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A new family of solutions for graph-restricted cooperative games: rethinking the weight of intermediary power

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Abstract

In graph-restricted cooperative games, a group of agents, represented by the nodes of a graph, work together to make a profit. However, two agents can cooperate within a coalition only if they are connected by the graph in the coalition. Several allocation rules have been proposed for these games, but there is something in common in all of them: if a player is an indispensable intermediary to communicate with the players in a coalition, this player will receive, at least, the same share of the profit generated by the coalition as the players who belong to it. In other words, intermediation power is valued, at least, as much as active cooperation. In some situations, this is neither fair nor realistic. In this paper, we introduce a family of values for graph-restricted games that value intermediary power less than active cooperation.

Keywords: cooperative games; Shapley value; graph-restricted games; Myerson value; multichoice games

1. Introduction

Cooperative game theory provides a framework for analyzing strategic interactions among players aiming to achieve mutually beneficial outcomes. When dealing with cooperative games, it is often assumed that the players can cooperate without any restrictions. However, in many situations, this is not the case. There can be different types of limitations on cooperation. In this paper, we will address limitations in communication. Myerson (1977) introduced graph-restricted games, in which players are represented by the nodes of a graph, and there exists an edge between two nodes if and only if the corresponding players can communicate directly. In the context of cooperative games (with transferable utility), the allocation of payoff among the players becomes

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a central concern. In the case of graph-restricted games, a reasonable allocation rule must take into account not only the role of each player in the game but also their intermediation power, that is, their ability to enable communication between other players. Myerson introduced an allocation rule and characterized this solution in terms of fairness and efficiency. Since then, multiple solutions for graph-restricted games have been described in the literature. Among these solutions, we point out the position value (Meessen, 1988; Borm et al., 1992), the F-value (Hamiache, 1999), the mean value (Hamiache, 2004), the average tree value (Herings et al., 2010), or the value proposed by Hamiache and Navarro (2020). Remarkable insights into solutions for graph-restricted games are provided by Borm et al. (1994) and Navarro (2019). The aim of this paper is to introduce a family of new solutions for graph-restricted games. The distinguishing feature of these new solutions compared to the Myerson value or the other values introduced in the literature is that they provide a lesser reward for intermediation power. Just as there are situations where the reward for intermediation established by the Myerson value or the other solutions may be appropriate, we believe that there are cooperative situations where the reward for intermediation should be lower in relation to the reward for actively cooperating in the game itself. The goal of this paper is to introduce suitable solutions for these situations. To this end, we will introduce a novel weighting mechanism through multichoice games. Hsiao and Raghavan (1992) introduced cooperative multichoice games, which are games in which players have a finite number of levels of action at which they can cooperate. They (Hsiao and Raghavan, 1993) proposed a solution for multichoice games inspired by the Shapley value for classic cooperative games. In this paper, we propose to use the value for multichoice games introduced by Hsiao and Raghavan to adjust the weight of intermediation, thus obtaining a new family of solutions for graph-restricted games.

The paper is organized as follows. In Section 2, we recall some preliminaries concerning cooperative games and graph-restricted games. In Section 3, we propose a multichoice approach to obtain a new family of values for graph-restricted games. We also obtain a characterization of these values in Section 4.

2. Preliminaries

Throughout this paper, N denotes a fixed finite set. The subsets of N will be called coalitions. The family of all coalitions will be denoted by 2^N.

2.1. Cooperative games

A *cooperative game (with transferable utility)* on N is a function (called characteristic function) $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Given a coalition E, $v(E)$ is the worth of E, and it is interpreted as the maximal gain that the players in E can achieve when they cooperate. The family of cooperative games on N is denoted by $\mathcal{G}^N$. This set is a $(2^{|N|} - 1)$-dimensional real vector space. One basis of this space is the collection $\{u_F : F \subseteq N,\ F \neq \emptyset\}$, where for a coalition $F \in 2^N \setminus \{\emptyset\}$ the *unanimity* *game* u_F is defined as $u_F(E) = 1$ if $F \subseteq E$ and $u_F(E) = 0$ otherwise. Every game $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games:

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \Delta_v(E) u_E, \text{ with } \Delta_v(E) = \sum_{F \subseteq E} (-1)^{|E|-|F|} v(F), \tag{1}$$

where $\Delta_v(E)$ is named the dividend of the coalition E in the game v. A game $v \in \mathcal{G}^N$ is *monotone* if $v(F) \leqslant v(E)$ for every $F \subseteq E \subseteq N$. A player $i \in N$ is a *null player* in $v \in \mathcal{G}^N$ if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$. A player $i \in N$ is a *necessary player* in $v \in \mathcal{G}^N$ if $v(E) = 0$ for every $E \subseteq N \setminus \{i\}$.

A value on $\mathcal{G}^N$ is a function $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that assigns to each game $v \in \mathcal{G}^N$ a vector $(\psi_i(v))_{i \in N}$ where the real number $\psi_i(v)$ is the payoff to player i in the game v. Multiple values have been defined in the literature. The best known of them is the *Shapley value* (Shapley, 1953), which is defined, for every $v \in \mathcal{G}^N$ and every $i \in N$, as

$$Sh_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E[v(E) - v(E \setminus \{i\})], \text{ where } p_E = \frac{(|N| - |E|)!(|E| - 1)!}{|N|!}.$$

Van den Brink (1994) characterized the Shapley value as the unique value ψ that satisfies:

EFFICIENCY. If $v \in \mathcal{G}^N$, then $\sum_{i \in N} \psi_i(v) = v(N)$.
ADDITIVITY. If $v_1, v_2 \in \mathcal{G}^N$, then $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)$.
NULL PLAYER PROPERTY. If $i \in N$ is a null player in $v \in \mathcal{G}^N$, then $\psi_i(v) = 0$.
NECESSARY PLAYER PROPERTY. If i is a necessary player in a monotone game $v \in \mathcal{G}^N$, then $\psi_i(v) \geq \psi_j(v)$ for all $j \in N$.

