Centers with Degenerate Infinity and their Commutators

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We study polynomial systems with degeneracy at infinity and a center-focus equilibrium at the origin. We give some general properties related to the existence of polynomial commutators and use these properties in order to characterize uniformly isochronous polynomial centers with polynomial commutator and, also, we show that the commutator of the centers of the analytic systems whose angular speed is constant can be chosen of radial form. Finally, we characterize the systems $(-y+P_s+\sum_{j=k}^{n-1} xH_j, x+Q_s+\sum_{j=k}^{n-1} yH_j)^t$ with polynomial commutator, with P_j, Q_j, H_j and K_j homogeneous polynomials.

Key Words: Centers; Isochronous Centers; Commutators.

1. INTRODUCTION.

The problem of characterizing isochronous centers have attracted the attention of several authors. However, it is far from being completely resolved, even for specific families of vector fields. Consequently, setting up weaker problems whose solutions enable us to get information about the general problem that is equivalent to the existence of an analytic commutator of such vector field, more precisely, for any analytic system with linear part $(-y, x)^t$, the existence of an analytic commutator with linear part $(x, y)^t$ is a necessary and sufficient condition for the origin to be an isochronous center (see Algaba *et al.* [1]). Sabatini [25] proves that the commutator can be transversal to the vector field and with null linear part. In this context, we are interested in to find the vector fields of a specific family that have a polynomial commutator.

There are only a few families of polynomial differential systems in which a complete classification of the isochronous centers is known, and almost all of them have polynomial commutator. From a chronological point of view, we should begin mentioning the quadratic isochronous centers, characterized by Loud [18]. In Pleshkan [22], cubic isochronous centers with homogeneous nonlinear part are settled. In Christopher, Devlin [9], the isochronous centers of the Kukles family are obtained. All these centers are reversible (i.e. have symmetrical phase portrait with respect to a straight line passing through the origin, changing time direction) and moreover, all they have polynomial commutator. Commutators of quadratic centers are computed in Sabatini [24] (degrees three, four and five); commutators of cubic systems with homogeneous nonlinear part can be found in Gasull *et al.* [14] (degrees two, three and five); commutators for the Kukles system can be seen in Volokitin, Ivanov [28] (they have degree four). Mardesic *et al.* [19] find a family of isochronous cubic systems which contain Kolmogorov's isochronous cubic systems. The first example of a polynomial isochronous center without polynomial commutator is found in Devlin [13]. It is a quartic system, with homogeneous nonlinear part, where an isochronous center at the origin and others two non isochronous centers coexist.

Because of their relevance in Mechanic, we mention the so called Newton equations. These equations are given by

$$\ddot{x} + \sum_{i=0}^{n} q_i(x)\dot{x}^i = 0,$$

where the q_i are polynomials of arbitrary degree with $q_0(0) = 0$, $q_0^{'}(0) = 1$ and, if $n \geq 1$, $q_1(0) = 0$. For $n = 0$ (potential equations), Urabe [27] proved that the origin is an isochronous center if and only if $q_0(x) = x$. For $n = 1$ (Liénard equations) all isochronous centers are conjectured to be known (see Algaba *et al.* [1], Christopher, Lloyd [10] and Sabatini [26]), without distinguishing between those with and without polynomial commutator. If $n = 2$ or $n \geq 4$, Volokitin and Ivanov ([28]) prove that an isochronous center can't have a polynomial commutator. For $n = 3$ (Abel's equations), the same work characterizes the polynomial centers which have a polynomial commutator.

In this work, we study the existence of polynomial commutators for the plane systems with degeneracy at infinity and a center-focus equilibrium at the origin. Such systems, up to a linear change of variable, are given by

$$\left\{ \begin{array}{lllll} \dot{x} & = & P(x,y) & = & -y + P_2(x,y) + P_3(x,y) + \cdots + P_n(x,y), \\ \dot{y} & = & Q(x,y) & = & x + Q_2(x,y) + Q_3(x,y) + \cdots + Q_n(x,y), \end{array} \right. \qquad (1)$$

with P_i, Q_i homogeneous polynomials and $xQ_n(x,y) = yP_n(x,y)$. The name of these systems comes from this last condition, since it implies that all points at infinity are critical.

In the cubic case all centers have been characterized, and some isochronous centers different from those given by Collins are found. All of them are reversible and a few have polynomial commutator. We should mention, about these systems, the papers of Chavarriga *et al.* [7], [8] and Lloyd *et al.* [17].

A particular case of this family are the plane polynomial systems which have a center-focus equilibrium at the origin and whose angular speed is constant. In these systems, the origin is the only finite equilibrium and if it is a center, it will be automatically isochronous, in fact, uniformly isochronous. These systems, up to a linear change of variable, have the following form:

$$\begin{cases} \dot{x} = -y + xH(x,y), \\ \dot{y} = x + yH(x,y), \end{cases} \qquad (2)$$

where H is a polynomial which vanishes at the origin.

The interest in studying this family is due, on the one hand, to the importance of these systems in the general problem of the isochronicity, since any polynomial system with linear part $(-y,x)^t$ has an isochronous center if and only if it is posible to transform it by means of specific analytic change of type $(x \to x + P(x^2), y \to y + Q(x,y))$ to a system of the form (2) (see Rudenok ([23]). On the other hand, these systems can be written as a single equation in generalized Abel form:

$$\frac{dr}{d\theta} = \sum_{k=1}^{n-1} H_k(cos\,\theta, sin\,\theta) r^{k+1}, \tag{3}$$

with r, θ polar coordinates and where H_k are the homogeneous polynomials of degree k of H. The study of this equation gives us information about the systems, and vice versa, since the constant solution $r = 0$ of (3) corresponds to the critical point $x = y = 0$ of (2), and the periodic solutions of (3) correspond to closed orbits of (2), (see [15], [4], [5] and [16]).

