A new algorithm for determining the monodromy of a planar differential system

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Abstract

We give a new algorithmic criterium that determines wether an isolated degenerate singular point of a system of differential equations on the plane is monodromic. This criterium involves the conservative and dissipative parts associated to the edges and vertices of the Newton diagram of the vector field.

Keywords: monodromy, characteristic orbits

1. Introduction.

We are interested in the behavior of the trajectories in a neighborhood of a singular point of the planar analytic differential system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

and, in particular, we try to establish when a singular point (we can assume the origin to be the singular point) is surrounded by orbits of the system (monodromic singular point), i.e. each trajectory by lying on a vicinity of a monodromic singular point is either a spiral or an oval. Moreover, from the finiteness theorem for the number of limit cycles, a monodromic point of an analytic planar vector field can be only either a focus or a center, see Il'yashenko [17]. So, the monodromy problem is a previous step to solve the center problem of a vector field which is one of the open classical problems in the qualitative theory of planar differential systems, see [2, 12, 13, 14, 15, 16, 18, 20].

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If the eigenvalues of the matrix of the linear part at the origin are conjugate complex or the matrix is nilpotent, the monodromy is a problem solved (Poincaré [20], Andreev [7]). However, there are only partial results when the quoted matrix is identically null (Medvedeva [19], Gasull *et al.* [12], Mañosa [18], García *et al.* [11]). All of them use the blow-up procedure introduced by Dumortier [10] which consists of performing a series of changes to desingularize the point.

Tang *et al.* [21] use a method of generalized normal sectors to determine orbits in exceptional directions near high degenerate equilibria.

Currently, the algorithms of Medvedeva [19] (in general) and García *et al.* [11] (for particular cases) are the criteria used in order to determine the monodromy of the origin.

Here we give an alternative algorithm which improves the Medvedeva algorithm in two aspects basically: it uses the expressions of the conservative and dissipative terms (reducing computational efforts) and it does not require to apply blow-up changes when the conservative quasi-homogeneous term of minor degree has trivial factors.

The paper is organized as follows. In the next section, we show the conservative-dissipative decomposition of a quasi-homogeneous vector field and recall some concept related to the Newton diagram of a vector field. In Section 3, we state the monodromy algorithm. In Section 4, we give the concept of characteristic orbit and show its relation with the monodromy problem. This section also contains some auxiliaries results in order to prove the correctness of the algorithm (Theorem 2). Last on, as an application of our algorithm, in Section 5, we obtain the systems

$$\dot{x} = ay^3 + cxy^2 + gx^2y + ex^5, \qquad \dot{y} = dy^3 + hxy^2 + fx^4y + bx^7,$$

with $ab(g^2+h^2) < 0$, whose origin is a monodromic singular point (Theorem 13). Medvedeva [19] studied the case $c = -d$, $g = 0$ and did not give all cases of monodromy.

2. Quasi-homogeneous vector field and Newton diagram

Next on, we give some definitions and concepts that we will use throughout the paper. Fixed a type $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 coprime natural numbers, which can be arbitrarily chosen, a function f of two variables is quasi-homogeneous of type $\mathbf{t}$ and degree k if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. We will also consider the limit cases $\mathbf{t} = (1, 0)$ and $\mathbf{t} = (0, 1)$, being $\mathcal{P}_k^{(1,0)} = x^k \mathbf{R}[[y]]$ and $\mathcal{P}_k^{(0,1)} = y^k \mathbf{R}[[x]]$, where $\mathbf{R}[[y]]$ (resp. $\mathbf{R}[[x]]$) are the algebras of the power series of y (resp. of x) with coefficients in $\mathbf{R}$.

A vector field $\mathbf{F} = (F_1, F_2)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $F_1 \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k.

The expansions of vector fields into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees are usually considered in the analysis of the topological determination of the singularity by means of the blow-up technique (see Bruno [9], Brunella and Miari [8] and Dumortier [10]). This concept also has been used by Algaba *et al.* [3] as an application of Normal Form Theory, and for the study of the integrability and center problems of systems with a degenerate singular point, i.e. systems whose matrix of the linear part evaluated in the singular point is identically null, see Algaba *et al.* [4, 6].

Next, we recall the decomposition of a quasi-homogeneous vector field as a sum of two quasi-homogeneous fields, a conservative one (having zero-divergence) and a dissipative one (in the sense of the non-conservative part that fully captures the divergence of the vector field), which will be useful in what follows and it will play a main role in our analysis. Throughout this paper, Hamiltonian system associated to the $\mathcal{C}^1$ function f is denoted by $\mathbf{X}_f$, i.e. $\mathbf{X}_f = (-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})^T$. Algaba *et. al.* [6] proved that any quasi-homogeneous vector field $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_j = \mathbf{X}_{h_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0, \tag{2}$$

where $\mathbf{D}_0(x, y) := (t_1 x, t_2 y)^T \in \mathcal{Q}_0^{\mathbf{t}}$, $\mu_j := \frac{1}{j+|\mathbf{t}|}\operatorname{div}(\mathbf{F}_j) \in \mathcal{P}_j^{\mathbf{t}}$ (the divergence of $\mathbf{F}_j$), $h_{j+|\mathbf{t}|} := \frac{1}{j+|\mathbf{t}|}(t_1 x Q_{j+t_2} - t_2 y P_{j+t_1}) \in \mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$ (the wedge product of $\mathbf{D}_0$ and $\mathbf{F}_j$) and $|\mathbf{t}| = t_1 + t_2$.

