Nonlinear Analysis 234 (2023) 113304

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Analytically integrable system orbitally equivalent to a semi-quasihomogeneous system

Antonio Algaba [a], Cristóbal García [a,1], Manuel Reyes [a], Jaume Giné [b,*]

[a] *Departamento de Ciencias Integradas, Center of Advanced Studies in Physics, Mathematics and Computation, Universidad de Huelva, 21071, Spain*
[b] *Departament de Matemàtica, Universitat de Lleida, Av. Jaume II, 69, 25001 Lleida, Catalonia, Spain*

ARTICLE INFO

Article history:
Received 23 December 2022
Accepted 24 April 2023
Communicated by Enrico Valdinoci

MSC:
34C20
34C14

Keywords:
Analytically integrability
Orbitally equivalent
Semi-quasihomogeneous system

ABSTRACT

For perturbations of integrable non-Hamiltonian quasi-homogeneous planar vector field whose origin is a non-degenerate singular point, orbital linearization and analytic integrability are equivalent. We show a class of analytically integrable vector fields whose origin is a degenerate singular point which is orbitally equivalent to a semi-quasi-homogeneous system, that is, it is not orbital equivalent to its lowest-degree quasi-homogeneous term.

1. Introduction

We are interested in studying the local analytical integrability of a planar differential system, and so we study the problem of determining whether a differential system is locally orbitally equivalent to the leading term of its quasi-homogeneous expansion. We recall that a quasi-homogeneous expansion is a expansion in quasi-homogeneous polynomials also called weight-homogeneous polynomials [21], see also the definition in Section 2. It is known that the planar differential systems that are integrable and non-Hamiltonian are $\mathcal{C}^1$ orbitally equivalent to a linear differential system in a full Lebesgue measure subset of the domain of definition of the differential system, in general, not containing the singular points and other curves of zero measure, see [20]. The result was generalized to n-dimensional systems in [23]. Moreover considering the local integrability, by the flow box theorem [14, Cauchy–Arnold Theorem], the differential systems are locally analytically integrable around a regular point.

For a differential system at certain singular point (we can assume the origin) the problem remains open.

* Corresponding author.
E-mail address: jaume.gine@udl.cat (J. Giné).
[1] Cristóbal García died in april 14th, 2023.

https://doi.org/10.1016/j.na.2023.113304

Let us do now a more detailed review of the works related to this problem. An analytic differential system

$$\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{F}_n(\mathbf{x}) + \text{h.o.t.} \tag{1.1}$$

and with $\mathbf{F}_n(\mathbf{x})$ a homogeneous polynomial vector field of degree n, is *homogenizable* if it is orbitally equivalent to its leading homogeneous part $\mathbf{F}_n(\mathbf{x})$ where *h.o.t.* are higher order terms. In the case that $n = 1$ we say that is *linearizable*. More specifically, by means of a near-identity change of variable $\mathbf{x} = \phi(\mathbf{y})$ and a formal rescaling of the time $\mathrm{d}t/\mathrm{d}\tau = \eta(\mathbf{x})$, with $\eta(\mathbf{0}) = 1$, system (1.1) is transformed into $\mathbf{y}' = \mathrm{d}\mathbf{y}/\mathrm{d}\tau = \mathbf{F}_n(\mathbf{y})$. This definition is also known as *orbitally linearizable* and it is studied its relation with the Lie symmetries and the generalization to $\mathbb{R}^n$ differential systems, see [17–20,26].

For $\mathbf{F}_1 \neq 0$ and if the origin is an isolated singular point of $\mathbf{F}_1$, the nodes and saddle–nodes points are not analytically integrable and the nondegenerate monodromic singular points and the saddle singular points are local analytically integrable if, and only if, they are linearizable, see for instance [22,27]. Several methods are proposed by different authors to compute necessary and sufficient conditions of analytic integrability for such singular points, see for instance [15,16,25].

If $\mathbf{F}_1$ is zero and $\mathbf{F}_2$ non-zero (or $\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{0}$ and $\mathbf{F}_3$ non-zero) and assuming that the origin is an isolated singular point of $\mathbf{F}_2$ (or $\mathbf{F}_3$, respectively) it has been proved that system (1.1) is analytically integrable if, and only if, it is homogenizable, see [1,8,10,11]. Nevertheless this result is false for $n \geq 4$.

For systems having a nilpotent singular point, we have a similar result. They are analytically integrable if, and only if, its leading quasi-homogeneous term is integrable and the vector field is orbital equivalent to this leading term (in such a case, we say that the system is *quasi-homogenizable*), see [2,6,7].

For systems whose leading quasi-homogeneous term is Hamiltonian, Algaba et al. [4] prove that are analytically integrable if, and only if, are orbitally equivalent to a Hamiltonian system and, therefore, in general, they are not quasi-homogenizable. For instance, Theorem 3.20 of [8] provides a Hamiltonian system sum of quartic and quintic homogeneous terms that is analytically integrable and not homogenizable. In [1,5] more families of systems non-orbitally equivalent to their leading term are presented. Indeed the family in [5] is not orbitally equivalent to its quasi-homogeneous leading term. However in all these systems the leading quasi-homogeneous term is Hamiltonian.

In this work we present a wide family of analytically integrable systems, whose leading quasi-homogeneous term is non-Hamiltonian, which are orbitally equivalent to a semi-quasi-homogeneous system (sum of two quasi-homogeneous terms), and not orbitally equivalent to its leading quasi-homogeneous term. Concretely, we prove that the analytic systems whose quasi-homogeneous expansion respect of the type $\mathbf{t} = (2, 3)$ is

$$\dot{x} = -4y^3 + 2dx^3y + \cdots, \qquad \dot{y} = -6x^5 + 3dx^2y^2 + \cdots, \tag{1.2}$$

are analytically integrable if and only if they are orbitally equivalent to a semi-quasi-homogeneous system (which has been conveniently chosen)

$$\dot{x} = -4y^3 + 2dx^3y + \tfrac{14(d-2)}{d+14}\beta_9 x^4 y, \quad \dot{y} = -6x^5 + 3dx^2y^2 - 7\beta_9 x^6 + \tfrac{28(d+2)}{d+14}\beta_9 x^3 y^2, \tag{1.3}$$

with $d \in \mathbb{Q} \cap (-2, 2)$ and β_9 a real number introduced by Proposition 3.3.

As far as we know, this is the first example of a family of analytically integrable systems, whose leading term is non-Hamiltonian, which are not quasi-homogenizable.