Let us do now a more detailed review of the works related to the family (2). Conti [11] has characterized the case in which H is an homogeneous polynomial of arbitrary degree and Algaba *et al.* [1] give a polynomial commutator for each isochronous center of this subfamily. In Mardesic *et al.* [20] we find linearizating changes for reversible systems of type (2) with $H = H_1 + H_2$. In fact, as it is proved later in Collins [12], all the centers of this subfamily are reversible. Up to rotation, we can take $H = x\sigma(y)$ with $\sigma(y)$ polynomial in the variable y.

Algaba *et al.* [2] and Chavarriga *et al.* [6] study, through independent ways, the case $H = H_1 + H_2 + H_3$, with $H_1^2 + H_2^2 \neq 0$ and $H_3 \neq 0$. All these centers are reversible and, in the first work, it is proved that those where, up to rotation, H takes the form $x\sigma(y)$ are the only ones having polynomial commutator. Algaba *et al.* [3] analyze the case $H = H_1 + H_2 + H_3 + H_4$ with $H_1^2 + H_2^2 + H_3^2 \neq 0$ and $H_4 \neq 0$. Whenever the nonlinear part has two homogeneous components of different degrees, all centers are reversible.

Moreover, for the whole family, centers with polynomial commutator are characterized, appearing for the first time a case in which H doesn't take the form $x\sigma(y)$.

Next, we detail the structure of the present paper and the results obtained. It is divided into five sections. In the second section, we study properties, basically in relation to the polynomial cofactors of an invariant curve, polynomial inverse integrating factors and polynomial commutators, if they exist, of the plane systems (1) with degeneracy at infinity and a center-focus equilibrium at the origin. We end the section with Theorem 2.1, that gives conditions which the homogeneous parts of a polynomial commutator must satisfy.

In the third section, we characterize the uniformly isochronous centers with polynomial commutator and show that the commutator of the centers of the analytic systems whose angular speed is constant can be chosen of radial form.

In the fourth section, we characterize the systems $(-y+P_s+\sum_{j=k}^{n-1} xH_j, x+Q_s+\sum_{j=k}^{n-1} yH_j)^t$ with polynomial commutator, being $P_j,\ Q_j, H_j, K_j$ homogeneous polynomials.

Finally, in the fifth section, we quote some relevant consequences of the results obtained and set up open problems.

2. SYSTEMS WITH DEGENERACY AT INFINITY WITH POLYNOMIAL COMMUTATOR

We first recall briefly the notions of invariant algebraic curves, inverse integrating factor and commutator of a vector field. Let us start with a polynomial system (1).

DEFINITION 1. *An invariant algebraic curve of system* (1) *is an algebraic curve* $f(x,y)=0$ *satisfying* $P\partial_x f + Q\partial_y f = Kf$.
$K(x,y)$ *is called the cofactor associated with* f. *Notice that the cofactor's degree is less than or equal to* $n-1$.

DEFINITION 2. *An inverse integrating factor of system* (1) *is any function* $V(x,y)$ *satisfying* $P\partial_x V + Q\partial_y V = (\partial_x P + \partial_y Q)V$.

This name comes from the fact that function $1/V$ is an integrating factor of the system; that is, $\partial_x(P/V) + \partial_y(Q/V) = 0$. Notice that an algebraic inverse integrating factor is an invariant algebraic curve whose cofactor is given by the divergence of the system.

DEFINITION 3. *Let* X *and* Y *be vector fields. We say that* X *and* Y *commute if their Lie bracket vanishes, that is, if* $[X,Y] := DX\cdot Y - DY\cdot X = 0$. *Moreover, if* X *and* Y *are transversal we say that* Y *is a commutator of* X.

LEMMA 1. *Let* $K = \sum_{j=1}^{n-1} K_j$ *be a cofactor of an invariant algebraic curve of degree* m *of system* (1), *with* $P_n = xH_{n-1}$ *and* $Q_n = yH_{n-1}$. *Then*

$K_{n-1} = mH_{n-1}$.

Proof. We consider $f = \sum_{j=0}^{m} f_j$, an invariant algebraic curve of the system (1), and its associated cofactor $K = \sum_{j=0}^{n-1} K_j$. By definition, we have

$$(-y + P_2 + P_3 + \cdots + P_n)\partial_x f + (x + Q_2 + Q_3 + \cdots + Q_n)\partial_y f = Kf.$$

This equality is a polynomial equation of degree $m+n-1$ over the variables x and y. From the higher degree terms, we deduce that

$$H_{n-1}(x\partial_x f_m + y\partial_y f_m) = K_{n-1} f_m.$$

By Euler's Theorem for homogeneous functions we have that $mH_{n-1} f_m = K_{n-1} f_m$, what leads us to $K_{n-1} = mH_{n-1}$. ∎

LEMMA 2. *Any polynomial inverse integrating factor of the system* (1), *with* $P_n = xH_{n-1}$ *and* $Q_n = yH_{n-1}$, *has degree* $n+1$.

Proof. Let f be a polynomial inverse integrating factor of degree m. By definition, f is an algebraic invariant curve of (1) whose associated cofactor is the divergence of the system. From Lemma 1, the higher order terms satisfy

$$mH_{n-1} = \partial_x(xH_{n-1}) + \partial_y(yH_{n-1}),$$

We conclude from Euler's Theorem that $mH_{n-1} = (n+1)H_{n-1}$, and this clearly forces $m = n+1$. ∎

LEMMA 3. *Any polynomial commutator of the system* (1) *with* $P_n = xH_{n-1}$ *and* $Q_n = yH_{n-1}$, *has the same degree that the system.*

Proof. Let $(U,V)^t = (\sum_{j=1}^{l} U_j, \sum_{j=1}^{l} V_j)^t$ be a polynomial commutator of (1). The function $V^* = PV - QU$ is an inverse integrating factor of system (1), of degree $n+l$ at most. But from Lemma 2 we know it has degree $n+1$; therefore, terms of degree higher than $n+1$ are zero. In particular, the term of degree $n+l$, $xH_{n-1}V_l - yH_{n-1}U_l$, is zero, what implies that $xV_l = yU_l$.