We will write the components of the vector field $\mathbf{F}$ in the form $P(x, y) = \sum a_{ij} x^i y^{j-1}$ and $Q(x, y) = \sum b_{ij} x^{i-1} y^j$.
The *support* of (1) and also of $\mathbf{F}$, denoted by $supp(\mathbf{F})$, is the set of pairs (i, j) with $(a_{ij}, b_{ij}) \neq (0, 0)$. The vector (a_{ij}, b_{ij}) is called *vector coefficient* of (i, j) in the support. Consider set

$$\bigcup_{(i,j)\in supp(\mathbf{F})} \left((i, j) + \mathbb{R}_+^2\right),$$

where $\mathbb{R}^2_+$ is the first quadrant and the union is taken over all points (i,j) in the support. The boundary of the convex hull of this set is made up of two open rays and a polygon, which can be just one point. The polygon together with the rays that do not lie on a coordinates axes, if they exist, is called *Newton diagram* of the vector field $\mathbf{F}$. The component parts of Newton diagram are called *edges* and their endpoints are the *vertices* of Newton diagram.
If a vertex of Newton diagram does not lie on any coordinates axis, then it is said to be *inner*; otherwise, it is an *exterior* vertex.
The *exponent* of a bounded edge ℓ of Newton diagram is a positive rational number t_2/t_1, equal to the tangent of the angle between the edge and the ordinate axis, and pair $\mathbf{t} = (t_1, t_2)$ is called *type* of edge ℓ. If Newton diagram contains an unbounded horizontal edge then we set its exponent equal to ∞ and its type is $(0,1)$, and if there is a vertical edge, it has exponent 0 and its type is $(1,0)$. We will denote α_ℓ and $\alpha_{\tilde{\ell}}$, with $\alpha_\ell < \alpha_{\tilde{\ell}}$, as the exponents of the edges upper ℓ and lower $\widetilde{\ell}$ adjoining the vertex V.

3. Monodromy algorithm

We now give two concepts which play a main role in our algorithm.

Definition 1. Let $h_{r+|\mathbf{t}|} \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ and $\mu_r \in \mathcal{P}^{\mathbf{t}}_r$ polynomials associated to lowest-degree quasi-homogeneous term of type $\mathbf{t}$ of $\mathbf{F}$.
We say that a polynomial of the form $y^{t_1} - \lambda x^{t_2}$, $\lambda \neq 0$, is a *strong factor associated to the type* $\mathbf{t}$ if it satisfies one of the following properties:

- **(i)** it is a factor of $h_{r+|\mathbf{t}|}$ of odd multiplicity order,
- **(ii)** it is a factor of $h_{r+|\mathbf{t}|}$ of even multiplicity order $(2m)$ and, either it is a no factor of μ_r with $\mu_r \not\equiv 0$ or is a factor of μ_r with even multiplicity order $(2n)$ with $0 < n < m$.

Definition 2. Give an inner vertex V of Newton diagram of system (1) such that $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$ are the types of its upper and lower adjacent edges, respectively, i.e. $\alpha_\ell = t_2/t_1 < s_2/s_1 = \alpha_{\tilde{\ell}}$, with $h_{r_{\mathbf{t}}+|\mathbf{t}|} h_{r_{\mathbf{s}}+|\mathbf{s}|} \not\equiv 0$, we define the constant associated to the vertex V as

$$\beta_V = \tilde{c}_{j_0} c_{i_0}, \tag{3}$$

where $i_0 = \min\{i \geq 0 \,|\, c_i \neq 0\}$, $j_0 = \min\{j \geq 0 \,|\, \tilde{c}_j \neq 0\}$ and c_i and $\tilde{c}_j$ being the coefficients of the polynomials $h_{r_{\mathbf{t}}+|\mathbf{t}|}$ and $h_{r_{\mathbf{s}}+|\mathbf{s}|}$, ordered from the highest to the lowest exponent in x and y, respectively.

The following result provides a series of necessary conditions which must verify system (1) for the origin to be monodromic.

Proposition 1. *If origin of system (1) is monodromic then the following properties hold:*

(A) *Newton diagram of system (1) has two exterior vertices. Moreover, if $(a, 0)$ and $(0, b)$ are the vector coefficients of the exterior vertices of Newton diagram, then $ab < 0$,*

(B) *all vertices of Newton diagram of (1) have even coordinates.*

Proposition 1 is proved in Section 4.

Next, we state our algorithmic criterium, which we use for characterizing the monodromy of the origin.

Theorem 2. *Let system (1) be verifying conditions* **(A)** *and* **(B)**. *Then origin of system (1) is monodromic if and only if system (1) satisfies, step to step, all conditions of the following process:*
Step 1. If all inner vertices V of its Newton diagram verifies
(1) $\beta_V > 0$.
Step 2. For each edge of Newton diagram with exponent t_2/t_1, polynomial $h_{r+|\mathbf{t}|}$ verifies
(2a) $h_{r+|\mathbf{t}|} \not\equiv 0$, *and*
(2b) $h_{r+|\mathbf{t}|}$ *does not have any strong factors.*
Step 3. For each factor of $h_{r+|\mathbf{t}|}$ of the form $y^{t_1} - \tilde{a}x^{t_2}$ with $\tilde{a} \neq 0$,
• *if t_1 is odd, applying the directional blow-up $x = u^{t_1}$, $y = u^{t_2}(\bar{y} + \tilde{a}^{1/t_1})$ and the reparameterization $dt = (t_1/u^r)d\tau$, we obtain system*

$$\begin{aligned} u' &= u \sum_{j=0}^{\infty} P_{r+j+t_1}(1, \bar{y} + \tilde{a}^{1/t_1})u^j, \\ \bar{y}' &= \sum_{j=0}^{\infty} (r + |\mathbf{t}| + j)h_{r+j+|\mathbf{t}|}(1, \bar{y} + \tilde{a}^{1/t_1})u^j. \end{aligned} \tag{4}$$

• *if t_1 is even, applying the directional blow-up $x = u^{t_1}(\bar{x} + \tilde{a}^{-1/t_2})$, $y = u^{t_2}$ and the reparameterization $dt = (t_2/u^r)d\tau$, we obtain system*

$$\begin{aligned} u' &= -u \sum_{j=0}^{\infty} Q_{r+j+t_2}(\bar{x} + \tilde{a}^{-1/t_2}, 1)u^j, \\ \bar{x}' &= \sum_{j=0}^{\infty} (r + |\mathbf{t}| + j)h_{r+j+|\mathbf{t}|}(\bar{x} + \tilde{a}^{-1/t_2}, 1)u^j. \end{aligned} \tag{5}$$

4.1. Characteristic orbits

Fixed a type $\mathbf{t}$, it defines the *generalized trigonometric functions,* $Cs(\theta)$ and $\mathrm{Sn}(\theta)$, as the unique solution of the initial value problem

$$\frac{d\mathbf{x}}{d\theta} = \mathbf{X}_H(\mathbf{x}), \qquad \mathbf{x}(\mathbf{0}) = (1,0)^T,$$

where H is the Hamiltonian $H(x,y) = x^{2t_2} + y^{2t_1} \in \mathcal{P}^{\mathbf{t}}_{2t_1t_2}$. These functions are periodic and T will denote their minimal period. Moreover, they satisfy the equality $\mathrm{Cs}^{2t_2}(\theta) + \mathrm{Sn}^{2t_1}(\theta) = 1$. For more details, see Dumortier [10]. We can already introduce the *generalized polar coordinates*, u and θ of the real plane $(x,y) \in \mathbb{R}^2$, as

$$x = u^{t_1}\mathrm{Cs}(\theta), \qquad y = u^{t_2}\mathrm{Sn}(\theta), \tag{6}$$

with $u \geq 0$ and $\theta \in [0,T)$.