2.2. Graph-restricted games and the Myerson value

In order to model cooperative situations with communication restrictions, Myerson (1977) introduced graph-restricted games. A graph-restricted game consists of a finite set of players N, a game $v \in \mathcal{G}^N$, and an undirected graph $g = (N, L)$. In this model, g represents the communication possibilities between the players, in the sense that a coalition is feasible only if all their players are connected, according to g, within the coalition. Myerson introduced a value for these games. Given a graph-restricted game (v, g), he defined $v_g : 2^N \to \mathbb{R}$ as

$$v_g(S) = \sum_{T \in S/g} v(T)$$

for every $S \in 2^N$, where S/g denotes the set of connected components of the induced subgraph $g[S]$. Then, Myerson defines

$$Y(v, g) = Sh(v_g).$$

Myerson proves that Y is characterized by the following properties:

COMPONENT EFFICIENCY. *If* $v \in \mathcal{G}^N$, $g \in \Gamma^N$, *and* $T \in N/g$, *then*

$$\sum_{i \in T} Y_i(v, g) = v(T).$$

FAIRNESS. *If* $v \in \mathcal{G}^N$, $g = (N, L) \in \Gamma^N$, *and* $ij \in L \in N$, *then*

$$Y_i(v, g) - Y_i(v, g_{-ij}) = Y_j(v, g) - Y_j(v, g_{-ij}),$$

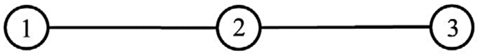
where g_{-ij} denotes the graph obtained by removing in g the edge between ij.

3. The *r*-fair communication value

3.1. Motivating example

The following example shows that the Myerson value does not satisfactorily solve all graph-restricted cooperative situations.

Example 1. Let $N = \{1, 2, 3\}$ and consider the graph $g \in \Gamma^N$ represented by the diagram below.



It is easy to check that

$$Y(u_{\{1,3\}}, g) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Notice that player 2 is a null player in the game $u_{\{1,3\}}$. However, it receives the same payoff as the rest of the players, who are necessary players in the game. In other words, a player with a passive role of enabling communication obtains the same payoff as the players who actively generate all the profit. In almost any realistic context, this distribution of profits would be out of place. Our goal in this paper is to introduce a solution for graph-restricted games that allows us to weigh the value of intermediary power. To this end, we will use multichoice games.

3.2. Multichoice games

Hsiao and Raghavan (1993) introduced multichoice games to model cooperative situations in which each player, when cooperating within a coalition, can choose between different levels of cooperation. Each player is allowed to take $(m + 1)$ actions $\sigma_0, \sigma_1, \ldots, \sigma_m$, where σ_0 is the action of doing nothing and σ_k is the action of cooperating at level k for each $k = 1, \ldots, m$. If $\beta = \{0, 1, \ldots, m\}$ then the action space is given by β^N. Each element $\mathbf{x} \in \beta^N$ is called an action of N, and $x_i = k$ if and only if player i takes action σ_k. A *multichoice cooperative game* over N with levels β is a function $V : \beta^N \to \mathbb{R}$ satisfying $V(\mathbf{0}) = 0$. The set of multichoice games over N with levels β is denoted by $\mathcal{MG}^{\beta^N}$. A value on $\mathcal{MG}^{\beta^N}$ is a mapping $\Psi : \mathcal{MG}^{\beta^N} \to \mathbb{R}^{\{1,\ldots,m\}\times N}$. If $V \in \mathcal{MG}^{\beta^N}$, $k \in \beta \setminus \{0\}$, and $i \in N$, the number $\Psi_{k,i}(V)$ is interpreted as the payoff that is allocated to player i for taking action

σ_k. It is considered that $\Psi_{0,i}(V) = 0$. In order to obtain a value for multichoice games, Hsiao and Raghavan consider weights $0 = w(0) \leqslant w(1) \leqslant \cdots \leqslant w(m)$ for the different actions that the players can take. These weights determine the ratios that are considered to be fair when allocating the payment obtained by a group of players who take different actions within a coalition. Once these weights are fixed, Hsiao and Raghavan introduce the value Φ, defined as

$$\Phi^w_{k,i}(V) = \sum_{\substack{l=1 \\ w(l)>0}}^{k} \sum_{\substack{\mathbf{x}\in\beta^N \\ x_i=l}} \sum_{F\subseteq N\setminus(N_m(\mathbf{x})\cup\{i\})} \frac{(-1)^{|F|}w(l)}{\|\mathbf{x}\|_w + \sum_{j\in F}(w(x_j+1)-w(x_j))}(V(\mathbf{x}) - V(\mathbf{x}-\mathbf{e}^i)),$$

where the following notation has been used: if $\mathbf{x} \in \beta^N$ and $l \in \beta$ then $N_l(\mathbf{x}) = \{j \in N : x_j = l\}$, $\|\mathbf{x}\|_w = \sum_{j\in N} w(x_j)$ and $\mathbf{e}^i \in \{0,1\}^N$ is given by $e^i_i = 1$ and $e^i_j = 0$ for every $j \in N \setminus \{i\}$. The number $\Phi^w_{k,i}(V)$ is the payoff to player i when she takes action σ_k in game V. Hsiao and Raghavan proved that this value is the unique value Ψ on $\mathcal{MG}^{\beta^N}$ that satisfies the following properties:

- (H1) if $\mathbf{y} \in \beta^N$ and $V \in \mathcal{MG}^{\beta^N}$ satisfies $V(\mathbf{x}) = V(\mathbf{x}\wedge\mathbf{y})$ for every $\mathbf{x} \in \beta^N$, then $\sum_{i\in N}\Psi_{y_i,i}(V) = V(m,\ldots,m)$.
- (H2) $\Psi(V_1 + V_2) = \Psi(V_1) + \Psi(V_2)$ for every $V_1, V_2 \in \mathcal{MG}^{\beta^N}$.
- (H3) if $\mathbf{y} \in \beta^N$ and $V \in \mathcal{MG}^{\beta^N}$ satisfies $V(\mathbf{x}) = 0$ for every $\mathbf{x} \not\geqslant \mathbf{y}$, then $\Psi_{k,i}(V) = 0$ for every $i \in N$ and $k < y_i$.
- (H4) if $\mathbf{y} \in \beta^N$, $c > 0$ and $V \in \mathcal{MG}^{\beta^N}$ is defined by $V(\mathbf{x}) = c$ if $\mathbf{x} \geqslant \mathbf{y}$ and $V(\mathbf{x}) = 0$ otherwise, then $(\Psi_{y_i,i}(V))_{i\in N}$ is proportional to $(w(y_i))_{i\in N}$.