2. Necessary condition of analytic integrability of systems (1.2)

We introduce some notation and concepts. Given $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 natural numbers without common factors, a scalar function f of two variables is *a quasi-homogeneous function* of type or weight exponent $\mathbf{t}$ and degree j if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^j f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree j is denoted by $\mathcal{P}_j^{\mathbf{t}}$. A vector field $\mathbf{F} = (P, Q)^T$ is *a quasi-homogeneous vector field* of type $\mathbf{t}$ and degree j if $P \in \mathcal{P}_{j+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{j+t_2}^{\mathbf{t}}$. We denote the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree j by $\mathcal{Q}_j^{\mathbf{t}}$. An analytic vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F}$ can be written in the form

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots ,$$

for some integer r, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$.

From [4, Prop. 2.7], every $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ can be uniquely written as $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0^{\mathbf{t}}$ with $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ and $\mu \in \mathcal{P}_r^{\mathbf{t}}$ where $\mathbf{D}_0^{\mathbf{t}} = (t_1 x, t_2 y)^T \in \mathcal{Q}_0^{\mathbf{t}}$ (dissipative quasi-homogeneous vector field) and $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to the polynomial h). Throughout the paper, we denote the differential operator associated to the vector field $\mathbf{F}$ by F, that is, $F := P\partial_x + Q\partial_y$.

We recall the concept of invariant curve and its associated cofactor. A function $C \in \mathbf{C}[[x, y]]$ (algebra of formal power series in x, y over $\mathbf{C}$), with $C(\mathbf{0}) = 0$, is an invariant curve at the origin of the vector field $\mathbf{F}$, if there exists $K \in \mathbf{C}[[x, y]]$, named cofactor of C, such that $F(C) = KC$. Moreover, if $K \equiv 0$, the vector field $\mathbf{F}$ is integrable and C is a formal first integral of $\mathbf{F}$.

Here, we study the analytic integrability of the systems (1.2), that is, the systems $\dot{\mathbf{x}} = \mathbf{F}_7 + \cdots$, $\quad \mathbf{F}_7 = \mathbf{X}_{y^4-x^6} + dx^2y\mathbf{D}_0^{\mathbf{t}}$ with d a real number.

A necessary condition of analytic integrability of a vector field is the polynomial integrability of its lowest-degree quasi-homogeneous term. This fact allows to give the following result.

Proposition 2.1. *If system (1.2) is analytically integrable then d is a rational number and $|d| < 2$. Moreover, in such a case, if we write $d = \frac{2(m_2-m_1)}{m_1+m_2}$ with m_1, m_2 coprime natural numbers, then an analytic primitive first integral of system (1.2) is of the form $I = (y^2 - x^3 + \cdots)^{m_1}(y^2 + x^3 + \cdots)^{m_2}$.*

Proof. If the system has an analytic first integral, then $\mathbf{F}_7$ is polynomially integrable. From [9, Theorem 16] if the origin of system (1.2) is an isolated singular point and $\mathbf{F}_7$ is integrable then for each factor of $y^4 - x^6$ (*i.e.* $y^2 - x^3$ and $y^2 + x^3$) there exists an unique invariant curve at the origin starting by $y^2 - x^3$ and $y^2 + x^3$. By [9, Theorem 17] a primitive first integral is $I = (y^2 - x^3 + \cdots)^{m_1}(y^2 + x^3 + \cdots)^{m_2}$, with m_1, m_2 coprime natural numbers. So, a polynomial primitive first integral of $\mathbf{F}_7$ is, of existing, $I_M = (y^2 - x^3)^{m_1}(y^2 + x^3)^{m_2}$ with $M = 6(m_1 + m_2)$. Taking into account that $F_7(y^2 - x^3) = 6(d + 2)x^2y(y^2 - x^3)$ and $F_7(y^2 + x^3) = 6(d - 2)x^2y(y^2 + x^3)$, the equation $F_7(I_M) = 0$ becomes

$$0 = 6\left[m_1(d + 2) + m_2(d - 2)\right](y^2 - x^3)^{m_1}(y^2 + x^3)^{m_2}x^2y.$$

Therefore, $d = \frac{2(m_2-m_1)}{m_1+m_2}$. Thus, d is a rational number satisfying $|d| < 2$. ■

In what follows, to study the analytic integrability problem of system (1.2), without lost of generality by Proposition 2.1, we will deal with the analytic systems whose quasi-homogeneous expansion respect of the type $\mathbf{t} = (2, 3)$ is

$$\dot{\mathbf{x}} = \mathbf{F}_7 + \cdots , \qquad \mathbf{F}_7 = \mathbf{X}_{y^4-x^6} + dx^2y\mathbf{D}_0^{\mathbf{t}},\ d = \frac{2(m_2-m_1)}{m_1+m_2} \in \mathbb{Q} \cap (-2, 2), \tag{2.4}$$

and we will denote by $I_M = (y^2 - x^3)^{m_1}(y^2 + x^3)^{m_2} \in \mathcal{P}_M^{\mathbf{t}}$ the polynomial first integral of $\mathbf{F}_7$ where $M = 6(m_1 + m_2)$.

3. Orbital normal form for systems (2.4)

Here, we will use the following orbital normal form provided by Algaba *et al.*[6].

Theorem 3.2. *Let* $\mathbf{F}=\mathbf{F}_r+\sum_{j\geq 1}\mathbf{F}_{r+j}$ *with* $\mathbf{F}_r=\mathbf{X}_h+\mu\mathbf{D}_0^{\mathbf{t}}\in\mathcal{Q}_r^{\mathbf{t}}$ *and* $\mathbf{F}_{r+j}\in\mathcal{Q}_{r+j}^{\mathbf{t}}$. *If* $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}\right)=\{0\}$ *for all* $j\in\mathbb{N}$ *then* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G}=\mathbf{F}_r+\sum_{j>0}\mathbf{G}_{r+j},\ \ \textit{with}\ \mathbf{G}_{r+j}=\mathbf{X}_{\delta_{r+j+|\mathbf{t}|}}+\eta_{r+j}\mathbf{D}_0^{\mathbf{t}}\in\mathcal{Q}_{r+j}^{\mathbf{t}},$$

where $\delta_{r+j+|\mathbf{t}|}\in\mathrm{Cor}\left(\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}\right)$, *a complementary subspace of the range of the operator* $\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}:\Delta_{j+|\mathbf{t}|}\longrightarrow\Delta_{r+j+|\mathbf{t}|}$ *is the linear operator defined by*

$$\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}(\delta)=\mathrm{Proy}_{\Delta_{r+j+|\mathbf{t}|}}(\mathbf{F}_r-\tfrac{div(\mathbf{F}_r)}{r+j+|\mathbf{t}|}D_0^{\mathbf{t}})(\delta) \tag{3.5}$$

where $\Delta_{j+|\mathbf{t}|}$ *is a complementary subspace to* $h\mathcal{P}_{j-r}^{\mathbf{t}}$, *and* $\eta_{r+j}\in\mathrm{Cor}\left(\ell_{r+j}\right)$, *a complementary subspace of the range of the operator*

$$\begin{aligned}\ell_{r+j}:\mathcal{P}_j^{\mathbf{t}}&\longrightarrow\mathcal{P}_{r+j}^{\mathbf{t}}\\ p_j&\rightarrow F_r(p_j).\end{aligned} \tag{3.6}$$

Next, we apply Theorem 3.2 to obtain an orbital normal form of system (2.4).

