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p-continuous vector–valued functions

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p-CONTINUOUS VECTOR-VALUED FUNCTIONS

Memoria presentada por
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para optar al grado de Doctor
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por la Universidad de Huelva.

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To Silvia

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Contents

Introduction

Background and summary of the thesis

The origin of this thesis is based on two well-known concepts in Functional Analysis (in general). They are compactness and the Banach space of continuous vector-valued functions.

In relation with the first one, in finite-dimensional normed spaces, the notion of compact set is equivalent to bounded and closed set. In the case of infinite dimension, that is not true. The most well-known characterization of compact sets in infinite-dimensional Banach spaces is the Grothendieck compactness principle [43]. He characterized the compact sets of a Banach space as those sets which lie in the closed convex hull of a null sequence. In 2002, Sinha and Karn [88] presented the notion of p-compact set (for $1 \leq p \leq \infty$). A subset K of a Banach space X is *relatively p-compact* if there exists a sequence $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that $K \subset \{\sum_n \alpha_n x_n : \sum_n |\alpha_n|^{p'} \leq 1\}$, where p' is the conjugate index of p (i.e., $1/p + 1/p' = 1$, with $1/\infty = 0$). From this point of view, the notion of compactness can be seen as the particular case of ∞-compactness. This was not the first attempt to generalize the notion of compactness in a similar way (see, e.g., [3] for a historical review and references).

Between the properties of p-compact sets, there is one that is one of the keys of the problem we thought about originally. If $1 \leq p \leq q \leq \infty$, then p-compact sets are q-compact. In particular, p-compact sets are compact sets (i.e., are ∞-compact) (see, e.g., [88, p. 20]).

The other key concept is the space of continuous vector-valued functions. It is difficult to fix the beginning of the study of vector-valued function Banach spaces. We maybe have to go back to 30-s and 40-s of the last century to find its origin in the works of Birkhoff, Bochner, Dunford, Pettis, ... continued by Grothendieck, Dinculeanu, Diestel, Uhl, and many others. It is well known that continuous functions map compact sets to compact sets. At this point, we can show the question which motivated this thesis.

> How are the continuous vector-valued functions which map compact sets to p-compact sets?

We call them *p-continuous vector-valued functions* (Definition 1.1.1) and the set of all of them is denoted by $\mathcal{C}_p(\Omega, X)$, where Ω is a compact Hausdorff space and X is a Banach space.

Our first aim was to characterize this set and to study how classical results of continuous vector-valued functions $\mathcal{C}(\Omega, X)$ work in $\mathcal{C}_p(\Omega, X)$. However, this target became wider soon. The space $\mathcal{C}_p(\Omega, X)$ (endowed with an appropriate norm) can be identified with a tensor product. Namely, $\mathcal{C}_p(\Omega, X)$ is isometrically isomorphic to $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$ (see Chapter 1), where d_p denotes the right Chevet–Saphar tensor norm. This yielded us to enlarge our aims.

1. The original aim: the behaviour of classical results of $\mathcal{C}(\Omega, X)$ in the new space $\mathcal{C}_p(\Omega, X)$.
2. Expand these results (as far as possible) to more general spaces like $Z\hat{\otimes}_{d_p}X$, $\mathcal{C}(\Omega)\hat{\otimes}_\alpha X$, or $Z\hat{\otimes}_\alpha X$ (where Z is a Banach space and α is a tensor norm).

There is another important fact. By Grothendieck's classics [43] (see, e.g., [84, pp. 49–50]), we know that $\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X$, where ε denotes the injective tensor norm. But Saphar [87, p. 99] has shown that d_∞ coincides with ε on $\mathcal{C}(\Omega) \otimes X$. Therefore, we could obtain new properties (and some old ones as a by-product) for the classical space $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$ by studying properties of $\mathcal{C}_p(\Omega, X)$.

The goal was clear. During the trip, we dealt with tensor products, vector measures, operators ideals, ... This made us to deepen in these topics in order to see what kind of results would be expected. Each chapter of this thesis is almost self-contained.

The thesis has been organized as follows.