3.3. Proposed methodology

We aim to define a new value for graph-restricted games. To this end, we will follow a procedure widely used in the literature to obtain values for cooperative games with a combinatorial structure. First, gathering the information from the characteristic function and the combinatorial structure, a new characteristic function is obtained. And, second, a solution concept is applied to this new game. In our case, the key point is that this game will not be a cooperative game in coalitional form, but a multichoice game. We detail the procedure below.

Given a graph-restricted game (v, g), we will define a multichoice game M^g_v. In M^g_v, when a coalition is formed, players in N will be able to take three different actions. The action σ_0 consists of doing nothing, that is, neither actively cooperating nor allowing communication between other players. The action σ_1 consists of not actively cooperating but allowing communication. The action σ_2 consists of both actively cooperating and allowing communication. Therefore, the action space will be 3^N, where $3 = \{0, 1, 2\}$. The payment that will be obtained by the multichoice coalition $\mathbf{x} \in 3^N$ will be equal to

$$M^g_v(\mathbf{x}) = \sum_{S\in(N_1(\mathbf{x})\cup N_2(\mathbf{x}))/g} v(S \cap N_2(\mathbf{x}))$$

for every $\mathbf{x} \in 3^N$. Let us justify the definition of M_v^g. Suppose that a coalition $\mathbf{x}$ is formed and we aim to calculate the profit generated. The players that will actively cooperate are those in $N_2(\mathbf{x})$, but we must take into account the restrictions in communication. The players that will allow communication are those in $N_1(\mathbf{x}) \cup N_2(\mathbf{x})$, so we must consider the connected components of the subgraph induced by this union. Notice that in each of those connected components, only the players in $N_2(\mathbf{x})$ will cooperate actively. This leads us to the definition given above for M_v^g.

Now, we will apply the multichoice value Φ^w to the game M_v^g. To this end, we will use the weights $w(0) = 0$, $w(1) = r \in [0, 1]$ and $w(2) = 1$. We are interested in the payoffs that the players will receive when they decide to actively cooperate. In this way, we obtain the payoff vector $\Phi^w_{2,i}(M_v^g)$ given by

$$\begin{aligned}\Phi^w_{2,i}(M_v^g) =& \\ &= \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}\, r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)) \\ &\quad + \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)),\end{aligned}$$

where the following notation has been used: if $\mathbf{x} \in 3^N$, $l \in \{0, 1, 2\}$, and $H \subseteq N$, then $H_l(\mathbf{x}) = \{j \in H : x_j = l\}$, $\|\mathbf{x}\|_w = \sum_{j\in N} w(x_j)$, and $\mathbf{e}^i \in \{0, 1\}^N$ is given by $e^i_i = 1$ and $e^i_j = 0$ for every $j \in N \setminus \{i\}$.

Definition 1. The r-fair communication value is the value Λ^r for graph-restricted games defined as

$$\Lambda_i^r(v, g) = \Phi^w_{2,i}\big(M_v^g\big)$$

for every $(v, g) \in \mathcal{G}^N \times \Gamma^N$ *and every* $i \in N$.

Once Λ^r is defined, the question arises of what values are obtained for $r = 0$ and $r = 1$. The following two propositions answer this.

Proposition 1. *The 1-fair communication value is equal to the Myerson value, that is,* $\Lambda^1 = Y$.

Proof. Let $v \in \mathcal{G}^N$, $g \in \Gamma^N$, and $i \in N$. We have that

$$\begin{aligned}\Lambda_i^1(v, g) &= \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_0(\mathbf{x})|}(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)) \\ &\quad + \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_0(\mathbf{x})|}(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)) \\ &= \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_0(\mathbf{x})|}(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i))\end{aligned}$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+|F_0(\mathbf{x})|}(M_v^g(\mathbf{x}+\mathbf{e}^i)-M_v^g(\mathbf{x}))$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+|F_0(\mathbf{x})|}(M_v^g(\mathbf{x}+\mathbf{e}^i)-M_v^g(\mathbf{x}-\mathbf{e}^i)).$$

Given $\mathbf{x}\in 3^N$ with $x_i=1$, if we write each $F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}$ as $F=H\cup R$ with $H\subseteq N_0(\mathbf{x})\setminus\{i\}$ and $R\subseteq N_1(\mathbf{x})\setminus\{i\}$, we obtain

$$\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+|F_0(\mathbf{x})|}=\sum_{H\subseteq N_0(\mathbf{x})\setminus\{i\}}\frac{(-1)^{|H|}}{\|\mathbf{x}\|_w+|H|}\sum_{R\subseteq N_1(\mathbf{x})\setminus\{i\}}(-1)^{|R|}.$$