The new system (4) or (5) (which axis $u = 0$ is invariant for) verifies:
(3a) *there is a point in support of system (4) or (5) lying on axis $u = 0$, and*
(3b) *all vertices of Newton diagram of (4) or (5) have even ordinates.*
We continue the process from Step 1 for system (4) or (5).

Theorem 2 is proved in Section 4.

The process above is finite since, by Dumortier [10], see [1, pages 80-82], a degenerate singular point of a vector field on the plane can be desingularized by means of a finite number of polar blow-ups (σ-process).

We name $\mathbf{t}$-*characteristic orbit* of system (1) to an orbit curve of system (1) defined in a neighborhood of the origin W which is transformed by means of the change (6) in $\gamma(\tau) = (u(\tau), \theta(\tau))$, such that

1. $\lim_{\tau\to+\infty} u(\tau) = 0$ (or $\lim_{\tau\to-\infty} u(\tau) = 0$), and
2. there exists $\lim_{\tau\to+\infty} \theta(\tau) = \theta^* \in [0,T)$ (or $\lim_{\tau\to-\infty} \theta(\tau) = \theta^* \in [0,T)$).

The following result proved in [5], ranks the different orbits of (1) that tend to the origin and give the relationship among them. It is an easy extension of the well-known result given by Zhang *et al.* [22, Theorem 3.10].

Theorem 3. *Let $\mathbf{x}(t)$ be an orbit of analytic system (1) verifying $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. Then it is either a spiral orbit of (1) (in such a case, origin is a focus) or it is a $\mathbf{t}$-characteristic orbit of (1) for any type $\mathbf{t}$.*
Otherwise, if there is no orbit of (1) such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$, then origin is a center.

4.2. Detecting characteristic orbits from the lowest-degree quasi-homogeneous vector field

The system (1) by means of the change (6) and rescaling the time by $dt = (2t_1t_2/u^r)d\tau$, becomes

$$\begin{aligned} \dot{u} &= \mathrm{d}u/\mathrm{d}\tau &= u\sum_{j=0}^{\infty}\left[-h'_{r+j+|\mathbf{t}|}(\theta) + 2t_1t_2\mu_{r+j}(\theta)\right]u^j, \\ \dot{\theta} &= \mathrm{d}\theta/\mathrm{d}\tau &= \sum_{j=0}^{\infty}(r+j+|\mathbf{t}|)h_{r+j+|\mathbf{t}|}(\theta)u^j, \end{aligned} \tag{7}$$

where we have denoted

$$h_{r+j+|\mathbf{t}|}(\theta) = h_{r+j+|\mathbf{t}|}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)), \qquad \mu_{r+j}(\theta) = \mu_{r+j}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)),$$

see Algaba *et al.* [2].

We emphasize that the quoted transformation does not change the direction of the orbits, since $\mathrm{d}t/\mathrm{d}\tau > 0$, and it carries the region of the plane (u, θ) given by rectangle $R = \{0 \leq u \leq \epsilon,\ 0 \leq \theta < T\},\ \epsilon > 0$ to the neighborhood of the origin $W = \{(x, y) \in \mathbb{R}^2,\ x^{2t_2} + y^{2t_1} \leq \epsilon^{2t_1 t_2}\}$.

The next results are proved in Algaba *et al.* [5], which are obtained easily from expression of system (7).

Proposition 4. *Let us assume that the lowest-degree quasi-homogeneous term of type* $\mathbf{t}$ *of* $\mathbf{F}$ *is* $\mathbf{F}_r = \mu_r \mathbf{D}_0$, *(i.e.* $h_{r+|\mathbf{t}|} \equiv 0$ *and* $\mu_r \not\equiv 0$*). Then, the origin of system (1) is a node (origin is surrounded of parabolic sectors).*

Proposition 5. *Let* $h_{r+|\mathbf{t}|} \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *be the Hamiltonian associated to the lowest-degree quasi-homogeneous term of type* $\mathbf{t}$ *of* $\mathbf{F}$*. If* $h_{r+|\mathbf{t}|}(\theta) \neq 0$*, for all* θ*, then the origin of system (1) is monodromic.*

From above proposition, polynomial $h_{r+|\mathbf{t}|}(\theta)$ must have some root so that the system have characteristic orbits. Therefore, if there were $\mathbf{t}$-characteristic orbits, polynomial $h_{r+|\mathbf{t}|}(x, y)$ should have factors $x,\ y$ or $y^{t_1} - \lambda x^{t_2}$, with $\lambda \neq 0$.

Proposition 6. *If there exists a strong factor associated to a type* $\mathbf{t}$*, then there exists a* $\mathbf{t}$*-characteristic orbit of system (1).*

With some abuse of the language, we will say that the characteristic orbit is associated to the strong factor.

The following property is a consequence of Proposition 6. It is a condition which must hold the support point of a vertex of system (1) so that there exist characteristic orbits different from the coordinate axes.

Proposition 7. *Let* (m, n) *a vertex of Newton diagram of system (1). If axis* $x = 0$ *(or* $y = 0$*) is not invariant and* m *(or* n*) is odd, then there is a characteristic orbit of (1) different from the coordinate axes.*

4.3. Characteristic orbits and blow-up of vertices of the Newton diagram.

We start giving a class of blow-up, which we name *blow-up of vertices*, and plays a main role in our study.

