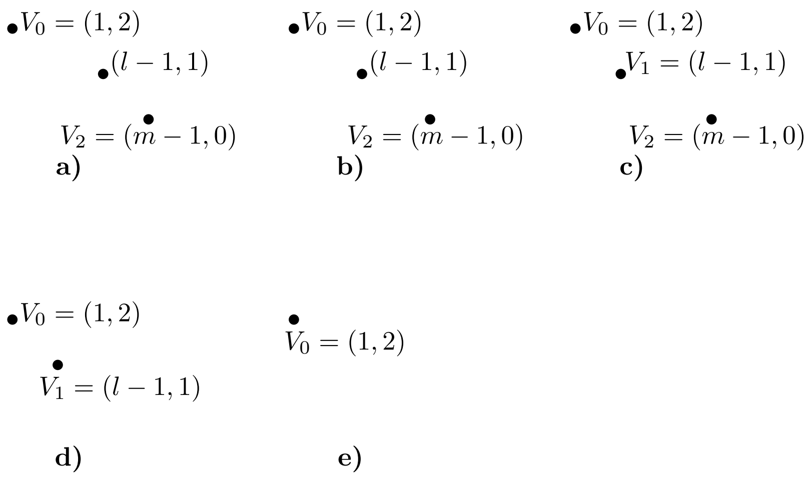

Figure 1: Different situations, Newton diagram in black line. **a)** Case $m < 2(l-1)$. **b)** Case $m = 2(l-1)$. **c)** Case $2(l-1) < m$. **d)** Case $l < +\infty$, $m = +\infty$. **e)** Case $l = m = +\infty$

4 The main results

First we study the case when system (2.5) is reducible.

Definition 4.10 *We say that the vector field* $\mathbf{F}$ *is reducible if there exists a scalar function* f*, with* $f(0) = 0$*, such that* $\mathbf{F} = f \cdot \mathbf{G}$.

Theorem 4.11 *Assume that system (2.5) is integrable and let* $\mathbf{F} = (x^2 + y^2)(-y, x)^T + \cdots$ *its associated vector field. System (2.5) has a first integral of the form* $I = (x^2 + y^2) + \cdots$ *if and only if* $\mathbf{F}$ *is reducible.*

Proof. We assume that the vector field $\mathbf{F}$ is reducible. Hence we it can be written $\mathbf{F} = (x^2 + y^2 + \cdots)[(-y, x)^T + \cdots] = f \cdot \mathbf{G}$. The vector field $\mathbf{G}$ is integrable because $\mathbf{F}$ it is integrable. Moreover a first integral of $\mathbf{G}$ is of the form $I = (x^2 + y^2) + \cdots$. Therefore, $I = (x^2 + y^2) + \cdots$ is a first integral of system (2.5).

Now we assume that $I = (x^2+y^2)+\cdots$ is a first integral of $\mathbf{F}$. By the Moser Lemma, see [24, Lemma 2.3], there exist a change of variables $(x, y) = \phi(u, v)$ such that $I(\phi(u, v)) = h(u, v)$, where we denote $h = x^2 + y^2$, being $I(\phi(u, v)) = h(u, v)$ a first integral of the transformed vector field $\mathbf{G} = \phi_* \mathbf{F}$.
We write $\mathbf{G}$ as $\mathbf{G} = (x^2 + y^2)(-y, x)^T + \sum_{j>2} \mathbf{G}_j$. We claim that $\mathbf{G}_j = f_j \cdot (-y, x)^T$, $\forall j > 2$. If h is a first integral of $\mathbf{G}$, then is a first integral for each $\mathbf{G}_j$. $\nabla h \cdot \mathbf{G} = 0 \Rightarrow \nabla h \cdot (x^2 + y^2)(-y, x)^T + \sum_{j>2} \nabla h \cdot \mathbf{G}_j = 0$ which implies $\nabla h \cdot \mathbf{G}_j = 0$, $\forall j > 2$.
Now, we write each $\mathbf{G}_j$ as $\mathbf{G}_j = \mathbf{X}_{g_j} + \mu_j \mathbf{D}_0 + \lambda_j (-y, x)^T$, with $g_j \in \Delta_j$, $\forall j > 2$, (see definition in Proposition 3.6). Then we have $\nabla h \cdot \mathbf{G}_j = \nabla h \cdot \mathbf{X}_{g_j} + 2\mu_j h = 0$; therefore, h is an invariant curve of $\mathbf{X}_{g_j}$ and we can write $g_j = f_j \cdot h$, $\forall j > 2$. But this is in contradiction with $g_j \in \Delta_j$; then we have $g_j = 0$, $\forall j > 2$. Consequently, $\nabla h \cdot \mathbf{G}_j = 2\mu_j h = 0$, from where $\mu_j = 0$, $\forall j > 2$. From here $\mathbf{G}_j = \lambda_j (-y, x)^T$, $\forall j > 2$. Thus we have $\mathbf{G} = (x^2+y^2)(-y, x)^T + \sum_{j>2} \lambda_j (-y, x)^T = (x^2 + y^2 + \sum_{j>2} \lambda_j)(-y, x)^T$. In short we can write the vector field $\mathbf{G} = f \cdot (-y, x)^T$. Undoing the change ϕ we obtain $\mathbf{F} = \tilde{f} \cdot [(-y, x)^T + \cdots]$, where $\tilde{f} = \phi^{-1}(f) = x^2 + y^2 + \cdots$.
Finally, we have write $\mathbf{F}$ as $\mathbf{F} = (x^2 + y^2 + \cdots) \cdot [(-y, x)^T + \cdots]$, thus the vector field $\mathbf{F}$ is reducible. ∎

In the following we assume that the vector field $\mathbf{F}$ associated to system (3.12) is irreducible. Now we recall the following result proved in [6] that we have adapted to our purpose.

Theorem 4.12 *The system* $\dot{\mathbf{x}} = \mathbf{F}_r = (P, Q)^T$, *with* $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$, P, Q *coprimes,* $PQ \not\equiv 0$ *and* $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$ *is integrable if, and only if,*

$$(\mathbf{D}_0 \wedge \mathbf{F}_r)(x,y) = c x^{\delta_x} y^{\delta_y} \prod_{i=1}^{m} (y^{t_1} - \lambda_i x^{t_2}), \tag{4.15}$$

with $r + |\mathbf{t}| = t_1\delta_x + t_2\delta_y + t_1 t_2 m$, $c \neq 0$, $\delta_x, \delta_y \in \{0,1\}$, $\delta_x + \delta_y + m \geq 2$ *and* $\lambda_1, ..., \lambda_m$ *distinct complex numbers not zero and exist* n_x, n_y, n_i, $i = 1, \cdots, m$ *non-negative integers, not all zero, verifying*

$$\begin{cases} Res[\eta(x,1), 0] = -\frac{1}{t_2} + \frac{(n_x+1)(r+|\mathbf{t}|)}{t_2 \widetilde{M}} & \text{if } \delta_x = 1, \\ Res[\eta(1,y), 0] = \frac{1}{t_1} - \frac{(n_y+1)(r+|\mathbf{t}|)}{t_1 \widetilde{M}} & \text{if } \delta_y = 1, \\ Res[\eta(1,y), \lambda_i^{1/t_1}] = \frac{1}{t_1} - \frac{(n_i+1)(r+|\mathbf{t}|)}{t_1 \widetilde{M}} & i = 1, \cdots, m, \end{cases} \tag{4.16}$$

where $\widetilde{M} = t_1(n_x+1)\delta_x + t_2(n_y+1)\delta_y + t_1 t_2 \sum_{j=1}^{m}(n_j+1)$ *and* $\eta = \dfrac{\operatorname{div}(\mathbf{F}_r)}{\mathbf{D}_0 \wedge \mathbf{F}_r}$.

