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Algebraic integrability of nilpotent planar vector fields

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ABSTRACT

We characterize, using normal forms of quasi-homogeneous expansions, the analytic vector fields at nilpotent singular point having an algebraic first integral over the ring $\mathbb{C}[[x,y]]$. As a consequence, we provide a link between the algebraic integrability problem and the existence of a formal inverse integrating factor which is null at the singular point.

1. Introduction

We consider an autonomous system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ where $\mathbf{F} = (P, Q)^T$ is a planar vector field defined in a domain $U \subset \mathbb{C}^2$ whose components P and Q are analytic in a neighborhood of the origin and coprimes.

Let $\Sigma \subset U$ be with $U \setminus \Sigma$ open, a set of integral curves of the system, we say that a continuous function $H : U \setminus \Sigma \to \mathbb{C}$ is a *first integral* of $\mathbf{F}$ if H is constant on each solution of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ contained in $U \setminus \Sigma$, and H is non-constant on any open subset of $U \setminus \Sigma$. If Σ is a empty set, H is a *strong* first integral; otherwise, H is a *weak first integral.* Thus, the first integrals computed by Darboux [15] in 1878 for polynomial systems are weak first integrals, in general.

Here, we study the existence of an *algebraic* first integral, *i.e.* a first integral which is solution of a polynomial with coefficients over $\mathbb{C}[[x,y]]$ (ring of the formal series in x, y with coefficients on $\mathbb{C}$). As we will see below (Proposition 4, the existence of an algebraic first integral is equivalent to the existence of a first integral that is a ratio of two coprime formal functions.

A function $H = \frac{f}{g}$ with f and g analytic or formal functions, is a first integral of $\mathbf{F}$ if H is a weak first integral on the open set $\Omega_g = \{(x,y) \in U : g(x,y) \neq 0\}$. Obviously, if H is a first integral, then Lie derivative of H by $\mathbf{F}$ is zero in Ω_g, i.e. $\mathbf{F}(H) = 0$ in Ω_g, where $\mathbf{F} := P\partial x + Q\partial y$ denotes the differential operator associated to the vector field $\mathbf{F}$. It will cause no confusion if we use the same letter to designate the vector field and its associated differential operator.

If the open U does not have any singular point, from flows box's theorem [12, Cauchy-Arnold Theorem], the vector field is analytically integrable, *i.e.* there always exists an analytic first integral of $\mathbf{F}$. Consequently, we will assume that U has a singular point of the vector field.

We recall a classification of the singular points of a vector field according to the expression of $D\mathbf{F}(\mathbf{0})$, the linear part of a vector field evaluated at the singular point (we will suppose that the origin is the singular point). We say that the origin is an *elementary* singular point if at least one eigenvalue of $D\mathbf{F}(\mathbf{0})$ is non-zero. Particularly, if both eigenvalues are non-zero, we say that the elementary singular point is *nondegenerate*. Otherwise, both eigenvalues are zero, the origin is *degenerate*. A special case of degenerate singular point is the *nilpotent* singular point ($D\mathbf{F}(\mathbf{0})$ is non-zero and the eigenvalues are zero).

The integrability problem consists in determining if the planar vector field has a first integral. In a general framework, it is an important question because the existence of a first integral determines completely its phase portrait. For example, a *nondegenerate singular point type center-focus* (its linear part has imaginary eigenvalues) is a *center* if, and only if, it has an analytic first integral around the singular point, see [27,29].

Several authors have studied the existence of first integrals according to the type of singularity. Poincaré [29] was interested in the rational integrability. Namely, he gave necessary conditions so that a nondegenerate elementary singular point has a rational first integral. The conditions provided are conditions of resonance of the eigenvalues associated to the singular point. Li et al. [25], using the Poincaré-Dulac normal form theory, generalized the results of Poincaré to the analytic two-dimensional systems with elementary singular points, and obtained necessary and sufficient conditions in order that the vector field has generalized rational local first integrals (ratio of two coprime analytic functions). Subsequently, Cong et al. [14] and Llibre et al. [26] extended this result for the vector fields with arbitrary dimension.

Furta [18] proves that if the eigenvalues associated to the origin do not satisfy any resonance condition then the vector field does not have any analytic first integrals at origin. See also Yoshida [33],

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Goriely [22] and Nowicki [28]. Shi [32] studies the nonexistence of rational first integral for semiquasihomogeneous systems.

Two analytic differential systems are called *orbitally equivalent* if there exists a transformation of the phase space which takes complex phase curves of the first to those of the second one or equivalently, there exist a formal near-identity change of variables and a formal re-parameterization of the time-variable that transform one into the other one.

By [17,23], the *nondegenerate center-focus type* and *resonant saddles* singular points, have an analytic local first integral if, and only if, they are orbitally linearizable.

In this paper, we deal with analytic differential systems

$$\dot{x} = y + X(x, y), \qquad \dot{y} = Y(x, y), \tag{1}$$

with X and Y analytic functions without constants and linear terms. Thus, the linear part valued at the origin is non-zero and the eigenvalues of the linear part of the vector field are zero at that point, i.e. the origin is a nilpotent singular point.

The study of this class of vector fields has attracted the attention of several authors. For example, Sadovskii [31], Berthier and Moussu [11], Andreev et al. [10], Giacomini et al. [21], García et al. [20], Algaba et al. [4] have characterized the nilpotent centers. The center problem has been solved for some families, see [9,16,24]. Particularly, Chavarriga et al. [13] and García and Giné [19] study the analytically integrable nilpotent centers.

The *analytic* integrability problem around a nilpotent singularity is already solved. Algaba et al. [1] study the vector fields whose leader quasi-homogeneous term is Hamiltonian, see [1, Theorem 3.19]. Recently, Algaba et al. [3] have solved the analytic integrability problem around a nilpotent singularity of a planar vector field under generic conditions (the origin of the leader quasi-homogeneous vector field is an isolated singular point), see [3, Theorem 1.2]. Later, Algaba et al. [2] have solved the remaining case, completing the analytic integrability problem for such singularity, see [2, Theorem 1.4]. In all the cases, the vector field has an analytic local first integral if, and only if, it is orbitally quasi-homogenizable (orbitally equivalent to its quasi-homogeneous leader term).

Here, we study the nilpotent vector fields which have an *algebraic* first integral on $\mathbb{C}[[x, y]]$. Because the eigenvalues are zero, Poincaré-Dulac normal form theory is not applicable and so we need an appropriate normal form for the study of this problem.

We introduce some notation and concepts to present the main results. Given $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 natural numbers without common factors, a scalar function f of two variables is *a quasi-homogeneous function* of type or weight exponent $\mathbf{t}$ and degree j if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^j f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree j is denoted by $\mathcal{P}_j^{\mathbf{t}}$. A vector field $\mathbf{F} = (P, Q)^T$ is *a quasi-homogeneous vector field* of type $\mathbf{t}$ and degree j if $P \in \mathcal{P}_{j+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{j+t_2}^{\mathbf{t}}$. We denote the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree j by $\mathcal{Q}_j^{\mathbf{t}}$. An analytic vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F}$ can be written in the form

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots,$$

for some integer r, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$.

As mentioned before, the nilpotent vector fields are analytically integrable if, and only if, they are orbitally equivalent to a polynomially integrable quasi-homogeneous vector fields, see [1–3]. In this paper, we solve the remaining case, completing the algebraic integrability problem for such singularity; that is, we address our study when the vector field has an algebraic first integral but it does not have any analytic first integral and we obtain the following result.

Theorem 1. *(Algebraic and non-analytic integrability of nilpotent vector fields). We assume that* $\mathbf{F}$ *is an analytic vector field whose origin is a nilpotent singularity and* $\mathbf{F}$ *is not analytically integrable. The vector field* $\mathbf{F}$ *has an algebraic first integral if, and only if, after a polynomial change of variables (if necessary), is formally orbitally equivalent to*

$$(y + dx^{n+1})\partial_x + (n+1)(x^{2n+1} + dx^n y)\partial_y,$$

with n *a natural number and* d *a rational number with* $|d| > 1$, *i.e. a quasi-homogeneous vector field of weight* $\mathbf{t} = (1, n+1)$ *and degree* n.

Next result allows to extend the results of [17,23,25] on algebraic integrability for vector fields whose origin is a nondegenerate or elementary singular point to the nilpotent vector fields.

Theorem 2. *(Algebraic integrability of nilpotent vector fields). Let* $\mathbf{F}$ *be an analytic vector field with a nilpotent singularity at the origin. The vector field* $\mathbf{F}$ *has an algebraic first integral if, and only if, it is orbitally equivalent to a rationally integrable quasi-homogeneous vector field.*

We also provide a characterization of algebraic integrability (analytic or not) through the existence of an inverse integrating factor, *i.e.* a formal function V, zero at origin, such that $(P/V, Q/V)^T$ is a Hamiltonian vector field in $U \setminus \{V = 0\}$.

The following result is a tool we will use to obtain necessary and sufficient conditions of algebraical integrability.

Theorem 3. *Let* $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ *be an analytic vector field whose origin is an isolated nilpotent singularity. Assume that* $\mathbf{F}_r = (P_{r+t_1}, Q_{r+t_2})^T$ *is a rationally integrable vector field and* $t_1 x Q_{r+t_2} - t_2 y P_{r+t_1} \in \mathcal{P}_{r+t_1+t_2}^{\mathbf{t}}$ *has only simple factors on* $\mathbb{C}[x, y]$. *Then* $\mathbf{F}$ *is algebraically integrable if, and only if,* $\mathbf{F}$ *has a formal inverse integrating factor* $V = \sum_{j \geq r+t_1+t_2} V_j$, $V_j \in \mathcal{P}_j^{\mathbf{t}}$ *with* $V_{r+t_1+t_2} = t_1 x Q_{r+t_2} - t_2 y P_{r+t_1}$.