In Chapter 1, we present the space of p-continuous X-valued functions, $\mathcal{C}_p(\Omega, X)$, and we prove that it is isometrically isomorphic to $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$, where d_p is the right Chevet–Saphar tensor norm. We also introduce the space of *unconditionally p-continuous vector-valued functions* in a natural way and we characterize it by a tensor product too. Tensor products play an important role in this chapter. Both characterizations rely, among others, on an important result about α-nuclear operators, which is proved. This chapter is based on [57].

Chapter 2 collects some topological properties of $\mathcal{C}_p(\Omega, X)$. Namely, we obtain some results related to density of simple vector-valued functions in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$ (where $\mathcal{B}(\Sigma)$ denotes the space of all bounded Borel-measurable scalar functions defined on Ω), complemented embeddings of $\mathcal{C}(\Omega)$ and X in $\mathcal{C}_p(\Omega, X)$, and sequences in $\mathcal{C}_p(\Omega, X)$. In this chapter, we also study the weak and weak* convergences of sequences in $\mathcal{C}_p(\Omega, X)$.

Chapter 3 focuses on a classical result of Analysis: integral representation of operators defined on continuous functions. In particular, we establish two integral representation theorems: one for operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ (which extends the classical Bartle–Dunford–Schwartz representation theorem (see, e.g., [27, p. 152, Theorem 1])) and another for operators $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ (which extends the classical Dinculeanu–Singer representation theorem (see, e.g., [27, p. 182])). We provide an alternative simpler proof of the latter result using the first one. We also build the needed integration theory. This chapter is based on [60].

Chapter 4 deals with the associated operator $U^\#$ defined in Chapter 3. Every operator $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ has an associated operator $U^\# \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ defined in a natural way. In this chapter, we study the problem of the existence of an operator $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ such that $U^\# = S$ for a given operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$, solving a long-standing conjecture by Dinculeanu [30]. This chapter is based on [58].

Chapter 5 is devoted to the study of absolutely (r, q)-summing operators from $\mathcal{C}_p(\Omega, X)$ to Y. We study the interplay between U, its associated operator $U^\#$, and its representing measure (built in Chapter 3). Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, this encompasses not only the classical Swartz theorem about absolutely summing operators from $\mathcal{C}(\Omega, X)$ to Y [90] but also its existing extensions, providing an improvement even to the Swartz theorem. Counterexamples are exhibited to indicate sharpness of our results. This chapter is based on [59].

Antecedentes y resumen de la tesis

El origen de esta tesis está basado en dos conceptos bien conocidos del Análisis Funcional (en general). Son el de compacidad y el de espacio de Banach de funciones vectoriales continuas.

En relación con el primero de ellos, en espacios normados de dimensión finita, la noción de conjunto compacto es equivalente a la de conjunto cerrado y acotado. En el caso de dimensión infinita, eso no es cierto. La caracterización más conocida de conjunto compacto en un espacio de Banach de dimensión infinita es el principio de compacidad de Grothendieck [43]. Él caracterizó los conjuntos compactos de un espacio de Banach como aquellos conjuntos contenidos en la envolvente convexa cerrada de una sucesión nula. En 2002, Sinha and Karn [88] presentaron la noción de conjunto p-compacto (para $1 \leq p \leq \infty$). Un subconjunto K de un espacio de Banach X es *relativamente p-compacto* si existe una sucesión $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ si $p = \infty$) tal que $K \subset \{\sum_n \alpha_n x_n : \sum_n |\alpha_n|^{p'} \leq 1\}$, donde p' es el índice conjugado de p (i.e., $1/p + 1/p' = 1$, con $1/\infty = 0$). Desde este punto de vista, la noción de compacidad puede ser vista como el caso particular de ∞-compacidad. Éste no fue el primer intento de generalizar la noción de compacidad de un modo similar (véase, p.ej., [3] para un resumen histórico y referencias).

Entre las propiedades de los conjuntos p-compactos, hay una que constituye una de las claves del problema en el que pensamos originalmente. Si $1 \leq p \leq q \leq \infty$, entonces los conjuntos p-compactos son q-compactos. En particular, los conjuntos p-compactos son conjuntos compactos (i.e., son ∞-compactos) (véase, p.ej., [88, p. 20]).