On the other hand, terms of higher degree in the Lie bracket between the system and the commutator $\left[(xH_{n-1},\ yH_{n-1})^t, (U_l,\ V_l)^t\right]$ are null:

$$\begin{pmatrix} (H_{n-1} + x\partial_x H_{n-1})U_l + x\partial_y H_{n-1} V_l - xH_{n-1}\partial_x U_l - yH_{n-1}\partial_y U_l \\ y\partial_x H_{n-1} U_l + (H_{n-1} + y\partial_y H_{n-1})V_l - xH_{n-1}\partial_x V_l - yH_{n-1}\partial_y V_l \end{pmatrix}.$$

As $xV_l = yU_l$ we have

$$\left[(xH_{n-1},\ yH_{n-1})^t, (U_l,\ V_l)^t\right] = (n-l)H_{n-1}(U_l,\ V_l)^t = (0,\ 0)^t.$$

From this, we deduce that $U_l = V_l = 0$, and finally that $U_j = V_j = 0$, for $j = n+1, \ldots, l$. ∎

LEMMA 4. *If $(U,V)^t = (\sum_{j=1}^{n} U_j, \sum_{j=1}^{n} V_j)^t$ is a polynomial commutator of the system* (1) *with $P_j = xH_{j-1}$, $Q_j = yH_{j-1}$, $j = k, \ldots, n$, then $xV_j = yU_j$, $j = k, \ldots, n$.*

Proof. From Lemma 2, the inverse integrating factor $V^* = \sum_{j=1}^{2n} V_j^* = PV - QU$ has degree $n+1$, what implies that terms of degrees $n+k, n+k+1, \ldots, 2n$ are null. As the term of degree $2n$ of the inverse integrating factor is $V_{2n}^* = xH_{n-1}V_n - yH_{n-1}U_n$, we have $xV_n = yU_n$. The one of degree $2n-1$ is given by

$$V^*_{2n-1} = H_{n-1}(xV_{n-1} - yU_{n-1}) + H_{n-2}(xV_n - yU_n) = 0,$$

and since $xV_n = yU_n$ we have $xV_{n-1} = yU_{n-1}$. In general, V^*_{n+i} with $k \le i \le n$ is null and is given by

$$V^*_{n+i} = H_{n-1}(xV_i - yU_i) + H_{n-2}(xV_{i+1} - yU_{i+1}) + \cdots + H_{i-1}(xV_n - yU_n).$$

Consequently it follows that $xV_j = yU_j$, $j = k, \ldots, n$. ∎

THEOREM 2.1. *If* $(U,V)^t = (\sum_{j=1}^n U_j, \sum_{j=1}^n V_j)^t$ *is a polynomial commutator of system* (1) *with* $P_j = xH_{j-1}$, $Q_j = yH_{j-1}$, $U_j = xK_{j-1}$, $V_j = yK_{j-1}$, $j = k, \ldots, n$ *then:*

1. $H_pK_q = H_qK_p$ *for every pair* p, q *with* $k-1 \le p \le n-1$, $k-1 \le q \le n-1$.
2. $H_{n-1}(xV_j - yU_j) = K_{n-1}(xQ_j - yP_j)$, *for every* j *with* $2 \le j \le k-1$.
3. *Let* $s = max\{j,\ yP_j - xQ_j \ne 0\} < k$. *If* $P_j = Q_j = 0$, $j = s+1$, ..., $k-1$, *then* $U_j = V_j = 0$, $j = s+1$, ..., $k-1$.

Proof. We prove the first part.

If we compute and develop the terms of degree greater or equal than $n+k-1$ of the Lie bracket between the field and the polynomial commutator we easily get the following relations:

$$\sum_{j=k'}^{n} (n + k' - 2j) H_{n+k'-1-j} K_{j-1} = 0, \quad \text{with } k \le k' \le n-1.$$

From the higher degree term ($k' = n-1$), we have $H_{n-1}K_{n-2} = H_{n-2}K_{n-1}$; for $k' = n-2$ we deduce that $H_{n-1}K_{n-3} = H_{n-3}K_{n-1}$; for $k' = n-3$ we have the expression $3H_{n-1}K_{n-4}+H_{n-2}K_{n-3}-H_{n-3}K_{n-2}-3H_{n-4}K_{n-1} = 0$, which can be simplified, using the previous expressions, to $H_{n-1}K_{n-4} = H_{n-4}K_{n-1}$. Following in this fashion, we prove the first part.

In order to prove the second part, we use the fact that terms of orders $n+2, \ldots, n+k-1$ of the inverse integrating factor V^* are zero.

As $V^*_{n+k-1} = H_{n-1}(xV_{k-1} - yU_{k-1}) + K_{n-1}(yP_{k-1} - xQ_{k-1})$, we have $H_{n-1}(xV_{k-1} - yU_{k-1}) = K_{n-1}(xQ_{k-1} - yP_{k-1})$.

As V^*_{n+k-2} is zero, it follows that

$$H_{n-1}(xV_{k-2} - yU_{k-2}) + H_{n-2}(xV_{k-1} - yU_{k-1})$$
$$+K_{n-2}(yP_{k-1} - xQ_{k-1}) + K_{n-1}(yP_{k-2} - xQ_{k-2}) = 0,$$

and since $H_{n-1}K_{n-2} = K_{n-1}H_{n-2}$, we have $H_{n-1}(xV_{k-1} - yU_{k-1}) = K_{n-1}(xQ_{k-1} - yP_{k-1})$.

Finally, since $V^*_{n+2} = 0$ we have $H_{n-1}(xV_2 - yU_2) = K_{n-1}(xQ_2 - yP_2)$.