Notice that if $N_1(\mathbf{x})\setminus\{i\}\neq\emptyset$ then $\sum_{R\subseteq N_1(\mathbf{x})\setminus\{i\}}(-1)^{|R|}=0$. Therefore,

$$\Lambda_i^1(v,g)=\sum_{\substack{\mathbf{x}\in 3^N\\ N_1(\mathbf{x})=\{i\}}}\sum_{F\subseteq N_0(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+|F|}(M_v^g(\mathbf{x}+\mathbf{e}^i)-M_v^g(\mathbf{x}-\mathbf{e}^i))$$

which, identifying each $\mathbf{x}\in 3^N$ such that $N_1(\mathbf{x})=\{i\}$ with the set $E=N_2(\mathbf{x})\cup\{i\}$, is equal to

$$=\sum_{\{E\subseteq N:i\in E\}}\sum_{F\subseteq N\setminus E}\frac{(-1)^{|F|}}{|E|+|F|}\left(\sum_{S\in E/g}v(S)-\sum_{T\in(E\setminus\{i\})/g}v(T)\right)$$

$$=\sum_{\{E\subseteq N:i\in E\}}\sum_{F\subseteq N\setminus E}\frac{(-1)^{|F|}}{|E|+|F|}(v_g(E)-v_g(E\setminus\{i\})). \tag{2}$$

We calculate[1]

$$\sum_{F\subseteq N\setminus E}\frac{(-1)^{|F|}}{|E|+|F|}=\sum_{r=0}^{n-|E|}\binom{n-|E|}{r}\frac{(-1)^r}{|E|+r}=\frac{(|E|-1)!(n-|E|)!}{n!}. \tag{3}$$

Substituting in (2), we obtain $\Lambda_i^1(v,g)=Y_i(v,g)$. □

Proposition 2. *If $v\in\mathcal{G}^N$ and $A\in\Gamma^N$, then*

$$\Lambda^0(v,g)=\sum_{T\in N/g}Sh(v|_T)$$

[1]The equality follows from the expression of the beta function, for all $a,b\in\mathbb{N}$:

$$\frac{(a-1)!(b-1)!}{(a+b-1)!}=\beta(a,b)=\sum_{r=0}^{b-1}\frac{(-1)^r}{a+r}\binom{b-1}{r}.$$

where $v|_T \in \mathcal{G}^N$ is defined as $v|_T(E) = v(E \cap T)$ for every $T \in N/g$. In particular, if g is connected, then

$$\Lambda^0(v, g) = Sh(v).$$

Proof. Let $v \in \mathcal{G}^N$, $A \in \Gamma^N$, and $i \in N$. We have

$$\Lambda_i^0(v, g) = \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_1(\mathbf{x})|}(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x} - \mathbf{e}^i)).$$

Following a reasoning similar to that used in Proposition 1, we can obtain that if $\mathbf{x} \in 3^N$, $x_i = 2$, and $N_0(\mathbf{x}) \neq \emptyset$, then

$$\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_1(\mathbf{x})|} = 0.$$

Therefore,

$$\begin{aligned}
\Lambda_i^0(v, g) &= \sum_{\substack{\mathbf{x}\in 3^N\\ N_0(\mathbf{x})=\emptyset\\ x_i=2}} \sum_{F\subseteq N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F|}(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x} - \mathbf{e}^i))\\
&= \sum_{\{E\subseteq N: i\in E\}} \sum_{F\subseteq N\setminus E} \frac{(-1)^{|F|}}{|E| + |F|}\left(\sum_{T\in N/g} v(T\cap E) - \sum_{T\in N/g} v(T\cap (E\setminus\{i\}))\right)\\
&= \sum_{T\in N/g} \sum_{\{E\subseteq N: i\in E\}} \sum_{F\subseteq N\setminus E} \frac{(-1)^{|F|}}{|E| + |F|}(v|_T(E) - v|_T(E\setminus\{i\}))
\end{aligned}$$

which, by (3), is equal to

$$\sum_{T\in N/g} \sum_{\{E\subseteq N: i\in E\}} \frac{(|E|-1)!(n-|E|)!}{n!}(v|_T(E) - v|_T(E\setminus\{i\})) = \sum_{T\in N/g} Sh_i(v|_T).$$

□

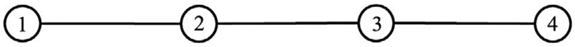
Example 2. Let us go back to the motivating example given at the beginning of this section. Let us take $r = \frac{1}{10}$. This value for r establishes a more appropriate weighting of intermediary power, in most contexts, than the value 1 (corresponding to the Myerson value, according to the previous proposition). It is easy to check that

$$\Lambda^{\frac{1}{10}}(u_{\{1,3\}}, g) = \left(\frac{10}{21}, \frac{1}{21}, \frac{10}{21}\right).$$

Note that the payoff received by player 2 is significantly lower than the payoff received by the players that actively generate the profits.

One might wonder if Λ^r is some convex combination of Λ^0 and Λ^1. It is easy to provide an example to dismiss this idea.

Example 3. Let $N = \{1, 2, 3, 4\}$ and consider the graph $g \in \Gamma^N$ represented by the diagram below.



Consider the game $v = u_{\{1,3\}} + u_{\{1,4\}} \in \mathcal{G}^N$. We have that

$$Y(v, g) = \Lambda^1(v, g) = \left(\frac{7}{12}, \frac{7}{12}, \frac{7}{12}, \frac{3}{12}\right),$$

$$\Lambda^0(v, g) = \left(1, 0, \frac{1}{2}, \frac{1}{2}\right),$$

$$\Lambda^{\frac{1}{10}}(v, g) = \left(\frac{430}{462}, \frac{43}{462}, \frac{241}{462}, \frac{210}{462}\right).$$

We can see that the value Λ^1 (the Myerson value) rewards intermediary power as much as active profit generation, the value Λ^0 does not reward the intermediary role, and the value $\Lambda^{\frac{1}{10}}$ weighs intermediary power and the ability to generate profit in a way that is not a convex combination of Λ^0 and Λ^1.

3.4. Characterization of Λ^r

Let Ψ be a value for graph-restricted games. In order to characterize the r-fair communication value. we will consider the following properties:

ADDITIVITY. *If* $v, w \in \mathcal{G}^N$ and $g \in \Gamma^N$, then $\Psi(v + w, g) = \Psi(v, g) + \Psi(w, g)$.