El otro concepto clave es el de espacio de funciones vectoriales continuas. Es difícil fijar el comienzo del estudio de los espacios de Banach de funciones vectoriales. Quizás tenemos que retroceder a los años 30 y 40 del pasado siglo para encontrar su origen en los trabajos de Birkhoff, Bochner, Dunford, Pettis, ... continuados por Grothendieck, Dinculeanu, Diestel, Uhl, y muchos otros. Es bien sabido que las funciones continuas transforman conjuntos compactos en conjuntos compactos. En este punto, podemos mostrar la pregunta que motivó esta tesis.

> ¿Cómo son las funciones vectoriales continuas que transforman conjuntos compactos en conjuntos p-compactos?

Nosotros las llamamos *funciones vectoriales p-continuas* (Definición 1.1.1) y el conjunto de todas ellas se denota por $\mathcal{C}_p(\Omega, X)$, donde Ω es un espacio de Hausdorff compacto y X es un espacio de Banach.

Nuestro primer objetivo fue caracterizar este conjunto y estudiar como los resultados clásicos para funciones vectoriales continuas $\mathcal{C}(\Omega, X)$ funcionaban en $\mathcal{C}_p(\Omega, X)$. Sin embargo, esto objetivo se amplió pronto. El espacio $\mathcal{C}_p(\Omega, X)$ (dotado con una norma apropiada) puede ser identificado con un producto tensorial. A saber, $\mathcal{C}_p(\Omega, X)$ es isométricamente isomorfo a $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$ (véase Capítulo 1), donde d_p denota la norma tensorial de Chevet–Saphar a derecha. Esto nos llevó a ampliar nuestros objetivos.

1. El objetivo original: el comportamiento de los resultados clásicos de $\mathcal{C}(\Omega, X)$ en el nuevo espacio $\mathcal{C}_p(\Omega, X)$.

2. Extender estos resultados (tanto como sea posible) a espacios más generales como $Z\hat{\otimes}_{d_p}X$, $\mathcal{C}(\Omega)\hat{\otimes}_\alpha X$ o $Z\hat{\otimes}_\alpha X$ (donde Z es un espacio de Banach y α es una norma tensorial).

Hay otro hecho importante. Gracias a los trabajos clásicos de Grothendieck [43] (véase, p.ej., [84, pp. 49–50]), sabemos que $\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X$, donde ε denota la norma tensorial inyectiva. Pero Saphar [87, p. 99] ha probado que d_∞ coindice con ε en $\mathcal{C}(\Omega) \otimes X$. Por lo tanto, podemos obtener nuevas propiedades (y algunas antiguas como consecuencia) para el espacio clásico $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$ estudiando propiedades de $\mathcal{C}_p(\Omega, X)$.

La meta estaba clara. Durante este viaje, tratamos con productos tensoriales, medidas vectoriales, ideales de operadores, ... Esto nos hizo profundizar en estos temas para ver qué tipo de resultados se podrían esperar. Cada capítulo de esta tesis es casi independiente.

La tesis ha sido organizada como sigue.

En el Capítulo 1, presentamos el espacio de las funciones p-continuas con valores en X, $\mathcal{C}_p(\Omega, X)$, y probamos que es isométricamente isomorfo a $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$, donde d_p es la norma tensorial de Chevet–Saphar a derecha. También introducimos el espacio de las *funciones vectoriales incondicionalmente p-continuas* de un modo natural y lo caracterizamos también mediante un producto tensorial. Los productos tensoriales juegan un papel importante en este capítulo. Ambas caracterizaciones se basan, entre otros, en un importante resultado sobre operadores α-nucleares, que demostramos. Este capítulo se basa en [57].

El Capítulo 2 recoge algunas propiedades topológicas de $\mathcal{C}_p(\Omega, X)$. A saber, obtenemos resultados relativos a la densidad de funciones vectoriales simples en $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$ (donde $\mathcal{B}(\Sigma)$ denota el espacio de las funciones escalares medibles Borel definidas en Ω), inclusiones complementadas de $\mathcal{C}(\Omega)$ y X en $\mathcal{C}_p(\Omega, X)$, y sucesiones en $\mathcal{C}_p(\Omega, X)$. En este capítulo, también estudiamos las convergencias débil y débil* de sucesiones en $\mathcal{C}_p(\Omega, X)$.

