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Mathematics teachers' specialized knowledge mobilized through problem transformation

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ABSTRACT

In this study we address two issues related to problem-posing tasks in teacher education: (i) the characterization of the specialized knowledge mobilized by prospective teachers when carrying out these tasks and (ii) the identification of the prospective teachers' pedagogical intentions in making adaptations to textbook problems. We asked prospective teachers to outline their suggestions for transforming a multiplicative problem so as to "promote the understanding" of their potential pupils. We then carried out a content analysis of their responses using the Mathematics Teachers' Specialized Knowledge model of teachers' specialized knowledge and identified their pedagogical intentions by means of the constant comparison method. The results show that prospective primary teachers mobilized both mathematical and pedagogical content knowledge in their responses to the problem reformulation task. Further, four distinct pedagogical intentions emerged that drew on different interpretations of the task prompt, and this influenced the type of transformation the prospective primary teachers suggested and the knowledge they mobilized in their answers.

1. Introduction

Problem posing has been considered a way of teaching mathematics with potential for constructing mathematical knowledge for more than 30 years (Cai et al., 2022; Kilpatrick, 1987; Silver, 1994; Singer et al., 2015). There is a broad range of literature on problem posing exploring different perspectives associated with the approach, such as the high cognitive demands involved (Silver, 1994), its connection to creativity (Bicer et al., 2020; Matsko & Thomas, 2015; Singer & Voica, 2015) and affect (Cai & Leikin, 2020), and the processes and phases involved (Baumanns & Rott, 2022; Cai et al., 2022). The majority of these approaches consider problem posing as an approach to mathematics teaching and learning in which students adopt an active role in generating problems as opposed to approaches in which students tackle problems provided by the teacher. In this regard, the kind of problem poser on which these studies have tended to focus has been pupils at a certain level of education in subjects aimed at constructing elements of mathematical knowledge. Nevertheless, there is a growing interest in the teacher as manager of the students' problem-solving and consequently problem-posing processes (Carrillo et al., 2021; Crespo, 2003; Leavy & Hourigan, 2022; Silver, 2013).

Studies of problem posing focused on the teacher can be grouped into three perspectives: the teacher as manager of the dynamics of problem posing (e.g., Leung, 2013, 2016; Martín-Díaz et al., 2020); the potential of problem posing for teacher education (e.g., Osana

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& Pelczer, 2015; Tichá & Hošpesová, 2013); and the teacher as creator of problems for their students to solve (e.g., Koichu, 2020; Leavy & Hourigan, 2022). Of these, the latter two perspectives are of particular interest to this study given the relevance of problem posing to initial teacher education. On the one hand, the approach has the potential to promote the construction of pedagogical content knowledge (Tichá & Hošpesová, 2013) as well as to help prospective teachers deepen their knowledge of mathematics from a professional perspective (Carrillo et al., 2021). On the other hand, it represents a starting point from which prospective teachers can develop the skills required to propose worthwhile problems to their students in their future professional practice (Leavy & Hourigan, 2021, 2022; Norton & Kastberg, 2012). Thus, the relationship between the teacher's knowledge and their ability to pose problems merits attention given that their problem-posing decision making requires them to mobilize their cognitive resources (Montes et al., 2022).

It is widely accepted that managing learning effectively requires teachers to have a specific kind of knowledge (Ball et al., 2008; Bromme, 1994; Carrillo et al., 2018; Chapman, 2015; Shulman, 1986; Tatto et al., 2008). It follows that developing this knowledge likewise requires a specific kind of task. Managing problem-solving tasks, in particular, demands a certain kind of specialized knowledge drawing on both mathematical and pedagogical expertise (Carrillo et al., 2018; Chapman, 2015). This is equally so in the realm of problem posing, where the demands on professional knowledge are also significant. It is thus important for initial teacher education programs to consider prospective teachers' competence in problem posing and to focus on the kind of knowledge that might be mobilized and developed in the course of initial training tasks. However, there has been little research into teachers' intentions—defined as a kind of self-instruction encapsulating the "underlying motivation (Rogers, 1983) or commitment (Sheeran and Webb, 2016) to act" (Conner & Norman, 2022)—in posing problems and less still into the contextual contingencies shaping teachers' intentions in problem posing (Chico et al., nd). We consider that training for problem posing, whether in initial or in-service contexts, should involve the participants in tasks specifically designed to develop the requisite abilities (Chico et al., nd; Montes et al., 2022). To achieve this, the variables and prompts associated with the problems should include an element of reflection on the pedagogical implications of the task. Thus, what is needed is twofold: first, to reflect on how these prompts can be categorized in terms of the intentions that underly them as one problem-posing variable from the teacher's perspective (Chico et al., nd); second, to explore the link between teachers' problem posing and the cognitive resources they mobilize while doing it (Chapman, 2015).

Our study has two objectives. First, we aimed to identify the kind of specialized knowledge required of prospective primary teachers (PPTs) to carry out a lesson-oriented task—that is, involving a degree of pedagogical intention—based on the transformation of multiplication problems. Second, we sought to identify the essential characteristics of the task which mobilized the requisite professional knowledge associated with problem posing for schools. Our research questions were as follows: What professional knowledge do the PPTs mobilize when tackling a task focused on transforming a multiplication problem to facilitate comprehension? What pedagogical intentions do prospective teachers show when they interpret the prompt "promote understanding of contents" with respect to a problem-transformation task?