Moreover a first integral is of the form

$$I = x^{\delta_x(n_x+1)} y^{\delta_y(n_y+1)} \prod_{i=1}^{m} (y^{t_1} - \lambda_i x^{t_2})^{n_i+1}. \tag{4.17}$$

We denote by $\mathbf{G}$ the vector field associated to system (3.13). Recall that if system (3.12) is formal integrable then system (3.13) also is formal integrable.

In the following result we give necessary integrability conditions of system (3.12), (a normal form of system (2.5)). In its proof we will use Theorem 4.12.

Theorem 4.13 *Let* $\mathbf{F}$ *be the associated vector field of system (3.12) and consider* m, l, k *defined in (3.14). We assume* $\mathbf{F}$ *irreducible. If* $\mathbf{F}$ *is formal integrable then the following conditions are satisfied.*

- $\min\{l, m\} < 2k$, $k < +\infty$,
- $m \leq 2(l-1)$, $l < +\infty$,
- *If* $m = 2(l-1)$, $m < +\infty$, *then* $\dfrac{(l-2)^2}{16l^2}(c_l - i\sigma d_l)^2 + (a_m - i\sigma b_m) \neq 0$

Proof. We will prove the converse result. If it is verified any of the following conditions, then system (3.12) is not formal integrable.

- $\min\{l, m\} \geq 2k$, $k < +\infty$,
- $m > 2(l-1)$, $l < +\infty$,
- $m = 2(l-1)$, $m < +\infty$ and $\dfrac{(l-2)^2}{16l^2}(c_l - i\sigma d_l)^2 + (a_m - i\sigma b_m) = 0$

We assume that $\min\{l, m\} \geq 2k$. If $\mathbf{F}$ is the associated vector field of system (3.12), then we have $\mathbf{F} = \mathbf{F}_2 + \mathbf{F}_{2k} + \cdots$ with $\mathbf{F}_2 = \frac{1}{2} h \mathbf{X}_h$ and $\mathbf{F}_{2k} = (e_{2k} h^k + \mu_l)\mathbf{D}_0 + \mathbf{X}_{g_m}$, where

- $e_{2k} \neq 0$,
- $\mu_l = c_l x^l + d_l x^{l-1} y$ if $2k = l$, otherwise $\mu_l = 0$,
- $g_m = a_m x^{m+2} + b_m x^{m+1} y$ if $2k = m$, otherwise $g_m = 0$.

If the system was integrable a first integral would be of the form $I = h^p + \sum_{j>2p} I_j$ with $I_j \in \mathbf{P}_j^{(1,1)}$, by Theorem 4.11 we have that $p \geq 2$. The integrability condition ($\nabla I \cdot \mathbf{F} = 0$) is satisfied and at the lower degree $2p + 2k$ is

$$\begin{aligned} 0 &= \nabla h^p \cdot \mathbf{F}_{2k} + \nabla I_{2p+2k-2} \cdot \mathbf{F}_2 \\ &= 2p e_{2k} h^{p+k} + 2p \mu_l h^p + p h^{p-1} \nabla h \cdot \mathbf{X}_{g_m} + h \nabla I_{2p+2k-2} \cdot \mathbf{X}_{\frac{h}{2}} \\ &= h[2p e_{2k} h^{p+k-1} + 2p \mu_l h^{p-1} - 2h^{p-2} \nabla g_m \cdot \mathbf{X}_{\frac{h}{2}} + \nabla I_{2p+2k-2} \cdot \mathbf{X}_{\frac{h}{2}}] \end{aligned}$$

which is compatible only if $e_{2k} = 0$, giving a contradiction.

We assume now that $m > 2(l-1)$. In this case the Newton diagram of (3.13) (after applying the blow-up to system (3.12)), would have two compact edges (if $m < +\infty$) or one (if $m = +\infty$). In any case, we focus in the edge of type $t = (1, l-2)$, common to both situations. This edge has associated the vector field $\mathbf{G}_r = (2ux_2 + (c_l - i\sigma d_l)u^{l-1}, -4x_2^2)^T$, where $r = l-2$.

If $\mathbf{F}$ is formal integrable then $\mathbf{G}$ is formal integrable and $\mathbf{G}_r$ is also is formal integrable. To apply Theorem 4.12, first we compute the Hamiltonian function

$$h = \frac{1}{r+|\mathbf{t}|}(u,(l-2)x_2)^T \wedge \mathbf{G}_r = -\frac{2l}{2l-3}ux_2[x_2 - \lambda u^{l-2}],$$

where $\lambda = \frac{2-l}{2l}(c_l - i\sigma d_l)$. On the other hand,

$$\mu = \frac{1}{r+|\mathbf{t}|}div(\mathbf{F}_r) = -\frac{6}{2l-3}[x_2 - \frac{l-1}{6}(c_l - i\sigma d_l)u^{l-2}].$$

Applying Theorem 4.12 and computing the residue expressions, we have

$$\begin{aligned}
Res[\eta(u,1),0] &= \frac{3}{l} = \frac{-\widetilde{M} + (2l-3)(n_x+1)}{(l-2)\widetilde{M}},\\
Res[\eta(1,x_2),0] &= \frac{l-1}{2-l} = \frac{\widetilde{M} - (2l-3)(n_y+1)}{\widetilde{M}},\\
Res[\eta(1,x_2),\lambda] &= \frac{-l^2-2l+6}{l(2-l)} = \frac{\widetilde{M} - (2l-3)(n_1+1)}{\widetilde{M}},
\end{aligned}$$

where $\widetilde{M} = (n_x+1) + (l-2)(n_y+1) + (l-2)(n_1+1)$. From the second equation we get $n_x + 1 = -(l-2)(n_1+1)$ and taking into account that $l \geq 3$, we would have that n_x should take negative values which implies a contradiction. Hence $\mathbf{G}_r$ is not formal integrable.