El Capítulo 3 se centra en un resultado clásico del Análisis: la representación integral de operadores definidos sobre funciones continuas. En particular, establecemos dos teoremas de representación integral: uno para operadores $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ (que extiende el teorema clásico de representación de Bartle–Dunford–Schwartz (véase, p.ej., [27, p. 152, Theorem 1])), y otro para operadores $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ (que extiende el teorema clásico de representación de Dinculeanu–Singer (véase, p.ej., [27, p. 182])). Nosotros proporcionamos una demostración alternativa más simple de este último resultado usando el primero. También construimos la teoría de integración necesaria. Este capítulo se basa en [60].

El Capítulo 4 trata del operador asociado $U^{\#}$ definido en el Capítulo 3. Todo operador $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ tiene un operador asociado $U^{\#} \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ definido de un modo natural. En este capítulo, estudiamos el problema de la existencia de un operador $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ tal que $U^{\#} = S$ para un operador $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ dado, resolviendo una antigua conjetura de Dinculeanu [30]. Este capítulo se basa en [58].

El Capítulo 5 está dedicado al estudio de los operadores absolutamente (r, q)-sumantes de $\mathcal{C}_p(\Omega, X)$ en Y. Estudiamos la relación entre U, su operador asociado $U^{\#}$, y su medida representante (construida en el Capítulo 3). Ya que $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, esto engloba no sólo el teorema clásico de Swart sobre operadores absolutamente sumantes de $\mathcal{C}(\Omega, X)$

en Y [90] sino también las extensiones que existen, proporcionando una mejora incluso al teorema clásico de Swartz. Se muestran varios contraejemplos para indicar la precisión de nuestros resultados. Este capítulo se basa en [59].

Notation

Our notation is standard.

We consider Banach spaces over the same, either real or complex, field $\mathbb{K}$. A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical isometric embedding $j_X : X \to X^{**}$. The identity operator on X is denoted by I_X. The closed unit ball of X is denoted by B_X. The closure of a set $A \subset X$ is denoted by $\overline{A}$.

Let X be a Banach space and let Ω be a compact Hausdorff space. The space of continuous functions from Ω into X ($\mathbb{K}$, respectively) is denoted by $\mathcal{C}(\Omega, X)$ ($\mathcal{C}(\Omega)$, respectively). We denote by Σ the σ-algebra of Borel subsets of Ω. We denote the characteristic function of $E \in \Sigma$ by χ_E. The space of Σ-simple functions with values in X and the Banach space of all bounded Σ-measurable functions with values in X (i.e., the space of functions from Ω into X which are the uniform limit of a sequence of Σ-simple functions) are denoted by $\mathcal{S}(\Sigma, X)$ and $\mathcal{B}(\Sigma, X)$, respectively. In the case $X = \mathbb{K}$, we abbreviate them to $\mathcal{S}(\Sigma)$ and $\mathcal{B}(\Sigma)$, respectively. It is well known that $\mathcal{C}(\Omega) \subset \mathcal{B}(\Sigma) \subset \mathcal{C}(\Omega)^{**}$ and, more generally, $\mathcal{C}(\Omega, X) \subset \mathcal{B}(\Sigma, X) \subset \mathcal{C}(\Omega, X)^{**}$ as closed subspaces.

Let $1 \leq p \leq \infty$. We denote by p' the conjugate index of p (i.e., $1/p + 1/p' = 1$ with the convention $1/\infty = 0$).

For $1 \leq p < \infty$, we denote by $\ell_p(X)$ and $\ell_p^w(X)$ the Banach spaces of *absolutely p-summable X-valued sequences* and *weakly p-summable X-valued sequences*, respectively (see, e.g., [26, pp. 32–33]). For $(x_n) \in \ell_p(X)$, we denote by $\|(x_n)\|_p$ its natural norm, i.e.,

$$\|(x_n)\|_p = \Big(\sum_n \|x_n\|^p \Big)^{1/p}.$$

For $(x_n) \in \ell_p^w(X)$, we denote by $\|(x_n)\|_p^w$ its natural norm, i.e.,

$$\|(x_n)\|_p^w = \sup\Big\{\Big(\sum_n |\langle x_n, x^*\rangle|^p\Big)^{1/p} \;:\; x^* \in B_{X^*}\Big\}.$$

Denote by $\ell_p^u(X)$ the Banach space of all *unconditionally p-summable sequences* in X, which is the closed subspace of $\ell_p^w(X)$ formed by the sequences $(x_n) \in \ell_p^w(X)$ satisfying $(x_n) = \lim_{N\to\infty}(x_1, \ldots, x_N, 0, 0, \ldots)$ in $\ell_p^w(X)$ (see [37] or, e.g., [18, 8.2, 8.3]).