Proposition 3.3. *An orbital normal form of system (2.4) is*

$$\dot{\mathbf{x}}=\mathbf{F}_7+\mu_8\mathbf{D}_0^{\mathbf{t}}+\mathbf{F}_9+\sum_{j\geq 10}\mu_j\mathbf{D}_0^{\mathbf{t}} \tag{3.7}$$

with $\mathbf{F}_7=\mathbf{X}_{y^4-x^6}+dx^2y\mathbf{D}_0^{\mathbf{t}},\ \mathbf{F}_9=\mathbf{X}_{\beta_9(y^2-x^3)x^4}+\alpha_9x^3y\mathbf{D}_0^{\mathbf{t}}$ *and* $\mu_j\in\mathrm{Cor}\left(\ell_j\right)$.

Proof. The linear operator (3.5) for system (2.4) is

$$\begin{aligned}\ell^{\mathrm{c}}_{j+12}:\Delta_{j+5}&\longrightarrow\Delta_{j+12}\\ p_{j+5}&\rightarrow\mathrm{Proy}_{\Delta_{j+12}}(X_{y^4-x^6}+\tfrac{dj}{j+12}x^2yD_0^{\mathbf{t}})(p_{j+5}),\end{aligned}$$

where Δ_{j+5} is a complementary subspace to $h\mathcal{P}_{j-7}^{\mathbf{t}}$ and $h=y^4-x^6$. We prove that the hypothesis of Theorem 3.2 are satisfied, that is $\mathrm{Ker}\left(\ell^c_{j+12}\right)=\{\mathbf{0}\}$ for any j. Indeed, if $j=2$, we choose $\Delta_7=\mathrm{span}\left\{x^2y\right\}$ and $\Delta_{14}=\mathrm{span}\left\{x^7,x^4(y^2-x^3)\right\}$. It has that $\ell^c_{14}\left(x^2y\right)=(d-14)x^7+dx^4(y^2-x^3)$. So, $\mathrm{Ker}\left(\ell^c_{14}\right)=\{0\}$ and we choose $\mathrm{Cor}\left(\ell^c_{14}\right)=\mathrm{span}\left\{x^4(y^2-x^3)\right\}$. For $j\neq 2$, both subspaces Δ_{j+5} and Δ_{j+12} have dimension 2 and it is easy to check that the associated matrix to the operator ℓ^{c}_{j+12} is not singular and therefore $\mathrm{Ker}\left(\ell^c_{j+12}\right)=\{0\}$. Consequently, $\mathrm{Cor}\left(\ell^c_{j+12}\right)$ is also a trivial set.

We obtain an expression of $\mathrm{Cor}\left(\ell_9\right)$. A basis of $\mathcal{P}_2^{\mathbf{t}}$ is $\mathcal{B}_2=\{x\}$ and a basis of $\mathcal{P}_9^{\mathbf{t}}$ is $\mathcal{B}_9=\left\{x^3y,\ y^3\right\}$. The transformed of the basis $\mathcal{B}_2$ is $\ell_9(x)=2dx^3y-4y^3$. So, we can choose $\mathrm{Cor}\left(\ell_9\right)=\mathrm{span}\left\{x^3y\right\}$. Applying Theorem 3.2, we have the orbital normal form (3.7). ■

The following result is direct.

Lemma 3.4. *If* $j=kM$, *the kernel of the operator* ℓ_{j+7} *is* $Ker(\ell_{kM+7})=\mathrm{span}\left\{I_M^k\right\}$. *Otherwise,* $Ker(\ell_{j+7})=\{0\}$.

In remark after Proposition 4.8 we will see that the way to choose the expression of the corranges $\mathrm{Cor}\left(\ell^c_{14}\right)$ and $\mathrm{Cor}\left(\ell_9\right)$ will be essential in our study.

The following result provides a cyclicity between the corranges of the operators (3.6) for system (3.7).

Proposition 3.5. *Consider system (3.7). Then we can choose a complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *for* $k \geq 7$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$.

Proof. On the one hand, subspaces $\mathrm{Cor}(\ell_{k+M})$ and $I_M\mathrm{Cor}(\ell_k)$, have the same dimension since for any l and $\tilde{l}$ it has that $\dim(\mathrm{Cor}(\ell_{6l+i})) = \dim(\mathrm{Cor}(\ell_{6\tilde{l}+i}))$ and $M = 6(m_1 + m_2)$. On the other hand, we see that $I_M\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently we prove that $I_M\mathrm{Cor}(\ell_k) \cap \mathrm{Range}(\ell_{k+M}) = \{0\}$. Indeed, we apply *reductio ad absurdum.* Let $p_k \in \mathrm{Cor}\,(\ell_k) \setminus \{0\}$ such that $p_k I_M \in \mathrm{Range}\,(\ell_{k+M})$, then there exists $p_{k+M-7} \in \mathcal{P}_{k+M-7} \setminus \{0\}$ such that $\ell_{k+M}(p_{k+M-7}) = p_k I_M$, that is, $\ell_{k+M}(p_{k+M-7})$ is multiple of I_M. As $m_1\frac{(k+M-7)}{j} > M,\ j = 1,\ldots,m_1-1$ and $m_2\frac{(k+M-7)}{j} > M,\ j = 1,\ldots,m_2-1$, by applying Lemma 3.7 we have that $p_{k+M-7} \in \langle (y^2-x^3)^{m_1}\rangle \cap \langle (y^2+x^3)^{m_2}\rangle$, thus $p_{k+M-7} = p_{k-7}I_M$ with $p_{k-7} \in \mathcal{P}_{k-7} \setminus \{0\}$ and consequently

$$p_k I_M = F_7(p_{k+M-7}) = F_7(p_{k-7}I_M) = I_M F_7(p_{k-7}).$$

Hence $p_k = F_7(p_{k-7})$, *i.e.* $p_k \in \mathrm{Range}\,(\ell_k) \cap \mathrm{Cor}\,(\ell_k)$ which gives a contradiction. Thus, the proof is completed. ■

The following results are used in the proof of above proposition. The first one is [6, Lemma 3.22].