We see the third part. If $P_j = Q_j = 0$, $j = s+1$, ..., $k-1$, being $s = max\{j,\ yP_j - xQ_j \neq 0\} < k$, then $P_j = xH_{j-1}$, $Q_j = yH_{j-1}$, $j = s+1, \ldots, n$, therefore, from Lemma 4, $U_j = xK_{j-1}$, $V_j = yK_{j-1}$, $j = s+1, \ldots, n$, and applying the first part with $p = n-1$, $q = j$, it has that $K_{j-1} = 0$, that is, $U_j = V_j = 0, j = s+1$, ..., $k-1$. ∎

3. CONSTANT ANGULAR SPEED SYSTEMS.

Let's consider the system

$$\begin{cases} \dot{x} &= -y + x(H_1 + H_2 + \cdots + H_{n-1}), \\ \dot{y} &= x + y(H_1 + H_2 + \cdots + H_{n-1}), \end{cases} \tag{4}$$

with $H_i = H_i(x,y)$ homogeneous polynomial of degree i. Using the previous results we know that if a polynomial commutator exists, it will take the form

$$(U,V)^t = (xK, yK)^t = (x\sum_{j=0}^{n-1} K_j, y\sum_{j=0}^{n-1} K_j)^t \tag{5}$$

with $K_i = K_i(x,y)$ homogeneous polynomial of degree i.

LEMMA 5. *The vector field* (5) *is a polynomial commutator of the system* (4) *if and only if the polynomials* K_j *verify:*

1. $H_pK_q = H_qK_p$ *for any pair* p, q *with* $1 \leq p \leq n-1$, $1 \leq q \leq n-1$.
2. (a) If $K_0 \neq 0$, then $-jH_j + x\partial_y K_j - y\partial_x K_j = 0$ *with* $1 \leq j \leq n-1$,

   (b) If $K_0 = 0$, then $x\partial_y K_j - y\partial_x K_j = 0$ *with* $1 \leq j \leq n-1$.

Proof. Applying the Theorem 2.1 we get the first part.

In the case $K_0 \neq 0$. Expanding and simplifying the Lie bracket of the field with the commutator we have

$$(x,\, y)^t(x\partial_x H + y\partial_y H - (x\partial_y K - y\partial_x K) + K(x\partial_x H + y\partial_y H) - H(x\partial_x K + y\partial_y K)).$$

If we denote the homogeneous part of degree j of the polynomial P by $(P)_j$, we will have

$$\sum_{j=1}^{n-1}(x\partial_x H + y\partial_y H)_j - \sum_{j=1}^{n-1}(x\partial_y K - y\partial_x K)_j$$

$$+\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(K_j(x\partial_x H + y\partial_y H)_i - H_i(x\partial_x K + y\partial_y K)_j).$$

That is, $\sum_{j=1}^{n-1} jH_j - x\sum_{j=1}^{n-1}\partial_y K_j + y\sum_{j=1}^{n-1}\partial_x K_j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(i-j)H_iK_j = 0$. Taking into account the terms degree by degree we get 2.(a).

If $K_0 = 0$, the Lie bracket is $\sum_{j=1}^{n-1} jH_j - x\sum_{j=1}^{n-1}\partial_y K_j + y\sum_{j=1}^{n-1}\partial_x K_j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(i-j)H_iK_j = 0$. Therefore $x\partial_y K_j - y\partial_x K_j = 0,\ 1 \leq j \leq n-1$.

Reciprocal implication is obvious, since under conditions *1* and *2* the field (5) is a commutator of the system (4). ∎

These relations allow us to compute polynomials K_j using polynomials H_j. Notice that if j is odd the polynomial K_j is unique, but if j is even, K_j is settled module the vector space generated by $(x^2+y^2)^{j/2}$.

From Lemma 5, it has the following result.

COROLLARY 3.1. *The system* (4) *has polynomial commutator with null linear part if and only if it has the form*

$$\begin{cases} \dot{x} &= -y + xP_{2l}(x,y)\sum_{j=0}^{r} a_j(x^2+y^2)^j, \\ \dot{y} &= x + yP_{2l}(x,y)\sum_{j=0}^{r} a_j(x^2+y^2)^j, \end{cases} \qquad (6)$$

with $P_{2l}(x,y)$ *homogeneous polynomial of degree* $2l, l \geq 0$, *and* $a_j,\ j = 0, \cdots, r$, *arbitrary real numbers. In this case, the commutator* (5) *is given* *by*

$$(x\sum_{j=0}^{r}a_j(x^2+y^2)^{j+l}, y\sum_{j=0}^{r}a_j(x^2+y^2)^{j+l})^t. \qquad (7)$$

We now study the case when the commutator has linear part $(x,y)^t$. Next, we prove a result that will be used in the proof of the main theorem.

LEMMA 6. *Let M, N be homogeneous polynomials of orders m and n respectively, with $m \leq n$, verifying*

$$mM(x\partial_y N - y\partial_x N) = nN(x\partial_y M - y\partial_x M). \qquad (8)$$

Then M divides to N.

Proof. The relation (8) can be expressed as

$$\frac{\frac{1}{n}(x\partial_y N - y\partial_x N)}{N} = \frac{\frac{1}{m}(x\partial_y M - y\partial_x M)}{M},$$

that is, $\frac{1}{n}(x\partial_y - y\partial_x)ln\, N = \frac{1}{m}(x\partial_y - y\partial_x)ln\, M$, so, $(x\partial_y - y\partial_x)ln\, \frac{N^m}{M^n} = 0$. Consequently $ln\, \frac{N^m}{M^n} = f(x^2+y^2)$ and therefore $N^m = M^n e^{f(x^2+y^2)}$ being f an analytic real function.

Since N^m and M^n are polynomials, we have that $e^{f(x^2+y^2)}$ is constant. So, $N^m = \gamma M^n$ with γ a constant. Since M and N are polynomials, that can only take place if M divides to N. ▮

THEOREM 3.2. *The system* (4) *with $H_1 = H_2 = \ldots = H_{j-1} = 0$, $H_j \neq 0$ ($j \geq 1$) has polynomial commutator with linear part $(x,y)^t$ if and only if there are polynomials α_l, β_l of order l ($l \leq j$, l divides to j) verifying*

$x\partial_y\beta_l - y\partial_x\beta_l = l\alpha_l$, such that the system (4) *is given by*

$$\left\{\begin{array}{rcl} \dot{x} & = & -y + x\alpha_l \sum_{k=j/l-1}^{r-1} a_k\beta_l^k, \\ \dot{y} & = & x + y\alpha_l \sum_{k=j/l-1}^{r-1} a_k\beta_l^k, \end{array}\right. \tag{9}$$

with a_k, $k = j/l - 1, \cdots, r-1$, arbitrary real numbers and $r = [\frac{n-1}{l}]$. In this case, the commutator (5) *is given by*

$$\left\{\begin{array}{rcl} U & = & x + x\sum_{k=j/l-1}^{r-1} a_k\beta_l^{k+1}, \\ V & = & y + y\sum_{k=j/l-1}^{r-1} a_k\beta_l^{k+1}. \end{array}\right. \tag{10}$$

Proof. If the field (4) is homogeneous, it is enough to take $l = j$.