FAIRNESS FOR NULL PLAYERS. *Let* $v \in \mathcal{G}^N$, $g = (N, L) \in \Gamma^N$, *and* $ij \in L$ *be such that* i, j *are null players in* v. *Then*,

$$\Psi_i(v, g) - \Psi_i(v, g_{-ij}) = \Psi_j(v, g) - \Psi_j(v, g_{-ij}).$$

FAIRNESS FOR NECESSARY PLAYERS. *Let* $v \in \mathcal{G}^N$, $g = (N, L) \in \Gamma^N$, *and* $ij \in L$ *be such that* i, j *are necessary players in* v. *Then*,

$$\Psi_i(v, g) - \Psi_i(v, g_{-ij}) = \Psi_j(v, g) - \Psi_j(v, g_{-ij}).$$

r-FAIRNESS ($r \in [0, 1]$). *Let* $v \in \mathcal{G}^N$, $g = (N, L) \in \Gamma^N$, *and* $ij \in L$ *be such that* i *is a null player in* v *and* j *is a necessary player in* v. *Then*,

$$\Psi_i(v, g) - \Psi_i(v, g_{-ij}) = r\big(\Psi_j(v, g) - \Psi_j(v, g_{-ij})\big).$$

To interpret the equality above, note that the differences $\Psi_i(v, g) - \Psi_i(v, g_{-ij})$ and $\Psi_j(v, g) - \Psi_j(v, g_{-ij})$ represent the additional gains that players i and j, respectively, will obtain when they establish bilateral communication with each other. Now, observe that, on the one hand, the additional gain of player i solely rewards action σ_1 of this player, that is, their capacity as an intermediary, since i, being a null player, cannot contribute through active cooperation. On the other hand, the additional gain of player j rewards action σ_2 of this player, because j, being necessary, cannot generate any gain through mediation alone. Therefore, note that if r-fairness holds, there exists a fixed weighting for the reward for action σ_1 in relation to the reward for σ_2. Finally, note that the requirement for r to belong to [0,1] is entirely reasonable, as, on the one hand, it would not make sense to penalize intermediation power, and, on the other hand, it would also not make sense for action σ_1 to be rewarded more than action σ_2, since the latter encompasses the former.

In the following theorem, we show a characterization of the r-fair communication value.

Theorem 1. *Let $r \in (0, 1]$. A value for graph-restricted games is equal to the r-fair communication value if and only if it satisfies the properties of component efficiency, additivity, fairness for null players, fairness for necessary players, and r-fairness.*

Proof. First, we will prove that Λ^r satisfies the four properties mentioned in the theorem.

Component efficiency. Let $v \in \mathcal{G}^N$, $g \in \Gamma^N$, and $T \in N/g$. Let $V_T, V_{N\setminus T} \in \mathcal{MG}^{3^N}$ defined as

$$V_T(\mathbf{x}) = \sum_{\{S \in (N_1(\mathbf{x}) \cup N_2(\mathbf{x}))/g : S \subseteq T\}} v(S \cap N_2(\mathbf{x}))$$

$$V_{N\setminus T}(\mathbf{x}) = \sum_{\{S \in (N_1(\mathbf{x}) \cup N_2(\mathbf{x}))/g : S \subseteq N\setminus T\}} v(S \cap N_2(\mathbf{x}))$$

for every $\mathbf{x} \in 3^N$. Notice that V_T (resp., $V_{N\setminus T}$) is, loosely speaking, the restriction of V to T (resp., $N \setminus T$), since it assigns to each coalition the profit achievable in V by the coalition with the restriction that players outside T (resp., $N \setminus T$) cannot cooperate. From the fact that $T \in N/g$, it can be easily derived that

$$M_v^g = V_T + V_{N\setminus T}. \tag{4}$$

Moreover, it is easy to check that

$$\Phi^w_{2,i}(V_T) = 0 \qquad \text{for every } i \in N \setminus T \tag{5}$$

and

$$\Phi^w_{2,i}(V_{N\setminus T}) = 0 \qquad \text{for every } i \in T. \tag{6}$$

By (4) and the additivity of $\Phi^w_{2,i}$, we have that

$$\sum_{i\in T} \Lambda_i^r(v, g) = \sum_{i\in T} \Phi^w_{2,i}(M_v^g) = \sum_{i\in T} \Phi^w_{2,i}(V_T) + \sum_{i\in T} \Phi^w_{2,i}(V_{N\setminus T}),$$

which, by (5), (6), and the property (H1) of Φ_2^w, is equal to

$$\sum_{i\in N}\Phi_{2,i}^w(V_T)=V_T(2,2,\ldots,2)=v(T).$$

ADDITIVITY. Let $v, w\in\mathcal{G}^N$, $g\in\Gamma^N$, and $i\in N$. It is clear that $M_{v+w}^g=M_v^g+M_w^g$. If we also use the additivity of Φ_2^w, we obtain

$$\begin{aligned}\Lambda_i^r(v+w,g)&=\Phi_{2,i}^w\big(M_{v+w}^g\big)=\Phi_{2,i}^w(M_v^g+M_w^g)\\&=\Phi_{2,i}^w(M_v^g)+\Phi_{2,i}^w(M_w^g)=\Lambda_i^r(v,g)+\Lambda_i^r(w,g).\end{aligned}$$