Finally we assume that $m = 2(l-1)$ and $\dfrac{(l-2)^2}{16l^2}(c_l - i\sigma d_l)^2 + (a_m - i\sigma b_m) = 0$. In this case, $\mathbf{G}_r = (2ux_2 + (c_l - i\sigma d_l)u^{l-1}, -4x_2^2 + 2l(a_m - i\sigma b_m)u^{2l-4})^T = (P,Q)^T$. Moreover we have

$$\begin{aligned}
h &= \frac{-2l}{2l-3}u(x_2 + \frac{l-2}{4l}(c_l - i\sigma d_l)u^{l-2})^2 := \alpha\, u \cdot f^2\\
\mu &= \frac{-6}{2l-3}(x_2 - \frac{l-1}{6}(c_l - i\sigma d_l)u^{l-2})
\end{aligned}$$

We will see that $\mathbf{G}_r$ is irreducible. Otherwise $\mathbf{G}_r = g \cdot \widehat{\mathbf{G}}_s$, with $\widehat{\mathbf{G}}_s \in \mathcal{Q}_s^{\mathbf{t}}$ and $g \in \mathcal{P}_{r-s}^{\mathbf{t}}$.

Thus we have $\mathbf{D}_0 \wedge \mathbf{G}_r = (r + |\mathbf{t}|)\, h = g\mathbf{D}_0 \wedge \widehat{\mathbf{G}}_s$; therefore, the irreducible factors of g are irreducible factors of h. That is the possible reducible factors of $\mathbf{G}_r$ are u o f. But u is not a factor of Q and f is not a factor of $P = 2u(x_2 + \frac{(c_l - i\sigma d_l)}{2}u^{l-2})$; otherwise $(x_2 + \frac{(c_l - i\sigma d_l)}{2}u^{l-2}) = f$, from where $7l = -2$, which gives a contradiction.

By Theorem 4.12, as $\mathbf{G}_r$ is irreducible and $\mathbf{D}_0 \wedge \mathbf{G}_r$ has not simple factors, $\mathbf{G}_r$ is not integrable and therefore system (3.13) is not integrable. ■

In the following result we give necessary conditions of integrability for system (2.5).

Theorem 4.14 *Let* $\mathbf{F}$ *be the associated vector field of system (3.12) irreducible. Consider* m *and* l *defined in (3.14). If* $\mathbf{F}$ *is integrable, then it is verified*

- **(a)** *If* $m < 2(l-1)$, m *odd, then a first integral is of the form* $I = (x^2+y^2)^2 + \cdots$.
- **(b)** *If* $m < 2(l-1)$, m *even, then a first integral is of the form* $I = f_1 f_2$ *where* $f_1 \not\equiv f_2$ *y* $f_i = (x^2+y^2) + \cdots$ *for* $i = 1, 2$.
- **(c)** *If* $m = 2(l-1)$ *and* $a_m - i\sigma b_m \neq \frac{1}{2l}(c_l - i\sigma d_l)^2$ *then exist* $M \in \mathbb{N}$ *and* $N \in \mathbb{N}$, *with* $M \neq N$, *such that*

$$\begin{aligned}
a_m &= \tfrac{1}{16l^2}\left((l+2)^2\tfrac{(M+N)^2}{(M-N)^2} - (l-2)^2\right)(c_l^2 - d_l^2),\\
b_m &= \tfrac{1}{8l^2}\left((l+2)^2\tfrac{(M+N)^2}{(M-N)^2} - (l-2)^2\right)c_l d_l.
\end{aligned}$$

In this case a first integral is of the form $I = f_1^M f_2^N$ *where* $f_1 \not\equiv f_2$ *and* $f_i = (x^2+y^2) + \cdots$ *for* $i = 1, 2$.

The proof of Theorem 4.14 is given in the next section.

Remark. The case $m = 2(l-1)$ with $a_m - i\sigma b_m = \frac{1}{2l}(c_l - i\sigma d_l)^2$ which corresponds to the case where the principal part of system (3.13) is reducible remains open.

5 Proof of Theorem 4.14

i) If $m < 2(l-1)$, the Newton diagram of system (3.13) has a unique compact of type $(2, m-2)$ associated to the vector field $\mathbf{G}_{m-2} = (2ux_2, -4x_2^2 + (m+2)(a_m - i\sigma b_m)u^{m-2})^T$ with the Hamiltonian function and function μ given by

$$\begin{aligned} h &= \frac{1}{r+|\mathbf{t}|}\mathbf{D}_0 \wedge \mathbf{G}_{m-2} = -\frac{m+2}{m-1}u\left[x_2^2 - (a_m - i\sigma b_m)u^{m-2}\right], \\ \mu &= \frac{1}{r+|\mathbf{t}|}div(\mathbf{G}_{m-2}) = -\frac{3}{m-1}x_2. \end{aligned} \tag{5.18}$$

Here we distinguish if m is odd or even.

- If m is odd, the Hamiltonian function has the irreducible factor $\left[x_2^2 - \lambda u^{m-2}\right]$ because m is odd, where $\lambda = (a_m - i\sigma b_m)$. Now applying Theorem 4.12 and we compute the residue

$$\begin{aligned} Res[\eta(u,1),0] &= \frac{3}{m+2}\frac{-\widetilde{M} + 2(m-1)(n_x+1)}{(m-2)\widetilde{M}}, \\ Res[\eta(1,x_2),\lambda^{1/2}] &= \frac{3}{2(m+2)} = \frac{\widetilde{M} - 2(m-1)(n_1+1)}{2\widetilde{M}}, \end{aligned}$$

where $\widetilde{M} = 2(n_x+1) + 2(m-2)(n_1+1)$. Equating the expressions of the residue and taking into account the value of $\widetilde{M}$, we obtain

$$(n_x+1) = 4(n_1+1).$$

Hence, by Theorem 4.12, a first integral of $\mathbf{G}_{m-2}$ is of the form $I = u^{n_x+1}(x_2^2 - \lambda u^{m-2})^{n_1+1} = u^{4(n_1+1)}(x_2^2 - \lambda u^{m-2})^{n_1+1} = (u^4(x_2^2 - \lambda u^{m-2}))^{n_1+1} = (u^4(x_2^2 - \lambda u^{m-2}))^N$, where $N = n_1 + 1$. Therefore, $I = (u^4(x_2^2 - \lambda u^{m-2}))^N + \cdots$ is a first integral of system (3.13). Since the first component of the vector field is integrable, we can write $I = (u^4(x_2^2 - \lambda u^{m-2}) + \cdots)^N \cdot U$, where U is a unity in the ring of the formal series of the variables u, x_2. Consequently a first integral for system (3.13) is of the form

$$I = u^4(x_2^2 - \lambda u^{m-2}) + \cdots. \tag{5.19}$$

On the other hand, if system (2.5) is integrable, a first integral is of the form $I = (x^2+y^2)^n + \cdots$, with $n \in \mathbb{N}$. If we apply the change of variables described in Lemma 3.9, this first integral becomes $I = u^{2n}x_2^n(x_2 - 2\sigma i)^n + O(u^{2n+1})$, that we can write as

$$I = u^{2n}x_2^n + O(u^{2n+1}). \tag{5.20}$$

Considering the type $\mathbf{t} = (3,1)$ in system (3.13), the first quasi-homogeneous component of the vector field is $\mathbf{G}_1 = (2ux_2, -4x_2^2)^T$. Ordering respect this type the first integral (5.19) we get $I = u^4x_2^2 + \cdots$.. Now we consider (5.20) with respect to the type $\mathbf{t} = (3,1)$ and we have $I = u^{2n}x_2^n + \cdots$. Equating both expressions we deduce that $n = 2$. Therefore, for system (2.5) a first integral is of the form $I = (x^2+y^2)^2 + \cdots$. and this corresponds to case **(a)**.