To answer these questions, the PPTs were presented with a task based on the transformation of a multiplication problem with the aim of promoting comprehension of the content the PPTs were to identify in the problem. We carried out a content analysis of the commentary accompanying each transformation using the Mathematics Teachers' Specialized Knowledge (MTSK) model developed by

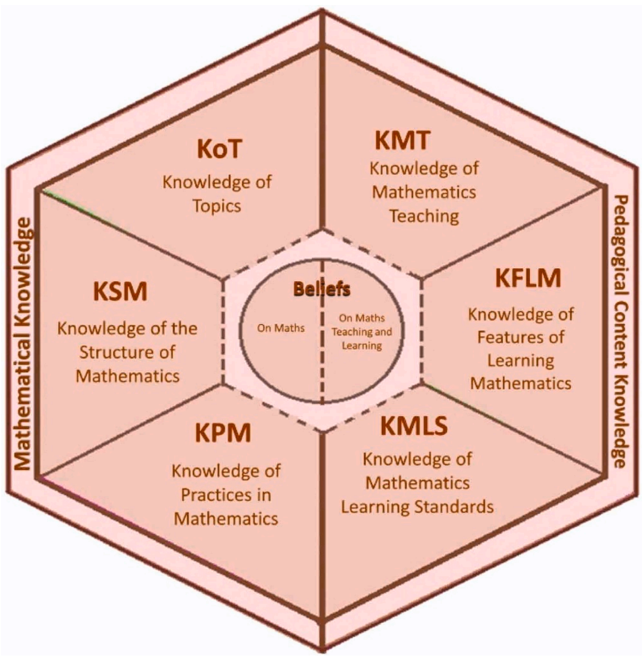

Fig. 1. The MTSK model (Carrillo et al., 2018).

Carrillo et al. (2018) in addition to carrying out an emergent analysis of the distinct pedagogical intentions reflecting the interpretations they made of the learning purpose of the task.

2. Theoretical background

In this section, we first introduce the domains and subdomains making up the MTSK model which we used as analytical lenses through which to identify and list the professional knowledge mobilized by the PPTs while transforming the problems. We then present a literature review of task perspectives in teacher education underpinning our study, particularly those tasks centred on problem posing.

2.1. Professional knowledge

As its name suggests, the MTSK model (Carrillo et al., 2018) employed in this study is designed to analyze the specialized knowledge of mathematics teachers. This model is a reconceptualization of the work of Shulman (1986) and Ball et al. (2008) with a reinterpretation of the idea of specialization (Scheiner et al., 2019). It takes as specialized all knowledge potentially useful in preparing, managing, and evaluating contexts in which mathematics teaching and learning take place, including teachers' reflections (Scheiner et al., 2019). The MTSK model was chosen for this study for its recognized capacity to delineate the knowledge teachers draw on in lessons and other lesson-like situations (Zakaryan & Ribeiro, 2019) and for its nonevaluative approach to the analytical process. Following the analytical approaches developed by Shulman (1986) and Ball et al. (2008), the MTSK model (Fig. 1) differentiates between the two chief domains of mathematical knowledge (MK) and pedagogical content knowledge (PCK). These come under the influence of the domain of beliefs, which carry a significant cognitive charge and permeate teacher's knowledge (Carrillo et al., 2018).

The first of these two domains, MK, is divided into three subdomains. The first of these, Knowledge of Topics (KoT), encompasses the teacher's knowledge of content from a local perspective. Hence, this subdomain considers knowledge of definitions, properties, procedures, registers of representation, and the phenomenology of certain concepts. The second subdomain, Knowledge of the Structure of Mathematics (KSM), focuses on the teacher's knowledge of the relationships and associations between different concepts. These associations can be based on an increased complexity between one concept and another or conversely a simplification of the cross-curricular applications of a concept or on the support offered by one concept to another (Montes et al., 2016). Finally, the third subdomain, Knowledge of Practices in Mathematics (KPM), includes knowledge of the syntactic aspects (Schwab, 1978) of mathematics, such as demonstrations, exemplification, and heuristics for problem solving along with the characteristic features of these aspects.

The second of the two domains, PCK, draws on the organizational structures developed in the work of Shulman (1986) and Ball et al. (2008) and is likewise divided into several subdomains: Knowledge of Mathematics Teaching (KMT), Knowledge of the Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Learning Standards (KMLS). Finally, there is a third domain of beliefs and conceptions of mathematics and its teaching and learning (Carrillo et al., 2018), which is not relevant to the current study. Each of the subdomains outlined above has a set of associated categories, with a view to facilitating analysis. The categories used in the analysis described in this paper are listed in Table 1, showing only the subdomains that appeared in the analysis.

The MTSK model has been used widely in exploring the knowledge of different mathematical topics and teaching skills and as a means of organizing tasks in teacher education across a range of educational levels (Moriel-Junior et al., 2022).

2.2. Problem-posing tasks in teacher education

Initial teacher education tasks are an important area of research in mathematics education (Grevholm et al., 2009). Grevholm et al. (2009) described tasks as having function, form, and focus. They have an aim in relation to the learning expected from student teachers, they are given a form to inspire, challenge and motivate students, and they have a specific foci chosen by the constructors of the task. (p. 1).

Table 1
The Subdomains Comprising MTSK and Constituent Categories.