Let V be an inner vertex of Newton diagram of system (1) with α_ℓ and $\alpha_{\tilde{\ell}}$ the exponents of the adjacent edges, upper and lower, respectively. Let $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$ be, with $\alpha_\ell = t_2/t_1 < s_2/s_1 = \alpha_{\tilde{\ell}}$ and $h_{r_{\mathbf{t}}+|\mathbf{t}|}$,

$h_{r_{\mathbf{s}}+|\mathbf{s}|}$ the Hamiltonians associated to the lowest-degree quasi-homogeneous terms of $\mathbf{F}$ of type $\mathbf{t}$ and $\mathbf{s}$, with $h_{r_{\mathbf{t}}+|\mathbf{t}|}h_{r_{\mathbf{s}}+|\mathbf{s}|} \not\equiv 0$.
If we define the sets

$$W_{\mathbf{t},\mathbf{s}}^{(\sigma_1,\sigma_2)} = \{(x,y) \in \mathbb{R}^2 \mid \epsilon x^{s_2/s_1} \leq y \leq \frac{1}{\epsilon} x^{t_2/t_1},\ (-1)^{\sigma_1} x \geq 0,\ (-1)^{\sigma_2}\epsilon > 0\},$$

with $\sigma_1, \sigma_2 \in \{0,1\}$, it is easy to see that the blow-up

$$x = u^{t_1 s_2} v^{t_1 s_1}, \qquad y = u^{t_2 s_2} v^{t_1 s_2},$$

transforms the rectangle of the $(u,v)-$plane $\{0 \leq u \leq \delta_1,\ 0 \leq v \leq \delta_2\}$ in the region of the first quadrant $W_{\mathbf{t},\mathbf{s}}^{(0,0)}$ with $\epsilon\delta_1^{t_1 s_2(s_2/s_1 - t_2/t_1)} = \epsilon\delta_2^{t_1 s_1(s_2/s_1 - t_2/t_1)} = 1,\ \epsilon > 0$.
In what follows, the characteristic orbits lying on $W_{\mathbf{t},\mathbf{s}}^{(\sigma_1,\sigma_2)}$ will be named characteristic orbits associated to vertex V.
We now study the existence of characteristic orbits contained in $W_{\mathbf{t},\mathbf{s}}^{(0,0)}$ and later we extend the results for the remaining quadrants.

Proposition 8. *Let V be an inner vertex of Newton diagram of system (1) with $h_{r_{\mathbf{t}}+|\mathbf{t}|}h_{r_{\mathbf{s}}+|\mathbf{s}|} \not\equiv 0$. The region $W_{\mathbf{t},\mathbf{s}}^{(0,0)}$ is a parabolic (resp. hyperbolic) sector of the origin if and only if $\beta_V < 0$ (resp. $\beta_V > 0$).*

The following result makes easy the study of the existence of characteristic orbits defined in the remaining quadrants.

Proposition 9. *Let system (1) be with axis $y = 0$ non-invariant. We assume that its Newton diagram has a inner vertex $V = (m,n)$ with $\beta_V > 0$ and $h_{r_{\mathbf{t}}+|\mathbf{t}|}h_{r_{\mathbf{s}}+|\mathbf{s}|} \not\equiv 0$.*
If axis $x = 0$ is not invariant, then there exist characteristic orbits associated to V in a quadrant different from the first one if it satisfies at least one of the following properties:

(a) *n or m are odd,*

(b) *$h_{r_{\mathbf{t}}+|\mathbf{t}|}$ or $h_{r_{\mathbf{s}}+|\mathbf{s}|}$ have a factor of the form $x,\ y,\ y^{t_1} - \lambda x^{t_2}$ or $y^{s_1} - \lambda x^{s_2},\ \lambda \neq 0$, which has odd multiplicity order.*

If axis $x = 0$ is invariant, then there exist characteristic orbits associated to V in the forth quadrant, if n is odd or there is a factor different from x satisfying **(b)**.

Next, we cite the following result which provides necessary conditions of existence of characteristic orbits.

Proposition 10. *If system (1) has a characteristic orbit in the first quadrant different from $x = 0$ and $y = 0$, then it has one, and only one, of the following situations:*
(e1) *There exists an edge of Newton diagram of (1) with type $\mathbf{t} = (t_1, t_2)$ such that $h_{r_\mathbf{t}+|\mathbf{t}|}(x, y) \equiv 0$.*
(e2) *There exists an edge of Newton diagram of (1) with type $\mathbf{t} = (t_1, t_2)$ and a certain $\tilde{a}$ real non-zero such that $y^{t_1} - \tilde{a}x^{t_2}$ is a factor of $h_{r_\mathbf{t}+|\mathbf{t}|}(x, y)$.*
(v) *There exists an inner vertex V of Newton diagram of (1) with $\beta_V < 0$.*

Proof of Proposition 1. On the one hand, if not **(B)**, by Proposition 7, the system has a characteristic orbit, hence the origin is not monodromic. On the other hand, if system (1) does not have two points in the support lying on the axis, then there is an invariant axis, thus the origin is not monodromic. Lastly, if a and b have the same sign, system (1), parameterizing the time and rescaling the state variables, becomes

$$\begin{array}{rcl} \dot{x} &=& y^{2m-1} + y^{2m}\Phi_1(y) + x\Phi_2(x, y), \\ \dot{y} &=& x^{2n-1} + x^{2n}\Psi_1(x) + y\Psi_2(x, y), \end{array} \tag{8}$$