The condition on the expression of the lowest-degree term of the inverse integrating factor can not be removed. We remark that there exist nilpotent vector fields which are not algebraically integrable and nevertheless have an algebraic (and formal) inverse integrating factor. For example, the polynomial $(x^4 - 2y^2)^2$ is an inverse integrating factor of the vector field $(y + x^2(x^4 - 2y^2))\partial_x + (x^3 + 2xy(x^4 - 2y^2))\partial_y$, which is not algebraically integrable, see [8, Theorem 2].

The rest of the paper is organized as follows: In Section 2, we provide a normal pre-form of a nilpotent vector field with an algebraic first integral, Proposition 5. Here we also include some properties of the invariant curves of this normal pre-form. Section 3.1 contains a normal form of an analytic planar vector field using quasi-homogeneous expansion, see Theorem 8. We include this section trying to do the paper self-contained. The results shown and their proofs, we can find them in [3]. The results of subsection 3.1 are new, we give a orbitally equivalent normal form for the vector fields whose quasi-homogeneous leader term is rationally integrable but no polynomially integrable, see Theorems 9 and 10. Section 4 is the proof of Theorem 1, 2 and 3 and Section 5 contains some technical results we use for proving the above results. Last on, in Section 6, we solve our problem for a polynomial family by providing the algebraic first integrals.

2. Necessary conditions of algebraical and non-analytical integrability of nilpotent vector fields

We first give necessary conditions of algebraic integrability. We recall that an invariant curve at the origin of a vector field $\mathbf{F}$ is a formal curve $C(x, y) = 0$, null at the origin, satisfying $\mathbf{F}(C) = KC$ with K formal function, named cofactor of C.

Proposition 4. *Consider a non-analytically integrable vector field. If it has an algebraic first integral at the origin then it has a first integral that is ratio of two coprime formal functions. Moreover, it has two formal invariant curves at the origin with the same cofactor.*

Proof. We assume that the analytic vector field has an algebraic first integral and it is not analytically integrable. From [30, Propositions 1 and 2], it also admits a first integral V of the specific form $V=\varphi_1^{d_1}\cdots\varphi_s^{d_s}$, with $\varphi_i\in\mathbb{C}[[x,y]]$, non-invertible, irreducible invariants and rational numbers d_i non-zero. So, if we write $d_i=m_i/n_i$ and denote $N=\mathrm{lcm}(\{|n_1|,\ldots,|n_s|\})$ and $M=\gcd(\{N|\frac{m_1}{n_1}|,\ldots,N|\frac{m_s}{n_s}|\})$, then $V^{N/M}$ is also a first integral and has the expression $\prod_{i=1}^{s}\varphi_i^{k_i}$, with $k_0,\ldots,k_s$ non-zero integer numbers (no all k_i have the same sign), *i.e.* it is a ratio of two coprime formal series, W_1 and W_2, with W_1 and W_2 functionally independent and $W_1(\mathbf{0})=W_2(\mathbf{0})=0$. We prove the second part. As W_1/W_2 is a first integral we have that $\mathbf{F}(\frac{W_1}{W_2})\equiv 0$ in Ω_{W_2} and therefore $0\equiv W_2\mathbf{F}(W_1)-W_1\mathbf{F}(W_2)$ in Ω_{W_2}. As W_1 and W_2 are functionally independent, we have that $\mathbf{F}(W_1)=KW_1$ and $\mathbf{F}(W_2)=KW_2$ in Ω_{W_2}. This property also is satisfied in Ω_{W_1} since it is enough to consider the first integral $\frac{W_2}{W_1}$. Therefore, the property is true on $\mathbb{C}^2$ since $\Omega_{W_1}\cap\Omega_{W_2}=\{\mathbf{0}\}$. Thus, W_1 and W_2 are formal invariant curves with the same cofactor. □

We give a normal pre-form of nilpotent vector fields with an algebraic first integral. This result allows to obtain necessary conditions of algebraic integrability.

Proposition 5. *We assume that* $\mathbf{F}$ *is an analytic vector field whose origin is a nilpotent isolated singularity and has an algebraic first integral. There exists a polynomial change of variables* ϕ *such that* $\phi_*\mathbf{F}=\mathbf{F}_r+\cdots$, $\mathbf{F}_r\in\mathcal{Q}_r^{\mathbf{t}}$, *where* $\mathbf{F}_r$ *is one of the following vector fields:*

A) $\mathbf{F}_r=y\partial_x+ax^{2n}\partial_y$, $a\neq 0$, *i.e.* $r=2n-1$, $\mathbf{t}=(2,2n+1)$.
Moreover, in this case, $\mathbf{F}$ *is analytically integrable.*
B) $\mathbf{F}_r=y\partial_x+bx^ny\partial_y$, $b\neq 0$, *i.e.* $r=n$, $\mathbf{t}=(1,n+1)$.
Moreover, in this case, $\mathbf{F}$ *is analytically integrable.*
C) $\mathbf{F}_r=(y+dx^{n+1})\partial_x+(n+1)(x^{2n+1}+dx^ny)\partial_y$, $d\in\mathbb{Q}$, *i.e.* $r=n$, $\mathbf{t}=(1,n+1)$. *Moreover:*
For $|d|\le 1$, *in this case,* $\mathbf{F}$ *is analytically integrable.*
For $|d|>1$, *in this case,* $\mathbf{F}$ *is not analytically integrable.*

Proof. A normal pre-form of analytically integrable nilpotent systems can be found in [3]. We study the algebraic and non-analytic integrability.

According to the kind of the Newton diagram of a vector field and therefore depending on the type and degree of the quasi-homogeneous leader term, a vector field $\mathbf{F}$ whose origin is a nilpotent singularity can be transformed, by means of a polynomial change of variables into $\phi_*\mathbf{F}=\mathbf{F}_r+\cdots$, with $\mathbf{F}_r\in\mathcal{Q}_r^{\mathbf{t}}$ being one, and only one, of the following quasi-homogeneous vector fields:

- $\mathbf{F}_r=(y+yf(x))\partial_x+y^2g(x)\partial_y$, i.e. $\mathbf{t}=(0,1)$ and $r=1$. In this case, $\phi_*\mathbf{F}=(y+y\tilde f(x,y))\partial_x+y^2\tilde g(x,y)\partial_y$, that is, the components of the vector field are not coprimes (the origin is a not isolated singular point) and it is always analytically integrable by the flow box's theorem, see [12].
- $\mathbf{F}_r=y\partial_x+ax^{2n}\partial_y$, $a\neq 0$, i.e. $\mathbf{t}=(2,2n+1)$ and $r=2n-1$. In this case, $\phi_*\mathbf{F}$ has one invariant curve at origin, $\frac{1}{2}y^2+\frac{a}{2n+1}x^{2n+1}+\cdots$. Therefore, by Proposition 4, $\phi_*\mathbf{F}$ does not have any algebraic and non-analytic first integral (case **A)**). The analytic integrability of the vector field $\Phi_*\mathbf{F}$ has been studied in [1].
- $\mathbf{F}_r=y\partial_x+bx^ny\partial_y$, $b\neq 0$, i.e. $\mathbf{t}=(1,n+1)$ and $r=n$. The vector field $\phi_*\mathbf{F}$ has, at most, two irreducible invariant curves at origin of the form $(n+1)y-bx^{n+1}+\cdots$ and $y+\cdots$, whose cofactors to degree n are 0 and bx^n, respectively. Therefore, these curves do not have the same cofactor, by Proposition 4, $\phi_*\mathbf{F}$ does not have any algebraic and non-analytic first integral (case **B)**). As a consequence, if there exists an algebraic first integral then the vector field is analytically integrable.
- $\mathbf{F}_r=(y+dx^{n+1})\partial_x+(n+1)(cx^{2n+1}+dx^ny)\partial_y$, with $c\in\{-1,0,1\}$, $d\neq 0$, i.e. $\mathbf{t}=(1,n+1)$ and $r=n$.
If $c=-1$, there exists an unique irreducible analytic invariant curve at origin starting by y^2+x^{2n+2} and therefore $\phi_*\mathbf{F}$ does not have any algebraic first integral.
If $c=0$, the vector field $\Phi_*\mathbf{F}$ is not analytically integrable since its Hamiltonian part has multiple factors, see [7, Theorem 3.2]. In this case, the invariant curves, of existing, are invariant curves of the form $y+\cdots$. We assume that the vector field has two invariant curves f_1 and f_2 with the same cofactor. This fact arrives to a contradiction since f_1-f_2 is also an invariant curve of $\mathbf{F}_r$ and it does not star with y. Thus, the vector field $\Phi_*\mathbf{F}$ does not have any algebraic first integral.
Otherwise, $c=1$. We distinguish two cases: if $|d|=1$, the curve $y+dx^{n+1}$ is a curve of singular point of $\mathbf{F}_r$. The change of variables $(x,v)=(x,y+dx^{n+1})$, transforms the vector field into **B)**. If $|d|\neq 1$, from [6, Theorem 22], has, at most, two irreducible analytic invariant curves at origin starting with $f:=y-x^{n+1}$ and $g:=y+x^{n+1}$, respectively, whose cofactors start with $(n+1)(d-1)$ and $(n+1)(d+1)$, respectively.
The formal invariant curves of $\phi_*\mathbf{F}$ (if they exist) are $f^{m_1}+\cdots$, $m_1\in\mathbb{N}$, and $g^{m_2}+\cdots$, $m_2\in\mathbb{N}$, whose cofactors are $(n+1)(d-1)m_1+\cdots$ and $(n+1)(d+1)m_2+\cdots$, respectively. So, if $\phi_*\mathbf{F}$ has two irreducible analytic invariant curves at origin with the same cofactor, then m_1 and m_2 hold $(d-1)m_1=(d+1)m_2$, therefore, d is a rational number with $|d|>1$. From Proposition 4 the result follows (case **C)**).