- If m is even, we can factorizes h; from the expression of (5.18) we have

$$h = -\frac{m+2}{m-1}u[x_2 - \sqrt{a_m - i\sigma b_m}u^{\frac{m-2}{2}}][x_2 + \sqrt{a_m - i\sigma b_m}u^{\frac{m-2}{2}}].$$

Consider $R^2 = \sqrt{a_m^2 + b_m^2}$ and α such that $R^2\cos(\alpha) = a_m$ and $R^2\sin(\alpha) = -\sigma b_m$. Thereby $\sqrt{a_m - i\sigma b_m} = R(\cos(\alpha) + i\sin(\alpha)$. Then we have

$$
\begin{aligned}
h &= -\frac{m+2}{m-1}u[x_2 - \lambda_1 u^{\frac{m-2}{2}}][x_2 - \lambda_2 u^{\frac{m-2}{2}}],\\
\mu &= -\frac{6}{m-1}x_2
\end{aligned}
$$

where $\lambda_1 = R\cos(\alpha) + iR\sin(\alpha)$ and $\lambda_2 = -R\cos(\alpha) - iR\sin(\alpha)$.
Consider the type $\mathbf{t} = (1, \frac{m-2}{2})$ and applying Theorem 4.12 we obtain

$$
\begin{aligned}
Res[\eta(u,1),0] = \frac{6}{m+2} &= \frac{-2\widetilde{M} + 2(m-1)(n_x+1)}{(m-2)\widetilde{M}},\\
Res[\eta(1,x_2),\lambda_i] = \frac{3}{m+2} &= \frac{\widetilde{M} - (m-1)(n_i+1)}{\widetilde{M}},\ i = 1,2,
\end{aligned}
$$

where $\widetilde{M} = (n_x+1) + \frac{m-2}{2}(n_1+1) + \frac{m-2}{2}(n_2+1)$.
We define $M = n_1 + 1$, $N = n_2 + 1$ and solving the equations we get

$$
\begin{aligned}
M &= N = \frac{1}{4}(n_x+1),\\
\widetilde{M} &= \frac{1}{4}(m+2)(n_x+1).
\end{aligned}
$$

Hence by Theorem 4.12, a first integral of $\mathbf{G}_{m-2}$ is of the form $I = u^{n_x+1}(x_2 - \lambda_1 u^{\frac{m-2}{2}})^{n_1+1}(x_2 - \lambda_2 u^{\frac{m-2}{2}})^{n_2+1}$, that we write as $I = (u^4(x_2 - \lambda_1 u^{\frac{m-2}{2}})(x_2 - \lambda_2 u^{\frac{m-2}{2}}))^N = \left[(u^2x_2 - \lambda_1 u^{m-2})(u^2x_2 - \lambda_2 u^{m-2})\right]^N$. Therefore, reasoning as the previous case, as the first quasi-homogeneous component is integrable, we can prove that a first integral of system (3.13) is of the form

$$
I = (u^2x_2 - \lambda_1 u^{m-2} + \cdots)(u^2x_2 - \lambda_2 u^{m-2} + \cdots).
$$

As in the case m odd, we can consider the type $\mathbf{t} = (3,1)$ in system (3.13), and we deduce that a first integral of system (2.5) is of the form $I = (x^2 + y^2 + \cdots)(x^2 + y^2 + \cdots)$. This corresponds to case **(b)**.

ii) if $m = 2(l-1)$, the Newton diagram of system (3.13) has a unique compact edge of type $(1, l-2)$ associated to the vector field $\mathbf{G}_{l-2} = (2ux_2 + (c_l - i\sigma d_l)u^{l-1}, -4x_2^2 + 2l(a_m - i\sigma b_m)u^{2l-4})^T$. As $a_m - i\sigma b_m \neq \frac{1}{2l}(c_l - i\sigma d_l)^2$, the vector field associated to system (3.13) is irreducible since its first quasi-homogeneous component it is irreducible. Therefore we can apply Theorem 4.12, where the Hamiltonian function and the dissipative term of $\mathbf{G}_{l-2}$ are

$$
\begin{aligned}
h &= \frac{-2l}{2l-3}u\Bigg[\left(x_2 + \frac{l-2}{4l}(c_l - i\sigma d_l)u^{l-2}\right)^2\\
&- \left(\frac{(l-2)^2}{16l^2}(c_l - i\sigma d_l)^2 + (a_m - i\sigma b_m)\right)u^{2l-4}\Bigg].\\
\mu &= \frac{-6}{2l-3}\left[x_2 - \frac{l-1}{6}(c_l - i\sigma d_l)u^{l-2}\right],
\end{aligned}
$$

By Theorem 4.13 we have that $\frac{(l-2)^2}{16l^2}(c_l - i\sigma d_l)^2 + (a_m - i\sigma b_m) \neq 0$, hence we can write h as

$$
h = \frac{-2l}{2l-3}u\left[x_2 - \lambda_1 u^{l-2}\right]\left[x_2 - \lambda_2 u^{l-2}\right],
$$

where

$$
\begin{aligned}
\lambda_1 &= -\frac{l-2}{4l}c_l - R\cos(\alpha) + \left(\frac{l-2}{4l}\sigma d_l - R\sin(\alpha)\right)i,\\
\lambda_2 &= -\frac{l-2}{4l}c_l + R\cos(\alpha) + \left(\frac{l-2}{4l}\sigma d_l + R\sin(\alpha)\right)i,
\end{aligned}
$$

and α is an angle such that $\left(a_m + \frac{(l-2)^2}{16l^2}(c_l^2 - d_l^2)\right) = R^2\cos(2\alpha)$, $-\left(b_m + \frac{(l-2)^2}{8l^2}c_l d_l\right)\sigma = R^2\text{sen}(2\alpha)$ and $R^2 = \sqrt{\left(a_m + \frac{(l-2)^2}{16l^2}(c_l^2 - d_l^2)\right)^2 + \left(b_m + \frac{(l-2)^2}{8l^2}c_l d_l\right)^2}$. Now applying Theorem 4.12 there exist n_1, n_2, n_x non–negative integers, not all zero, verifying

$$
\begin{aligned}
Res[\eta(u,1),0] &= \frac{3}{l} = \frac{-\widetilde{M} + (2l-3)(n_x+1)}{(l-2)\widetilde{M}}, \\
Res[\eta(1,x_2),\lambda_1] &= \frac{3[\lambda_1 - \frac{l-1}{6}(c_l - i\sigma d_l)]}{l(\lambda_1 - \lambda_2)} = 1 - \frac{(2l-3)(n_1+1)}{\widetilde{M}}, \\
Res[\eta(1,x_2),\lambda_2] &= \frac{3[\lambda_2 - \frac{l-1}{6}(c_l - i\sigma d_l)]}{l(\lambda_2 - \lambda_1)} = 1 - \frac{(2l-3)(n_2+1)}{\widetilde{M}},
\end{aligned}
$$

where $\widetilde{M} = (n_x+1) + (l-2)[(n_1+1) + (n_2+1)]$. From this equations we get

$$c_l\sin(\alpha) + \sigma d_l\cos(\alpha) = 0, \tag{5.21}$$

$$\frac{1}{4R}\frac{l+2}{l}(c_l\cos(\alpha) - \sigma d_l\sin(\alpha)) = -\frac{M-N}{M+N}, \tag{5.22}$$

where $M = n_1 + 1$, $N = n_2 + 1$ and $n_x + 1 = 2(M+N)$.
We distinguish four cases:

1) Case $c_l \neq 0$, $d_l = 0$. Form (5.21) we have $\sin\alpha = 0$ and $|\cos(\alpha)| = 1$, hence $\cos(2\alpha) = 1$ and $\sin(2\alpha) = 0$. From the expression $-\left(b_m + \frac{(l-2)^2}{8l^2}c_l d_l\right)\sigma = R^2\sin(2\alpha)$, we deduce $b_m = 0$. From the expression $\left(a_m + \frac{(l-2)^2}{16l^2}(c_l^2 - d_l^2)\right) = R^2\cos(2\alpha)$, we deduce $a_m = R^2 - \frac{(l-2)^2}{16l^2}c_l^2$. Condition (5.22) can be written as $(l+2)c_l\cos(\alpha)(M+N) = -4lR(M-N)$. We multiply by $\cos\alpha$ and we isolate $R\cos(\alpha) = -\frac{l+2}{4l}\frac{M+N}{M-N}c_l$ ($M \neq N$). We have $a_m = R^2 - \frac{(l-2)^2}{16l^2}c_l^2 = R^2\cos^2(\alpha) - \frac{(l-2)^2}{16l^2}c_l^2$, and replacing $R\cos\alpha$, we obtain the expression given in the statement.
2) Case $c_l = 0$, $d_l \neq 0$. Analogously to the previous case. From (5.21) we have $\cos\alpha = 0$ and $|\sin(\alpha)| = 1$, hence $\cos(2\alpha) = -1$ and $\sin(2\alpha) = 0$. From the expression $-\left(b_m + \frac{(l-2)^2}{8l^2}c_l d_l\right)\sigma = R^2\sin(2\alpha)$, we deduce $b_m = 0$. From the expression $\left(a_m + \frac{(l-2)^2}{16l^2}(c_l^2 - d_l^2)\right) = R^2\cos(2\alpha)$, we deduce $a_m = \frac{(l-2)^2}{16l^2}d_l^2 - R^2$. Condition (5.22) can be written as $(l+2)\sigma d_l\sin(\alpha)(M+N) = 4lR(M-N)$. We multiply by $\sin\alpha$ and we isolate $R\sin(\alpha) = \frac{l+2}{4l}\frac{M+N}{M-N}\sigma d_l$ ($M \neq N$). We have $a_m = \frac{(l-2)^2}{16l^2}\sigma d_l^2 - R^2 = \frac{(l-2)^2}{16l^2}\sigma d_l^2 - R^2\sin^2(\alpha)$ and replacing $R\sin\alpha$, we obtain the expression given in the statement.
3) Case $c_l^2 + d_l^2 \neq 0$, $c_l^2 - d_l^2 = 0$. From (5.21) we have $\cos^2\alpha = \sin^2\alpha$. From the expression $\left(a_m + \frac{(l-2)^2}{16l^2}(c_l^2 - d_l^2)\right) = R^2\cos(2\alpha)$, we deduce $a_m = 0$. In (5.21) we can isolate $c_l = -\frac{\sigma d_l\cos(\alpha)}{\sin(\alpha)}$, and replacing in (5.22) we get $\frac{(l+2)\sigma d_l}{4lR\sin(\alpha)} = \frac{M-N}{M+N}$. As $M \neq N$, we can isolate $R\sin(\alpha) = \frac{(l+2)\sigma d_l}{4l}\frac{M+N}{M-N}$. On the other hand, if we multiply (5.21) by $\sin\alpha$, we can isolate $\sin^2\alpha = -\frac{\sigma d_l\cos(\alpha)\sin(\alpha)}{c_l}$, and substituting into the last equality we obtain $-R^2\frac{\sigma d_l\cos(\alpha)\sin(\alpha)}{c_l} = \frac{(l+2)^2 d_l^2}{16l^2}\frac{(M+N)^2}{(M-N)^2}$. From here we can isolate $R^2\cos(\alpha)\sin(\alpha) = -\frac{(l+2)^2}{16l^2}\frac{(M+N)^2}{(M-N)^2}\sigma c_l d_l$. Substituting this last equality in the expression $-\left(b_m + \frac{(l-2)^2}{8l^2}c_l d_l\right)\sigma = R^2\sin(2\alpha) = 2R^2\cos(\alpha)\sin(\alpha)$, we can isolate b_m and we obtain the expression given in the statement.
4) Case $c_l^2 + d_l^2 \neq 0$, $c_l^2 - d_l^2 \neq 0$. From (5.21) we have $\tan\alpha = -\sigma\frac{d_l}{c_l}$. Moreover $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)} = \frac{-2\sigma c_l d_l}{c_l^2 - d_l^2}$. On the other hand,

$$\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{-(b_m + \frac{(l-2)^2}{8l^2}c_l d_l)\sigma}{a_m + \frac{(l-2)^2}{16l^2}(c_l^2 - d_l^2)}.$$

   Equating both expressions and isolating a_m, we obtain $a_m = \frac{c_l^2 - d_l^2}{2c_l d_l}b_m$. Reasoning as in the previous case, we obtain the expression of b_m given in the statement, that substituted in the last equality gives the expression of a_m given in the statement.

Hence in this case results that a first integral is of the form $I = u^{2(M+N)}(x_2 - \lambda_1 u^{l-2})^M (x_2 - \lambda_2 u^{l-2})^N$. Reasoning as in previous cases, since the first quasi-homogeneous component of the vector field is integrable, we can write $I = u^{2(M+N)}(x_2 - \lambda_1 u^{l-2} + \cdots)^M (x_2 - \lambda_2 u^{l-2} + \cdots)^N$, i.e., $I = (u^2 x_2 - \lambda_1 u^l + \cdots)^M (u^2 x_2 - \lambda_2 u^l + \cdots)^N$. Ordering respect to the type $\mathbf{t} = (3, 1)$, this first integral becomes $I = (u^2 x_2 + \cdots)^M (u^2 x_2 + \cdots)^N = (u^2 x_2)^{M+N} + \cdots$.

On the other hand, we know that a first integral for system (2.5) is of the form $I = (x^2 + y^2)^n + \cdots$, $n \in \mathbb{N}$. Applying the change described in Lemma 3.9, it can be written as $I = u^{2n} x_2^n + \cdots = (u^2 x_2)^n + \cdots$. Equating both expression, we deduce that $n = M + N$. Therefore, a first integral for system (2.5) is of the form $I = (x^2 + y^2 + \cdots)^M (x^2 + y^2 + \cdots)^N$. This corresponds to the case **(c)**.