For $p = \infty$, we denote by $\ell_\infty(X)$ the Banach space of *bounded X-valued sequences*, and by $c_0(X)$ its closed subspace of *X-valued null sequences*. For $(x_n) \in \ell_\infty(X)$, we denote by $\|(x_n)\|_\infty$ its natural norm, i.e.,

$$\|(x_n)\|_\infty = \sup_n \|x_n\|.$$

Remark that $\ell_\infty^w(X) = \ell_\infty(X)$ (see, e.g., [26, p. 33]) and $\ell_\infty^u(X) = c_0(X)$ as Banach spaces (see [37, p. 351] or, e.g., [18, 8.2]).

We denote by $\mathcal{L} = (\mathcal{L}, \|\cdot\|)$, $\mathcal{K} = (\mathcal{K}, \|\cdot\|)$, and $\mathcal{F} = (\mathcal{F}, \|\cdot\|)$ the operator ideals of bounded, compact, and finite-rank linear operators, respectively.

Preliminaries

There are some notions that are used from the beginning and throughout this thesis like tensor products or Banach operator ideals. In this section we collect some results related to them.

Tensor products

Let X, Y, and Z be Banach spaces. Denote by $B(X, Y; Z)$ the space of bilinear maps from $X \times Y$ into Z. In the case $Z = \mathbb{K}$, we abbreviate the space of bilinear forms defined on $X \times Y$ to $B(X, Y)$. For every $x \in X$ and $y \in Y$, let $x \otimes y : B(X, Y) \to \mathbb{K}$ denote the corresponding *elementary tensor*, which is defined by

$$(x \otimes y)(A) = \langle A, x \otimes y\rangle = A(x, y)$$

for all $A \in B(X, Y)$. Then, the *tensor product* $X \otimes Y$ of the Banach spaces X and Y is defined by

$$X \otimes Y := \operatorname{span}\{x \otimes y \;:\; x \in X,\; y \in Y\}$$

in the algebraic dual of $B(X,Y)$. From its definition, it is clear that elementary tensors share some properties with the bilinear maps like

1) $(x_1+x_2)\otimes y=(x_1\otimes y)+(x_2\otimes y)$;

2) $x\otimes(y_1+y_2)=(x\otimes y_1)+(x\otimes y_2)$;

3) $k(x\otimes y)=kx\otimes y=x\otimes ky$;

where $x,x_1,x_2\in X$, $y,y_1,y_2\in Y$, and $k\in\mathbb{K}$. Therefore, every tensor $u\in X\otimes Y$ admits a representation like

$$u=\sum_{i=1}^{n}k_ix_i\otimes y_i,$$

with $(k_i)\subset\mathbb{K}$, $(x_i)\subset X$, and $(y_i)\subset Y$, which is not unique in view of the above properties.

Every tensor $u=\sum_{i=1}^{n}x_i^*\otimes y_i\in X^*\otimes Y$ can be algebraically identified with an operator $T\in\mathcal{F}(X,Y)$ by

$$Tx=\Big(\sum_{i=1}^{n}x_i^*\otimes y_i\Big)(x)=\sum_{i=1}^{n}x_i^*(x)y_i,\quad x\in X.$$

For every bilinear map $A\in B(X,Y;Z)$, it can be defined a linear map $\tilde{A}:X\otimes Y\to Z$ by

$$\tilde{A}(x\otimes y)=A(x,y).$$

This definition gives us a bijective relation between bilinear maps on $X\times Y$ into Z and linear maps on $X\otimes Y$ into Z (see, e.g., [84, pp. 5–6]). Even more, this relation is an isomorphism (see, e.g., [84, p. 6, Proposition 1.4]). The operator $\tilde{A}$ is called the *linearization* of the bilinear map A.

Let X_1, X_2, Y_1, and Y_2 be Banach spaces. Let $T_1:X_1\to Y_1$ and $T_2:X_2\to Y_2$ be linear maps. Then, the *tensor product map* $T_1\otimes T_2:X_1\otimes X_2\to Y_1\otimes Y_2$ is defined as the linearization of the bilinear map $A:X_1\times X_2\to Y_1\otimes Y_2$, $(x_1,x_2)\mapsto(T_1x_1)\otimes(T_2x_2)$.