with $n, m \in \mathbb{N}$. We suppose that the origin of (8) is monodromic. We denote by $V_i,\ i = 0, ..., p$, to the vertex of Newton diagram of (8) ordered according their abscissas, i.e. $V_0 = (0, 2m)$ and $V_p = (2n, 0)$, and denote by ℓ_i to the edges, that is, the exponents satisfy $\alpha_{\ell_1} < \cdots < \alpha_{\ell_p}$. Also, we denote by h_i to the lowest-degree hamiltonian associated to the type $\mathbf{t}^{(i)} = (t_1^{(i)}, t_2^{(i)})$ of the edge ℓ_i. As origin is monodromic, from Proposition 6, all real factors of h_i have even multiplicity order; so, $h_i(x, y) = x^{2m_i}y^{2n_i}\tilde{h}_i(x, y),\ m_i, n_i \in \mathbb{N}_0$.
By the one hand, both $\tilde{h}_i(1, 0)$ and $\tilde{h}_i(0, 1)$ have the same sign since otherwise h_i there is a factor with odd order multiplicity. By the other hand, from Proposition 8, $\beta_{V_i} = \tilde{h}_i(1, 0)\tilde{h}_{i+1}(0, 1)$ is positive, $i = 1, ..., p-1$. Therefore, all coefficients of $\tilde{h}_i$ with greater exponent in x and y have the same sign. But, this leads a contradiction since $\tilde{h}_1(0, 1) = -\frac{t_2^{(1)}}{r_1+|\mathbf{t}^{(1)}|}$ and $\tilde{h}_p(1, 0) = \frac{t_1^{(p)}}{r_p+|\mathbf{t}^{(p)}|}$ have different sign. ∎

Proof of Theorem 2. *Necessity.* The conditions **(A,B)** must be verified by Proposition 1. The condition **(1)** holds from Proposition 8. From Proposition 4, **(2a)** holds. If $h_{r+|\mathbf{t}|} \not\equiv 0$, for the strong factor associated to the type

$\mathbf{t}$, from Proposition 6, it has that **(2b)** holds. The condition **(3a)** holds, since, otherwise it has an invariant axis different from $u = 0$. And condition **(3b)** must be satisfied from Proposition 7.

Sufficiency. We see that if there is a characteristic orbit of (1), then at least one condition of the algorithm is not satisfied.

In fact, if such an orbit is a solution of the form $y = \tau(x)$ (or $x = \tau(y)$) with τ flat at origin, then it is easy to prove that $y = 0$ (or $x = 0$) is invariant and hence condition **(A)** does not hold.

From Proposition 10, if the characteristic orbit lies on the first quadrant then this orbit will be associated either to an inner vertex or to an edge of diagram Newton of (1). On the one hand, if it is associated to the inner vertex V, then **(1)** is not satisfied. On the other hand, if it is associated to an edge, from Proposition 4, $h_{r+|\mathbf{t}|} \equiv 0$, and from Proposition 10, there exists a real number $\tilde{a}$ non-zero such that the orbit is associated to the factor $y^{t_1} - \tilde{a}x^{t_2}$, factor of $h_{r+|\mathbf{t}|}$. We distinguish two possibilities: it is a characteristic factor of $h_{r+|\mathbf{t}|}$, that is **(2b)** does not hold. Otherwise, we apply the changes described in Step 3 and the process continues.

If the orbit is associated to an edge of Newton diagram but it lies another quadrant different from the positive, the change $x = (-1)^{\sigma_1}u$, $y = (-1)^{\sigma_2}v$ with $\sigma_1, \sigma_2 \in \{0, 1\}$, it moves to the first quadrant. The following situations can appear:

1. the new associated Hamiltonian $g_{r+|\mathbf{t}|}$ is identically null. It is easy to prove that in such a case the Hamiltonian above is also $h_{r+|\mathbf{t}|} \equiv 0$, thus, **(2a)** does not hold.
2. the new associated Hamiltonian is non-null and there exists a real number $\tilde{a}$ non-zero such that the characteristic orbit is associated to $v^{t_1} - \tilde{a}u^{t_2}$, factor of $g_{r+|\mathbf{t}|}(u, v)$. It is easy to see that this factor is transformed in the factor $y^{t_1} - \tilde{a}(-1)^{\sigma_2 t_1 + \sigma_1 t_2} x^{t_2}$ of $h_{r+|\mathbf{t}|}(x, y)$ with the same multiplicity order. Moreover, such a factor conserves, after making the indicated change, the multiplicity order as factor of the divergence of the lowest-degree quasi-homogeneous term of type $\mathbf{t}$. So, the transformed factor verifies **(2b)** if and only if the original factor also verified it. In such a case, we apply the described changes in Step 3, checking conditions **(3a)** and **(3b)** for new system and the process continues.

If the orbit is associated to an inner vertex V but it does not lie on the first quadrant, after reflecting it into the first quadrant, such an orbit continues analyze separately both cases in order to study the existence of characteristic orbits of (9) associated to their factors.

Proposition 11. *We assume that $\Delta_1 = 0$ and $d \neq c$. There are not characteristic orbits of (9) associated to the factor $y - \frac{d-c}{2a}x$ of $h_4^{(1,1)}$ if and only if*

$$h = \frac{d(c-d)}{2a}, \;\; g = \frac{c^2-d^2}{4a} \qquad (10)$$

and one of the following conditions is verified:

i. $(3d-c)(e-f) > 0$, $(3d-c)(c-d) < 0$;

ii. $c = 3d$, $d(e-f) < 0$;

iii. $e = f$, $4e^2 \leq (c-d)(5d-c)b/a$.

associated to the same vertex V of Newton diagram and from Proposition 9 at least one of conditions **(B)** or **(2b)** is not satisfied. ■

5. Application

We consider system

$$\begin{aligned} \dot{x} &= ay^3 + cxy^2 + gx^2y + ex^5 \\ \dot{y} &= dy^3 + hxy^2 + fx^2y + bx^7, \quad ab(g^2+h^2) < 0. \end{aligned} \qquad (9)$$

Its Newton diagram consists of two exterior vertices $A = (0,4)$ associated to $(ay^3, 0)^T$ and $E = (8,0)$ associated to vector field $(0, bx^7)^T$ and an inner vertex $C = (2,2)$ associated to $(gx^2y, hxy^2)^T$.