Subdomains		Categories associated with the subdomain Knowledge of:
Mathematical Knowledge	Knowledge of Topics (KoT)	Procedures Definitions, properties, and foundations Registers of representation Phenomenology and applications
Pedagogical Content Knowledge	Knowledge of Features of Learning Mathematics (KFLM)	Theories of mathematical learning Strengths and weaknesses in learning mathematics Ways pupils interact with mathematical content Interests and expectations
	Knowledge of Mathematics Teaching (KMT)	Theories of mathematical teaching Teaching resources (physical and digital) Strategies, techniques, tasks, and examples

In our view, the function, form, and focus of a task in teacher education can be structured according to the theoretical position from which it is posed (e.g., Carrillo et al., 2020). Hence, the function of tasks focusing on problem posing in initial primary teacher education has been varied. It is very common to find tasks in which problem posing and problem solving are linked and in which there is an evaluative function with respect to the knowledge of the prospective teachers (Xie & Masingila, 2017) or with respect to the characteristics of the problems posed by prospective and practicing teachers (Chen et al., 2011; Montes et al., 2022). Such studies take an approach to problem posing which is a priori decontextualized from the future work the teachers will carry out. Other studies focus on aspects linked to the teaching profession, such as attention given to responding to errors and the subsequent solving process (Crespo, 2003); managing the cognitive demands of future problem solvers (Leavy & Hourigan, 2021); nourishing professional knowledge, especially PCK (Tichá & Hošpesová, 2013); and highlighting aspects associated with problems and teaching mathematics through the transformation of problems (Milinkovic, 2015). Transformation of problems is here understood as posing a problem from a prior problem by modifying several elements (Crespo, 2003; Milinkovic, 2015; Stoyanova & Ellerton, 1996).

In all these cases, the form and characteristics of the tasks were made amply explicit, although prompts have received less attention. Cheeseman et al. (2017) drew attention to two types of prompts in primary level tasks: enabling prompts and extending prompts. The former are aimed at altering some aspect of the task demands (e.g., the representation, numbers, or number of steps) to meet the needs of pupils struggling to find a solution, whereas the latter are aimed at encouraging pupils to go deeper into some aspect of the original task. In both cases, the learning aims are understood to shape the prompts (Russo, 2018). Consequently, problem-posing tasks must include a prompt that lets posers know what they are expected to do (Cai et al., 2022; Cai & Hwang, nd).

In the case of primary teacher education tasks designed to construct or mobilize professional knowledge for problem posing, we recognize that the prompts serve to orient the characteristics of the professional knowledge to be mobilized. Hence, an instruction such as "modify this problem so as to convert it into an open problem" would be an invitation to mobilize MK but not PCK because the prompt focuses solely on aspects of content. Conversely, the instruction "modify the representation in order to boost the pupils' learning" focuses on PCK, although it also requires a degree of knowledge of the possible ways the content in question could be represented. Hence, the way in which the task is oriented, in addition to the pedagogical intention projected by the prompt, become variables which enable specific aspects of professional knowledge to be targeted. This means that if the task design and prompts are structured in terms of a particular model of professional knowledge, it is possible to direct teacher education tasks towards the desired aspects (Montes et al., 2019).

At the same time, there is an opportunity with any initial teacher training task to formulate the prompt such that a particular pedagogical intention (Conner & Norman, 2022) is foregrounded or left implicit or indeed absent. In the first of these options, the task is framed with explicit instructions that make the purpose in setting it very clear so that the prospective teacher can take ownership of the problem (e.g., Chico et al., nd). In the second option, the purpose of the task is inherent in the professional and sociomathematical norms governing the teacher education program (Van Zoest et al., 2012), whereas in the third option, the problem posing is non-pedagogical and makes no claim to be directed towards classroom use.

3. Method

We designed a series of five tasks to be carried out by the PPTs, all involving the transformation of multiplicative structures based on a problem from a second year primary textbook widely used throughout Spain. We selected a textbook problem because textbooks are a much used resource by teachers responsible for teaching mathematics at the primary level (Chico & Montes, 2023; Hadar, 2017). All five tasks involved some form of transformation of the problem with the aim, as might be reasonably expected from a teacher managing a mathematical activity, to "promote understanding of the mathematical content." The first task required the prospective teachers to transform the problem with solely this instruction—that is, to "promote understanding of the mathematical content" identified in the problem statement (Fig. 2). In the second and third tasks, the PPTs were asked to transform the problem image, and in the fourth and fifth, they were asked to modify the problem with respect to understanding division and multiplication, respectively. The tasks, then, were intended to replicate the kind of challenge the PPTs would face in their day-to-day life as practicing teachers, that of adapting a problem from a textbook to promote understanding of the content. In this paper, we present the results from the first of these tasks, the most open-ended in the series in that it made no specification regarding either the element to be transformed or the content. This open-endedness was intentional and aimed to give the PPTs the freedom to make their own decisions about how to approach the task according to their knowledge and beliefs. From a research perspective, it had the additional benefit of providing a

Modify the word problem below so as to promote understanding of the mathematical content you identify. For each modification, indicate in the table: what kind of transformation it is; why you selected it; and how it helps the students' mathematical learning.

Bruno has 10 boxes containing marbles. Each box contains 5 marbles. In total, how many marbles does Bruno have?

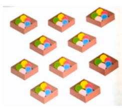

Fig. 2. Statement of the task.

significantly enriched variety of problems consistent with the exploratory nature of this investigation.

The problem to be transformed in the first task (Fig. 2) consisted of a multiplication task structured in terms of an isomorphism of measure (Vergnaud, 1983) in which the problem statement offered a picture illustrating the equal groups involved in the multiplication. The task was designed to promote the mobilization of both mathematical and pedagogical knowledge and oriented towards amplifying PPTs' understanding so as to involve them in the task from the perspective of the day-to-day work of the teacher. The problem was chosen so as to provide the trainees with various elements that could be transformed, ranging from numerical to syntactic and semantic aspects in addition to visual support, among others.

We restricted the mathematical content to multiplication because this is a crucial area in primary education and because the participating PPTs had already had some contact with the topic. The task was given to three groups of PPTs studying at the University of Huelva towards the end of their second year of training in 2020–21. The PPTs were given a week to submit their responses individually in writing. In total, 159 PPTs participated, providing 220 transformations, with several of them providing more than one transformation.