Remark. Theorem 4.14 is useful because in its absence to demonstrate the integrability of a vector field $\mathbf{F} = (x^2 + y^2)(-y, x)^T + \cdots$ we would have to try with first integrals $I = (x^2 + y^2)^n + \cdots$ for any $n \in \mathbb{N}$. Theorem 4.14 not only gives us the value of n but also gives us some initial conditions to have formal integrability.

Corollary 5.15 *System(2.8) is not formal integrable.*

Proof. In the computation of the coefficients of the normal formal, the first we find is $b_3 = \frac{1}{5} \neq 0$. Hence taking into account Theorem 4.14, and that we are in the case $m = 3 < 2(l-1)$ for all $l \geq 3$, we have that if system (2.8) is integrable then it has a first integral of the form $I = (x^2 + y^2)^2 + \cdots$. However, imposing the condition of integrability $\nabla I \cdot \mathbf{F} = 0$, this is not verified at degree 8. Therefore, system (2.8) is not integrable. ■

6 Applications

First we show a method to find a quasi-homogeneous normal form based in Lie transformations. Here we present a resume of the procedure, for more details see [2]. The main idea is the use of a generator $\mathbf{U}(\mathbf{x})$ for the change of variables (i.e., the change $\mathbf{x} = \phi(\mathbf{y}, \varepsilon)$ is the unique solution of the Cauchy problem with initial values $\frac{\partial}{\partial \varepsilon}\phi(\mathbf{y}, \varepsilon) = \mathbf{U}(\phi(\mathbf{y}, \varepsilon)), \phi(\mathbf{y}, 0) = \mathbf{y}$). The transformed vector field $\mathbf{G}$ of $\mathbf{F}$, using the change of variables $\mathbf{x} = \Phi(\mathbf{y}, \varepsilon)$ in function of the generator $\mathbf{U}$, see [1], is given by

$$\mathbf{G} = \mathbf{F} + [\mathbf{F}, \mathbf{U}] + \frac{1}{2!}[[\mathbf{F}, \mathbf{U}], \mathbf{U}] + \frac{1}{3!}[[[\mathbf{F}, \mathbf{U}], \mathbf{U}]\mathbf{U}] + \cdots$$

We consider an expansion in quasi-homogeneous terms of the generator $\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2 + \cdots$ and of the vector field $\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots$. Thus we can write the term of degree $r + k$ as

$$[\mathbf{F}, \mathbf{U}]_{r+k} = \sum_{j=1}^{k} [\mathbf{F}_{r+k-j}, \mathbf{U}_j],$$

$$[[\mathbf{F}, \mathbf{U}], \mathbf{U}]_{r+k} = \sum_{j=1}^{k-1} [[\mathbf{F}, \mathbf{U}]_{r+k-j}, \mathbf{U}_j].$$

Using this equalities we have a recursive method to compute the term $\mathbf{G}_{r+k}$ as

$$\mathbf{G}_{r+k} = \sum_{j=0}^{k} \frac{1}{j!}\mathbf{V}_{r+k,j},$$

where

$$\mathbf{V}_{r+k,0} = \mathbf{F}_{r+k}, k \geq 0, \quad \mathbf{V}_{r+k,l} = \sum_{j=1}^{k-l+1} [\mathbf{V}_{r+k-j,l-1}, \mathbf{U}_j], \quad 1 \leq l \leq k.$$

To compute $\mathbf{G}_{r+k}$, we use the Lie triangle. Here we only consider the first two rows, in [2] is fully developed.

$\mathbf{F}_r$	
$\mathbf{F}_{r+1}$	$[\mathbf{F}_r, \mathbf{U}_1]$

(6.23)

This method can be adapted to study the formal orbital equivalence. Consider the system

$$\mathbf{x}' = \frac{d\mathbf{x}}{dT} = \sum_{k\geq 0}(\mathbf{F}_{r+k}(\mathbf{x}) - \sum_{j=1}^{k}\mu_j \mathbf{F}_{r+k-j}),$$

where we have reparametrized the time in the initial system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ by

$$\frac{dt}{dT} = 1 - \sum_{j\geq 1}\mu_j(\mathbf{x}), \mu_j \in Cor(\ell_j).$$

The first elements of the Lie triangle, described in [2], are

$\mathbf{F}_r$		
$\mathbf{F}_{r+1} - \mu_1\mathbf{F}_r$	$[\mathbf{F}_r, \mathbf{U}_1]$	
$\mathbf{F}_{r+2} - \mu_1\mathbf{F}_{r+1} - \mu_2\mathbf{F}_r$	$[\mathbf{F}_{r+1} - \mu_1\mathbf{F}_r, \mathbf{U}_1] + [\mathbf{F}_r, \mathbf{U}_2]$	$[[\mathbf{F}_r, \mathbf{U}_1], \mathbf{U}_1]$

(6.24)

The main idea is that we can choose $\mathbf{U}_k$ y μ_k in order to simplify the quasi-homogeneous terms of degree $r + k$. For each degree, we can solve an homological equation which gives us a quasi-homogeneous normal form up to this degree.

Using this procedure we have the following result.

Lemma 6.16 *Consider the system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 \\ b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4 \end{pmatrix} + q - h.h.o.t. \tag{6.25}$$

The first coefficients of the normal form (3.12) for system (6.25) are:

$$\begin{aligned} a_3 &= \frac{1}{5}(a_{31} - a_{13} + b_{40} - b_{22} + b_{04}), \quad & b_3 &= \frac{1}{5}(-a_{40} + a_{22} - a_{04} + b_{31} - b_{13}), \\ c_3 &= \frac{1}{5}(4a_{40} - 2a_{22} + b_{31} - 3b_{13}), \quad & d_3 &= \frac{1}{5}(3a_{31} - a_{13} + 2b_{22} - 4b_{04}). \end{aligned}$$

6.1 Example 1

Consider the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^3y \\ bx^4 + cx^2y^2 \end{pmatrix}, \tag{6.26}$$

where $a, b, c \in \mathbb{R}$.

Theorem 6.17 *System (6.26) has a first integral of the form $I = x^2 + y^2 + \cdots$ if, and only if, one of the following conditions is satisfied*

(1) $a = 0$, $c = b$.

(2) $b = -a$, $c = 0$.

Proof. First we assume that $I = x^2 + y^2 + \cdots$ is a first integral of system (6.26). Imposing the condition of integrability $\nabla I \cdot \mathbf{F} = 0$ and making an analysis for each degree we obtain the conditions of the statement.

Another way to obtain the necessary conditions is applying Theorem 4.11, that is, $\mathbf{F}$ must be reducible. As the first component is reducible, the second component must be reducible. Thus we obtain $a = 0$, $c = b$, and in this case $x^2 + y^2$ is the factor of reducibility or $c = 0$ and $b = -a$ in which case the vector field $(-y, x)$ is the common factor of both components.