□

So, by above proposition, the algebraic integrable nilpotent vector fields without any analytic first integrals, after a polynomial change of variables, if necessary, are of the form

$$(y+dx^{n+1})\partial_x+(n+1)(x^{2n+1}+dx^ny)\partial_y+\cdots, \tag{2}$$

with n a natural number and d a rational number with $|d|>1$.

The existence of invariant curves of system (2) is provided by Algaba et al. [6].

Proposition 6. *([6, Theorem 19]) Let* $\mathbf{F}$ *be the vector field (2). It holds:*

(i) if $d\in\mathbb{Q}$ *and* $d>1$, $\mathbf{F}$ *has a unique irreducible analytic invariant curve at the origin starting with* $y-x^{n+1}$.

Moreover, it has an irreducible analytic invariant curve at the origin starting with $y+x^{n+1}$ *if* $d\ge 2n+3$.

(ii) if $d\in\mathbb{Q}$ *and* $d<-1$, $\mathbf{F}$ *has a unique irreducible analytic invariant curve at the origin starting with* $y+x^{n+1}$.

Moreover, it has an irreducible analytic invariant curve at the origin starting with $y-x^{n+1}$ *if* $d\le -2n-3$.

We emphasize that if C is a formal or analytic invariant curve at the origin, then for any unit function $u\in\mathbb{C}[[x,y]]$, $u(\mathbf{0})=1$, Cu is also a formal invariant curve since

$$\mathbf{F}(Cu)=C\mathbf{F}(u)+u\mathbf{F}(C)=Cu\frac{\mathbf{F}(u)}{u}+CuK=\left(\frac{\mathbf{F}(u)}{u}+K\right)Cu,$$

with K cofactor of C and $u(\mathbf{0})=1$. As a consequence of [6, Theorem 5], at least one of the formal irreducible invariant curves of (2) starting with $y-x^{n+1}$ (or $y+x^{n+1}$) is analytic.

The following result provides a canonical expression of the invariant curves starting with $y-x^{n+1}$ or $y+x^{n+1}$.

Proposition 7. *If* $C=C_{n+1}+C_{n+2}+\cdots$ *is an invariant curve of (2), with* $C_{n+1}=y-x^{n+1}$ *(or* $C_{n+1}=y+x^{n+1}$*) and* $C_{n+1+j}\in\mathcal{P}_{n+1+j}^{\mathbf{t}}$ *with*

$\mathbf{t}=(1,n+1)$, *then there exists a unit function* u *such that* $Cu=C_{n+1}+c_{n+2}x^{n+2}+c_{n+3}x^{n+3}+\cdots$ *is an invariant curve of (2).*

Proof. We suppose that $C=C_{n+1}+\cdots$ is an invariant curve of $\mathbf{F}$ with $C_{n+1}=y-x^{n+1}$ (the proof is analogous for $C_{n+1}=y+x^{n+1}$).

We consider the sum $\mathcal{P}^{\mathbf{t}}_{n+1+j}=\langle x^{n+1+j}\rangle\bigoplus(y-x^{n+1})\mathcal{P}^{\mathbf{t}}_{j}$.

We know that $Cu=(y-x^{n+1}+C_{n+2}+\cdots)(1+u_1+u_2+\cdots)$ is an invariant curve of $\mathbf{F}$ for any unit function u.

The quasi-homogeneous term of degree $n+2$ of Cu is $(Cu)_{n+2}=(y-x^{n+1})u_1+C_{n+2}$. By writing $C_{n+2}=\lambda_{n+2}+(y-x^{n+1})\nu_1$ with $\lambda_{n+2}\in\langle x^{n+2}\rangle$ and by choosing $u_1=-\nu_1$, we have that $(Cu)_{n+2}=\lambda_{n+2}=c_{n+2}x^{n+2}$. So we choose, step to step, the $u_1,u_2,\ldots,u_{j-1}$. The term of degree $n+1+j$ of Cu is

$$(Cu)_{n+1+j}=(y-x^{n+1})u_j+C_{n+2}u_{j-1}+\cdots+C_{n+j}u_1+C_{n+1+j},$$

and we choose u_j such that $-(y-x^{n+1})u_j$ is the projection of $C_{n+2}u_{j-1}+\cdots+C_{n+j}u_1+C_{n+1+j}$ on $(y-x^{n+1})\mathcal{P}^{\mathbf{t}}_{j}$ and thus $(Cu)_{n+1+j}\in\langle x^{n+1+j}\rangle$. □

3. Normal form of quasi-homogeneous expansions under orbital equivalence

We include this section trying to do the paper self-contained. In the first part of this section, we provide a normal form of an analytic planar vector field using quasi-homogeneous expansion, Theorem 8. Its proof we can find it in [3].

We consider a general vector field $\mathbf{F}=\mathbf{F}_r+\cdots$, where the lowest degree quasi-homogeneous term $\mathbf{F}_r$ verifies that $h:=\mathbf{D}_0\wedge\mathbf{F}_r\in\mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, conservative part of $\mathbf{F}_r$, is non-zero. The key in the problem of obtaining a normal form of the vector field $\mathbf{F}$ is to analyze the effect that a near-identity transformation $\mathbf{x}=\mathbf{y}+\mathbf{P}_j(\mathbf{y})$ and a reparametrization of the time by $\frac{\mathrm{dt}}{\mathrm{dT}}=1+\tau_j(\mathbf{x})$ has on the system $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$, where $\mathbf{P}_j\in\mathcal{Q}^{\mathbf{t}}_j$ and $\tau_j\in\mathcal{P}^{\mathbf{t}}_j$ with $j\geq 1$.

The quasi-homogeneous terms of the transformed system $\mathbf{y}'=\frac{\mathrm{d}\mathbf{y}}{\mathrm{dT}}=\mathbf{G}(\mathbf{y})$ agree with the original ones up to degree $r+j-1$ and for the degree $r+j$ it holds

$$\begin{aligned}\mathbf{G}_{r+j}&=\mathbf{F}_{r+j}-(D\mathbf{P}_j\mathbf{F}_r-D\mathbf{F}_r\mathbf{P}_j)+\tau_j\mathbf{F}_r=\mathbf{F}_{r+j}-[\mathbf{P}_j,\mathbf{F}_r]+\tau_j\mathbf{F}_r\\&=\mathbf{F}_{r+j}-\mathcal{L}_{r+j}(\mathbf{P}_j,\tau_j),\end{aligned}$$

where we have introduced the *homological operator under orbital equivalence*

$$\begin{aligned}\mathcal{L}_{r+j}:\ &\mathcal{Q}^{\mathbf{t}}_j\times\mathcal{P}^{\mathbf{t}}_j\longrightarrow\mathcal{Q}^{\mathbf{t}}_{r+j}\\&(\mathbf{P}_j,\tau_j)\to\mathcal{L}_{r+j}(\mathbf{P}_j,\tau_j)=[\mathbf{P}_j,\mathbf{F}_r]-\tau_j\,\mathbf{F}_r,\end{aligned}$$

and $[\mathbf{P}_j,\mathbf{F}_r]:=D\mathbf{P}_j\mathbf{F}_r-D\mathbf{F}_r\mathbf{P}_j$ is the Lie bracket of $\mathbf{P}_j$ and $\mathbf{F}_r$. Following the ideas of the conventional normal form theory, by means of a sequence of time-reparameterizations and near-identity transformations, the system $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ can be formally reduced to normal form under orbital equivalence, i.e. the system can be transformed into

$$\mathbf{y}'=\frac{\mathrm{d}\mathbf{y}}{\mathrm{dT}}=\mathbf{G}(\mathbf{y})=\mathbf{G}_r(\mathbf{y})+\mathbf{G}_{r+1}(\mathbf{y})+\cdots,$$

with $\mathbf{G}_r\not\equiv\mathbf{0}$ and $\mathbf{G}_{r+j}\in\mathrm{Cor}(\mathcal{L}_{r+j})\subseteq\mathcal{Q}^{\mathbf{t}}_{r+j}$ where $\mathrm{Cor}(\mathcal{L}_{r+j})$ is any complementary subspace to the range of the homological operator $\mathcal{L}_{r+j}$. We note that such space is not unique, in general.

Here, we give an expression of $\mathcal{L}_{r+j}$ we will use to obtain an expression of $\mathrm{Cor}(\mathcal{L}_{r+j})$, or equivalently, we provide a normal form of the vector field.

A main role plays the linear operator ℓ_{r+j}, Lie-derivative operator of $\mathbf{F}_r$, i.e.

$$\begin{aligned}\ell_{r+j}:\ &\mathcal{P}^{\mathbf{t}}_j\longrightarrow\mathcal{P}^{\mathbf{t}}_{r+j}\\&p_j\to\mathbf{F}_r(p_j).\end{aligned}\tag{3}$$

The operator $\mathcal{L}_{r+j}$ restricted to $\mathcal{Q}^{\mathbf{t}}_j\times\mathrm{Cor}(\ell_j)$ and the operator $\mathcal{L}_{r+j}$ have the same range. In this way, by keeping the notation, we consider to $\mathcal{L}_{r+j}$ restricted to $\mathcal{Q}^{\mathbf{t}}_j\times\mathrm{Cor}(\ell_j)$ as the homological operator under orbital equivalence.