FAIRNESS FOR NULL PLAYERS. Let $v\in\mathcal{G}^N$, $g=(N,L)\in\Gamma^N$, and $ij\in L$ be such that i, j are null players in v. We have that

$$\begin{aligned}&\Lambda_i^r(v,g)-\Lambda_i^r(v,g_{-ij})=\\&\quad=\sum_{\substack{\mathbf{x}\in 3^N\\x_i=1}}\sum_{F\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x})-M_v^g(\mathbf{x}-\mathbf{e}^i)\big)\\&\qquad+\sum_{\substack{\mathbf{x}\in 3^N\\x_i=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x})-M_v^g(\mathbf{x}-\mathbf{e}^i)\big)\\&\quad-\sum_{\substack{\mathbf{x}\in 3^N\\x_i=1}}\sum_{F\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\big(M_v^{g_{-ij}}(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\big)\\&\qquad-\sum_{\substack{\mathbf{x}\in 3^N\\x_i=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\big(M_v^{g_{-ij}}(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\big)\end{aligned}$$

which, taking into account that i is a null player in v, is equal to

$$\begin{aligned}&=\sum_{\substack{\mathbf{x}\in 3^N\\x_i=1}}\sum_{F\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x})-M_v^g(\mathbf{x}-\mathbf{e}^i)\big)\\&\quad-\sum_{\substack{\mathbf{x}\in 3^N\\x_i=1}}\sum_{F\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\big(M_v^{g_{-ij}}(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\big)\end{aligned}$$

which, from the fact that $M_v^g(\mathbf{y})=M_v^{g_{-ij}}(\mathbf{y})$ for every $\mathbf{y}\in 3^N$ such that $y_i=0$, is equal to

$$=\sum_{\substack{\mathbf{x}\in 3^N\\x_i=1}}\sum_{F\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\big)$$

which, from the fact that $M_v^g(\mathbf{y}) = M_v^{g-ij}(\mathbf{y})$ for every $\mathbf{y} \in 3^N$ such that $y_j = 0$, is equal to

$$= \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+ \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=2}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$= \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+ \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F\cup\{j\}|} r}{\|\mathbf{x}\|_w + r|(F\cup\{j\})_0(\mathbf{x})| + (1-r)|(F\cup\{j\})_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+ \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=2}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$= \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+ \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|+1} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)(|F_1(\mathbf{x})|+1)} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+ \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=2}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

which, taking into account that j is a null player in v, is equal to

$$= \sum_{\substack{\mathbf{x}\in 3^N \\ x_i=1 \\ x_j=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|+1} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)(|F_1(\mathbf{x})|+1)}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}-\mathbf{e}^j) - M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^j)\big)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|+1} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)(|F_1(\mathbf{x})|+1)}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x}+\mathbf{e}^j)\cup N_1(\mathbf{x}+\mathbf{e}^j))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}+\mathbf{e}^j\|_w + r|F_0(\mathbf{x}+\mathbf{e}^j)| + (1-r)|F_1(\mathbf{x}+\mathbf{e}^j)|}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|+1} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)(|F_1(\mathbf{x})|+1)}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + 1 - r + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i,j\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big).$$

It suffices to notice that in the last expression i and j are interchangeable.

Fairness for necessary players. Let $v \in \mathcal{G}^N$, $g = (N, L) \in \Gamma^N$ and $ij \in L$ be such that i, j are necessary players in v. We have that

$$\Lambda_i^r(v, g) - \Lambda_i^r(v, g_{-ij})$$

$$= \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)\big)$$

$$+ \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)\big)$$

$$- \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^{g_{-ij}}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\big)$$

$$- \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^{g_{-ij}}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\big)$$

which, taking into account that i is a necessary player in v, is equal to

$$= \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big)$$

which, since j is a necessary player in v, is equal to

$$= \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2\\ x_j=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\big),$$

where we can interchange i and j.

r-FAIRNESS. Let $v \in \mathcal{G}^N$, $g = (N, L) \in \Gamma^N$, and $ij \in L$ be such that i is a null player in v and j is a necessary player in v. We aim to prove that

$$\Psi_i(v, g) - \Psi_i(v, g_{-ij}) = r\big(\Psi_j(v, g) - \Psi_j(v, g_{-ij})\big).$$

On the one hand, we have

$$\Lambda_i^r(v, g) - \Lambda_i^r(v, g_{-ij})$$

$$= \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}} \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|} r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)\big)$$

$$+ \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\big(M_v^g(\mathbf{x}) - M_v^g(\mathbf{x}-\mathbf{e}^i)\big)$$

$$-\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g_{-ij}}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\right)$$

$$-\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}}\ \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g_{-ij}}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\right)$$

which, taking into account that i is a null player in v, is equal to

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x}) - M_v^{g}(\mathbf{x}-\mathbf{e}^i)\right)$$

$$-\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g_{-ij}}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\right)$$

which, from the fact that $M_v^g(\mathbf{y}) = M_v^{g_{-ij}}(\mathbf{y})$ for every $\mathbf{y}\in 3^N$ such that $y_i = 0$, is equal to

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\right)$$

which, since j is a necessary player in v, is equal to

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x})\right).$$

On the other hand, we have that

$$\Lambda_j^r(v,g) - \Lambda_j^r(v,g_{-ij})$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{j\}} \frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x}) - M_v^{g}(\mathbf{x}-\mathbf{e}^j)\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_j=2}}\ \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x}) - M_v^{g}(\mathbf{x}-\mathbf{e}^j)\right)$$

$$-\sum_{\substack{\mathbf{x}\in 3^N\\ x_j=1}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{j\}} \frac{(-1)^{|F|}r}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^{g_{-ij}}(\mathbf{x}) - M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^j)\right)$$

$$-\sum_{\substack{\mathbf{x}\in 3^N\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g-ij}(\mathbf{x})-M_v^{g-ij}(\mathbf{x}-\mathbf{e}^j)\right)$$

which, taking into account that j is a necessary player in v, is equal to

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=0\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

which, from the fact that $M_v^g(\mathbf{x})=M_v^{g-ij}(\mathbf{x})$ for every $\mathbf{x}\in 3^N$ such that $x_i=0$, is equal to