Proof. If $\Delta_1 = 0$ and $d \neq c$, polynomial $y - \frac{d-c}{2a}x$ is a factor of $h_4^{(1,1)}$ with multiplicity order two. It is not a strong factor if it is a simple factor of $\mu_2^{(1,1)}$. So, (10) holds. Following the monodromy algorithm, we perform the blow-up

$$x = u, \quad y = u(\bar{y} + \frac{d-c}{2a}u), \quad dt = \frac{1}{u^2}d\tau.$$

The system is transformed into

$$\begin{aligned} u' &= \frac{d(d-c)}{2a}\bar{y}u - \frac{3d-c}{2}\bar{y}^2u + a\bar{y}^3u + eu^3, \\ \bar{y}' &= -a\bar{y}^4 + (c-d)\bar{y}^3 - \frac{1}{4a}(d-c)^2\bar{y}^2 + (f-e)\bar{y}u^2 \\ &\quad + bu^4 + \frac{1}{2a}(c-d)(e-f)u^2. \end{aligned} \qquad (11)$$

The Newton diagram of (11) has an inner vertex $\tilde{V}_0 = (1,2)$ associated to $(\frac{d(d-c)}{2a}u\bar{y}, -\frac{(c-d)^2}{4a}\bar{y}^2)^T$ and an exterior vertex. If $e \neq f$ the exterior vertex is $\tilde{V}_1 = (3,0)$, associated to $(0, \frac{1}{2a}(e-f)(c-d)u^2)^T$, and if $e = f$ the exterior vertex is $\tilde{V}_2 = (5,0)$ associated to the vector field $(0, bu^4)^T$.
So, Newton diagram of (11) has a unbounded vertical edge of type $(1,0)$ whose hamiltonian is

$$\tilde{h}_1^{(1,0)}(u,\bar{y}) = u\bar{y}^2[-\frac{1}{4a}(c-d)^2 + (c-d)\bar{y} - a\bar{y}^2].$$

First, we suppose that $f \neq e$. In such a case, the bounded edge of Newton diagram of (11) is of type $(1,1)$ and its associated hamiltonian is

$$\tilde{h}_3^{(1,1)}(u,\bar{y}) = \frac{1}{12a}(c-d)u[(3d-c)\bar{y}^2 + 2(e-f)u^2].$$

We assume that $c \neq 3d$. Thus, $\beta_{\tilde{V}_0} = -\frac{1}{48a^2}(c-d)^3(3d-c)$.
If $(3d-c)(e-f) < 0$, there are characteristic orbits of (11) since $\tilde{h}_3^{(1,1)}$ has simple factors. If $(3d-c)(e-f) > 0$, there are not non-trivial factors. Moreover, if $(c-d)(3d-c) < 0$, then $\beta_{\tilde{V}_0} > 0$ and hence there are not characteristic orbits different from $u = 0$; otherwise, there exists a characteristic orbit associated to $\tilde{V}_0$.
We now assume that $c = 3d$. Thus, $\tilde{h}_3^{(1,1)} = \frac{1}{3a}d(e-f)u^3$ and $\beta_{\tilde{V}_0} = -\frac{d^3}{3a^2}(e-f)$.

So, if $d(e-f)>0$ there are not characteristic orbits; otherwise, there is a characteristic orbit associated to the vertex $\tilde{V}_0$.

On the other hand, if $e=f$, the bounded edge of Newton diagram of (11) is of type $(1,2)$ and its associated hamiltonian is

$$\tilde{h}_5^{(1,2)}(u,\bar{y})=\tfrac{u}{5}[\tfrac{1}{4a}(5d-c)(c-d)\bar{y}^2-2eu^2\bar{y}+bu^4].$$

The discriminant of $\tilde{h}_5^{(1,2)}$ is $\Delta=4e^2-(c-d)(5d-c)b/a$. We distinguish the following cases:
If $\Delta>0$, the polynomial has simple factors and therefore there are characteristic orbits of system (11).
If $\Delta<0$ (that is, $(c-d)(5d-c)<0$), then $\beta_{\tilde{V}_0}=-\frac{1}{80a^2}(c-d)^3(5d-c)>0$, therefore there is not any characteristic orbit.

If $\Delta=0$ and $e\neq 0$, then $(c-d)(5d-c)<0$ and by reasoning a similar way, we deduce that there is not any characteristic orbit.

Last, if $\Delta=0$ and $e=0$, then $\tilde{h}_5^{(1,2)}=\frac{b}{5}u^5$ and thus $\beta_{\tilde{V}_0}=-\frac{b}{20a}(c-d)^2$, i.e. there is not any characteristic orbit different from $u=0$. ∎

Proposition 12. *We assume that $\Delta_2=0$ and $f\neq 3e$. There are not characteristic orbits of (9) associated to the factor $y+\frac{2b}{f-3e}x^3$ of $h_8^{(1,3)}$ if and only if*

$$h=\tfrac{f^2-9e^2}{4b},\qquad g=\tfrac{e(f-3e)}{2b} \tag{12}$$

and one of the following conditions is verified:

i. $(d-3c)(f-5e)<0,(f-3e)(f-5e)>0,$ *ii.* $f=5e,\ e(d-3c)<0.$

Proof. As $\Delta_2=0$ and $f\neq 3e$, then $y+\frac{2b}{f-3e}x^3$ is a double factor of $h_8^{(1,3)}$. Therefore, it will not be a strong factor if it is a simple factor of $\mu_4^{(1,3)}$. Imposing these conditions, (12) holds.

In order to establish if characteristic orbits of (9) associated to such a factor exist, we do the blow-up change

$$x=u,\quad y=u^3(\bar{y}-\tfrac{2b}{f-3e}),\quad dt=\tfrac{1}{u^4}d\tau$$

which transforms system (9) into

$$u' \;=\; \tfrac{e(f-3e)}{2b}u\bar{y}+\tfrac{4cb^2}{(f-3e)^2}u^3-\tfrac{4cb}{f-3e}u^3\bar{y}+cu^3\bar{y}^2-\tfrac{8ab^3}{(f-3e)^3}u^5$$

$$
\begin{aligned}
& +\tfrac{12ab^2}{(f-3e)^2}u^5\bar{y} - \tfrac{6ab}{f-3e}u^5\bar{y}^2 + au^5\bar{y}^3, \\
\bar{y}' \;=\; & \tfrac{(f-3e)^2}{4b}\bar{y}^2 - \tfrac{8b^3(d-3c)}{(f-3e)^3}u^2 + \tfrac{12b^2(d-3c)}{(f-3e)^2}u^2\bar{y} - \tfrac{6b(d-3c)}{f-3e}u^2\bar{y}^2 \\
& -\tfrac{48ab^4}{(f-3e)^4}u^4 + (d-3c)u^2\bar{y}^3 + \tfrac{96ab^3}{(f-3e)^3}u^4\bar{y} - \tfrac{72ab^2}{(f-3e)^2}u^4\bar{y}^2 \\
& +\tfrac{24ab}{f-3e}u^4\bar{y}^3 - 3au^4\bar{y}^4. \qquad (13)
\end{aligned}
$$