The perspective of the study was largely qualitative, with the aim of exploring, describing, and accounting for (McMillan & Schumacher, 2005) key aspects of the PPTs' specialized knowledge and their pedagogical intentions when transforming the problem statement in Fig. 2. An interpretative framework was adopted for analysis of the PPTs' responses, which stressed that evaluation of their modifications and explanations was not a goal of the study.

The PPTs' responses were subjected to a content analysis (Krippendorff, 2018) to i) identify the MTSK subdomains (Carrillo et al., 2018; see Table 1) mobilized by the PPTs in explaining their transformations and ii) establish empirically based categories (Flick, 2007) of pedagogical intention. Given the implicitness of intentions, we used the commentaries of each transformation to identify them. In both cases, the theoretical sensitivity towards the data (Strauss & Corbin, 1998) shown by the researchers and the validation procedure via expert triangulation (Flick, 2007) are crucial for drawing inferences from the data and for minimizing the risk of interpretative bias in the analysis. We illustrate below the analysis carried out based on one of the transformations proposed by PPT 17..

In the first paragraph of the commentary, we identify the pedagogical intention "focus the problem on a particular multiplicative content." Specifically, PPT 17 seeks to promote the use of the multiplication (instead of counting) in solving the problem. This intention oriented the professional knowledge mobilized. In this case, the PPT suggested changing the image, thus mobilizing knowledge of the representations of multiplication (KoT), the use of this resource in teaching (KMT), and the way in which pupils interact with the image by counting marbles (KFLM).

In the second paragraph, we identify the intention "facilitate understanding of the problem statement" in making the context more familiar to primary students ("*to help pupils to get into the situation*"). In their comments, PPT 17 mobilized knowledge related to the way in which pupils interact with the content (KFLM), assuming that when the context of the problem is closer to pupils' everyday life, they can see better the utility of the operations.

The categorization of the PPTs' pedagogical intentions was carried out by the method of constant comparison (Strauss & Corbin, 1998) according to the following coding process: open coding, axial coding, and selective coding. Open coding consists of reducing the data to preliminary codes directly associated with the text. This resulted in an emergent set of codes, such as making the context more familiar or simplifying the numbers, which were refined and renamed over the course of the analysis. The process of axial coding then refines and delimits the codes that emerged from the previous phase and begins the process of exploring interrelationships between codes to establish preliminary hierarchies. Finally, at the selective coding phase, the process initiated in the previous phase is continued until the main categories emerging from the data are established along with the subcategories pertaining to each, described below (Table 2). Although it is standard practice to initiate the analysis with open coding and to finalize it with selective coding, the processes are circular in nature, allowing researchers to cycle through one coding to the next and to combine codes as necessary (Flick, 2007). The coding process (open, axial, and selective) was thus repeated until no new codes emerged from the data (Flick, 2007; Strauss & Corbin, 1998). Table 2 shows the main categories, subcategories, and some open codes that emerged from the analysis.

To reduce the risk of bias inherent in the interpretative analysis of the data, the a priori assignation of data to the corresponding MTSK subdomains, and the inductive process of coding the PPTs' pedagogical intentions, the analysis was conducted in three stages:

Bruno has a collection of marbles which he keeps distributed in boxes so as not to lose any. He has 10 boxes in total and in each box there are 5 marbles. How many marbles has Bruno got in his collection all together?

Commentary:

I substituted the original illustration showing 10 boxes with a new illustration, so that only one box is shown. I consider this a necessary change as students could resort to counting the marbles in the illustration instead of performing the relevant operation, in this case multiplication.

Change the problem statement, providing a more detailed context to help the pupils get into the situation. This transformation fosters better understanding of the problem and brings it closer to their everyday life. When the pupils understand that the problem is about their day-to-day life, they begin to see the utility of these operations in situations outside the classroom.

Fig. 3. Response by PPT 17.

Table 2
Open Codes, Subcategories, and Categories of Pedagogical Intentions.

Categories	Subcategories	Open codes
Focus on content	Multiplicative thinking	Work on partitive/quotative division
		Promote the use of multiplication
		Foster the commutative property
		Combine operations with multiplication
	Introduce new content	Combine with addition, fractions, etc.
		Introduce probability, measurement, addition, etc.
Vary the complexity	Increase	Work on more complex tables or numbers
		Introducing difficulties, irrelevant data, or more complex content
	Decrease	Simplify numbers or problem-solving strategies
		Easing content
Facilitate understanding of the problem statement	Context	Make the context more familiar
		Extend the context
		Change the main character
	Text	Change order of information
		Introduce connectors
	Image	Change the color of the marbles
		Increase the distance between marbles
Evaluate understanding	Revise contents	Remember or apply previous content
		Reinforce contents
	Checking understanding	Check the learning of content

(1) independent coding, by which two researchers coded the data independently of each other; (2) comparison of the interpretations, by which the same two researchers compared their results and negotiated meanings and interpretations, which were then registered in a memorandum; and (3) date triangulation, in which the four authors discussed all instances from the previous stages that had not been resolved and likewise negotiated the interrelationships between the codes and the main categories.

Once consensus had been reached regarding the MTSK subdomains identified and the establishment of the categories of the PPTs' pedagogical intentions, the researchers carried out a review of all the instances of each in the data.

4. Results

In this section, we first present the results of the identification of the subdomains of professional knowledge structured by the MTSK model. We give an overview of the professional knowledge extracted from the data set, illustrated with excerpts from the PPTs' responses. These examples account for at least 8% of the sample and thus represent a significant part of the data set. We then present the results of the categorization of the PPTs' pedagogical intentions, giving a detailed analysis of each category and subcategory with illustrations.