The following subspaces of $\mathcal{Q}^{\mathbf{t}}_j$ will be useful in the study of the homological operator under orbital equivalence. Notice that $\mathcal{P}^{\mathbf{t}}_{j-r}=\{0\}$ if $j<r$ and $\mathcal{P}^{\mathbf{t}}_0=\mathrm{span}\{1\}$.

$$\begin{array}{lll}\mathcal{C}^{\mathbf{t}}_j&:=&\left\{\mathbf{X}_{\delta_{j+|\mathbf{t}|}}:\delta_{j+|\mathbf{t}|}\in\Delta_{j+|\mathbf{t}|},\text{ a complementary subspace to }h\mathcal{P}^{\mathbf{t}}_{j-r}\right\},\\ \mathcal{D}^{\mathbf{t}}_j&:=&\left\{\eta_j\mathbf{D}_0:\eta_j\in\mathcal{P}^{\mathbf{t}}_j\right\},\\ \mathcal{F}^{\mathbf{t}}_j&:=&\left\{\lambda_{j-r}\mathbf{F}_r:\lambda_{j-r}\in\mathcal{P}^{\mathbf{t}}_{j-r}\right\}.\end{array}\tag{4}$$

If $h\neq 0$, it has that $\mathcal{Q}^{\mathbf{t}}_j=\mathcal{C}^{\mathbf{t}}_j\bigoplus\mathcal{D}^{\mathbf{t}}_j\bigoplus\mathcal{F}^{\mathbf{t}}_j$, for all $j\in\mathbb{N}$.

This decomposition of the quasi-homogeneous vector fields allows to define the corresponding projectors Π^{c}, Π^{d} and Π^{f}. Also, we can identify $\mathcal{Q}^{\mathbf{t}}_j=\mathcal{C}^{\mathbf{t}}_j\bigoplus\mathcal{D}^{\mathbf{t}}_j\bigoplus\mathcal{F}^{\mathbf{t}}_j\equiv\mathcal{C}^{\mathbf{t}}_j\times\mathcal{D}^{\mathbf{t}}_j\times\mathcal{F}^{\mathbf{t}}_j$.

So, if $\mathbf{P}_j\in\mathcal{Q}^{\mathbf{t}}_j$, there exist $\delta\in\Delta_{j+|\mathbf{t}|}$, $\eta\in\mathcal{P}^{\mathbf{t}}_j$ and $\lambda\in\mathcal{P}^{\mathbf{t}}_{j-r}$ such that

$$\mathbf{P}_j=\mathbf{X}_\delta+\eta\mathbf{D}_0+\lambda\mathbf{F}_r,$$

we denote $\Pi^{\mathrm{c}}\left(\mathbf{P}_j\right)=\mathbf{P}^{\mathrm{c}}_j:=\mathrm{Proj}_{\mathcal{C}^{\mathbf{t}}_j}\left(\mathbf{P}_j\right)=\mathbf{X}_\delta$, $\Pi^{\mathrm{d}}\left(\mathbf{P}_j\right)=\mathbf{P}^{\mathrm{d}}_j:=\mathrm{Proj}_{\mathcal{D}^{\mathbf{t}}_j}\left(\mathbf{P}_j\right)=\eta\mathbf{D}_0$ and $\Pi^{\mathrm{f}}\left(\mathbf{P}_j\right)=\mathbf{P}^{\mathrm{f}}_j:=\mathrm{Proj}_{\mathcal{F}^{\mathbf{t}}_j}\left(\mathbf{P}_j\right)=\lambda\mathbf{F}_r$.

With this notation the homological operator under equivalence can be written as

$$\mathcal{L}_{r+j}:\mathcal{C}^{\mathbf{t}}_j\times\mathcal{D}^{\mathbf{t}}_j\times\mathcal{F}^{\mathbf{t}}_j\times\mathrm{Cor}\left(\ell_j\right)\longrightarrow\mathcal{C}^{\mathbf{t}}_{r+j}\times\mathcal{D}^{\mathbf{t}}_{r+j}\times\mathcal{F}^{\mathbf{t}}_{r+j},$$

such as

$$\mathcal{L}_{r+j}\left(\mathbf{P}^{\mathrm{c}}_j,\mathbf{P}^{\mathrm{d}}_j,\mathbf{P}^{\mathrm{f}}_j,\nu_j\right)=$$

$$\left(\mathbf{X}_{\left(\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}(\delta)\right)},-[\mathbf{F}_r,\mathbf{X}_\delta]^{\mathrm{d}}+\ell_{r+j}\left(\eta_j\right)\mathbf{D}_0,-[\mathbf{F}_r,\mathbf{X}_\delta]^{\mathrm{f}}-n\eta_j\mathbf{F}_r+\ell_j\left(\lambda_{j-r}\right)\mathbf{F}_r-\nu_j\mathbf{F}_r\right).$$

where $\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}:\Delta_{j+|\mathbf{t}|}\longrightarrow\Delta_{r+j+|\mathbf{t}|}$ is the linear operator defined by

$$\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}(\delta)=\mathrm{Proj}_{\Delta_{r+j+|\mathbf{t}|}}(\mathbf{F}_r-\frac{\mathrm{div}(\mathbf{F}_r)}{r+j+|\mathbf{t}|}\mathbf{D}_0)(\delta).\tag{5}$$

Under certain conditions on $\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}$, we can give an orbitally equivalent normal form of an analytic vector field.

Theorem 8. *Given* $\mathbf{F}=\mathbf{F}_r+\sum_{j\geq 1}\mathbf{F}_{r+j}$ *with* $\mathbf{F}_{r+j}\in\mathcal{Q}^{\mathbf{t}}_{r+j}$. *If* $\mathrm{Ker}\left(\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}\right)=\{0\}$ *for all* $j\in\mathbb{N}$ *then* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G}=\mathbf{F}_r+\sum_{j>0}\mathbf{G}_{r+j},\ \text{with }\mathbf{G}_{r+j}=\mathbf{X}_{\delta_{r+j+|\mathbf{t}|}}+\eta_{r+j}\mathbf{D}_0\in\mathcal{Q}^{\mathbf{t}}_{r+j},$$

where $\delta_{r+j+|\mathbf{t}|}\in\mathrm{Cor}\left(\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}\right)$ *and* $\eta_{r+j}\in\mathrm{Cor}\left(\ell_{r+j}\right)$, *the operators* $\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}$ *and* ℓ_{r+j} *are defined in (5) and (3), respectively.*

Proof. By using the decomposition in direct summa of initial and final spaces of the operator $\mathcal{L}_{r+j}$ and writting $\mathcal{F}^{\mathbf{t}}_{r+j}=\mathrm{Range}\left(\ell_j\right)\mathbf{F}_r\bigoplus\mathrm{Cor}\left(\ell_j\right)\mathbf{F}_r$, we obtain the following matrix diagonal by blocks of the operator $\mathcal{L}_{r+j}$:

$$\begin{array}{ccccc}\mathcal{C}^{\mathbf{t}}_j&\mathcal{D}^{\mathbf{t}}_j&\mathcal{F}^{\mathbf{t}}_j&\mathrm{Cor}\left(\ell_j\right)&\\ \mathbf{X}_{\left(\ell^{\mathrm{c}}_{r+j+|\mathbf{t}|}(\delta)\right)}&0&0&0&\mathcal{C}^{\mathbf{t}}_{r+j}\\ -[\mathbf{F}_r,\mathbf{X}_\delta]^{\mathrm{d}}&\ell_{r+j}\left(\eta_j\right)\mathbf{D}_0&0&0&\mathcal{D}^{\mathbf{t}}_{r+j}\\ -[\mathbf{F}_r,\mathbf{X}_\delta]^{\mathrm{f}}&-r\eta_j\mathbf{F}_r&\ell_j\left(\lambda_{j-r}\right)\mathbf{F}_r&0&\mathrm{Range}\left(\ell_j\right)\mathbf{F}_r\\ &&0&-\nu_j\mathbf{F}_r&\mathrm{Cor}\left(\ell_j\right)\mathbf{F}_r\end{array}$$

From hypothesis $\mathrm{Ker}\left(\ell^c_{r+j+|\mathbf{t}|}\right) = \{0\}$, we can deduce that the upper left block of the matrix has maximum range. Thus, we can choose the following subspace complementary to range of $\mathcal{L}_{r+j}$,

$$\mathrm{Cor}\left(\mathcal{L}_{r+j}\right) = \mathbf{X}_{\mathrm{Cor}\left(\ell^c_{r+j+|\mathbf{t}|}\right)} \times \mathrm{Cor}\left(\ell_{r+j}\right)\mathbf{D}_0 \times \{0\} \times \{0\}$$

and, therefore, the result follows. □

Remark 1. Suppose that there is a j_0 such that $\mathrm{Ker}\left(\ell^c_{r+j_0+|\mathbf{t}|}\right)$ is not a trivial set. We consider the linear operator

$$L_{r+j_0} : \mathbf{X}_{\mathrm{Ker}\left(\ell^c_{r+j_0+|\mathbf{t}|}\right)} \times \mathcal{D}^{\mathbf{t}}_{j_0} \longrightarrow \mathcal{D}^{\mathbf{t}}_{r+j_0},$$

defined by $L(\delta, \eta_{j_0}) := \mathrm{Proj}_{\mathcal{D}^{\mathbf{t}}_{r+j_0}}\left(-[\mathbf{F}_r, \mathbf{X}_\delta]^{\mathrm{d}} + \ell_{r+j_0}\left(\eta_{j_0}\right)\mathbf{D}_0\right)$, *i.e.* the second component of the operator $\mathcal{L}_{r+j_0}$.

If L_{r+j_0} has full range, the quasi-homogeneous term of degree $r+j_0$ of the normal form is $\mathbf{G}_{r+j_0} = \mathbf{X}_\delta$ with $\delta \in \mathrm{Cor}\left(\ell^c_{r+j_0+|\mathbf{t}|}\right)$.