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F\cup\{i\}|}}{\|\mathbf{x}\|_w+r|(F\cup\{i\})_0(\mathbf{x})|+(1-r)|(F\cup\{i\})_1(\mathbf{x})|}\left(M_v^{g}(\mathbf{x})-M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|+1}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)(|F_1(\mathbf{x})|+1)}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

which, taking into account that i is a null player in v, is equal to

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|+1}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)(|F_1(\mathbf{x})|+1)}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2\\ x_j=2}}\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x}-\mathbf{e}^i)-M_v^{g_{-ij}}(\mathbf{x}-\mathbf{e}^i)\right)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}}\frac{(-1)^{|F|+1}}{\|\mathbf{x}\|_w+r|F_0(\mathbf{x})|+(1-r)(|F_1(\mathbf{x})|+1)}\left(M_v^g(\mathbf{x})-M_v^{g_{-ij}}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\ \sum_{F\subseteq N_0(\mathbf{x}+\mathbf{e}^i)\cup N_1(\mathbf{x}+\mathbf{e}^i)} \frac{(-1)^{|F|}}{\|\mathbf{x}+\mathbf{e}^i\|_w + r|F_0(\mathbf{x}+\mathbf{e}^i)| + (1-r)|F_1(\mathbf{x}+\mathbf{e}^i)|}\left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|+1}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)(|F_1(\mathbf{x})|+1)}\left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$+\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + 1 - r + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

$$=\sum_{\substack{\mathbf{x}\in 3^N\\ x_i=1\\ x_j=2}}\ \sum_{F\subseteq (N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}\left(M_v^g(\mathbf{x}) - M_v^{g-ij}(\mathbf{x})\right)$$

We have already seen that Λ^r satisfies the five properties mentioned in the theorem. Now we will show that such properties uniquely determine this value. Let Ψ be a value for graph-restricted games that satisfies the five axioms for some $r \in [0, 1]$. We must prove that $\Psi = \Lambda^r$.

Our first goal will be to show that if $c \in \mathbb{R}$, $E \in 2^N \setminus \{\emptyset\}$ and $g = (N, L) \in \Gamma^N$, then

$$\Psi(cu_E, g) = \Lambda^r(cu_E, g).$$

We will prove this by induction on $|L|$.

BASE CASE. If $|L| = 0$ then $N/g = \{\{i\} : i \in N\}$ and, since Ψ and Λ^r satisfy component efficiency, it is clear that $\Psi(cu_E, g) = \Lambda^r(cu_E, g)$.

INDUCTIVE STEP. Suppose that $|L| > 0$. Take $T \in N/g$. It suffices to prove that

$$\Psi_k(cu_E, g) = \Lambda^r_k(cu_E, g) \tag{7}$$

for every $k \in T$. Take $i \in T$. We can assume that $|T| \geqslant 2$ (otherwise, it suffices to use the property of component efficiency). Let us suppose that $\Lambda^r_i(cu_E, g) \geqslant \Psi_i(cu_E, g)$ (otherwise, the reasoning to follow is parallel to the one we will describe in our case). Let $j \in N$ be such that $ij \in L$. Let us distinguish cases:

(a) If $i, j \in E$, then, since Λ^r and Ψ satisfy the property of fairness for necessary players, we have that

$$\Lambda^r_i(cu_E, g) - \Lambda^r_i(cu_E, g_{-ij}) = \Lambda^r_j(cu_E, g) - \Lambda^r_j(cu_E, g_{-ij}), \tag{8}$$

$$\Psi_i(cu_E, g) - \Psi_i(cu_E, g_{-ij}) = \Psi_j(cu_E, g) - \Psi_j(cu_E, g_{-ij}). \tag{9}$$

Besides, by induction hypothesis,

$$\Lambda_i^r(cu_E, g_{-ij}) = \Psi_i(cu_E, g_{-ij}), \tag{10}$$
$$\Lambda_j^r(cu_E, g_{-ij}) = \Psi_j(cu_E, g_{-ij}). \tag{11}$$

From (8)–(11), it easily follows that

$$\Lambda_j^r(cu_E, g) - \Psi_j(cu_E, g) = \Lambda_i^r(cu_E, g) - \Psi_i(cu_E, g) \geqslant 0. \tag{12}$$

(b) If $i \in E$ and $j \in N \setminus E$, then, since Λ^r and Ψ satisfy the property of r-fairness, we have that

$$r\big(\Lambda_i^r(cu_E, g) - \Lambda_i^r(cu_E, g_{-ij})\big) = \Lambda_j^r(cu_E, g) - \Lambda_j^r(cu_E, g_{-ij}), \tag{13}$$
$$r\big(\Psi_i(cu_E, g) - \Psi_i(cu_E, g_{-ij})\big) = \Psi_j(cu_E, g) - \Psi_j(cu_E, g_{-ij}). \tag{14}$$

Besides, by induction hypothesis,

$$\Lambda_i^r(cu_E, g_{-ij}) = \Psi_i(cu_E, g_{-ij}), \tag{15}$$
$$\Lambda_j^r(cu_E, g_{-ij}) = \Psi_j(cu_E, g_{-ij}). \tag{16}$$

From (13)–(16), it easily follows that

$$\Lambda_j^r(cu_E, g) - \Psi_j(cu_E, g) = r\big(\Lambda_i^r(cu_E, g) - \Psi_i(cu_E, g)\big) \geqslant 0. \tag{17}$$

(c) If $i \in N \setminus E$ and $j \in E$, then, similarly to case b, we obtain that

$$\Lambda_j^r(cu_E, g) - \Psi_j(cu_E, g) = \frac{1}{r}\big(\Lambda_i^r(cu_E, g) - \Psi_i(cu_E, g)\big) \geqslant 0. \tag{18}$$

(d) If $i, j \in N \setminus E$, then, similarly to case (a), but using fairness for null players instead of fairness for necessary players, we obtain that

$$\Lambda_j^r(cu_E, g) - \Psi_j(cu_E, g) = \Lambda_i^r(cu_E, g) - \Psi_i(cu_E, g) \geqslant 0. \tag{19}$$

Notice that, in any case, we obtain that

$$\Lambda_j^r(cu_E, g) - \Psi_j(cu_E, g) \geqslant 0.$$

With the same reasoning, we can prove that $\Lambda_k^r(cu_E, g) - \Psi_k(cu_E, g) \geqslant 0$ for any $k \in T$ with $jk \in L$. By iteration, we obtain that

$$\Lambda_k^r(cu_E, g) - \Psi_k(cu_E, g) \geqslant 0 \tag{20}$$

for any $k \in T$. By component efficiency, we have that

$$\sum_{k \in T} \big(\Lambda_k^r(cu_E, g) - \Psi_k(cu_E, g)\big) = \sum_{k \in T} \Lambda_k^r(cu_E, g) - \sum_{k \in T} \Psi_k(cu_E, g) = v(T) - v(T) = 0,$$

and, taking into account (20), we conclude that $\Lambda^r_k(cu_E, g) = \Psi_k(cu_E, g)$ for any $k \in T$.