Reciprocally, we first assume that $a = 0$ and $c = b$. In this case system (6.26) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\left[\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ bx^2 \end{pmatrix}\right], \tag{6.27}$$

which is integrable with a first integral of the form $I = x^2 + y^2 + \cdots$.

We assume now that $b=-a$ and $c=0$. With this values system (6.33) yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2-ax^3)\begin{pmatrix} -y \\ x \end{pmatrix}, \tag{6.28}$$

which is integrable because is a non-degenerate center doing a scaling of time. Moreover, as it is reducible, applying Theorem 4.11, a first integral is of the form $I=x^2+y^2+\cdots$. ∎

Theorem 6.18 *System (6.26) with* $25a^2+8b^2+15ab\neq 0$, *is analytically integrable if, and only if, one of these following conditions is verified.*

(1) $c=-\frac{3}{2}a$, $5a+2b\neq 0$.

(2) $c=a+b$, $b\neq 0$, $5a-4b\neq 0$, $5a+2b\neq 0$ *and exist* $M,N\in\mathbb{N}$, *with* $M\neq N$, *such that* $b=-\frac{2M+3N}{2M}a$.

(3) $a=-\frac{2}{5}b$, $c=\frac{3}{5}b$, $b\neq 0$.

Moreover in the Cases 1 *and* 3 *a first integral is of the form* $I=(x^2+y^2)^2+\cdots$ *and in the Case* 2 *is of the form* $I=(x^2+y^2)^{M+N}+\cdots$.

Proof. By Lemma 6.16 the first coefficients of the normal form of system (6.26) are

$$\begin{aligned} a_3 &= \frac{1}{5}(a+b-c), \ \ b_3=0, \\ c_3 &= 0, \ \ d_3=\frac{1}{5}(3a+2c). \end{aligned} \tag{6.29}$$

We consider first the case $a_3\neq 0$, that is, $a+b-c\neq 0$, and we apply Theorem 4.14 statement **(a)**, since $m=3$ and $m<2(l-1)$ for all $l>3$. In this case we have that system (6.26) is integrable and it has a first integral of the form $I=(x^2+y^2)^2+\cdots$. Now we assume that the system is integrable and imposing the integrability condition $\nabla I\cdot\mathbf{F}=0$ we obtain that $c=-\frac{3}{2}a$. Reciprocally if it is verified such relation of statement (1), system (6.26) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^3y \\ bx^4-\frac{3}{2}ax^2y^2 \end{pmatrix}, \tag{6.30}$$

which is Hamiltonian and therefore integrable.

We now assume that $a_3=0$, that is, $c=a+b$. With this condition the following coefficients of the normal form for system (6.26) are

$$\begin{aligned} a_3 &= 0, \quad b_3=0, \\ c_3 &= 0, \quad d_3=\frac{1}{5}(5a+2b), \\ a_4 &= \frac{1}{150}b(5a-4b), \ \ b_4=0. \end{aligned}$$

We first consider the case $a_4\neq 0$ and $d_3\neq 0$. In this case we are under the conditions of Theorem 4.14 statement **(c)**, that is, $m=2(l-1)$ with $l=3$ y $m=4$. Therefore we have that

$$a_m=\frac{1}{16l^2}((l+2)^2\frac{(M+N)^2}{(M-N)^2}-(l-2)^2)(c_l^2-d_l^2).$$

In our case this implies

$$a_4=\frac{1}{144}[1-25\frac{(M+N)^2}{(M-N)^2}]d_3^2.$$

If we substitute the values of a_4 and d_3, we can isolate $b=-\frac{2M+3N}{2M}a$. Hence the relations given in statement (2) are satisfied.

On the other hand, doing the change $\{u=x, v=x^2+y^2\}$ and scaling the time, if statement (2) is satisfied, system (6.26) is transformed to the quasi-homogeneous system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -v+au^3 \\ 2(a+b)u^2v \end{pmatrix}. \tag{6.31}$$

Hence we can apply Theorem 4.12 and system (6.26) is integrable with a first integral of the form $I = (x^2+y^2)^M (x^2+y^2+\frac{2b-a}{3}x^3)^N$. This completes the proof of statement (2).

We now assume that $a_3 = 0$, $d_3 = 0$ and $a_4 \neq 0$, that is $a = -\frac{2}{5}b$, $c = \frac{3}{5}b$ and $b \neq 0$. We apply Theorem 4.14 statement **(b)** and we deduce that system (6.26) is integrable with a first integral of the form $I = (x^2+y^2+\cdots)(x^2+y^2+\cdots)$. Reciprocally, if it is satisfied statement (3), system (6.26) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} -\frac{2}{5}bx^3y \\ bx^4+\frac{3}{5}bx^2y^2 \end{pmatrix}, \tag{6.32}$$

which is Hamiltonian with a first integral of the form $I = (x^2+y^2)(x^2+y^2+\frac{4}{5}x^3)$. ∎

Lemma 6.19 *System (6.26) under the conditions of statement* (2) *of Theorem 6.18 is not orbitally equivalent to a Hamiltonian vector field.*

Proof. The result is proved seeing that in this case the coefficient of the normal form a_3 is not null. ∎

Remark. Notice that in the case $25a^2+8b^2+15ab = 0$ the first component of the associated vector field (3.13) to system (2.5) is reducible and this case remains open.

6.2 Example 2

Consider the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^4+bx^2y^2+cy^4 \\ dx^3y+exy^3 \end{pmatrix}. \tag{6.33}$$

Theorem 6.20 *System* (6.33) *has a first integral of the form* $I = x^2+y^2+\cdots$ *if, and only if, it is verified one the following conditions.*

(1) $a = 0$, $c = -e$, $d = -b$.

(2) $a = b-c$, $d = e$.

Proof. First we assume that $I = x^2+y^2+\cdots$ is a first integral of system (6.33). By imposing the integrability condition $\nabla I \cdot \mathbf{F} = 0$ and doing the analysis for each degree we obtain the conditions of the statement.

Reciprocally, we first assume that $a = 0$, $c = -e$ and $d = -b$. In this case, system (6.33) yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2-y(bx^2-ey^2))\begin{pmatrix} -y \\ x \end{pmatrix}, \tag{6.34}$$

which is integrable with a first integral of the form $I = x^2+y^2$.

Now we assume that $a = b-c$ and $d = e$. With this values of the parameters system (6.33) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\left[\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} (b-c)x^2+cy^2 \\ exy \end{pmatrix}\right], \tag{6.35}$$

which is integrable because is a non-degenerate center doing a scaling of time. Moreover, as it is reducible, applying Theorem 4.11, a first integral is of the form $I = x^2+y^2+\cdots$. ∎

Theorem 6.21 *System (6.33) with* $(a-c-e)(a-b+c) \neq 0$ *is analytically integrable if, and only if, one of the following conditions is verified.*

(1) $b = -\frac{3}{2}e$, $d = -4a$, $5a+c+\frac{5}{2}e \neq 0$.