4.1. Professional knowledge

At least one subdomain of the MTSK was identified in 88% of the responses. The more frequently occurring subdomains were Knowlege of Topics (KoT), Knowledge of the Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Teaching (KMT). Table 3 shows the raw percentages for the frequency of occurrence of each subdomain. As is evident in Table 3, many of the responses involved the mobilization of more than one subdomain.

It is worth noting the proportion of PPT responses without any clear evidence of any element of specialized knowledge (12%). In what follows, we illustrate these three subdomains and the relations between them.

4.1.1. Knowledge of topics (KoT)

The Knowledge of Topics (KoT) subdomain—associated with definitions, properties and foundations, registers of representations, and phenomenology and applications—was the most frequently activated subdomain in the data set (78%). Essentially, the PPTs' modifications involved changing the properties and visual representations of multiplication and division as well as varying the contexts in which these could be presented. There were also a few PPTs who applied their knowledge of the characteristics of mathematics problems, and in some instances the PPTs demonstrated knowledge of other topics by modifying the core content of the given problem.

Regarding knowledge of definitions, properties, and foundations, some of the PPTs' responses demonstrated knowledge of the different types of multiplication problems with the structure of the isomorphism of measures (Vergnaud, 1983). Such was the case of PPT 86, who carried out a transformation mobilizing their KoT with respect to the two kinds of division within the structure of the

Table 3
Frequency of the Subdomains of Knowledge Identified.

Subdomain	KoT	KFLM	KMT	No knowledge identified
Percentage	78%	49%	19%	12%

isomorphism of measures: quotative and partitive. The quotative sense of discovering the number of groups is given in the PPT's commentary: "*As we eliminate the marbles we distribute them in the number of boxes which Bruna has.*" The partitive sense is supplied by the suggestion "*All the marbles are dived into groups of five to be put in the boxes.*" Others, such as PPT 155, mobilized knowledge of the characteristics of problems involving multiplicative comparison, writing "*I've gone from rate multiplication to multiplicative comparison. The children will learn to handle different kinds of problem statements, handling new terms such as X times more.*"

In other cases, PPTs also mobilized knowledge related to the properties of multiplication. In one instance, PPT 51 modified the starting problem, featuring 10 boxes of 5 marbles, to "*5 boxes of 10 marbles all the same colour.*" The accompanying commentary clearly revealed knowledge of the property because it explicitly stated that the modification was intended to address "*learning about the commutative property.*"

Elsewhere, some of the PPTs mobilized knowledge connected with the characteristics of mathematics problems by adding stages to the solution process. An example of this is provided by PPT 2, who suggested the following problem: "*Bruno has 5 boxes, but his brother gives him 3 more boxes as a present. In each box there are 7 marbles. How many marbles has Bruno got all together*?" In their commentary, this PPT said that "*in order to achieve the result we want, the pupils have to go through another process, and work out the total* [number of boxes] *first.*" In this example, it can be seen that PPT 2 is aware that mathematics problems can consist of various stages in the solution process.

Finally, it is also worth noting that although there was a general tendency to propose transformations focused on division or multiplication, there were some instances of PPTs making modifications that addressed content different from the original. For example, PPT 49 mobilized knowledge related to the definition of squaring: "*The same number of boxes as marbles, so as to introduce the concept of squaring.*"

There was also evidence of KoT related to registers of representation. In these cases, the PPTs modified the visual representation provided based on the mathematical content. For example, PPT 152 proposed changing the problem information by substituting the illustration with one showing 10 empty boxes and 50 marbles (Fig. 4), requiring the pupils to distribute the marbles evenly into the boxes and helping them to arrive at the solution of placing 5 marbles in each box: "*I've modified the illustration in order to show more clearly what they need to do in this problem. They set out the 10 empty boxes and then afterwards take a bag of 50 marbles*." Based on their use of this representation, it can be inferred that the PPT was aware of a representation available to demonstrate multiplication and make it more meaningful (distributing into equal groups, facilitating the total for each group).

Similarly, PPT 88 suggested the word problem "*Bruno has got 20 boxes full of marbles. If there are 3 marbles in each box, how many marbles has Bruno got all together*?" The PPT discarded the original illustration and replaced it with one showing a neat line of marbles along which the problem solver must circle the groups ("*Substitute the picture of marbles in boxes and instead draw a line with lots of marbles. Each time they count off three marbles, they circle them and put them in a box*").

4.1.2. Knowledge of the features of learning mathematics (KFLM)

Knowledge of the Features of Learning Mathematics (KFLM) was the second most frequently activated subdomain of the MTSK model (49%). Specifically, the PPTs mobilized knowledge associated with the strengths and difficulties encountered by primary pupils when faced with specific mathematical content along with knowledge associated with the pupils' interactions with specific content.

Hence, in some of the suggested transformations, the PPTs directed attention to triggering or correcting an anticipated difficulty the pupils might have and demonstrating knowledge of the strengths and difficulties in their accompanying notes. For example, PPT 37 suggested the problem, "*Bruno has 7 boxes with marbles. Each box has 8 marbles in it. How many marbles has Bruno got all together*?" The alteration in the numerical data was prompted by the PPT's awareness that primary pupils often have difficulties with multiplying the numbers 7 and 8: "[Tackle] *a more difficult multiplication table, thus working on different numbers like 7 and 8, which give more trouble.*"

The PPTs also demonstrated awareness of how primary pupils interact with the problem and with the visual representation of the problem statement. For example, PPT 104 proposed the following: "*Tidying his bedroom, Bruno has decided to put away his marbles in various boxes. He wants to use 10 boxes, and put 5 marbles into each box. How many marbles has he got in total*?" In their notes, the PPT declared: "*As a result of this change they can learn to bring mathematics into their everyday life. Furthermore, with this slight change they might find fewer difficulties when it comes to understanding the problem*." This PPT was aware that pupils are more likely to understand the problem when it is framed in situations they are familiar with. It is for this reason we infer that PPT 104 demonstrated an aspect of specialist knowledge associated with primary pupils' interactions with content.