3.1. Normal form under orbital equivalence of nilpotent systems whose first quasi-homogmeous component are algebraically but non-analytically integrable

We give an orbitally equivalent normal form of (2) with d rational number and $|d| > 1$, nilpotent vector field whose lowest-degree quasi-homogeneous term is rationally and non-polynomially integrable. We distinguish two cases:

Theorem 9. *Assume d rational, $|d| > 1$ and $|d| \neq 1 + \frac{2(n+1)}{k}$, for all k natural number. The vector field (2) is orbitally equivalent to $\mathbf{G} = \mathbf{F}_n + \sum_{j>0} \mathbf{G}_{n+j}$, with $\mathbf{G}_{n+j} = \alpha_{n+j} x^{n+j}(x\partial_x + (n+1)y\partial_y)$, where j is not a multiple of $n+1$.*

Proof. The linear operator ℓ_{n+j} given by (3) is the operator $\hat{\ell}_{n+j}$ for $C = 0$ given by (10). Thus, applying Lemma 12 for $C = 0$, we have that a complementary subspace to the range of the linear operator ℓ_{n+j}, with $j = i_0 + l_0(n+1)$, $0 \le i_0 < n+1$, is $\mathrm{Cor}(\ell_{n+j}) = \{0\}$ if $i_0 = 0$, and $\mathrm{Cor}(\ell_{n+j}) = \langle x^{n+j}\rangle$ if $i_0 > 0$.

On the other hand, the linear operator $\ell^c_{j+2(n+1)}$ given by (5) is the operator $\hat{\ell}^c_{j+2(n+1)}$ for $C = -\frac{2(n+1)}{j+2(n+1)} d$ given by (11). By hypothesis, we have that $C + d = \frac{jd}{j+2(n+1)} \neq \pm 1$, for all j, thus from Lemma 14 for $C = -\frac{2(n+1)}{j+2(n+1)} d$, we have that $\mathrm{Cor}(\ell^c_{j+2(n+1)}) = \{0\}$ and $\mathrm{Ker}(\ell^c_{j+2(n+1)}) = \{0\}$. So, from Theorem 8, the result follows. □

Theorem 10. *Assume $|d| = 1 + \frac{2(n+1)}{k_0}$, for some k_0 natural number. The vector field (2) is orbitally equivalent to $\mathbf{G} = \mathbf{F}_n + \sum_{j>0} \mathbf{G}_{n+j}$, with $\mathbf{G}_{n+j} = \alpha_{n+j} x^{n+j}(x\partial_x + (n+1)y\partial_y)$ where j is not a multiple of $n+1$, $j \neq k_0$, and $\mathbf{G}_{n+k_0} = (k_0 + 2n + 2)\beta_{n+k_0} x^{k_0+2n+1}\partial_y$.*

Proof. Reasoning as before, we have that a complementary subspace to the Range of the linear operator ℓ_{j+n} given by (3), with $j = i_0 + l_0(n+1)$, $0 \le i_0 < n+1$, is $\mathrm{Cor}(\ell_{n+j}) = \{0\}$ if $i_0 = 0$, and $\mathrm{Cor}(\ell_{n+j}) = \langle x^{n+j}\rangle$ if $i_0 > 0$.

On the other hand, if $d > 0$ (for $d < 0$ the proof is analogous), the linear operator $\ell^c_{k_0+2(n+1)}$ given by (5) is the operator $\hat{\ell}^c_{k_0+2(n+1)}$ for $C = -\frac{2(n+1)}{k_0+2(n+1)} d$ given by (11). By hypothesis, we have that $C + d = \frac{k_0 d}{k_0+2(n+1)} = 1$, thus from Lemma 14 for $C = -\frac{2(n+1)}{k_0+2(n+1)} d$, we have that $\mathrm{Cor}(\ell^c_{k_0+2(n+1)}) = \langle x^{k_0+2(n+1)}\rangle$ and $\mathrm{Ker}(\ell^c_{k_0+2(n+1)}) = \langle x^{k_0+1}(y - x^{n+1})\rangle$.

By Remark 1, it is enough to prove that L_{n+k_0} has full range. Indeed,

$$-\left[\mathbf{F}_n, \mathbf{X}_{x^{k_0+1}(y-x^{n+1})}\right] = \left(Dx^{k_0+n} + Ex^{k_0-1}(y + x^{n+1})\right)\mathbf{D}_0 + Fx_0^k \mathbf{F}_n$$

$$D = \frac{2[(n+2)(n+1) + 2(n+2)k_0 + 2k_0^2}{k_0}$$
$$E = \frac{k_0(k_0+1)}{n+1}$$
$$F = -\frac{(k_0+1)(k_0+2n+2)}{n+1}.$$

So, from Lemma 11 for $C = 0$, the matrix of the linear operator L_{n+k_0}, with $k_0 = i_0 + (n+1)l_0$, $0 < i_0 < n$, becomes

$$\begin{pmatrix} D & d_0 & 0 & 0 & \cdots & \cdots & 0 \\ E & e_0 & d_1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & e_1 & d_2 & & & 0 \\ \vdots & \vdots & & \ddots & \ddots & & 0 \\ 0 & 0 & & & \ddots & d_{l_0-1} & 0 \\ 0 & 0 & & & & e_{l_0-1} & d_{l_0} \\ 0 & 0 & & & & & e_{l_0} \end{pmatrix}_{(l_0+2)\times(l_0+2)}$$

whose determinant is $e_1 e_2 \cdots e_{l_0}(De_0 - Ed_0)$, with $e_i = (n+1)(l_0 - i) + i_0 \neq 0$, $i = 1, \ldots, l_0$ and $De_0 - Ed_0 = \frac{2(k_0^3 + (5+4n)k_0^2 + 3(n+2)(n+1)k_0 + (n+2)(n+1)^2)}{k_0} > 0$, thus L_{n+k_0} has full range and $\mathbf{G}_{n+k_0} = \beta_{n+k_0}\mathbf{X}_{x^{k_0+2(n+1)}}$. □

4. Proofs of theorems 1, 2 and 3

Proof of Theorem 1. The sufficient condition is direct since the vector field $\mathbf{F}_n$ has a rational first integral and the near-identity change of variables and re-parameterization of the time-variable transform its integral first into a non-analytic function, it which is quotient of formal series.

Let prove the necessary condition. From Proposition 5, after a polynomial change of variables (if necessary) is transformed into (2).

From now on, without loss of generality, we will consider the vector field (2) with d a rational number and $d > 1$ since otherwise, $d < -1$, the change $(x, y, t) \to (-x, y, -t)$ for n even, or $(x, y, t) \to (x, -y, -t)$ for n odd, transforms the system (2) into

$$(y + \bar{d}x^{n+1})\partial_x + (n+1)(x^{2n+1} + \bar{d}x^n y)\partial_y + \cdots,$$

with $\bar{d} = -d > 1$. Also, the invariant curves $y - x^{n+1}$ and $y + x^{n+1}$ are transformed into $y + x^{n+1}$ and $y - x^{n+1}$, respectively.

We write the positive rational number $\frac{d+1}{d-1} = 1 + \frac{2}{d-1} := \frac{p}{q} > 1$ with $\gcd(p, q) = 1$. Thus, $d = \frac{p+q}{p-q}$ and $\mathbf{F}_n$ has the irreducible rational first integral $I_{\mathrm{rat}} = \frac{f^p}{g^q}$, where $f = y - x^{n+1}$ and $g = y + x^{n+1}$ are the irreducible invariant curves of $\mathbf{F}_n$.

From [6, Theorem 19], the vector field $\mathbf{F}$ has an unique analytic irreducible invariant curve C starting with f. Therefore, C^p is the unique analytic invariant curve starting with f^p. Moreover, the formal invariant curves starting with f^p are $C^p u$ with u unit function.

Let I an algebraic and non-analytic first integral of $\mathbf{F}$. By Proposition 4, we know that there exists a first integral of the form $I = W_1/W_2$ with $W_1 = f_{m_0} + \cdots$ and $W_2 = g_{n_0} + \cdots$, with W_1 and W_2 two formal invariant curves of $\mathbf{F}$. Therefore $f_{m_0} = f^p$ and $g_{m_0} = g^q$.

If I is a first integral of $\mathbf{F}$ in Ω_{W_2}, it holds that $\mathbf{F}(\frac{W_1}{W_2}) \equiv 0$ in Ω_{W_2} and therefore $0 \equiv W_2\mathbf{F}(W_1) - W_1\mathbf{F}(W_2)$. In particular, its lowest-degree quasi-homogeneous term is also zero. So,

$$0 \equiv g_{n_0}\mathbf{F}_r(f_{m_0}) - f_{m_0}\mathbf{F}_r(g_{n_0}) = g_{n_0}^2 \mathbf{F}_r\left(\frac{f_{m_0}}{g_{n_0}}\right) \text{ in } \Omega_g \cap \Omega_{g_{n_0}}.$$

By continuity, it extends to $\Omega_{g_{n_0}}$.

Hence $\frac{f_{m_0}}{g_{n_0}} = I_{\mathrm{rat}}$ and $I = \frac{C^p u}{g^q + \cdots}$ with C analytic invariant curve starting with $f, g^q + \cdots$ formal invariant curve and u unit function.

We express the positive rational number $\frac{d-1}{n+1}=\frac{a}{b}$ with $\gcd(a,b)=1$. According the value of d, we distinguish two cases:

4.1. Case $\frac{a}{b}\neq\frac{2}{j}$, for all j natural number

Under these conditions, by Theorem 9, the vector field (2) is orbitally equivalent to $\mathbf{G}=\mathbf{F}_n+\sum_{j>0}\mathbf{G}_{n+j}$, where $\mathbf{G}_{n+j}=\mu_{n+j}\mathbf{D}_0$ with $\mu_{n+j}=\alpha_{n+j}x^{n+j}$, with j not multiple of $n+1$.