Lemma 3.6. *Let* $f \in \mathcal{P}_s^{\mathbf{t}}$ *an irreducible quasi-homogeneous polynomial invariant curve of* $\mathbf{F}_r$, $K_r \in \mathcal{P}_r^{\mathbf{t}}$ *its cofactor and* $k, m \in \mathbb{N}$. *Assume that the vector fields of* $\mathcal{Q}_r^{\mathbf{t}}$, $(k-js)\mathbf{F}_r + jK_r\mathbf{D}_0^{\mathbf{t}}$, $0 \leq j \leq m-1$, *are irreducible. Then, if* $F_r(p_k) \in \langle f^m\rangle$ *with* $p_k \in \mathcal{P}_k^{\mathbf{t}}$, *it satisfies that* $p_k \in \langle f^m \rangle$.

We apply Lemma 3.6 for systems (2.4).

Lemma 3.7. *Consider system (2.4). The following statements are satisfied:*
(a) Assume that $M \neq m_1\frac{k}{j},\ j = 1,\ldots,m-1$ *with* $k \geq m$. *Then, if* $p_k \in \mathcal{P}_k^{\mathbf{t}}$ *such that* $F_7(p_k) \in \left\langle (y^2-x^3)^m\right\rangle$, *it has that* $p_k \in \left\langle (y^2-x^3)^m\right\rangle$.
(b) Assume that $M \neq m_2\frac{k}{j},\ j = 1,\ldots,m-1$ *with* $k \geq m$. *Then, if* $p_k \in \mathcal{P}_k^{\mathbf{t}}$ *such that* $F_7(p_k) \in \left\langle (y^2+x^3)^m\right\rangle$, *it has that* $p_k \in \left\langle (y^2+x^3)^m\right\rangle$.

Proof. We prove the statement (a), the proof of statement (b) is analogous.

The cofactor of $y^2 - x^3 \in \mathcal{P}_6^{\mathbf{t}}$ is $K_7 = 6(d+2)x^2y$. The components of the vector field $(k-6j)\mathbf{F}_7 + jK_7\mathbf{D}_0^{\mathbf{t}}$ are

$$2y[(kd+12j)x^3 + (12j-2k)y^2], \qquad 3x^2[(12j-2k)x^3 + (kd+12j)y^2].$$

These polynomials are coprime if and only if $k \neq \frac{24j}{2-d}$, that is, $M \neq m_1\frac{k}{j},\ j = 1,\ldots,m-1$. From Lemma 3.6, the result follows. ■

4. Characterization of analytic integrability of system (2.4)

We now provide necessary and sufficient conditions of formally integrability of system (3.7), the orbital normal form of the system (2.4).

Proposition 4.8. *System (2.4) is analytically integrable if, and only if, is orbitally equivalent to system (3.7) with* $\mu_8 = 0$ *and* $\mu_i = 0$ *for all* $i \geq 0$ *and* $\alpha_9 = \frac{8d}{d+14}\beta_9$. *Moreover, in such a case, a first integral of system (3.7) is*

$$J = (y^2-x^3)^{m_1}\left(y^2+x^3+\tfrac{7}{d+14}\beta_9x^4\right)^{m_2}.$$

Proof. The sufficient condition is trivial since it enough to check that J is a polynomial first integral of $\mathbf{F}_7+\mathbf{F}_9$ with $\alpha_9=\frac{8d}{d+14}\beta_9$.
Note that from $(F_7+F_9)(J)=0$, if we write $J=J_M+J_{M+2}+\cdots+J_{M+2m_2}$, we deduce the following relation

$$\begin{array}{l} F_7(J_M)=0,\\ F_7(J_{M+2j})+F_9(J_{M+2(j-1)})=0,\ 1\le j\le m_2,\\ F_9(J_{M+2m_2})=0. \end{array} \tag{4.8}$$

We are going to necessary condition. Assume that system (3.7) is formally integrable and we distinguish two cases:
We suppose that $\beta_9=0$. Denoting $\mu_9=\alpha_9$, the vector field associated to system (3.7) is $\mathbf{F}=\mathbf{F}_7+\sum_{j>7}\mu_j\mathbf{D}_0^{\mathbf{t}}$, with $\mu_j\in\operatorname{Cor}(\ell_j)$. We assume that not all the μ_j are zero. Let j_0 be defined by $j_0=\min\{j>7:\mu_j\not\equiv 0\}$. A formal first integral of $\mathbf{F}$ is of the form $I=I_M+\sum_{j>M}I_j$ with $I_j\in\mathcal{P}_j$. Imposing the integrability condition and taking into account that $\ell_{7+j}(p_j)=F_7(p_j)$ and $D_0^{\mathbf{t}}(p_j)=jp_j$ with $p_j\in\mathcal{P}_j^{\mathbf{t}}$, we have

$$\begin{aligned} 0=(F(I))_{j_0+M}&=F_7(I_{M+j_0-7})+(\mu_{j_0}D_0^{\mathbf{t}})(I_M)\\ &=\ell_{M+j_0}\left(I_{M+j_0-7}\right)+M\mu_{j_0}I_M, \end{aligned}$$

i.e. $\mu_{j_0}I_M\in\operatorname{Range}(\ell_{M+j_0})$. But by Proposition 3.5, $\mu_{j_0}I_M\in\operatorname{Cor}(\ell_{M+j_0})$ it which is a contradiction. Thus, $\mathbf{F}$ is not formally integrable. Now, we suppose that $\beta_9\neq 0$. From Proposition 2.1, a first integral of $\mathbf{F}$ is of the form $I=I_M+\sum_{i>M}I_i$ with $I_M=(y^2-x^3)^{m_1}(y^2+x^3)^{m_2}$ and $I_i\in\mathcal{P}_i^{\mathbf{t}}$.
Next, we impose that the equation $F(I)=0$ is satisfied degree to degree.
Integrability condition to degree $M+8$ is

$$0=(F(I))_{M+8}=F_7(I_{M+1})+F_8(I_M)=\overbrace{F_7(I_{M+1})}^{\in\operatorname{Range}(\ell_{M+8})}+\overbrace{M\mu_8I_M}^{\in\operatorname{Cor}(\ell_{M+8})}.$$

So, on the one hand, $\mu_8I_M\in\operatorname{Range}(\ell_{M+8})\cap\operatorname{Cor}(\ell_{M+8})=\{0\}$ which implies $\mu_8=0$. On the other hand, $I_{M+1}\in\operatorname{Ker}(\ell_{M+8})$. From Lemma 3.4, $\operatorname{Ker}(\ell_{M+8})=\{0\}$ i.e. $I_{M+1}=0$.