A norm α on $X\otimes Y$ is a *reasonable crossnorm* if

1) $\alpha(x\otimes y)\le\|x\|\|y\|$ for all $x\in X$ and $y\in Y$;

2) for all $x^* \in X^*$ and $y^* \in Y^*$, the linear functional $x^* \otimes y^*$ on $X \otimes Y$ is bounded, and $\|x^* \otimes y^*\| \leq \|x^*\| \, \|y^*\|$.

It is easy to show that the above conditions imply that the inequalities are in fact equalities. It is said that α is a *tensor norm* if

1) α is a reasonable crossnorm;

2) for every operators $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$, the operator $T_1 \otimes T_2 : X_1 \otimes X_2 \to Y_1 \otimes Y_2$ is bounded, and $\|T_1 \otimes T_2\| \leq \|T_1\| \, \|T_2\|$;

3) $\alpha_{X \otimes Y}(u) = \inf\{\alpha_{X_0 \otimes Y_0}(u) \, : \, u \in X_0 \otimes Y_0\}$, where the infimum is taken over all finite-dimensional subspaces X_0 and Y_0 of X and Y, respectively. (This property is referred in the literature saying that α is *finitely generated*.)

The algebraic tensor product $X \otimes Y$ of Banach spaces X and Y equipped with a tensor norm α will be denoted by $X \otimes_\alpha Y$, and its completion by $X \hat{\otimes}_\alpha Y$.

The *transpose* of a tensor norm α, denoted by α^t, is defined for each pair of Banach spaces X and Y, and each $u \in X \otimes Y$ by

$$\alpha^t(u) = \alpha(u^t),$$

where the transpose, u^t, of $u = \sum_{i=1}^n x_i \otimes y_i$ is given by $u^t = \sum_{i=1}^n y_i \otimes x_i$.

The *dual norm* of a tensor norm α, denoted by α', is defined by

$$\alpha'(u) = \inf\{\alpha'(u) \, : \, u \in X_0 \otimes Y_0\},$$

where the infimum is taken over all finite-dimensional subspaces X_0 and Y_0 of X and Y, respectively, whose tensor product contains u. In finite-dimensional Banach spaces, the definition of α' is clear:

$$X_0 \otimes_{\alpha'} Y_0 = (X_0^* \otimes_\alpha Y_0^*)^*.$$

That is to say,

$$\alpha'(u) = \sup\{|\langle u, v\rangle| \, : \, v \in X_0^* \otimes Y_0^*, \, \alpha(v) \leq 1\}.$$

Among all the tensor norms, the following ones are the most used throughout this thesis. Let $u \in X \otimes Y$.

(a) The *projective* tensor norm. It is denoted by π and defined as follows (see, e.g., [84, p. 16]):

$$\pi(u) = \inf\Big\{\sum_{i=1}^{n}\|x_i\|\,\|y_i\| \,:\, u = \sum_{i=1}^{n} x_i\otimes y_i\Big\}.$$

(b) The *injective* tensor norm. It is denoted by ε and defined as follows (see, e.g., [84, p. 45]):

$$\varepsilon(u) = \sup\Big\{\Big|\sum_{i=1}^{n} x^*(x_i)y^*(y_i)\Big| \,:\, u = \sum_{i=1}^{n} x_i\otimes y_i,\, x^*\in B_{X^*},\, y^*\in B_{Y^*}\Big\}.$$

(c) The *Chevet–Saphar* tensor norm. Let $1\le p\le\infty$. The *right Chevet–Shapar* tensor norm, denoted by d_p, and the *left Chevet–Saphar* tensor norm, denoted by g_p, are defined as follows (see, e.g., [84, p. 135]):

$$d_p(u) = \inf\Big\{\|(x_i)\|^w_{p'}\|(y_i)\|_p \,:\, u = \sum_{i=1}^{n} x_i\otimes y_i\Big\},$$

and

$$g_p(u) = \inf\Big\{\|(x_i)\|_p\|(y_i)\|^w_{p'} \,:\, u = \sum_{i=1}^{n} x_i\otimes y_i\Big\},$$

taking the infimums over all the representations of $u\in X\otimes Y$.