(2) $b = 2a-e$, $d = e$, $a-c-e \neq 0$.

(3) $d = a-b+c+e$ *and exist* $M, N \in \mathbb{N}$, *with* $M \neq N$, *such that* $2(a-c-e)M+3(a-b+c)N = 0$ *and* $c = -\frac{2M+3N}{3N^2}((2M+N)a-Nb)$.

(4) $b = -\frac{3}{2}e$, $c = -5a-\frac{5}{2}e$, $d = -4a$, $4a+e \neq 0$.

Moreover, in the cases 1 and 2 a first integral is of the form $I = (x^2+y^2)^2+\cdots$, *in the case 4 is of the form* $I = (x^2+y^2+\cdots)(x^2+y^2+\cdots)$ *and in the case 3 is of the form* $I = (x^2+y^2+\cdots)^M(x^2+y^2+\cdots)^N$.

Proof. Taking into account Lema 6.16, the first coefficients of the normal form are

$$\begin{aligned} a_3 &= 0, \quad b_3 = -\frac{1}{5}(a-b+c-d+e), \\ c_3 &= -\frac{1}{5}(2b-4a-d+3e), \quad d_3 = 0. \end{aligned} \tag{6.36}$$

First we assume that $b_3 \neq 0$; that is $d \neq a-b+c+e$. Applying Theorem 4.14 statement **(a)**, as $m = 3$ (odd), it is verified $m < 2(l-1)$ for all $l \geq 3$. Hence, if system (6.33) is integrable, and a first integral is of the form $I = (x^2+y^2)^2 + \cdots$.

Therefore, we assume that $I = (x^2+y^2)^2 + \cdots$ is a first integral of system (6.33). Imposing the integrability condition $\nabla I \cdot \mathbf{F} = 0$ for each degree, we obtain the conditions of statements (1) and (2).

Reciprocally, in Case (1) system (6.33) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^4 - \frac{3}{2}ex^2y^2 + cy^4 \\ -4ax^3y + exy^3 \end{pmatrix}, \tag{6.37}$$

which is Hamiltonian with a first integral given by $I = (x^2+y^2)^2 - 4ax^4y + 2ex^2y^3 - \frac{4}{5}ey^5$.

In Case (2) system (6.33) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^4 + (2a-e)x^2y^2 + cy^4 \\ ex^3y + exy^3 \end{pmatrix}. \tag{6.38}$$

Doing the changes $\{u = x^2, v = y\}$, $\{u \leftrightarrow v\}$, $\{x_1 = u, x_2 = v + u^2\}$, $\{x_1 = X, x_2^2 = Y\}$ and reparametrizing the time the system transforms to

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 1 + eX \\ 4aY + 4(c-a+e)X^4 \end{pmatrix}, \tag{6.39}$$

which is integrable by the box flow theorem.

Now we assume that $b_3 = 0$; that is, $d = a-b+c+e$. The following coefficients of the normal form of system (6.33) are

$$\begin{aligned} a_3 &= b_3 = 0, \\ c_3 &= \frac{1}{5}(5a-3b+c-2e), \quad d_3 = 0, \\ a_4 &= -\frac{1}{150}(b-2c-e)(-9b+5a+4e+13c), \quad b_4 = 0. \end{aligned} \tag{6.40}$$

The study follows analyzing the nullification or not of a_4 and c_3.

We assume first that $a_4 \neq 0$ and $c_3 \neq 0$. Applying Theorem 4.14 statement **(c)**, as $m = 4$ and $l = 3$, it is verified $m = 2(l-1)$. Moreover, we have that $a_m - i\sigma b_m \neq \frac{1}{2l}(c_l - i\sigma d_l)^2$, from where, $a_4 \neq \frac{1}{6}c_3^2$. Therefore, substituting the expressions of a_4 and c_3 we obtain $(a-c-e)(a-b+c) \neq 0$. Thus if system (6.33) is integrable then exist $M, N \in \mathbb{N}$ such that

$$a_4 = \frac{1}{144}\left[25\frac{(M+N)^2}{(M-N)^2} - 1\right]c_3^2 \tag{6.41}$$

and a first integral is of the form $I = (x^2+y^2+\cdots)^M(x^2+y^2+\cdots)^N$.

If we substitute the values of a_4 and c_3 we have that the resonance condition (6.41) becomes

$$2(a-c-e)N + 3(a-b+c)M = 0. \tag{6.42}$$

Therefore a candidate to be a first integral is of the form $I = (x^2+y^2)^{M+N} + \cdots$, that is, $I = (x^2+y^2+\cdots)^M(x^2+y^2+\cdots)^N$.

Note that with the fixed values up to now, $x^2+y^2 = 0$ is an invariant curve of system (6.33) with cofactor $K = 2x(ax^2+x^2y^2+ey^2+y^4)$. Hence, we can write the first integral as $I = (x^2+y^2)^M(x^2+y^2+\cdots)^N$.

Now we impose that system (6.33) has a first integral of the form $I = (x^2+y^2)^M(x^2+y^2+\sum_{j\geq 3} q_j)^N$, where q_j are generic polynomials of degree $j \geq 3$, that is, we impose the integrability condition $\nabla I \cdot \mathbf{F} = 0$ for each degree. At degree 9 we find the last necessary condition of integrability, which is

$$(2M+3N)(2M+N)a - N(2M+3N)b + 3N^2c = 0. \tag{6.43}$$

From here we can isolate the parameter c and we obtain the relations of Case (3).

To prove the sufficient condition it is enough to check that $I = (x^2+y^2)^M(x^2+y^2+Ax^2+By^3)^N$, with $A = -\frac{2(M+N)}{N}a$ and $B = \frac{2(M+N)}{3N^2}((2M+N)a - Nb)$, is a first integral of system (6.33). This completes the proof of Case (3).

Finally we assume that $c_3 = 0$, that is, $b_3 = 0$, $a_4 \neq 0$ and $c_3 = 0$, which is equivalent to $2a - b - e \neq 0$, $c = -5a + 3b + 2e$ and $d = -4a + 2b + 3e$. Applying Theorem 4.14 statement **(b)**, it is verified $m < 2(l-1)$ because $m = 4$ and $l > 3$. Hence, if system (6.33) is integrable it has a first integral of the form $I = (x^2+y^2+\cdots)(x^2+y^2+\cdots)$. Imposing the integrability condition $\nabla I \cdot \mathbf{F} = 0$, we have that $b = -\frac{3}{2}e$. Therefore substituting the found expressions of the parameters we obtain the relations of Case (4).

Reciprocally, if it is verified statement (4), system (6.33) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^4 - \frac{3}{2}ex^2y^2 + (-5a - \frac{5}{2}e)y^4 \\ -4ax^3y + exy^3 \end{pmatrix}, \tag{6.44}$$

which is Hamiltonian with a first integral $I = (x^2+y^2)(x^2+y^2-4ax^2y+2(e+2a)y^3)$. ■

Remark. Notice that in the case $(a-c-e)(a-b+c) = 0$ the first component of the associated vector field (3.13) to system (2.5) is reducible and this case remains open.

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