Taking into account that y^2-x^3 is an invariant curve of system (3.7), we have that $I=(y^2-x^3)^{m_1}(y^2+x^3+c_{21}x^2y+(c_{40}x^4+c_{12}xy^2)+\cdots)^{m_2}$. Thus, $I_M=(y^2-x^3)^{m_1}(y^2+x^3)^{m_2}$, $I_{M+1}=m_2(y^2-x^3)^{m_1}(y^2+x^3)^{m_2-1}c_{21}x^2y$. From $I_{M+1}=0$ we have that $c_{21}=0$. Moreover, $I_{M+2}=m_2(y^2-x^3)^{m_1}(y^2+x^3)^{m_2-1}(c_{40}x^4+c_{12}xy^2)$.

The integrability condition for degree $M+9$ is $F_7(I_{M+2})+F_9(I_M)=0$. Expanding the condition, we have that $F_7(I_{M+2})+F_9(I_M)=\frac{2}{m_1+m_2}y(y^2-x^3)^{m_1}(y^2+x^3)^{m_2-1}A(x,y)=0$ with

$$\begin{aligned} A(x,y)=&\left[3(m_1+m_2)^2\alpha_9+2(m_1+m_2)(2m_1-5m_2)\beta_9-6c_{12}m_2(m_1+m_2)\right.\\ &\left.+4m_2(m_1+2m_2)c_{40}\right]x^6\\ &-2c_{12}m_2(m_1+m_2)x^3y^2\\ &+\left[3(m_1+m_2)^2\alpha_9+4(m_1+m_2)^2\beta_9+4m_2(m_1+2m_2)c_{12}-8m_2(m_1+m_2)c_{40}\right]y^4. \end{aligned}$$

Solving the equation and taking into account that $\frac{m_2}{m_1}=\frac{2+d}{2-d}$, we have that

$$c_{12}=0,\ c_{40}=\tfrac{7}{d+14}\beta_9,\ \alpha_9=\tfrac{8d}{d+14}\beta_9.$$

We denote $\hat{\mathbf{F}}_9=\mathbf{X}_{\beta_9(y^2-x^3)x^4}+\frac{8d}{d+14}\beta_9x^3y\mathbf{D}_0^{\mathbf{t}}$. The analytic first integrals of $\mathbf{F}_7+\hat{\mathbf{F}}_9$ are $I=\Psi(J)=J+a_2J^2+a_3J^3+\cdots$ with Ψ any analytic function with $\Psi(\mathbf{0})=1$. Thus

$$F_7\left(\Psi(J)_{N+2}\right)+\hat{F}_9\left(\Psi(J)_N\right)=0,\quad\forall N\ge M. \tag{4.9}$$

We now prove that $\mu_i = 0$ for all $i > 9$. Otherwise, let $j_0 = \min\{i > 9 : \mu_i \neq 0\}$.
We suppose that $\mathbf{F}^* = \mathbf{F}_7+\hat{\mathbf{F}}_9+\mu_{j_0}\mathbf{D}_0^{\mathbf{t}}+\cdots$ is analytically integrable, satisfying $\mathcal{J}^{j_0-1}\mathbf{F}^* = \mathcal{J}^{j_0-1}(\mathbf{F}_7+\hat{\mathbf{F}}_9)$, where $\mathcal{J}^k\mathbf{F}$ denotes the k-jet quasi-homogeneous of $\mathbf{F}$, then there exists a first integral I^* of $\mathbf{F}^*$, such that

$$\begin{aligned}\mathcal{J}^{j_0-M-8}I^* &= \mathcal{J}^{j_0-M-8}(\Psi(J)),\\ \mathcal{J}^{j_0-M-7}I^* &= \mathcal{J}^{j_0-M-8}(\Psi(J)) + I^*_{j_0-M-7},\end{aligned}$$

and $\mathcal{J}^{j_0+M-7}I^*$ is a first integral of $\mathbf{F}_7+\hat{\mathbf{F}}_9$ up to order j_0+M-1. Moreover

$$F_7\left(I^*_{j_0+M-7}\right) + \hat{F}_9\left(\Psi(J)_{j_0+M-9}\right) + \mu_{j_0}D_0(I_M) = 0.$$

From (4.9), we have that $\hat{F}_9\left(\Psi(J)_{j_0+M-9}\right) = -F_7\left(\Psi(J)_{j_0+M-7}\right)$. So,

$$\overbrace{F_7\left(I^*_{j_0+M-7} - \Psi(J)_{j_0+M-7}\right)}^{\in \mathrm{Range}(\ell_{j_0+M})} + \overbrace{MI_M\mu_{j_0}}^{\in \mathrm{Cor}(\ell_{j_0+M})} = 0.$$

Therefore, $\mu_{j_0} = 0$ and we arrive to contradiction. ■

Remark. The choosing of an appropriate expression of Δ_{14} and $\mathrm{Cor}\,(\ell_9)$ is the key of our research. On the one hand, if the vector field $\mathbf{F}_9$ does not have any polynomial first integral, for example $\mathbf{F}_9 = \mathbf{X}_{\beta_9^* x^4y^2} + \alpha_9^* y^3\mathbf{D}_0^{\mathbf{t}}$, then $\mathbf{F}_7+\mathbf{F}_9$ is not polynomially integrable. Then in order that the normal form (3.7) be integrable some term μ_j must be non-zero. On the another hand, $\mathbf{F}_9$ it must be in such a way that the vector field $\mathbf{F}_7+\mathbf{F}_9$ has a polynomial first integral. Taking into account these considerations and as the invariant curves of $\mathbf{F}_7$ are y^2-x^3 and y^2+x^3, we choose an expression of Δ_{14} such that $\mathbf{F}_9$ has at least one of them as invariant curve, for example $\Delta_{14} = \mathrm{span}\left\{x^4(y^2-x^3)\right\}$, *i.e.* $\mathbf{F}_9 = \mathbf{X}_{\beta_9 x^4(y^2-x^3)} + \mu_9\mathbf{D}_0^{\mathbf{t}}$. Last on, we have chosen $\mathrm{Cor}\,(\ell_9) = \mathrm{span}\left\{x^3y\right\}$ so that $\mathbf{F}_9$ is polynomially integrable.

We give the main result of our study.