If it is not homogeneous, there is a polynomial H_i non null with $i > j$. Let m be the natural number such that $mj < i \leq (m+1)j$.

The proof consists, basically, of studying what has to satisfy the degree of H_i and stating the expression of H_i in terms of K_i.

From Lemma 5, the factors K_i, K_j corresponding to the terms of degree i, j of the commutator are also non null and have to verify

$$\frac{\frac{1}{i}(x\partial_y K_i - y\partial_x K_i)}{K_i} = \frac{\frac{1}{j}(x\partial_y K_j - y\partial_x K_j)}{K_j},$$

that is, $jK_j(x\partial_y K_i - y\partial_x K_i) = iK_i(x\partial_y K_j - y\partial_x K_j)$. Therefore, from Lemma 6, we have $K_i = K_j M_{i-j}$, with M_{i-j} homogeneous polynomial over x, y, of order $i - j$. If we substitute K_i, in the previous expression we will get $jK_j(x\partial_y M_{i-j} - y\partial_x M_{i-j}) = (i-j)M_{i-j}(x\partial_y K_j - y\partial_x K_j)$, that is, now M_{i-j}, K_j verify the hypothesis of Lemma 6. At which there are $M_{i-2j}, M_{i-3j}, \ldots, M_{i-mj}$, homogeneous polynomials of the degree given by the subscript, such that

$$M_{i-2j} = K_j M_{i-3j},\ M_{i-3j} = K_j M_{i-4j}, \ldots,\ M_{i-(m-1)j} = K_j M_{i-mj},$$

and so $K_i = M_{i-mj} K_j^m$.

We distinguish two situations:

- If $i = (m+1)j$, $K_j = M_{i-mj}$ up to a constant, at which $K_{(m+1)j}$ and $H_{(m+1)j}$ are K_j^{m+1} and $H_j K_j^m$ up to a constant, respectively. Notice therefore, that if the non null factors H_i take the form H_{mj} we have that $l = j$ and $\alpha_j = H_j$, $\beta_j = K_j$.

- The other way, $mj < i < (m+1)j$, applying Lemma 6 on polynomials K_j, M_{i-mj} the existence of a polynomial of degree $(m+1)j - i$, $B_{(m+1)j-i}$, such that $K_j = M_{i-mj} B_{(m+1)j-i}$. Substituting we have that $B_{(m+1)j-i}$ and M_{i-mj} satisfy the Lemma again. If the obtained polynomials have the same degree, that is, $2i = (2m+1)j$, they will be proportional, at which we will have $M_{j/2} = \gamma B_{j/2}$ and so $K_j = \gamma B_{j/2}^2$ and $H_j = \frac{1}{j}(x\partial_y \gamma B_{j/2}^2 - y\partial_x \gamma B_{j/2}^2) = \frac{1}{j/2}\gamma B_{j/2}(x\partial_y B_{j/2} - y\partial_x B_{j/2}) = A_{j/2} B_{j/2}$. That is, if there wasn't any other H_i non null, $l = j/2$, $\alpha_{j/2} = A_{j/2}$ and $\beta_{j/2} = B_{j/2}$.

  If they have different degrees, the one with the lowest degree will divide to the other one. That is, either $K_j = B_{(m+1)j-i}^2 M_{2i-(2m+1)j}$ or $K_j = B_{(2m+1)j-2i} M_{i-mj}^2$.

  If these factors don't have the same degree, it will lead us to the existence of polynomials B and M such that K_j is given by $B^3 M$,

B^2M^2 or BM^3.

Anyway, we have that K_j is the product of two factors, that is, $K_j = B^pM^q$ with B, M homogeneous polynomials verifying the hypothesis of Lemma 6 and p, q natural numbers and moreover both polynomials have the same degree, l. So, $M = \delta B$, what implies $K_j = \delta^q B^{p+q}$. That is, K_j has the form $K_j = \widetilde{\delta}B^{j/l}$ with B homogeneous polynomial of degree l, $\widetilde{\delta}$ constant and

$$H_j = \frac{1}{j}(x\partial_y\widetilde{\delta}B^{j/l} - y\partial_x\widetilde{\delta}B^{j/l}) = \widetilde{\delta}\frac{1}{l}(x\partial_y B - y\partial_x B)B^{j/l-1} = \widetilde{\delta}\alpha\beta^{j/l-1}.$$

If there is more than one H_i non null, for instance H_{i_1}, H_{i_2} (so K_{i_1}, K_{i_2} are not null), applying the previous reasoning we deduce the existence of two natural numbers l_1, l_2 and polynomials α_{l_1}, β_{l_1} and α_{l_2}, β_{l_2} of degrees l_1, l_2 respectively, such that, up to constants,

$$K_{i_1} = (\beta_{l_1})^{i_1/l_1}, \quad K_{i_2} = (\beta_{l_2})^{i_2/l_2}.$$

At which we will have two expressions for K_j and comparing both of them we get $\beta_{l_1}^{l_2} = \beta_{l_2}^{l_1}$.

If $l_1 < l_2$, since β_{l_1}, β_{l_2} are polynomials, we have that $\beta_{l_2} = \beta_{l_1}^{l_2/l_1}$, at which $K_{i_2} = (\beta_{l_1})^{\frac{l_2 i_2}{l_1 l_2}} = (\beta_{l_1})^{i_2/l_1}$. In this case $l = l_1$.