We suppose that $d-3c \neq 0$. The Newton diagram of (13) has an inner vertex $\hat{V}_0 = (1,2)$ associated to $(\frac{e(f-3e)}{2b}u\bar{y}, \frac{(f-3e)^2}{4b}\bar{y}^2)^T$, an exterior vertex $\hat{V}_1 = (3,0)$ associated to $(0, -\frac{8b^3(d-3c)}{(f-3e)^3}u^2)^T$ and one edge of type $\mathbf{t} = (1,1)$ whose hamiltonian is

$$\hat{h}_3^{(1,1)}(u,\bar{y}) = \tfrac{f-3e}{4b}u\left[(f-5e)\bar{y}^2 - \tfrac{32b^4(d-3c)}{(f-3e)^4}u^2\right]$$

whose multiple factors can be only trivial factors. So, if $(d-3c)(f-5e) > 0$ there are characteristic orbits of the origin different from $u=0$ since there are simple factors; if $(d-3c)(f-5e) < 0$ and $\beta_{\hat{V}_0} = \frac{(f-3e)^3(f-5e)}{16b^2} > 0$ there are not characteristic orbits associated to the factor $y + \frac{2b}{f-3e}x^3$; and if $f = 5e$ and $\beta_{\hat{V}_0} = -\frac{16b^2(d-3c)}{f-3e} > 0$, i.e. $e(d-3c) < 0$, there are no characteristic orbits associated either.

On the other hand, if $d = 3c$, Newton diagram of (13) has an inner vertex $\hat{V}_0 = (1,2)$ associated to $(\frac{e(f-3e)}{2b}u\bar{y}, \frac{(f-3e)^2}{4b}\bar{y}^2)^T$, an exterior vertex $\hat{V}_1 = (5,0)$ associated to $(0, -\frac{48ab^4}{(f-3e)^4}u^4)^T$ and one edge of type $\mathbf{t} = (1,2)$ whose hamiltonian is

$$\hat{h}_5^{(1,2)}(u,\bar{y}) = \tfrac{1}{5}u\left[\tfrac{(f-3e)(f-7e)}{4b}\bar{y}^2 - \tfrac{8cb^2}{(f-3e)^2}u^2\bar{y} + \tfrac{48ab^4}{(f-3e)^4}u^4\right].$$

Let $\Delta = \frac{16b^4}{(f-3e)^4}\left[c^2 - \frac{3a}{4b}(f-3e)(f-7e)\right]$, discriminant of $\hat{h}_5^{(1,2)}$. We distinguish the following cases, separately:

Case $\Delta > 0$. In this case there are characteristic orbits associated to the factor $y + \frac{2b}{f-3e}x^3$.

Case $\Delta < 0$. As $ab < 0$, then it must be $(f-3e)(f-7e) < 0$, and hence $\beta_{\hat{V}_0} = \frac{(f-3e)^3(f-7e)}{80b^2} < 0$. It follows that there are characteristic orbits associated to the factor $y + \frac{2b}{f-3e}x^3$.

Case $f = 7e$ and $\Delta = 0$. Thus, $c = 0$; in such a case, $\hat{h}_5^{(1,2)}$ has only a factor, u. Moreover, $\beta_{\hat{V}_0} = \frac{12ab^3}{(f-3e)^2} < 0$, i.e. there are characteristic orbits associated to the factor $y + \frac{2b}{f-3e}x^3$.

Case $\Delta = 0$ and $(f-7e) \neq 0$. It has that $(f-7e)(f-3e) = \frac{4bc^2}{3a} < 0$, that is, $\beta_{\hat{V}_0} = \frac{(f-3e)^3(f-7e)}{80b^2} < 0$. So, it has that there are characteristic orbits associated to the factor $y + \frac{2b}{f-3e}x^3$. ■

Theorem 13. *The origin of system (9) with $ab(g^2+h^2) < 0$ is monodromic if and only if one of the following conditions holds:*
(a) $(h-g)(h-3g) > 0$, $(d-c)^2 + 4a(h-g) < 0$, $(f-3e)^2 - 4b(h-3g) < 0$.
(b) $h = g$, $d = c$, $(f-3e)^2 + 8bg < 0$.
(c) $h = g$, $d = c$, $f = -e$, $2e^2 + bg = 0$, $ce < 0$.
(d) $h = 3g$, $f = 3e$, $(d-c)^2 + 8ag < 0$
(e) $h = \frac{(f+3e)(f-3e)}{4b}$, $g = \frac{e(f-3e)}{2b}$, $(d-c)^2 + \frac{a}{b}(f-3e)(f+e) < 0$, $(d-3c)(f-5e) < 0$, $(f+e)(f-5e) > 0$.
(f) $h = \frac{4e^2}{b}$, $g = \frac{e^2}{b}$, $f = 5e$, $(d-c)^2 + 12e^2\frac{a}{b} < 0$, $e(d-3c) < 0$.
(g) $d = -3c$, $h = -6c^2/a$, $g = -2c^2/a$, $f = 3e$, $ec > 0$.
(h) $d = -3c$, $h = -6c^2/a$, $g = -2c^2/a$, $e = f = 0$, $c \neq 0$.
(i) $(3d-c)(e-f) > 0$, $(3d-c)(3c+d) < 0$, $(f-3e)^2 + (c-d)(d+3c)b/a < 0$, *and (10).*
(j) $2d^2 = 2ha = ga$, $c = 3d$, $d(e-f) < 0$, $(f-3e)^2 + 20d^2b/a < 0$.
(k) $e = f$, $4e^2 \leq (c-d)(5d-c)b/a$, $4e^2 + (c-d)(d+3c)b/a < 0$, *and (10).*
(l) $f = 5e$, $d = -2c$, $ec > 0$, $hb = 4gb = 4e^2$, $3c^2b + 4e^2a = 0$.
(m) $(f+e)(f-5e) > 0$, $(3c+d)(c-3d) > 0$, $(c-3d)(f-e) > 0$, $(3c-d)(f-5e) > 0$ *and (10) and (12).*
(n) $c = 3d$, $f = -2e$, $de < 0$, $ag = 2ah = 2d^2$.