Theorem 4.9. *The vector field* $\mathbf{F}$*, associated vector field to system* (2.4)*, is analytically integrable if, and only if, it is orbitally equivalent to* $\mathbf{F}_7+\hat{\mathbf{F}}_9$ *where* $\hat{\mathbf{F}}_9 = \mathbf{X}_{\beta_9(y^2-x^3)x^4} + \frac{8d}{d+14}\beta_9x^3y\mathbf{D}_0^{\mathbf{t}}$.

Proof. We prove the necessity. We assume that $\mathbf{F}$ is analytically integrable. From Proposition 3.3, system (2.4) is orbitally equivalent to (3.7) and therefore system (3.7) is formally integrable. Applying Proposition 4.8, we have that system (3.7) is $\mathbf{F}_7+\hat{\mathbf{F}}_9$. We see the sufficient condition. We assume that $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_7+\hat{\mathbf{F}}_9$. From Proposition 4.8, $J = (y^2-x^3)^{m_1}(y^2+x^3+c_{4,0}x^4)^{m_2}$ with $c_{4,0} = \frac{7}{d+14}\beta_9$, is a polynomial first integral of $\mathbf{F}_7+\hat{\mathbf{F}}_9$. So, $I = (y^2-x^3+\cdots)^{m_1}(y^2+x^3+\cdots)^{m_2}$ is a formal first integral of $\mathbf{F}$. By [24, Theorem A], $\mathbf{F}$ has also an analytic first integral, see also [25]. ■

Remark. The first integral $I = (y^2-x^3)^{m_1}(y^2+x^3+ax^4)^{m_2}$ can be deduced using the theory of singularities [12,13], but from this information we cannot deduce system (3.7), the system that allows to construct an algorithm for the computations of the necessary conditions of integrability of system (2.4) which is the objective of this work.

5. Applications

We illustrate our techniques, studying the analytic integrability of two semi-quasihomogeneous families (2.4) for $d = -\frac{2}{3}$, *i.e.* $m_1 = 2,\ m_2 = 1$.
The coefficients of the normal form (3.7) of the systems have been obtained following the procedure given in [3].

Theorem 5.10. *The system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -4y^3 - \frac{4}{3}x^3y + a_{41}x^4y + a_{13}xy^3 \\ -6x^5 - 2x^2y^2 + b_{60}x^6 + b_{32}x^3y^2 + b_{04}y^4 \end{pmatrix}. \tag{5.10}$$

is analytically integrable if, and only if, one of the following conditions is satisfied:
(i) $b_{60} - 3b_{32} + b_{04} = a_{13} - 2b_{32} = 3a_{41} - 2b_{32} = 0$,
(ii) $16b_{60} - 22b_{32} + 55b_{04} = a_{13} + 3b_{04} = 8a_{41} + 2b_{32} + 11b_{04} = 0$,
(iii) $2b_{60} + 5b_{32} + 35b_{04} = a_{13} + 6b_{04} = a_{41} + b_{32} + 5b_{04} = 0$,
(iv) $12b_{32} - 5b_{04} = 36b_{60} - 7b_{04} = 2a_{13} - b_{04} = 6a_{41} - b_{04} = 0$,
(v) $3b_{32} + b_{04} = 9b_{60} + 20b_{04} = a_{13} + 2b_{04} = 3a_{41} + 2b_{04} = 0$.

Proof. System (5.10) is a system (2.4), $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ with $d = -\frac{2}{3}$ and $\mathbf{F} = \mathbf{F}_7 + \mathbf{F}_9$.

We prove the necessary condition of analytic integrability provided by Theorem 4.9. As $\mathbf{F}$ has only odd terms, a normal form (3.7) of system (5.10) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -4y^3 - \frac{4}{3}x^3y \\ -6x^5 - 2x^2y^2 \end{pmatrix} + \mathbf{X}_{\beta_9 x^4(y^2-x^3)} + \alpha_9 x^3 y \mathbf{D}_0^{\mathbf{t}} + \alpha_{11}x^4y\mathbf{D}_0^{\mathbf{t}} + \alpha_{15}x^6y\mathbf{D}_0^{\mathbf{t}} + \alpha_{17}x^7y\mathbf{D}_0^{\mathbf{t}} + \alpha_{19}x^8y\mathbf{D}_0^{\mathbf{t}} + \cdots$$

Following the decomposition given in [6], we do the following re-parameterization of the constants of the system

$$\begin{aligned} &a_{41} = 2(d_{31} - c_{42} - \tfrac{2}{3}\lambda_{01}), && a_{13} = 2(d_{03} - 2\lambda_{01}), \\ &b_{60} = 7c_{70} - 6\lambda_{01}, && b_{32} = 4c_{42} + 3d_{31} - 2\lambda_{01}, \\ &b_{04} = 3d_{03}. \end{aligned}$$

The coefficients α_9 and β_9 are $\alpha_9 = d_{31} - \frac{4}{11}c_{70} - \frac{1}{3}d_{03}$ and $\beta_9 = \frac{1}{22}(21c_{42} - c_{70})$. Applying Theorem 4.9, the first integrability condition is $\alpha_9 + \frac{2}{5}\beta_9 = 0$, *i.e.*

$$d_{31} = \tfrac{21}{55}c_{70} + \tfrac{1}{3}d_{03} - \tfrac{1}{55}c_{42}.$$

For this value of d_{31}, it has that, up to multiply by a constant,

$$\alpha_{11} = -3c_{70}(44\lambda_{01} - 550d_{03} + 315c_{70}) + c_{42}(945c_{42} + 1012\lambda_{01} - 4070d_{03}).$$

We consider the following cases:
• Assume $c_{42} = 0$. If $c_{70} = 0$, system (5.10) is case **(i)**.
Otherwise, $c_{70} \neq 0$ and $\lambda_{01} = -\frac{315}{44}c_{70} + \frac{550}{44}d_{03}$. In such a case, it has that α_{15} and α_{17} are simultaneously non-zero.
• Assume $c_{42} \neq 0$, $c_{70} = 0$. So, $\alpha_{11} = 0$ if $\lambda_{01} = -\frac{945}{1012}c_{42} + \frac{185}{46}d_{03}$. In such a case, it has that α_{15} and α_{17} are not zero simultaneously.
• Assume $c_{42}c_{70} \neq 0$. Then the coefficient α_{11} is zero if, and only if, there exists ρ such that

$$\begin{aligned} 945c_{42} + 1012\lambda_{01} - 4070d_{03} &= 3\rho c_{70}, \\ 44\lambda_{01} - 550d_{03} + 315c_{70} &= \rho c_{42}. \end{aligned}$$