If $\mathbf{F}$ has an algebraic and non-analytic first integral then $\mathbf{G}$ also has an algebraic and non-analytic first integral $\hat{I}$. In this case, f is an invariant curve of $\mathbf{G}$, thus f^p is the unique invariant curve starting with f^p. Therefore, $\hat{I}=\frac{f^p}{\bar{g}}$ with $\bar{g}=g^q+\sum_{j>q(n+1)}\bar{g}_j$.

On the other hand, $\mathbf{G}(f^p)=p(n+1)\big((d-1)x^n+\sum_{j>0}\mu_{n+j}\big)f^p$. By Proposition 4, f^p and $\bar{g}$ have the same cofactor, i.e.

$$\mathbf{G}(\bar{g})=p(n+1)\left((d-1)x^n+\sum_{j>0}\mu_{n+j}\right)\bar{g}. \tag{6}$$

For proving the necessary condition enough to check that all μ_{n+j} are zero. We imagine by reduction to the absurd one, that there exists a $j_0=\min\{j\in\mathbb{N},\ \ \mu_{n+j}\neq 0\}$, *i.e.*$\mu_{n+j_0}=\alpha_{n+j_0}x^{n+j_0}$, with $\alpha_{n+j_0}\neq 0$ and j_0 is not multiple of $n+1$.

Denoting $m=j_0+q(n+1)$, the equation (6) to degree $j_0+q(n+1)+n$ is

$$0=\mathbf{F}_n(\bar{g}_m)+\left(\mu_{n+j_0}\mathbf{D}_0\right)(g^q)-p(n+1)\left(\bar{g}_m(d-1)x^n+g^q\mu_{n+j_0}\right).$$

Using the Euler Theorem for quasi-homogeneous polynomials, $\mathbf{D}_0(p_j)=jp_j$ with $p_j\in\mathcal{P}_j^{\mathbf{t}}$, and re-sorting the terms, the equation (6) to order $j_0+q(n+1)+n$ is

$$0=\left(\mathbf{F}_n-\frac{p(n+1)(d-1)}{m}x^n\mathbf{D}_0\right)(\bar{g}_m)-(n+1)(p-q)\mu_{n+j_0}g^q.$$

i.e. $\bar{g}_m$ satisfies the functional equation

$$\ell_{n+m}^{(1)}(\bar{g}_m)=(n+1)(p-q)\alpha_{n+j_0}x^{n+j_0}g^q, \tag{7}$$

where $\ell_{n+m}^{(1)}$ is the linear operator given in Lemma 13. If (a is odd) or (a even and $j_0\neq b(2n+1)$), from Lemma 13, the right-side of (7) belongs to a complementary subspace of the range of $\hat{\ell}_{n+m}$ therefore we arrive to contradiction, *i.e.* $\alpha_{n+j_0}=0$. Otherwise, a even and $j_0=b(2n+1)$, thus $\mathbf{F}$ is orbitally equivalent to $\mathbf{G}=\mathbf{F}_n+\alpha_{n+j_0}x^{n+j_0}\mathbf{D}_0+\cdots$. The vector field $\mathbf{G}$ has the invariant curves $(y-x^{n+1})^p$ and $(y+x^{n+1})^q+\cdots$. We note that the curve $(y-x^{n+1})^p$ is the unique invariant curve starting with $(y-x^{n+1})^p$ and its cofactor is $(n+1)px^n((d-1)+\alpha_{n+j_0}x^{j_0})$. The curve $(y+x^{n+1})^q+\cdots$ could be not unique, *i.e.* could exist other invariant curve of the form $(y+x^{n+1})^q+\cdots$ and its cofactor would be $(n+1)qx^n((d+1)+\alpha_{n+j_0}x^{j_0})+\cdots$. Therefore, if both factors were equal, then the quasi-homogeneous terms of degree $n+j_0$ would be equal, thus $\alpha_{n+j_0}=0$.

4.2. Case $\frac{a}{b}=\frac{2}{k_0}$

For proving the necessary condition enough to check that all $\mathbf{G}_{n+j}$ are zero. We assume by reduction to the absurd one that there exists a $j_0=\min\{j\in\mathbb{N},\ \mathbf{G}_{n+j}\neq 0\}$.

We distinguish three cases:

- If $j_0<k_0$, from Theorem 10, the first term $\mathbf{G}_{n+j}$ non-zero with $j>0$ is $\mathbf{G}_{n+j_0}:=\alpha_{n+j_0}x^{n+j_0}(x\partial_x+(n+1)y\partial_y)$, with $\alpha_{n+j_0}\neq 0$.

If k_0 is even, then $a=1$ odd. So, by reasoning as before, by Lemma 13, a complementary subspace to the range of the operator linear $\ell_{n+j_0+(n+1)q}^{(1)}$ is $\langle x^{j_0+n}g^q\rangle$, we arrive to contradiction.

If k_0 is odd ($b=k_0$), $a=2$ even and $j_0\neq k_0(2n+1)>k_0$. Thus, by reasoning as before, $\alpha_{n+j_0}=0$.

- We suppose that $j_0=k_0$, the first term $\mathbf{G}_{n+j}$ non-zero with $j>0$ is $\mathbf{G}_{n+k_0}:=\beta_{n+k_0}X_{x^{k_0+2n+2}}:=(k_0+2n+2)\beta_{n+k_0}x^{k_0+2n+1}\partial_y$, with $\beta_{n+k_0}\neq 0$.

Let $C=f+\cdots$ the unique analytic invariant curve starting with f. From Proposition 7, there exists a formal invariant curve $\bar{C}=f+c_{n+2}x^{n+2}+c_{n+3}x^{n+3}+\cdots$ and its cofactor is $K:=K_n+K_{n+1}+\cdots$ with $K_n=(n+1)(d-1)x^n$. We now compute the first coefficient c_{j+n+1} non-zero.

It is easy to check that $c_{j+n+1}=0,\ \ 1<j<k_0$.

Equation $\mathbf{G}(\bar{C})-K\bar{C}=0$ to degree k_0+2n+1 is

$$\mathbf{F}_n(\bar{C}_{k_0+n+1})+\mathbf{G}_{k_0+n}(\bar{C}_{n+1})-\bar{C}_{n+1}K_{k_0+n}-\bar{C}_{k_0+n+1}K_n=0.$$

That is,

$$(y+dx^{n+1})(k_0+n+1)c_{k_0+n+1}x^{k_0+n}+(k_0+2n+2)\beta_{n+k_0}x^{k_0+2n+1} -f(K_{n+k_0}-(n+1)(d-1)c_{k_0+n+1}x^{k_0+n+1}.$$

By writing $y+dx^{n+1}=f+(d+1)x^{n+1},\ \ (d+1)k_0=2(k_0+n+1)$ and sorting the coefficients, we get

$$[2(k_0+2n+2)c_{k_0+n+1}+(k_0+2n+2)\beta_{n+k_0}]x^{k_0+2n+1} +[(k_0+n+1)c_{k_0+n+1}x^{k_0+n}-K_{n+k_0}]f=0.$$

Therefore, $c_{k_0+n+1}=-\frac{1}{2}\beta_{n+k_0}$ and $K_{n+k_0}=-\frac{1}{2}(k_0+n+1)\beta_{n+k_0}x^{n+k_0}$.

So, the cofactor of $\bar{C}^p$ is $p(n+1)(d-1)x^n-\frac{1}{2}\beta_{n+k_0}(k_0+n+1)px^{k_0+n}$.

By Proposition 4, the vector field has an algebraic first integral if $\bar{C}^p$ and $\bar{g}=g^q+\sum_{j>q(n+1)}\bar{g}_j$ have the same cofactor, i.e.

$$\mathbf{G}(\bar{g})=\left(p(n+1)(d-1)x^n-\frac{1}{2}\beta_{n+k_0}(k_0+n+1)px^{k_0+n}+\cdots\right)\bar{g}. \tag{8}$$

The above equation (8) to degree $m+n$ with $m=k_0+q(n+1)$ is

$$\begin{aligned}0&=\mathbf{F}_n(\bar{g}_m)+\mathbf{G}_{k_0+n}(g^q)-p(n+1)(d-1)x^n\bar{g}_m+\frac{1}{2}\beta_{n+k_0}(k_0+n+1)px^{k_0+n}g^q\\&=\mathbf{F}_n(\bar{g}_m)+q(k_0+2n+2)\beta_{n+k_0}x^{k_0+2n+1}g^{q-1}-p(n+1)(d-1)x^n\bar{g}_m\\&\quad+\frac{1}{2}\beta_{n+k_0}p(k_0+n+1)x^{k_0+n}g^q.\end{aligned}$$

Using the Euler Theorem for quasi-homogeneous polynomials and re-sorting the terms, we have that $\bar{g}_m$ satisfies the functional equation

$$\ell_{n+m}^{(1)}(\bar{g}_m)=-q(k_0+2n+2)\beta_{n+k_0}x^{k_0+2n+1}g^{q-1} -\frac{1}{2}p(k_0+n+1)\beta_{n+k_0}x^{k_0+n}g^q, \tag{9}$$

If k_0 is even, then $a=1$ odd. So, by Lemma 13, a complementary subspace to the range of the operator linear $\ell_{n+k_0+(n+1)q}^{(1)}$ is $\langle x^{k_0+n}g^q\rangle$. Moreover, the right-hand of (9) belongs to $\mathrm{Cor}(\ell_{n+m}^{(1)})$. Indeed, following Lemma 12, it is enough to prove that $A:=q(k_0+2n+2)\beta_{n+k_0}e_q+\frac{1}{2}p(k_0+n+1)\beta_{n+k_0}d_q\neq 0$. In this case, $p=k_0+n+1,\ \ q=n+1,d_q=2(n+1)(-n+q),\ \ e_q=k_0+(n+1)(n+1-q)$. Substituting, we get $A=-(n+1)^2\beta_{n+k_0}$. Thus, if $\beta_{n+k_0}\neq 0$, we have that the right-hand of (9) belongs to $\mathrm{Cor}(\ell_{n+m}^{(1)})$ and we arrive to contradiction. Therefore, $\beta_{n+k_0}=0$.