If there were r homogeneous no null polynomials $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$ of degrees $i_1, i_2, \ldots, i_r$, respectively (corresponding polynomials $K_{i_1}, K_{i_2}, \ldots, K_{i_r}$ are also non null) we would have at most r expressions of K_i as a power of homogeneous polynomials of degrees $l_1, l_2, \ldots, l_r$, and in this case, $l =$

$min(l_1, l_2, \cdots, l_r)$.

In order to prove the other implication, it is enough to check that the systems (9) and (10) verify the Lemma 5. ∎

Next, we are interested in the centers of the analytic systems with constant angular speed,

$$(\dot{x}, \dot{y})^t = X(x, y) = (-y + xH(x, y), x + yH(x, y))^t, \tag{11}$$

where H is an analytic function in a neighborhood of the origin and $H(0, 0) = 0$. From Lemma 4, it is proved that if H is a polynomial, the polynomial commutators, in case of existing, are of the type

$$Y(x, y) = (xK(x, y), yK(x, y))^t. \tag{12}$$

The following result generalizes this characterization to the analytic case.

THEOREM 3.3. *If the analytic system* (11) *has a center at the origin, then there exists an analytic commutator of the form* (12) *with* K *an analytic function around the origin.*

Proof. It is known that X and Y commute if and only if the local flows $\Phi_X(t, (x, y))$, $\Phi_Y(t, (x, y))$ with $\Phi_X(0, (x, y)) = (x, y)$ and $\Phi_Y(0, (x, y)) = (x, y)$, verify

$$\Phi_X(t, \Phi_Y(s, (x, y))) = \Phi_Y(s, \Phi_X(t, (x, y))),$$

for every t and s such that $\Phi_X(t, \Phi_Y(s, (x, y)))$ and $\Phi_Y(s, \Phi_X(t, (x, y)))$ exist, see Olver [21].

We assume that X is a center. Let Ψ be, the local flow which defines the solutions of the differential equation $\dot{x} = \alpha(x)$, with α analytic, $\alpha(0) = 0$ and $\alpha(x) < 0$ in $(0, \epsilon)$. Fixed (x, y) in a neighborhood at the origin, let s_0 be the minimum value $s \geq 0$ such that $\Phi_X(-s, (x, y))$ is on the x-axis. We already define the analytic flow Φ_Y as $\Phi_Y(t, (x, y)) = \Phi_X(s_0, (\Psi(t, \Phi_X(-s_0, (x, y)), 0))$. Since X is uniformly isochronous, Φ_Y does not depend on s_0. Moreover, it commutes with Φ_X and all the straight lines which pass on through the origin are invariant to the flow. That is, the associated vector field Y is analytic and of the form (12). ∎

4. VECTOR FIELDS $(-Y + P_S + \sum_{J=K}^{N-1} XH_J, X + Q_S + \sum_{J=K}^{N-1} YK_J)^T$ WITH POLYNOMIAL COMMUTATOR.

Let's consider the system

$$\begin{cases} \dot{x} = -y + P_s + x(H_k + H_{k+1} + \cdots + H_{n-1}), \\ \dot{y} = x + Q_s + y(H_k + H_{k+1} + \cdots + H_{n-1}), \end{cases} \tag{13}$$

with $2 \leq s \leq k$, P_s, Q_s homogeneous polynomial of degree s and $H_i = H_i(x, y)$ homogeneous polynomial of degree i. This family includes the systems studied in the previous section. From Theorem 2.1, if the system (13) have polynomial commutator, then either it has the form

$$(U, V)^t = (x + U_s + x \sum_{j=k}^{n-1} K_j, y + V_s + y \sum_{j=k}^{n-1} K_j)^t, \tag{14}$$

or

$$(U, V)^t = (U_s + x \sum_{j=k}^{n-1} K_j, V_s + y \sum_{j=k}^{n-1} K_j)^t, \tag{15}$$

with $(yP_s - xQ_s)K_j = (yU_s - xV_s)H_j,\ j = k,\ ...,\ n-1.$

The next lemma characterizes the terms $U_s,\ V_s$ of a polynomial commutator of (13).

Lemma 7. *If the vector field* (14) *is a polynomial commutator of the system* (13) *then* U_s, V_s *verify*

$$((y\partial_x - x\partial_y)^2 + id)(U_s) = -(s-1)(Q_s + y\partial_x P_s - x\partial_y P_s),$$
$$((y\partial_x - x\partial_y)^2 + id)(V_s) = -(s-1)(P_s - y\partial_x Q_s + x\partial_y Q_s),$$

with $y\partial_x U_s - x\partial_y U_s = V_s + (s-1)P_s,\ y\partial_x V_s - x\partial_y V_s = -U_s - (s-1)Q_s.$ *If the vector field* (15) *is a polynomial commutator of the system* (13) *then* U_s, V_s *verify*

$$((y\partial_x - x\partial_y)^2 + id)(U_s) = 0, \quad ((y\partial_x - x\partial_y)^2 + id)(V_s) = 0,$$

with $y\partial_x U_s - x\partial_y U_s = V_s,\ y\partial_x V_s - x\partial_y V_s = -U_s.$

Proof. The term of degree s of the Lies bracket between the field and the polynomial commutator (14) is given by $[(-y,x)^t, (U_s,V_s)^t] + [(P_s,Q_s)^t, (x,y)^t]$. Expanding and applying Euler's Theorem, it becomes

$$(-V_s + y\partial_x U_s - x\partial_y U_s + (s-1)P_s, U_s + y\partial_x V_s - x\partial_y V_s + (s-1)Q_s)^t.$$

As it must be null, we have that U_s and V_s must verify $y\partial_x U_s - x\partial_y U_s = V_s + (s-1)P_s$ and $y\partial_x V_s - x\partial_y V_s = -U_s - (s-1)Q_s$. Applying to both expressions the operator $y\partial_x - x\partial_y$ we arrive to

$$((y\partial_x - x\partial_y)^2 + id)(U_s) = -(s-1)(Q_s + y\partial_x P_s - x\partial_y P_s),$$
$$((y\partial_x - x\partial_y)^2 + id)(V_s) = -(s-1)(P_s - y\partial_x Q_s + x\partial_y Q_s),$$

that is, we can calculate in a explicit way the polynomials U_s V_s.