PROOF. If $\Delta_1 > 0$ or $\Delta_2 > 0$, the system has strong factors associated to $h_4^{(1,1)}$ or to $h_8^{(1,3)}$. If $(h-g)(h-3g) < 0$, then $\beta_{V_1} = \frac{(h-g)(h-3g)}{32} < 0$ and thus there is a characteristic orbit associated to V_1. Therefore, in both cases, the origin is not monodromic.

So, without loss of generality, we can assume that $\Delta_1 \leq 0$, $\Delta_2 \leq 0$ and $(h-g)(h-3g) \geq 0$.

Next, we analyze all the possibilities:
(1) If $\Delta_1 < 0$, $\Delta_2 < 0$ and $(h-g)(h-3g) > 0$, applying the algorithm, it has that the origin of system (9) is monodromic (case **(a)**).
(2) If $h = g$, as $\Delta_1 = (d-c)^2 \leq 0$, then $\Delta_1 = 0$; that is, $d = c$. So, $h_4^{(1,1)}(x,y) = -\frac{a}{4}y^4$ does not have non-trivial factors. Moreover, $\Delta_2 = (f-3e)^2 + 8bg$ and $\beta_{V_1} = \frac{ag}{16} \neq 0$.

We now differentiate two cases:
(2.i) If $\Delta_2 = (f-3e)^2 + 8bg < 0$, then $bg < 0$ and therefore $\beta_{V_1} > 0$ since $ab < 0$. Thus, the origin is monodromic (case **(b)**).
(2.ii) If $\Delta_2 = 0$ (in such a case, $f - 3e \neq 0$), then $y + \frac{2b}{f-3e}x^3$ is a factor of $h_8^{(1,3)}$ with multiplicity order two. From Proposition 12 the conditions $h = g$ and $d = c$ arrive to $f = -e$ and $ce < 0$ (case **(c)**).
(3) If $h = 3g$, as $\Delta_2 = (f-3e)^2 \leq 0$, then $\Delta_2 = 0$ and $f = 3e$. In this case, $h_8^{(1,3)} = \frac{b}{8}x^8$, i.e. it does not have non-trivial factors. Moreover, $\Delta_1 = (d-c)^2 + 8ag$ and $\beta_{V_1} = \frac{bg}{16} \neq 0$. Reasoning in a similar way, we distinguish two cases:
(3.i) If $\Delta_1 = (d-c)^2 + 8ag < 0$, then $ag < 0$ and therefore $\beta_{V_1} > 0$ since $ab < 0$. Thus, the origin is monodromic (case **(d)**).
(3.ii) If $\Delta_1 = 0$ (in such a case, $d - c \neq 0$), then $y + \frac{d-c}{2b}x$ is a double factor of $h_4^{(1,1)}$. Applying Proposition 11, it has that $h = 3g = -6c^2/a$, $d = -3c$ and $ec > 0$ (case **(g)**) or $e = f = 0$, $c \neq 0$ (case **(h)**).
(4) If $(h-g)(h-3g) > 0$, $\Delta_1 < 0$ and $\Delta_2 = 0$, the conditions $h - 3g \neq 0$ and $\Delta_2 = 0$ arrive to $f - 3e \neq 0$. So, $y + \frac{2b}{f-3e}x^3$ is a double factor of $h_8^{(1,3)}$. If $f \neq 5e$, from Proposition 12 it has that for that the system does not have characteristic orbits associated to this factor, it must hold (12), $(d-3c)(f-5e) < 0$, $(f-3e)(f-5e) > 0$. So, condition $(h-g)(h-3g) > 0$ is equivalent to $(f-3e)(f+e) > 0$. And the two conditions $(f-3e)(f-5e) > 0$ and $(f-3e)(f+e) > 0$ arrive to $(f+e)(f-5e) > 0$ (case **(e)**). And if $f = 5e$, from Proposition 12, the system must hold $h = \frac{4e^2}{b}$, $g = \frac{e^2}{b}$ and $e(d-3c) < 0$, (case **(f)**).
(5) If $(h-g)(h-3g) > 0$, $\Delta_2 < 0$ and $\Delta_1 = 0$, since $h - g \neq 0$ and $\Delta_1 = 0$, then $d - c \neq 0$. Thus, $y - \frac{d-c}{2b}x$ is a double factor of $h_4^{(1,1)}$. From Proposition 11 if the system does not have characteristic orbits associated to this factor, then (10) holds, hence $(h-g)(h-3g) > 0$ implies $(c-d)(d+3c) > 0$. On the other hand, since the system does not have characteristic orbits associated to the factor $y - \frac{d-c}{2b}x$, applying Proposition 11, the series of conditions lead to cases **(i)**, **(j)** and **(k)**.
(6) We suppose that $(h-g)(h-3g) > 0$, $\Delta_2 = 0$ and $\Delta_1 = 0$.
It is easy to check that $d \neq c$ and $f \neq 3e$, since otherwise $h = g$ or $h = 3g$. Therefore, applying Propositions 11 and 12, system satisfies (10) and (12). By (10) and (12), condition $(h-g)(h-3g) > 0$ becomes $(c-d)(d+3c) > 0$ and $(f+e)(f-3e) > 0$.
We suppose that $f = 5e$. By (10) and (12), $d = -2c$, from Proposition 12

case ii and Proposition 11 case i, we get case **(l)**. We note that conditions given by Proposition 11 cases ii and iii lead to $c = d = 0$ and $e = f = 0$, respectively.
If $f \neq 5e$, so that the origin be monodromic, the system also must hold $(d-3c)(f-5e) < 0$ and $(f-3e)(f-5e) > 0$, Proposition 12 case i.
From Proposition 11 case i and case ii, we obtain cases **(m)** and **(n)**, respectively.
Lastly, Proposition 11 case iii does not hold, otherwise $f = e$, i.e. $(f+e)(f-3e) < 0$ and this is a contradiction. ■

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