If k_0 is odd ($b=k_0$), $a=2$ even and $k_0\neq k_0(2n+1)$. Thus, from Lemma 12 and reasoning as before, $\beta_{n+j_0}=0$.

- if $j_0>k_0$, from Theorem 10, an orbitally equivalent normal form of $\mathbf{F}$ is $\mathbf{G}=\mathbf{F}_n+\sum_{j>0}\mathbf{G}_{n+j}$, where $\mathbf{G}_{n+j}=\alpha_{n+j}x^{n+j}(x\partial_x+(n+1)y\partial_y)$, with j is not multiple of $n+1$. By reasoning as the case $j_0<k_0$, we prove that all the $\mathbf{G}_{n+j}\equiv 0$.

Proof of Theorem 2. We see the necessary condition. If the vector field $\mathbf{F}$ is analytically integrable then there exists Φ, a polynomial change of variables, such that the lowest-degree quasi-homogeneous term $\Phi_*\mathbf{F}:=\mathbf{F}_r$ is polynomially integrable. We distinguish three cases: for $\mathbf{F}_r$ Hamiltonian vector field, we apply [1, Theorem 3.19]; for $\mathbf{F}_r$ non-Hamiltonian and irreducible, we apply [3, Theorem 1.2]; and for $\mathbf{F}_r$ non-Hamiltonian and reducible we apply [2, Theorem 1.4]. In all the cases, the analytic integrability leads to $\mathbf{F}$ orbitally equivalent to $\mathbf{F}_r$.

Otherwise, if $\mathbf{F}$ has an algebraic first integral but it is not analytic, we apply Theorem 1.

The sufficient condition is trivial.

Proof of Theorem 3. By Proposition 5, we can assume that, after a polynomial change of variables and a scaling, if necessary, $\mathbf{F} = \mathbf{F}_r + \cdots$ where $\mathbf{F}_r = P_{r+t_1}\partial_x + Q_{r+t_2}\partial_y$ is one of the following vector fields:

A) $y\partial_x + ax^{2n}\partial_y,\ a \neq 0,\ \mathbf{t} = (2, 2n+1),\ r = 2n-1,$

B) $y\partial_x + (n+1)x^n y\partial_y,\ \mathbf{t} = (1, n+1),\ r = n,$

C) $(y + dx^{n+1})\partial_x + (n+1)(x^{2n+1} + dx^n y)\partial_y,$ with d rational number.

For the case **A)**, the vector is a perturbation of the Hamiltonian vector field whose Hamiltonian function is $t_1xP_{r+t_1} - t_2yQ_{r+t_2} = 2ax^{2n+1} - (2n+1)y^2$. From [1], the vector field $\mathbf{F}$ is analytically integrable if, and only if, it is orbitally equivalent to $y\partial_x + ax^{2n}\partial_y$. This vector field has the inverse integrating factor $2ax^{2n+1} - (2n+1)y^2$. Thus, undoing the change of variables, it has a formal inverse integrating factor starting with $2ax^{2n+1} - (2n+1)y^2$.

For the case **B)**, we have $t_1xP_{r+t_1} - t_2yQ_{r+t_2} = y(x^{n+1} - (n+1)y)$. From [2, Theorem 1.5], the vector field $\mathbf{F}$ is analytically integrable if, and only if, it has a formal inverse integrating factor starting with $(n+1)y(x^{n+1} - y)$.

For the case **C)** the polynomial $t_1xP_{r+t_1} - t_2yQ_{r+t_2}$ is $(n+1)(y^2 - x^{2n+2})$. From [5, Theorem 11], these vector fields are orbitally equivalent to $\mathbf{F}_r$ if, and only if, they have a formal inverse integrating factor starting with $(n+1)(y^2 - x^{2n+2})$. So, from Theorem 2, the result follows.

Notice that, by Proposition 5, after a polynomial change of variables, if necessary, we can always assume that if the leader quasi-homogeneous term $\mathbf{F}_r$ is rationally integrable then it satisfies that $t_1xP_{r+t_1} - t_2yQ_{r+t_2}$ has only simple factors on $\mathbb{C}[x, y]$.

5. Auxiliary and technical results

Throughout this section,

$$\mathbf{F}_n = (y + dx^{n+1})\partial_x + (n+1)(x^{2n+1} + dx^n y)\partial_y,$$
$$\mathbf{t} = (1, n+1),\ i.\ e.\ \mathbf{D}_0 = x\partial_x + (n+1)y\partial_y,$$
$$f = y - x^{n+1},\quad g = y + x^{n+1}.$$

The following result is technical and its proof is direct.

Lemma 11. *Let $j = l_0(n+1) + i_0$ with $0 \le i_0 < n+1$. For each $i = 0, \ldots, l_0$, we denote $g_i^{(l_0)} := x^{(l_0-i)(n+1)}g^i$. We consider $\mathbf{G}_n = \mathbf{F}_n + Cx^n\mathbf{D}_0$ with $C \in \mathbb{R}$. Then,*

(i) if $i_0 = 0$,

$$\mathbf{G}_n(g_i^{(l_0)}) = d_i x^n g_i^{(l_0)} + e_i x^n g_{i+1}^{(l_0)},$$

with $d_i = (d + C - 1)j + 2(n+1)i,\ e_i = j - (n+1)i,\ i = 0, \ldots, l_0$.

(ii) if $i_0 > 0$,

$$\mathbf{G}_n(x^{i_0}g_i^{(l_0)}) = d_i x^{i_0-1}g_i^{(l_0+1)} + e_i x^{i_0-1}g_{i+1}^{(l_0+1)},$$

with $d_i = (d + C - 1)j + 2(n+1)i,\ e_i = j - (n+1)i,\ i = 0, \ldots, l_0$.

Fixed $C \in \mathbb{R}$, we consider the linear operator

$$\hat{\ell}_{n+j} : \mathcal{P}_j^{\mathbf{t}} \to \mathcal{P}_{n+j}^{\mathbf{t}} \tag{10}$$
$$p_j \quad \hat{\ell}_{n+j}(p_j) = (\mathbf{F}_n + Cx^n\mathbf{D}_0)(p_j)$$

We have the following result.

Lemma 12. *Consider the linear operator $\hat{\ell}_{n+j}$ with $j = l_0(n+1) + i_0,\ 0 \le i_0 < n+1$. Then, a complementary subspace to $\mathrm{Range}(\hat{\ell}_{n+j})$ is*

(i) if $i_0 = 0$,

$$\mathrm{Cor}(\hat{\ell}_{n+j}) = \begin{cases} \{0\}, & \text{if } d + C - 1 \neq -\frac{2i}{l_0},\ i = 0, \cdots, l_0, \\ \langle x^{n+j} \rangle, & \text{otherwise.} \end{cases}$$

(ii) if $i_0 > 0$, $\mathrm{Cor}(\hat{\ell}_{n+j}) = \langle x^{n+j} \rangle$.

Moreover, if $d_i = (n+1)((d + C - 1)l_0 + 2i) + (d + C - 1)i_0 \neq 0,\ i = 0, \ldots, l-1,$ with $l \le l_0$, then we can choose $\mathrm{Cor}(\hat{\ell}_{n+j}) = \langle x^{i_0-1+(l_0+1-l)(n+1)}g^l \rangle$.

Proof. We distinguish two cases:

Assume $i_0 = 0$, *i.e.* $j = l_0(n+1)$. In this case, $\mathcal{P}_j^{\mathbf{t}} = \langle g_i^{(l_0)},\ \ i = 0, \cdots, l_0 \rangle$ and $\mathcal{P}_{n+j}^{\mathbf{t}} = \langle x^n g_i^{(l_0)},\ \ i = 0, \cdots, l_0 \rangle$. if we choose these basis, from Lemma 11, the matrix of the linear operator $\hat{\ell}_{n+j}$ is the following square matrix and diagonal whose elements d_i are all different

$$\begin{pmatrix} d_0 & 0 & 0 & \cdots & \cdots & 0 \\ e_0 & d_1 & 0 & \cdots & \cdots & 0 \\ 0 & e_1 & d_2 & & & 0 \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & \ddots & d_{l_0-1} & 0 \\ 0 & & & & e_{l_0-1} & d_{l_0} \end{pmatrix}_{(l_0+1)\times(l_0+1)}$$

So, if there exists any d_{i_0} null, the range of the matrix is l_0. Therefore, the dimension of $\mathrm{Cor}(\hat{\ell}_{n+j})$ is one and we can choose $\mathrm{Cor}(\hat{\ell}_{n+j}) = \langle x^{j+n} \rangle$ since the elements $e_0, \cdots, e_{l_0-1}$ are all non-zero. Otherwise, if all d_i are non-zero, the operator has full range, that is $\mathrm{Cor}(\hat{\ell}_{n+j})$ is a trivial set.

Now assume $0 < i_0 < n+1$. In this case, $\mathcal{P}_j^{\mathbf{t}} = \langle x^{i_0}g_i^{(l_0)},\ \ i = 0, \cdots, l_0 \rangle$ and $\mathcal{P}_{n+j}^{\mathbf{t}} = \langle x^{i_0-1+n}g_i^{(l_0+1)},\ \ i = 0, \cdots, l_0+1 \rangle$, that is $\dim\left(\mathcal{P}_j^{\mathbf{t}}\right) = l_0 + 1 < l_0 + 2 = \dim\left(\mathcal{P}_{j+n}^{\mathbf{t}}\right)$. If we choose the basis given in Lemma 11, the matrix of the linear operator $\hat{\ell}_{n+j}$ is the following lower triangular matrix of order $(l_0+2)\times(l_0+1)$,

$$\begin{pmatrix} d_0 & 0 & 0 & \cdots & \cdots & 0 \\ e_0 & d_1 & 0 & \cdots & \cdots & 0 \\ 0 & e_1 & d_2 & & & 0 \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & \ddots & d_{l_0-1} & 0 \\ 0 & & & & e_{l_0-1} & d_{l_0} \\ 0 & & & & & e_{l_0} \end{pmatrix}_{(l_0+2)\times(l_0+1)}$$

thus we can choose $\mathrm{Cor}(\hat{\ell}_{n+j}) = \mathrm{span}\{x^{n+j}\}$ since $e_0, \cdots, e_{l_0}$ are all non-zero.