Analogously, it proves the case when the system (13) commutes with (15).

▮

The next result characterizes the systems (13) with $s = 2$ that have polynomial commutator.

THEOREM 4.1. *The system* (13) *with* $s = 2$, $P_2^2 + Q_2^2 \neq 0$, *has polynomial commutator if and only if either it has constant angular speed (see Corollary 3.1 and Theorem 3.2), or, up to rotation, it is of the form*

$$\left\{ \begin{array}{rcl} \dot{x} &=& -y + p_{20}x^2 - 2q_{20}xy + q_{20}p_{20}x^3 + ax^2y, \\ \dot{y} &=& x + q_{20}x^2 + p_{20}xy - q_{20}y^2 + q_{20}p_{20}x^2y + axy^2, \end{array} \right. \tag{16}$$

and its commutator is

$$\begin{array}{rcl} U &=& x + q_{20}x^2 + p_{20}xy - q_{20}y^2 + q_{20}p_{20}x^2y + axy^2, \\ V &=& y + 2q_{20}xy + p_{20}y^2 + q_{20}p_{20}xy^2 + ay^3, \end{array} \tag{17}$$

with p_{20}, q_{20}, a *real numbers and* $q_{20} \neq 0$.

Proof. We consider the system (13) with $P_s = P_2 = p_{20}x^2 + p_{11}xy + p_{02}y^2$, $Q_s = Q_2 = q_{20}x^2 + q_{11}xy + q_{02}y^2$. Making a rotation, we can suppose $p_{02} = 0$. The operator $(y\partial_x - x\partial_y)^2 + id$ on the set of the homogeneous polynomials of degree s in the variables x, y is invertible, and $((y\partial_x - x\partial_y)^2 + id)^{-1}$ comes given by

$$((y\partial_x - x\partial_y)^2 + id)^{-1}(m_{20}x^2 + m_{11}xy + m_{02}y^2)$$

$$= \tfrac{1}{3}(m_{20} + 2m_{02})x^2 - \tfrac{1}{3}m_{11}xy + \tfrac{1}{3}(2m_{20} + m_{02})y^2.$$

We first suppose that the system commutes with (14), applying Lemma 7, the quadratic terms of the polynomial commutator U_2, V_2 are unique and take the expression

$$U_2 = \frac{1}{3}(-p_{11} - q_{20} - 2q_{02})x^2 + \frac{1}{3}(2p_{20} + q_{11})xy + \frac{1}{3}(p_{11} - 2q_{20} - q_{02})y^2,$$

$$V_2 = \frac{1}{3}(p_{20} - q_{11})x^2 + \frac{1}{3}(-p_{11} + 2q_{20} - 2q_{02})xy + \frac{1}{3}(2p_{20} + q_{11})y^2,$$

furthermore, it is easy to show that they satisfy the conditions

$$y\partial_x U_2 - x\partial_y U_2 = V_2 + P_2, \quad y\partial_x V_2 - x\partial_y V_2 = -U_2 - Q_2.$$

The cubic term of the Lie bracket is

$$\begin{pmatrix} x \\ y \end{pmatrix} (y\partial_x K_2 - x\partial_y K_2 + 2H_2) + \left[\begin{pmatrix} P_2 \\ Q_2 \end{pmatrix}, \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} \right]. \tag{18}$$

If we multiply the first component by y, the second one by x, and subtracting both expressions, we obtain a quartic term that doesn't depend on the coefficients of H_2, K_2 and that must be identically null, i.e. we get necessary conditions that the coefficients of P_2, Q_2 must verify for the existence of the polynomial commutator. The simultaneous annulment of these coefficients leads us to

$$P_2 = p_{20}x^2 - 2q_{20}xy, \quad Q_2 = q_{20}x^2 + p_{20}xy - q_{20}y^2.$$

As $yP_2 - xQ_2 = -q_{20}x(x^2 + y^2)$ and $yU_2 - xV_2 = -q_{20}y(x^2 + y^2)$, from Theorem 2.1, the polynomials H_j, K_j verify $xK_j = yH_j$, for all j, i.e. $H_j = xM_{j-1}, K_j = yM_{j-1}, \ j = k, \ ..., \ n-1$. In this case, $xK_2 = yH_2$ we arrive to $H_2 = q_{20}p_{20}x^2 + axy$, $K_2 = q_{20}p_{20}xy + ay^2$.

It is easy to show that the Lie bracket of the fields

$$(-y+p_{20}x^2-2q_{20}xy+q_{20}p_{20}x^3+ax^2y, x+q_{20}x^2+p_{20}xy-q_{20}y^2+q_{20}p_{20}x^2y+axy^2)^t$$

and

$$(x+q_{20}x^2+p_{20}xy-q_{20}y^2+q_{20}p_{20}x^2y+axy^2, y+2q_{20}xy+p_{20}y^2+q_{20}p_{20}xy^2+ay^3)^t$$

is null, therefore both fields commute.

Next, we prove that if $(U,V)^t = (x+U_2+x\sum_{j=k}^{n-1} K_j, y+V_2+y\sum_{j=k}^{n-1} K_j)^t$

is a polynomial commutator of (13) then

$$\left[\begin{pmatrix} P_2 \\ Q_2 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} K_j\right] + \left[\begin{pmatrix} x \\ y \end{pmatrix} H_j, \begin{pmatrix} U_2 \\ V_2 \end{pmatrix}\right] = (j-2)q_{20}\begin{pmatrix} x \\ y \end{pmatrix}(x^2+y^2)M_{j-1}, \tag{19}$$

for all $j = k, \ ..., n-1$.