4.1.3. Knowledge of mathematics teaching (KMT)

Finally, 19% of the PPTs' responses mobilized Knowledge of Mathematics Teaching (KMT). With respect to this subdomain, the

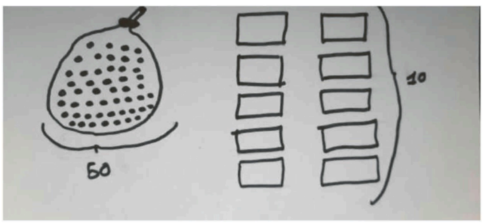

Fig. 4. Illustration modified by PPT 152.

PPTs mobilized knowledge associated with material resources, modifying either the illustration or the quantities involved in the problem. The transformations providing evidence of this subdomain concerned how the illustration might help pupils to identify the mathematical content inherent in the problem. PPT 62, for example, proposed the following problem: "*Bruno has got 10 boxes of marbles. Each box contains yellow and blue marbles. Which colour marble has Bruno got more of, yellow or blue*?" To enable the pupils to solve this problem, the colors of the marbles in the original illustration were altered to yellow and blue, a change in visual representation drawing on the prospective teacher's KMT. This change enabled the task to focus on a specific content area, in this case counting, as the PPT noted: "*We can also say this way we are getting the pupils to focus their attention on making a count in order to differentiate the marbles.*" In other cases, the PPTs eliminated the image so as to hinder the use of counting strategies, bringing into play their knowledge of visual representation as a mathematics teaching resource. This subdomain occurred mainly in combination with KoT or KFLM, as we shall explore in greater detail in the following section.

4.1.4. Relations between subdomains

In 33% of the responses, the PPTs demonstrated their knowledge of just one subdomain, chiefly KoT (24%; see Table 4). As discussed above, the knowledge the PPTs mobilized was largely associated with multiplication: multiplicative structures, the properties of multiplication and division, visual models, and so on, although other mathematical content was introduced in some cases. In the remaining 55% of the responses, the PPTs incorporated knowledge of two subdomains (43%) or more than two subdomains (12%) of specialized knowledge in their commentaries on their transformations. Table 4 shows the percentages of responses in with the PPTs mobilized only one subdomain (along the diagonal) or two subdomains.

The most frequently occurring combination of two subdomains was KoT and KFLM (see Table 4). The PPTs frequently mobilized their KFLM when considering how the pupils might interact with the content they were proposing or, in much the same way, when they considered the additional difficulty that solving the problem might represent for the pupils' capacities. These interactions or difficulties are directly related to the content (KoT) because they tend to introduce, for example, additional operations or content the PPTs considered additionally challenging in solving the problem. An instance of this can be seen in the problem proposed by PPT 75: "*After the problem statement mentioned before* [the original problem] *the text tells the pupil that a friend asks him for a marble from each box.*" The commentary notes the difficulty entailed in adding a subtraction to the problem, stating that.

> this little difficulty is added to the problem, which can be confusing for the pupils when you ask them to take away a marble from each box, or what amounts to the same, subtract the number of boxes from the total number of marbles.

This modification is an example of the association of KoT with KFLM mentioned above.

Regarding the combination of more than two subdomains, the mobilized triplet KoT, KMT, and KFLM occurred in 8% of the responses. This combination of subdomains is similar to the above in the sense that the PPTs mobilized their KoT according to the way the pupils understand a particular content area (KFLM), and, to make this content area more comprehensible, they drew on a resource or strategy from their KMT. For example, PPT 60 added the following problem statement: "*In the presentation of the activity, the distribution of the boxes will appear in in parallel rows of two by two*." This change, they go on to say, was included to achieve "*better understanding and support for dividing decimals with a number or quantity, by distributing two boxes per line and ten marbles in each one.*" Consequently, in addition to their knowledge of the decimal number system (KoT) and how pupils interact with mathematical content with respect to the given grouping and representation (KFLM), this PPT also utilized a specific representation (KoT) which they associated with a strategy to help the pupils better understand the problem statement (KMT). All of this is grounded in their KFLM given that they are aware that primary pupils will understand the mathematical content better if the modification to the problem is applied.

4.2. Pedagogical intention

This section considers the variety of pedagogical intentions underpinning the transformations suggested by the PPTs, which emerged from analysis of the various interpretations of the prompt to "promote understanding of the mathematical content" given in the task. We describe below the four main categories and subcategories along with the corresponding percentages of occurrence in the data (Table 5).

The PPTs mostly interpreted the prompt "promote understanding" as focusing the problem on content related to multiplicative thinking, increasing the complexity of the problem, reinforcing understanding of specific content, or bringing the context of the problem closer to the experiences of elementary school students to facilitate understanding. Although the system of categories is disjunctive, a single transformation may be subject to two or more intentions, as can be appreciated in Table 5. Table 6 displays the percentages of frequency with which the PPTs declared only one intention (along the diagonal) or two intentions.

63% of the responses demonstrated a single intention, chiefly "focus on content" (36%) and "vary complexity" (13%). The other responses demonstrated a combination of intentions manifested in changes to different aspects of the problem. As can be seen in

Table 4
Percentage of Responses Involving One or Two Subdomains.