Furthermore, if $d_i \neq 0$, for all $i = 0, \cdots l-1$, the polynomials $x^{i_0-1}g_i^{(l_0-1)}, i = 0, \cdots, l$, do not belong to $\mathrm{Range}(\hat{\ell}_{n+j})$. Thus, any one of them is a base of $\mathrm{Range}(\hat{\ell}_{n+j})$. $\square$

We use the following result in the proof of Theorem 1.

Lemma 13. *We denote the operator $\hat{\ell}_{n+j_0+q(n+1)}$ for $C = -\frac{p(n+1)(d-1)}{j_0+q(n+1)}$ and j_0 natural number, by $\ell^{(1)}_{n+j_0+q(n+1)}$. If d is a rational number greater than one, we can write $\frac{d-1}{n+1} = \frac{a}{b}$ with $\gcd(a, b) = 1$. Then, if one of the following conditions is satisfied:*

(i) a is odd,

(ii) a is even and $j_0 \neq (2n+1)b$,

then a complementary subspace to $\mathrm{Range}(\ell^{(1)}_{n+j_0+q(n+1)})$ is $\mathrm{Cor}(\ell^{(1)}_{n+j_0+q(n+1)}) = \langle x^{n+j_0+(q-i)(n+1)}g^i \rangle$, with $i \in \{0, \cdots, q\}$.

Proof. Following above proposition, the proof consists on to check that $d_i \neq 0,\ \ i = 0, \cdots, q-1$ with $d_i = (d + C - 1)(j_0 + q(n+1)) + 2(n+1)i = (d-1)[j_0 - (p-q)(n+1)] + 2(n+1)i.$

As $\frac{d+1}{d-1} = \frac{2b+a(n+1)}{a(n+1)}$, it has that $p = 2b + a(n+1)$ and $q = a(n+1)$. So, $d_i = 0$ if, and only if, $a(2b(n+1) - j_0) = 2bi$. Therefore, $j_0 = 2b(n+1) - \frac{2bi}{a} \in \mathbb{N}$, and $2bi = am$, with m natural. So, $a = 2i$ and $j = m$ since $\gcd(a,b) = 1$. Consequently, a is even and in such case $j_0 = 2b(n+1) - b = b(2n+1)$. □

Fixed $C \in \mathbb{R}$, we consider the following linear operator

$$\hat{\ell}^c_{j+2(n+1)} : \Delta_{j+n+2} \to \Delta_{j+2(n+1)} \qquad (11)$$
$$\delta \qquad \hat{\ell}^c_{j+2(n+1)}(\delta) = \mathrm{Proj}_{\Delta_{j+2(n+1)}}((\mathbf{F}_n + Cx^n\mathbf{D}_0)(\delta)),$$

where Δ_{j+n+2} and $\Delta_{j+2(n+1)}$ are complementary subspaces to $fg\mathcal{P}^{\mathbf{t}}_{j-n}$ and $fg\mathcal{P}^{\mathbf{t}}_{j}$, respectively.

We have the following result.

Lemma 14. *Consider the linear operator $\hat{\ell}^c_{j+2(n+1)}$. It has that:*

(i) If $C + d = -1$, $\mathrm{Cor}(\hat{\ell}^c_{j+2(n+1)}) = \langle x^{j+2(n+1)}\rangle$ and $\mathrm{Ker}(\hat{\ell}^c_{j+2(n+1)}) = \langle x^{j+1}g\rangle$.

(ii) If $C + d = 1$, $\mathrm{Cor}(\hat{\ell}^c_{j+2(n+1)}) = \langle x^{j+2(n+1)}\rangle$ and $\mathrm{Ker}(\hat{\ell}^c_{j+2(n+1)}) = \langle x^{j+1}f\rangle$.

(iii) If $C + d \neq \pm 1$, $\mathrm{Cor}(\hat{\ell}^c_{j+2(n+1)}) = \{0\}$ and $\mathrm{Ker}(\hat{\ell}^c_{j+2(n+1)}) = \{0\}$.

Proof. Denoting $\mathbf{G}_n = \mathbf{F}_n + Cx^n\mathbf{D}_0$, it has that

$$\mathbf{G}_n(x^{j+n+2}) = (j+n+2)(d+C-1)x^{j+2(n+1)} + (j+n+2)x^{j+n+1}g,$$

and

$$\mathbf{G}_n(x^{j+1}g) = (j+n+2)(d+C+1)x^{j+n+1}g + (j+1)x^j fg,$$

Hence choosing the bases $\Delta_{j+n+2} = \mathrm{span}\{x^{j+n+2}, x^{j+1}g\}$ and $\Delta_{j+2(n+1)} = \mathrm{span}\{x^{j+2(n+1)}, x^{j+n+1}g\}$. Notice that $\dim(\Delta_{j+n+2}) = \dim(\Delta_{j+2(n+1)})$, we obtain that the matrix of the operator $\ell^c_{j+2(n+1)}$ is similar to

$$\begin{pmatrix} d+C-1 & 0 \\ 1 & d+C+1 \end{pmatrix}$$

The result follows from the structure of the above matrix. □

6. Example

Consider the algebraic integrability problem of the following nilpotent system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y + 4x^2 + x(b_3x^3 + a_2x^2 + b_1xy + a_0y) \\ 2x^3 + 8xy + 2y(b_3x^3 + a_2x^2 + b_1xy + a_0y) \end{pmatrix} \qquad (12)$$

The quasi-homogeneous expansion respect to the type $\mathbf{t} = (1,2)$ of the associated vector field is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ with $\mathbf{F}_1 = (y + 4x^2)\partial_x + (2x^3 + 8xy)\partial_y \in \mathcal{Q}^{(1,2)}_1$, $\mathbf{F}_2 = (a_0y + a_2x^2)\mathbf{D}_0 \in \mathcal{Q}^{(1,2)}_2$, $\mathbf{F}_3 = (b_1xy + b_3x^3)\mathbf{D}_0 \in \mathcal{Q}^{(1,2)}_3$ and $\mathbf{D}_0 = x\partial_x + 2y\partial_y \in \mathcal{Q}^{(1,2)}_0$.

So, the vector field $\mathbf{F}$ is of the form (2) with $d = 4$ and $n = 1$.

The origin of system (12) has two irreducible formal invariant curves $C_1 = (y - x^2) + \cdots$ and $C_2 = (y + x^2) + \cdots$. Moreover $\mathbf{F}_1$ is rationally integrable and a rational first integral is $I_{\mathrm{rat}} = (y + x^2)^3(y - x^2)^{-5}$.

The following result solves the algebraic integrability problem for the family (12).

Theorem 15. *System (12) has an algebraic and non-analytic first integral if, and only if, the following conditions hold:*

$$a_2 - 4a_0 = a_0(b_3 - 4b_1) = 0.$$

Proof. From Theorem 3, the vector field (12) has an algebraic and non-analytic first integral if, and only if, there exists a formal inverse integrating factor $V = V_4 + V_5 + \cdots$ where $V_j \in \mathcal{P}^{(1,2)}_j$ with $V_4 := 2(x^2 - y)(x^2 + y)$. The proof consists in imposing degree to degree the vanishing of the coefficients of the expression of $\mathbf{G} = \mathbf{F}(V) - \mathrm{div}(\mathbf{F})V = \mathbf{G}_5 + \cdots$ because they prevent the algebraic integrability. To degree 5, it is easy to show that $\mathbf{G}_5$ is identically zero if $a_2 = 4a_0$ and, in such a case, $V = 2(x^2 - y)(x^2 + y) - 2a_0x^5 + 2a_0xy^2 + \cdots$.

To degree 6 there are not any condition and to degree 7, we have that $a_0(b_3 - 4b_1) = 0$.

We distinguish two cases: we assume that $b_3 = 4b_1$. If $b_1 = 0$, the vector field has the first integral $(y + x^2)^3(y - x^2)^{-5}(1 + a_0x)^4$. Otherwise, $b_1 \neq 0$. Now we consider three cases:

i) If $4b_1 - a_0^2 = 0$, the vector field has the first integral

$$(y + x^2)^3(y - x^2)^{-5}(a_0x + 2)^4 e^{-2/(a_0x+2)}.$$

ii) If $4b_1 - a_0^2 = a^2 > 0$, the vector field has the first integral

$$(y + x^2)^3(y - x^2)^{-5}[(a^2 + a_0^2)x^2 + 4a_0x + 4]^4 e^{\frac{4a_0}{a}\operatorname{artan}(\frac{1}{2a}[(a^2+a_0^2)x + 2a_0])}.$$

iii) If $4b_1 - a_0^2 = -a^2 < 0$, the vector field has the first integral

$$(y + x^2)^3(y - x^2)^{-5}[2 + (a + a_0)x]^{2(1+\frac{a_0}{a})}[2 - (a - a_0)x]^{2(1-\frac{a_0}{a})}.$$

Finally, if $a_0 = 0$, the quasi-homogeneous polynomial $(y - x^2)(y + x^2)$ is an inverse integrating factor and the vector field has the first integral

$$(y + x^2)^3(y - x^2)^{-5}[(4b_3 - b_1)x^2 + (4b_1 - b_3)y + 15]^2.$$

□

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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