Solving with respect to d_{03} and λ_{01}, it has that

$$\begin{aligned} d_{03} &= \tfrac{1}{8580}\left((3\rho + 7245)c_{70} - (23\rho + 945)c_{42}\right), \\ \lambda_{01} &= \tfrac{1}{3432}\left((15\rho + 11655)c_{70} - (37\rho + 4725)c_{42}\right). \end{aligned}$$

For these values, it has that $\alpha_{15} = (c_{42} - c_{70})(c_{42} + c_{70})Q_3(\rho, c_{70}, c_{42})$, where $Q_3(\rho, c_{70}, c_{42})$ is a polynomial in ρ, c_{70}, c_{42} of degree three in ρ. If $c_{42} = c_{70}$. System (5.10) is case **(ii)**. If $c_{42} = -c_{70}$. System (5.10) is case **(iii)**. Otherwise, $(c_{42} - c_{70})(c_{42} + c_{70}) \neq 0$ and $Q_3(\rho, c_{70}, c_{42}) = 0$. It has that

$$\alpha_{17} = (c_{42} - c_{70})(c_{42} + c_{70})Q_4(\rho, c_{70}, c_{42})$$
$$\alpha_{19} = (c_{42} - c_{70})(c_{42} + c_{70})Q_6(\rho, c_{70}, c_{42})$$

where $Q_4(\rho, c_{70}, c_{42})$ and $Q_6(\rho, c_{70}, c_{42})$ are polynomials in ρ, c_{70}, c_{42} of degree 4 and 6 in ρ, respectively. Both resultants of $Q_3(\rho, c_{70}, c_{42})$ and $Q_4(\rho, c_{70}, c_{42})$ and $Q_3(\rho, c_{70}, c_{42})$ and $Q_6(\rho, c_{70}, c_{42})$ with respect to c_{42} are

$$c_{70}^6(\rho + 6783)(\rho + 777)R_{15}(\rho), \qquad c_{70}^{10}(\rho + 6783)(\rho + 777)R_{25}(\rho),$$

where $R_{15}(\rho)$, $R_{25}(\rho)$ are polynomials in ρ of degree 15 and 25 respectively. We distinguish the following cases:
Assume $\rho = -6783$. In this case,

$$\alpha_{11} = \overbrace{(c_{42} - c_{70})(c_{42} + c_{70})}^{\neq 0}(8c_{42} - 3c_{70})(29593c_{42} - 10293c_{70}).$$

If $c_{42} = \frac{3}{8}c_{70}$, system (5.10) is case **(iv)**. If $c_{42} = \frac{10293}{29593}c_{70}$, it has that $\alpha_{15} \neq 0$.
Assume $\rho = -777$. In this case,

$$\alpha_{15} = \overbrace{(c_{42} - c_{70})(c_{42} + c_{70})}^{\neq 0}(8c_{42} - 3c_{70})(1387c_{42} - 87c_{70}).$$

If $c_{42} = \frac{3}{8}c_{70}$, system (5.10) is case **(v)**. If $c_{42} = \frac{87}{1387}c_{70}$, it has that $\alpha_{11} \neq 0$.
Otherwise, $(\rho + 6783)(\rho + 777) \neq 0$, we check that the resultant of the polynomials $R_{15}(\rho)$ and $R_{25}(\rho)$ with respect to ρ is non-null.

To prove the sufficiency we provide an analytic first integral for each case of Theorem 5.10.
For the case **(i)** with $b_{32} \neq 0$, a first integral is $I = (x^3 - y^2)^2(x^3 + y^2)(b_{32}x - 2)^{\frac{-3b_{04}}{b_{32}}}$.
For the case **(i)** with $b_{32} = 0$, a first integral is $I = (x^3 - y^2)^2(x^3 + y^2)\mathbf{e}^{\frac{3}{2}b_{04}x}$.
For the case **(ii)** a first integral is $I = (x^3 + y^2)(32x^3 - 32y^2 + 15b_{04}x^4 - 6b_{32}x^4 - 24b_{04}xy^2)^2$.
For case **(iii)** a first integral is $I = (x^3 - y^2)^2(8x^3 + 8y^2 + 21b_{04}x^4 + 3b_{32}x^4 + 12b_{04}xy^2)$.
For case **(iv)** a first integral is $I = (24x^3 - 24y^2 - b_{04}x^4)^2(48x^3 + 48y^2 - b_{04}x^4)(b_{04}x - 8)^{-12}$.
Finally, for case **(v)**, a first integral is $I = (12x^3 + 12y^2 + 5b_{04}x^4 + 6b_{04}xy^2)(6x^3 - 6y^2 + 2b_{04}x^4 - 3b_{04}xy^2)^2$. ∎

Theorem 5.11. *The system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -4y^3 - \frac{4}{3}x^3y + a_{50}x^5 + a_{22}x^2y^2 \\ -6x^5 - 2x^2y^2 + b_{41}x^4y + b_{13}xy^3 \end{pmatrix}. \tag{5.11}$$

is analytically integrable if, and only if, one of the following conditions is satisfied:
(i) $2b_{41} + 7a_{50} = 2a_{50} + b_{13} = a_{22} - 2a_{50} = 0$,
(ii) $b_{41} + 8a_{50} = b_{13} - 2a_{50} = a_{22} + 5a_{50} = 0$.

Proof. System (5.11) is $\dot{\mathbf{x}} = \mathbf{F}_7 + \mathbf{F}_8$ with

$$\mathbf{F}_7 = \mathbf{X}_{y^4 - x^6} - \frac{2}{3}x^2y\mathbf{D}_0^{\mathbf{t}}, \qquad \mathbf{F}_8 = \begin{pmatrix} a_{50}x^5 + a_{22}x^2y^2 \\ b_{41}x^4y + b_{13}xy^3 \end{pmatrix}.$$

We prove the necessary condition. A normal form of system (5.11) is

$$\dot{\mathbf{x}} = \mathbf{F}_7 + \left(\alpha_8^{(1)}x^4 + \alpha_8^{(2)}xy^2\right)\mathbf{D}_0^{\mathbf{t}} + \mathbf{X}_{\beta_9 x^4(y^2-x^3)} + \alpha_9 x^3 y \mathbf{D}_0^{\mathbf{t}} + \cdots$$

The first coefficients are

$$\begin{aligned}\alpha_8^{(1)} &= \tfrac{1}{760}(293a_{50} + 58b_{41} - 27a_{22} + 18b_{13}),\\ \alpha_8^{(2)} &= \tfrac{1}{190}(29a_{22} + 44b_{13} + 9a_{50} - 6b_{41}).\end{aligned}$$