MTSK subdomains	KoT	KFLM	KMT
KoT	24	33	6
KFLM	-	5	2
KMT	-	-	4

Table 5
Frequency of PPTs' Didactic Intentions.

Focus on content		Vary the complexity		Facilitate understanding of the problem statement			Evaluate understanding	
Multiplicative thinking	New content	Increase	Decrease	Context	Text	Image	Revise	Check
53	13	22	13	14	5	3	18	2
66		35		20			20	

Table 6
Percentages of Responses Involving One or Two Pedagogical Intentions.

Pedagogical intentions	Focus on content	Vary complexity	Facilitate understanding	Evaluate understanding
Focus on content	36	14	8	6
Vary complexity	-	13	3	3
Facilitate understanding	-	-	7	1
Evaluate understanding	-	-	-	7

Table 5, the most frequent composite was that of "content-complexity" (14%) followed by "content-facilitate" (8%). In the next section, we describe each intention in detail and illustrate the subcategories of the most frequent intentions with excerpts from the PPTs' responses.

4.2.1. Focus on content

"Focus on content" was the most frequent pedagogical intention among the responses (66%) and occurred as the only intention in 36% of the cases. The category emerged out of suggestions to reformulate the original problem to introduce or work on a specific content element. Mainly, the PPTs transformed the problem to address meanings, properties, or operations associated with multiplicative thinking (53%). Chief among these were those that aimed to: (i) transform the problem while retaining the original isomorphism of measures multiplicative structure; (ii) encourage the use of multiplication—in lieu of repeated addition—or division or a multiplication table; (iii) combine different operations; or (iv) combine certain properties of multiplication, such as the commutative and distributive properties. In the first group, the PPTs transformed the problem from one of multiplicative rate to one of quotative or partitive division by swapping the roles of the given and required information. For example, PPT 39 proposed the following quotative division problem: "*Bruno has got 50 marbles and he needs to divide them up so that there are 5 marbles in each box. How many boxes does he need*?" His stated aim was to "*change the problem to a quotative division problem. Hence, instead of using multiplication to find the result,* [the pupils] *will have to work out that the best way to solve the problem in this case is by division.*" In the second group, PPT 119 suggested illustrating the problem with "*just one box of five marbles*" so that the pupils "*come to the conclusion that they must use multiplication and not just add up all the marbles in the boxes.*" In this way, PPT 119 aimed to nudge the pupils into using multiplication to solve the problem by not illustrating the problem components in full and thereby removing the option to directly apply a counting strategy (Barmby et al., 2009). Lastly, in the third group, the PPTs largely combined multiplication with addition, subtraction, or division by introducing new stages to the problem. For example, PPT 10 offered the following: "*Mary has 3 boxes of sweets, and in each box there are 8 mint sweets and 7 strawberry sweets. How many sweets has she got altogether? How many has she got if we count just the strawberry ones*?" This PPT's rationale was that "*by changing the number of operations* [the pupils] *learn to combine addition and multiplication.*"

To a lesser extent (13%), PPTs transformed the problem to focus solely on new content not directly associated with multiplicative thinking, such as probability or measurement.

4.2.2. Vary the complexity of the problem

The second most frequent intention (35%) in the PPTs' responses aimed to "vary the complexity of the problem." This pedagogical intention includes interpretations of the prompt "promote understanding of content" in terms of increasing (22%) or decreasing (13%) the complexity of the problem.

To increase the complexity of the problem, the PPTs performed a variety of modifications: (i) varying the numbers, (ii) adding steps to the problem to involve more than one operation in the solution, (iii) modifying the image to avoid counting marbles, (iv) introducing irrelevant data in the word problem, or (v) varying the contents of the problem. The most common modifications to increase the complexity of problems were to vary the quantities and add a stage to the problem. For example, PPT 44 proposed changing 5 marbles to 8 marbles in the problem statement "*to make the problem-solving process a little more complicated since I think that for the students, the 8 times table is more complicated than the 5 times table.*" Meanwhile, PPT 22 added a stage to the problem "*with the intention of complicating the exercise by adding an addition sum.*"

The PPTs reduced the complexity of the problem by decreasing the number of elements involved. For example, PPT 46 proposed reducing the number of boxes to 5 and the number of marbles to 2 so as to "*reduce the difficulty of solving the problem statement, given that, if the student does not know how to do multiplication, they use repeated addition to make it shorter and less difficult.*"

4.2.3. Facilitate understanding of the problem statement

This category encompasses all the responses (20%) in which the PPTs stated their intention to be that of facilitating understanding of the problem statement. Their proposals included modification of the following: (i) the context, by adding characters, actions, descriptions, or relationships to the original context (14%); (ii) the problem statement, by introducing connectors or changing the order of the information provided (5%); and (iii) the image, by changing the colors or creating greater space between the marbles in the illustration (3%). With these transformations, the PPTs sought to facilitate primary students' understanding of and access to the problem situation presented to them. Hence, the new problems were closer to primary students' experiences or contained more detailed descriptions of certain information given in the problem. For example, PPT 33 proposed the problem "*Today is your birthday, and your mother gave you a box containing 5 marbles. How many marbles would you have if you were given 10 boxes in total?*" because.

> by making them the protagonists of the word problem, they understand the context better, and are therefore better able to solve it. On the other hand, I have presented the marbles in one box first, and then asked how many there would be in 10 boxes, because I think that in order to facilitate their understanding it is better for them to understand the problem from the smallest to the largest details.

In this excerpt, the instruction to promote understanding was interpreted by PPT 33 in terms of bringing it closer to the primary pupils' perceived mode of thought by changing the character and the order of the information given in the problem statement.