Expanding the sum of both brackets we have

$$\begin{pmatrix} x \\ y \end{pmatrix}(U_2\partial_x H_j + V_2\partial_y H_j - P_2\partial_x K_j - Q_2\partial_y K_j) + \begin{pmatrix} P_2K_j - U_2H_j \\ Q_2K_j - V_2H_j \end{pmatrix}. \tag{20}$$

The second summand of the right-side of the expression also has radial form since

$$y(P_2K_j - U_2H_j) - x(Q_2K_j - V_2H_j) = (yP_2 - xQ_2)K_j - (yU_2 - xV_2)H_j = 0,$$

from Theorem 2.1. In this case,

$$U_2\partial_x H_j + V_2\partial_y H_j - P_2\partial_x K_j - Q_2\partial_y K_j = (j-1)q_{20}(x^2+y^2)M_{j-1},$$

$$P_2K_j - U_2H_j = -q_{20}(x^2+y^2)M_{j-1},$$

therefore, it verifies (19).

For any j with $2 \le j < n-1$, using Theorem 2.1 and by (19), the term of degree $j+2$ of the Lie bracket of the field with the commutator comes given by

$$\begin{pmatrix} x \\ \\ y \end{pmatrix}(y\partial_x K_{j+1} - x\partial_y K_{j+1} + (j+1)H_{j+1} + (j-2)q_{20}(x^2+y^2)M_{j-1}), \quad (21)$$

with $2 \le j < n-1$. For $j = n-1$, the annulment of the term of degree $n+1$ of the Lie bracket leads us $(n-3)q_{20}(x^2+y^2)M_{n-2}$, being $H_{n-1} = xM_{n-2},\ K_{n-1} = yM_{n-2}$. That is, either $M_{n-2},\ M_{n-3},\ ...,\ M_{k-1}$ are nulls, therefore the system is (16) with $q_{20} \neq 0$, or $q_{20} = 0$, i.e. $P_2 = p_{20}x^2,\ Q_2 = p_{20}xy$, therefore the field has constant angular speed.

Finally, if the system commutes with (15), it has $U_2 = V_2 = 0$, since the operator $(y\partial_x - x\partial_y)^2 + id$ is invertible, and from Theorem 2.1 it has $yP_2 = xQ_2$, i. e. the system has constant angular speed. ∎

5. FINAL REMARKS.

As a consequence of our study, we remark the following results related to the systems with constant angular speed:

1. If the system (4) has nonlinear terms of degree n with $n-1$ prime,

the only non-homogeneous systems with polynomial commutator are those where H, up to rotation, take the form $x\sigma(y)$.

This fact can be extended to the case in which there are at least two nonlinearities of degrees n and m respectively, with $n-1$ and $m-1$ primes with each other.

2. All systems of degree less or equal to three, with a center at the origin, are reversible and have polynomial commutator. Up to rotation, these are

$$\left\{\begin{array}{rcl}\dot{x} & = & -y + x(a_2 y + b_2 xy),\\ \dot{y} & = & x + y(a_2 y + b_2 xy).\end{array}\right.$$

At the fourth degree, all the systems with homogeneous nonlinear part are centers and have polynomial commutator. Therefore, there are non-reversible systems with polynomial commutator.

3. The reversible homogeneous quartic systems are, up to rotation, $(\dot{x},\dot{y})^t = (-y + xy(c_2x^2 + c_4y^2), x + y^2(c_2x^2 + c_4y^2))^t$. All have polynomial commutator. Nevertheless, the reversible quartic systems with non-homogeneous nonlinear part,

$$\left\{\begin{array}{rcl}\dot{x} & = & -y + x(a_2 y + b_2 xy + c_2x^2y + c_4y^3),\\ \dot{y} & = & x + y(a_2 y + b_2 xy + c_2x^2y + c_4y^3).\end{array}\right.$$

with $c_2 \neq 2c_4$, haven't polynomial commutator, for example,

$$\left\{\begin{array}{rcl}\dot{x} & = & -y + xy + xy^3,\\ \dot{y} & = & x + y^2 + y^4.\end{array}\right.$$

4. The systems of the form

$$\begin{cases} \dot{x} &= -y + xH_j + xH_j(a_1K_j + ... + a_nK_j^n), \\ \dot{y} &= x + yH_j + yH_j(a_1K_j + ... + a_nK_j^n), \end{cases}$$

with $x\partial_y K_j - y\partial_x K_j = jH_j$ and $(\dot{x},\dot{y})^t = (-y + xH_j, x + yH_j)^t$ non-reversible and a_j real numbers, are non-reversible non-homogeneous systems with polynomial commutator. A example of isochronous, non-homogeneous, non reversible systems (4) is

$$\begin{cases} \dot{x} &= -y + 2x(y-x)(y^2+2xy-x^2)(6y^3-6xy^2+6x^2y+2x^3+1), \\ \dot{y} &= x + 2y(y-x)(y^2+2xy-x^2)(6y^3-6xy^2+6x^2y+2x^3+1). \end{cases}$$

Consequently, the concepts of reversibility, existence of polynomial commutator and isochronicity aren't coinciding concepts for the systems (4) with $n \geq 4$. Therefore, the isochronicity problem for polynomial systems with constant angular speed is still open. So far, all the centers of the systems (4) that we know either are reversible systems or have polynomial commutator. Is true the affirmation in general?

In the case of the systems with degenerate infinity, we raise following question: From Lemma 4, we know that if $(\sum_{j=1}^n U_j, \sum_{j=1}^n V_j)^t$ is a polynomial commutator of the system (1) with $P_j = xH_{j-1}$, $Q_j = yH_{j-1}$, $j = k,\ldots,n$, then $xV_j = yU_j$, $j = k,\ldots,n$. Can we generalize this result to the analytic case?, i.e. if the system of the form $(\dot{x},\dot{y})^t = (-y + P_2 + P_3 + ... + P_k + xH, x + Q_2 + Q_3 + ... + Q_k + yH)^t$, with H analytic function in the origin and $H_1 = ... = H_k = 0$, is a isochronous center, then it will exist an analytic commutator of the form $(x+U_2+U_3+...+U_k+xK, x+V_2+V_3+...+V_k+yK)^t$, with K analytic function in the origin and $K_1 = ... = K_k = 0$?

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