Clearly, $g_p = d_p^t$. Thus properties of $Y\hat{\otimes}_{d_p}X$ can be viewed as properties of $X\hat{\otimes}_{g_p}Y$ and vice versa. This concerns, in particular, the following description of these tensor products. If $u\in Y\hat{\otimes}_{d_p}X$, then there exist sequences $(y_n)\in\ell^w_{p'}(Y)$ and $(x_n)\in\ell_p(X)$ (or $(x_n)\in c_0(X)$ when $p=\infty$) such that $u=\sum_{n=1}^{\infty} y_n\otimes x_n$ in $Y\hat{\otimes}_{d_p}X$ (see, e.g., [84, Proposition 6.10]).

Recall that $\pi = d_1 = g_1$ (see, e.g., [84, Proposition 6.6]).

Recall (see, e.g., [84, p. 142]) that the dual space operator ideal (we follow the terminology of [66]) of the Chevet–Saphar tensor norm d_p coincides with $\mathcal{P}_{p'}$, i.e., $(Y\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Y,X^*)$ (where $\mathcal{P}_{p'} = (\mathcal{P}_{p'},\|\cdot\|_{\mathcal{P}_{p'}})$ denotes the Banach operator ideal of absolutely p'-summing operators (see below)) as Banach spaces, under the canonical identification

$$\langle\sum_{i=1}^{n} y_i\otimes x_i, A\rangle = \sum_{i=1}^{n}(Ay_i)(x_i),\;\; A\in(Y\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Y,X^*).$$

Banach operator ideals

Let X and Y be Banach spaces. Let $\mathcal{A}$ be a class of bounded linear operators between Banach spaces. Then $\mathcal{A}(X,Y) = \mathcal{A} \cap \mathcal{L}(X,Y)$ is a *component* of that ideal. It is said that $\mathcal{A}$ is an *operator ideal* if

1) $\mathcal{A}(X,Y)$ is a vector space for all Banach spaces X and Y;

2) $\mathcal{F}(X,Y) \subset \mathcal{A}(X,Y)$ for all Banach spaces X and Y (i.e., the finite-rank operators are elements of $\mathcal{A}(X,Y)$);

3) For all $T \in \mathcal{A}(X,Y)$, $T_1 \in \mathcal{L}(X_0,X)$, and $T_2 \in \mathcal{L}(Y,Y_0)$, it is verified that $T_2 T T_1 \in \mathcal{A}(X_0,Y_0)$ for all Banach spaces X, X_0, Y, and Y_0.

A *Banach operator ideal* is the pair $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$, where $\mathcal{A}$ is an operator ideal and the map $\|\cdot\|_{\mathcal{A}} : T \in \mathcal{A} \mapsto \|T\|_{\mathcal{A}} \in [0,\infty)$ is such that

1) $(\mathcal{A}(X,Y), \|\cdot\|_{\mathcal{A}})$ is a Banach space for all Banach spaces X and Y;

2) the one-dimensional operator $x^* \otimes y$ belongs to $\mathcal{A}(X,Y)$ for all $x^* \in X^*$ and $y \in Y$, and $\|x^* \otimes y\|_{\mathcal{A}} \leq \|x^*\|\,\|y\|$;

3) for all $T \in \mathcal{A}(X,Y)$, $T_1 \in \mathcal{L}(X_0,X)$, and $T_2 \in \mathcal{L}(Y,Y_0)$, it is verified that $T_2 T T_1 \in \mathcal{A}(X_0,Y_0)$, and $\|T_2 T T_1\|_{\mathcal{A}} \leq \|T_2\|\,\|T\|_{\mathcal{A}}\|T_1\|$.

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach operator ideal. The dual Banach operator ideal $\mathcal{A}^d$ is defined by

$$\mathcal{A}^d(X,Y) = \{T \in \mathcal{L}(X,Y) \,:\, T^* \in \mathcal{A}(Y^*,X^*)\}$$

for all Banach spaces X and Y.