This intention largely occurred in combination with other intentions, above all "focus on content." In these modifications, the PPTs generally focused their attention on facilitating understanding of the problem statement but also focused on some aspect of the mathematical content. For example, PPT 11 suggested the following: "*Bruno wants to share his 50 sweets with his classmates. Bruno knows he needs to distribute all the sweets into 10 bags. How many sweets can he give to each classmate*?" The accompanying explanation notes that.

> the alteration in the problem data is aimed to help pupils acquire the meaning of division as a sharing out of a total into different parts. The change in context facilitates this understanding as they are familiar with the problem and it should be easier for them to understand the problem statement.

In this instance, then, PPT 11 modified the information to introduce division as the crucial content item while at the same time modifying the context so as to facilitate understanding of the problem statement.

4.2.4. Evaluate understanding

Finally, in 20% of the responses, the PPTs stated that their intention was to steer the solution towards an evaluation task. Principally, this took the form of using the task to revise certain content (18%), and, to a lesser extent, check the pupils' understanding of the content involved in the new problem (2%). In this group of modifications, the PPTs typically included an additional stage involving addition or subtraction operations to arrive at the solution, with the stated intention of revising these content areas. For example, PPT 17 added a new question to the problem statement so as to "*prompt pupils to recall previous lessons which covered addition. There is also revision and consolidation of knowledge from previous lessons.*" Elsewhere, PPTs added stages with the intention of checking pupils' understanding of certain content areas. For example, PPT 41 proposed splitting the boxes between two people—"*6 boxes for Ana and 4 boxes for Bruno. Bruno's boxes have 5 marbles and Ana's have 4*"—on the premise that this change enables the teacher to check "*whether pupils have grasped the concept of multiplication. They will demonstrate their understanding of multiplication with more than one number.*"

5. Conclusions

The development of professional competence in problem posing is an area in need of attention in teacher education (Carrillo et al., 2021). The training process to meet this need must ensure the construction of the requisite professional knowledge, both mathematical and pedagogical, to enable prospective teachers to formulate problems appropriate to the contexts they will work in (Chapman, 2015; Tichá & Hošpesová, 2013). This construction can be carried out through the kind of activity illustrated in this paper in which PPTs tackle tasks focused on the transformation of problems for particular pedagogical ends.

Analysis of the PPTs' responses provided evidence of the mobilization of both MK and PCK. Furthermore, the MTSK model made it possible to identify, with a reasonable degree of detail, different aspects of both Knowledge of Topics (KoT), Knowledge of the Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Teaching (KMT). It is no surprise that many of the transformations were underpinned by evidence of KoT given the need for this kind of knowledge to formulate and transform problems (Cai et al., 2022). However, it is interesting that the mobilization of KFLM often involved bringing KoT elements into play, demonstrating that PPTs transcend the position of the solver and assume the teacher role (Tichá & Hošpesová, 2013). By allowing a great degree of freedom regarding the kind of transformation that could be carried out, we believe the task proved to be especially interesting. Future developments of this work could complement the implementation of the task with discussions of the fundamental issues involved in transforming problems in a group problem-posing session (Cai et al., 2022; Kilpatrick, 1987).

Likewise, the task prompt linked the intention of the task to the fostering of hypothetical pupils' understanding. This prompt was interpreted by the PPTs in different ways: focusing on aspects of the content, varying the complexity, facilitating comprehension of the problem statement, and evaluating understanding. Hence, the way the prompt is structured should be seen as a variable to take into account when planning problem-posing tasks from a professional perspective because it frames the orientation given to the problem and requires the mobilization of different aspects of professional knowledge. In this respect, we believe it is necessary to direct future studies towards gaining further insights into how professional knowledge interacts with PPTs' pedagogical intentions when designing problems so as to generate training tasks that take these relationships into account.

At the same time, the idea of pedagogical intentionality, rarely addressed in the field (e.g., Cruz, 2006), should, we believe, be taken into account in models that conceptualize stages in problem posing (e.g., Baumanns & Rott, 2022) and in the practical application of such models to actual cases of problem posing in lessons. On the other hand, regarding the study of problem-posing strategies (such as the *what if not* strategy; Lavy & Bershadsky, 2003), it is worth considering whether the mobilization of both mathematical and pedagogical knowledge suggests there are teaching strategies specific to pedagogical intentionality.

This paper advances the idea that, in the sphere of problem-solving tasks, both the prompts and the variables currently considered in the literature—variables such as content, context, structure, and heuristics (e.g., Cai et al., 2022)—can be broadened in initial teacher education programs to include pedagogically oriented prompts and variables that focus reflection on aspects of the teaching and learning of mathematics.

One limitation of the present study is that it considers only one of the tasks carried out by the PPTs. In future studies, we aim to analyze the complete set of five tasks (which includes different multiplicative structures), thereby complementing the results of this study in terms of both the characterization of professional knowledge and the pedagogical intentions demonstrated by the PPTs.

We plan to further the research presented in this paper with a follow-up study into the relationships between mobilized knowledge and pedagogical intentions based on a larger sample with the aim of studying the implications that PPTs' pedagogical intentional choices might have on the kind of knowledge that is (and isn't) mobilized in problem transformation tasks. We are particularly interested in the influence of pedagogical variables on PPTs' cognitive resources. This question has implications for the design of problem-posing tasks for prospective teachers as opposed to those tasks aimed at other educational sectors and levels.

CRediT authorship contribution statement

M. Montes: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **J. Chico**: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Writing – original draft, Writing – review & editing. **J.P. Martín-Díaz**: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Resources, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **E. Badillo**: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Resources, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

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Data availability

Data will be made available on request.

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