So we already know that $\Psi(cu_E, g) = \Lambda^r(cu_E, g)$ for all $c \in \mathbb{R}$, $E \in 2^N \setminus \{\emptyset\}$ and $g \in \Gamma^N$.

Finally, take $v \in \mathcal{G}^N$ and $g \in \Gamma^N$. Then,

$$\Psi(v, g) = \Psi\left(\sum_{\{E\subseteq N: E\neq\emptyset\}} \Delta_v(E)u_E, g\right) = \sum_{\{E\subseteq N: E\neq\emptyset\}} \Psi(\Delta_v(E)u_E, g)$$

$$= \sum_{\{E\subseteq N: E\neq\emptyset\}} \Lambda^r(\Delta_v(E)u_E, g) = \Lambda^r\left(\sum_{\{E\subseteq N: E\neq\emptyset\}} \Delta_v(E)u_E, g\right) = \Lambda^r(v, g).$$

□

Remark. Finally, we will show that if $|N| \geqslant 5$ then the axioms in Theorem 1 are logically independent. Suppose that $N = \{1, \ldots, n\}$ with $n \geqslant 5$.

(1) In order to obtain a value that satisfies all the axioms except component efficiency, it suffices to consider $\Psi^1(v, g) = \mathbf{0}$ for every $v \in \mathcal{G}^N$ and every $g = (N, L) \in \Gamma^N$.

(2) Let $\Psi^2 : \mathcal{G}^N \times \Gamma^N \to \mathbb{R}^N$ defined as

$$\Psi^2(v, g) = \begin{cases} n\,\mathbf{e}^1 & \text{if } v = \sum_{i\in N} u_{\{i\}} \text{ and } g \text{ is complete,} \\ \Lambda^r(v, g) & \text{otherwise,} \end{cases}$$

where $\mathbf{e}^1 \in \mathbb{R}^N$ is given by $e^1_1 = 1$ and $e^1_j = 0$ for every $j \in N \setminus \{1\}$. Evidently, Ψ^2 satisfies component efficiency. Taking into account that the game $\sum_{i\in N} u_{\{i\}}$ does not have either null or necessary players, it is clear that Ψ^2 satisfies fairness for null players, fairness for necessary players, and r-fairness. We conclude that Ψ^2 satisfies all the properties except additivity.

(3) For each $v \in \mathcal{G}^N$ and each $g = (N, L) \in \Gamma^N$, we define

$$\Psi^3(v, g) = \Delta_v(\{1, 2\})\Lambda^r\big(u_{\{1,2\}}, g_{-35}\big) + \sum_{\{E\in 2^N\setminus\{\emptyset\}: E\neq\{1,2\}\}} \Delta_v(E)\Lambda^r(u_E, g)$$

if $34, 45, 35 \in L$, and $\Psi^3(v, g) = \Lambda^r(v, g)$ otherwise. By construction, Ψ^3 is additive. Note that if $g = (N, L)$ and $34, 45, 35 \in L$, then $N/g = N/(g_{-35})$. This easily leads us to the fact that Ψ^3 satisfies component efficiency. Now notice that if $\Delta_v(\{1, 2\}) \neq 0$ then the set of necessary players of v is a subset of $\{1, 2\}$. Taking this into account, it is straightforward to prove that Ψ^3 satisfies fairness for necessary players and r-fairness. Consequently, Ψ^3 satisfies all the axioms except fairness for null players.

(4) Let $\Psi^4 : \mathcal{G}^N \times \Gamma^N \to \mathbb{R}^N$ defined as

$$\Psi^4(v, g) = \begin{cases} \Delta_v(N)\,\mathbf{e}^1 + \sum\limits_{\{E \in 2^N \setminus \{\emptyset\} : E \neq N\}} \Delta_v(E)\Lambda^r(u_E, g) & \text{if } g \text{ is complete,} \\ \\ \Lambda^r(v, g) & \text{otherwise,} \end{cases}$$

where $\mathbf{e}^1 \in \mathbb{R}^N$ is given by $e^1_1 = 1$ and $e^1_j = 0$ for every $j \in N \setminus \{1\}$. By construction, Ψ^4 is additive. It is simple to check that Ψ^4 satisfies component efficiency. Notice that if $\Delta_v(N) \neq 0$ then v does not have null players. Keeping this in mind, it is clear that Ψ^4 satisfies fairness for null players and r-fairness. We conclude that Ψ^4 satisfies all the axioms except fairness for necessary players.

(5) Let $\Psi^5 : \mathcal{G}^N \times \Gamma^N \to \mathbb{R}^N$ defined as

$$\Psi^5(v, g) = \Delta_v(\{1, 2\})\Lambda^0\big(u_{\{1,2\}}, g\big) + \sum_{\{E \in 2^N \setminus \{\emptyset\} : E \neq \{1,2\}\}} \Delta_v(E)\Lambda^1(u_E, g).$$

By construction, Ψ^5 is additive. Taking into consideration that Λ^0 and Λ^1 satisfy component efficiency, fairness for null players, and fairness for necessary players, it is easy to verify that Ψ^5 also satisfies these axioms. Finally, it is evident that $\Psi^5 \neq \Lambda^r$ for any $r \in [0, 1]$.

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