Imposing $\alpha_8^{(1)} = \alpha_8^{(2)} = 0$, we have that $b_{13} = -\frac{6}{7}a_{50} - \frac{4}{7}a_{22}$ and $b_{41} = \frac{9}{14}a_{22} - \frac{67}{14}a_{50}$.
For these values, α_9 and β_9 are

$$\begin{aligned}\alpha_9 &= \tfrac{1}{60368}(757a_{22}^2 + 2502a_{22}a_{50} - 8571a_{50}^2)\\ \beta_9 &= \tfrac{1}{120736}(-3911a_{22}^2 - 12888a_{22}a_{50} + 44115a_{50}^2).\end{aligned}$$

So, by Theorem 4.9, if system (5.11) has an analytic first integral then $\alpha_9 + \frac{2}{5}\beta_9 = -\frac{9}{21560}(-2a_{50}+a_{22})(5a_{50}+a_{22})$ is zero.
If $-2a_{50} + a_{22} = 0$ we obtain the case **(i)**. If $5a_{50} + a_{22} = 0$ we have the case **(ii)**.
To prove the sufficiency, we provide an analytic first integral for each family.
For the case **(i)**, an analytic first integral is $I = (x^3 + y^2)(8x^3 - 8y^2 - 3b_{13}x^2y)^2$.
For the case **(ii)**, an analytic first integral is $I = (x^3 - y^2)^2(4x^3 + 4y^2 + 3b_{13}x^2y)$. ∎

Acknowledgments

The first three authors are partially supported by *Ministerio de Economia y Competitividad, Plan Nacional* $I+D+I$ co-financed with FEDER funds, in the frame of the project PGC2018-096265-B-I00, and by *Consejería de Educación y Ciencia de la Junta de Andalucía* (FQM-276, P12-FQM-1658, UHU126150 and P20_00764). The last author is partially supported by the Agencia Estatal de Investigación, Spain grant PID2020-113758GB-I00 and an AGAUR (Generalitat de Catalunya) grant number 2021SGR 01618.

References

[1] A. Algaba, I. Checa, C. García, J. Giné, Analytic integrability inside a family of degenerate centers, Nonlinear Anal. RWA 31 (2016) 288–307.
[2] A. Algaba, M. Díaz, C. García, J. Giné, Analytic integrability around a nilpotent singularity: the non-generic case, Commun. Pure Appl. Anal. 19 (1) (2020) 407–423.
[3] A. Algaba, E. Freire, E. Gamero, C. García, Quasi-homogeneous normal forms, J. Comput. Appl. Math. 150 (2003) 193–216.
[4] A. Algaba, E. Gamero, C. García, The integrability problem for a class of planar systems, Nonlinearity 22 (2009) 395–420.
[5] A. Algaba, C. García, J. Giné, Analytic integrability for some degenerate planar vector fields, J. Differential Equations 257 (2) (2014) 549–565.
[6] A. Algaba, C. García, J. Giné, Analytic integrability around a nilpotent singularity, J. Differential Equations 267 (2019) 443–467.
[7] A. Algaba, C. García, M. Reyes, Existence of an inverse integrating factor, center problem and integrability of a class of nilpotent systems, Chaos Solitons Fractals 45 (2012) 869–878.
[8] A. Algaba, C. García, M. Reyes, Analytical integrability problem for perturbations of cubic Kolmogorov systems, Chaos Solitons Fractals 113 (2018) 1–10.
[9] A. Algaba, C. García, M. Reyes, Analytic invariant curves and analytic integrability of a planar vector field, J. Differential Equations 266 (2019) 1357–1376.
[10] A. Algaba, C. García, M. Reyes, Analytical integrability of perturbations of quadratic systems, Mediterr. J. Math. 18 (5) (2021) 8, paper 176.
[11] A. Algaba, C. García, M. Reyes, Analytically integrable centers of perturbations of cubic homogeneous, Qual. Theory Dyn. Syst. 20 (2) (2021) 17, Paper No. 43.

[12] V.I. Arnold, Normal forms of functions in neighborhoods of non-degenerate critical points, Russian Math. Surveys 29 (1974) 10–50.
[13] V.I. Arnold, A.N. Varchenko, S.M. Gusein-Zade, Singularities of Differentiable Mappings, in: Monographs in Mathematics, vols. 82, 83, Birkhäuser, Boston, 1985, 1988.
[14] A.D. Bruno, Local Methods in Nonlinear Differential Equations, Springer-Verlag, 1980.
[15] B. Ferčec, J. Giné, A blow-up method to prove formal integrability for some planar differential systems, J. Appl. Anal. Comput. 8 (6) (2018) 1833–1850.
[16] B. Ferčec, J. Giné, Blow-up method to compute necessary conditions of integrability for planar differential systems, Appl. Math. Comput. 358 (2019) 16–24.
[17] I.A. García, Orbital linearization around equilibria of vector fields in $\mathbb{R}^n$ and Lie symmetries, J. Math. Anal. Appl. 405 (1) (2013) 111–115.
[18] I.A. García, Complete integrability, orbital linearizability and independent normalizers for local vector fields in $\mathbb{R}^n$, J. Lie Theory 25 (1) (2015) 37–43.
[19] J. Giné, M. Grau, Linearizability and integrability of vector fields via commutation, J. Math. Anal. Appl. 319 (1) (2006) 326–332.
[20] J. Giné, J. Llibre, On the planar integrable differential systems, Z. Angew. Math. Phys. 62 (4) (2011) 567–574.
[21] A. Goriely, Integrability and nonintegrability of dynamical systems, in: Advanced Series in Nonlinear Dynamics, Vol. 19, World Scientific Publishing Co. Inc., River Edge, NJ, 2001.
[22] M. Han, K. Jiang, Normal forms of integrable systems at a resonant saddle, Ann. Differ. Equ. 14 (2) (1998) 150–155.
[23] J. Llibre, C. Valls, X. Zhang, The completely integrable differential systems are essentially linear differential systems, J. Nonlinear Sci. 25 (4) (2015) 815–826.
[24] J.F. Mattei, R. Moussu, Holonomie et intégrales premiéres, Ann. Sci. Ecole Norm. Super. 13 (1980) 469–523.
[25] V.G. Romanovski, D.S. Shafer, The Center and Cyclicity Problems: A Computational Algebra Approach, Birkhäuser Boston, Inc., Boston, MA, 2009.
[26] X. Zhang, A note on local integrability of differential systems, J. Differential Equations 263 (11) (2017) 7309–7321.
[27] Q. Zhang, Y. Liu, Integrability and generalized center problem of resonant singular point, Appl. Math. Lett. 40 (2015) 13–16.