Let $1 \leq q \leq r \leq \infty$. Recall that an operator $U \in \mathcal{L}(X,Y)$ is *absolutely (r,q)-summing* if there is a constant $C \geq 0$ such that

$$\|(Ux_i)_{i=1}^n\|_r \leq C\,\|(x_i)_{i=1}^n\|_q^w$$

for all finite systems $(x_i)_{i=1}^n \subset X$, $n \in \mathbb{N}$. The least constant C for which the previous inequality holds is denoted by $\|U\|_{\mathcal{P}_{(r,q)}}$. A straightforward verification shows that it suffices to consider finite systems (x_i) belonging to a dense subset V of X.

All absolutely (r,q)-summing operators between arbitrary Banach spaces form a Banach operator ideal, denoted by $\mathcal{P}_{(r,q)}$. Recall also that the Banach operator ideal $\mathcal{P}_q$ of *absolutely q-summing operators* is defined as $\mathcal{P}_q = \mathcal{P}_{(q,q)}$. It is well known (and easy to see) that $(\mathcal{P}_{(\infty,q)}, \|\cdot\|_{\mathcal{P}_{(\infty,q)}}) = (\mathcal{L}, \|\cdot\|)$. If $1 \le p < q < \infty$, then $\mathcal{P}_p(X,Y) \subset \mathcal{P}_q(X,Y)$ and $\|T\|_{\mathcal{P}_q} \le \|T\|_{\mathcal{P}_p}$ for all $T \in \mathcal{P}_p(X,Y)$ (see, e.g., [26, p. 39, Theorem 2.8]).

Compactness and p-compactness

Let X be a Banach space and let $1 \le p \le \infty$. Recall that a set $K \subset X$ is *relatively p-compact* (see [88, Definition 2.1]) if there exists $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that

$$K \subset p\text{-co}(x_n) := \{\sum_n a_n x_n \,:\, (a_n) \in B_{\ell_{p'}}\}.$$

Notice that, according to the Grothendieck compactness principle, relatively ∞-compact sets are precisely relatively compact sets. If $1 \le p \le q \le \infty$, then p-compact sets are q-compact; in particular, p-compact sets are compact (i.e., ∞-compact).

Let Y be a Banach space. An operator $T \in \mathcal{L}(Y,X)$ is called *p-compact* (see [88, Definition 2.2]) if T maps bounded sets to relatively p-compact sets; or equivalently, there exists $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that $T(B_Y) \subset p\text{-co}(x_n)$. The class of such operators is denoted by $\mathcal{K}_p(Y,X)$. Obviously, $\mathcal{K}_\infty(Y,X) = \mathcal{K}(Y,X)$, the class of compact operators. Again, if $1 \le p \le q \le \infty$, then $\mathcal{K}_p(Y,X) \subset \mathcal{K}_q(Y,X) \subset \mathcal{K}_\infty(Y,X) = \mathcal{K}(Y,X)$.

It is known (see [88, Theorem 4.2]) that $\mathcal{K}_p$ is a Banach operator ideal. In the last seven years, p-compact sets and p-compact operators have been studied quite intensively (see, e.g., [1,2,5,16,19,21,22,39,49,64,72,73,89]). A suitable formula for the Banach operator ideal norm in $\mathcal{K}_p(Y,X)$ was given by Delgado, Piñeiro, and Serrano [21] (see, e.g., [1, Theorem 3.4 and Remark 3.7]) as follows: for every $T \in \mathcal{K}_p(Y,X)$,

$$\|T\|_{\mathcal{K}_p} = \inf \|(x_n)\|_p \ ,$$

where the infimum is taken over all sequences $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$) such that $T(B_Y) \subset p\text{-co}(x_n)$.

Chapter 1

p-Continuous vector-valued functions

In this chapter, we present the space of p-continuous X-valued functions, $\mathcal{C}_p(\Omega, X)$, and we prove that it is isometrically isomorphic to $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$, where d_p is the right Chevet–Saphar tensor norm. We also introduce the space of unconditionally p-continuous vector-valued functions in a natural way and we characterize it by a tensor product too. Tensor products play an important role in this chapter. Both characterizations rely, among others, on an important result about α-nuclear operators, which is proved. This chapter is based on [57].

1.1 Introduction

Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. In this chapter we present the main notion of the thesis: the space of p-continuous vector-valued functions.

Definition 1.1.1. We define the space of *p-continuous X-valued functions* $\mathcal{C}_p(\Omega, X)$ as follows:

$$\mathcal{C}_p(\Omega, X) = \{f : \Omega \to X \mid f \in \mathcal{C}(\Omega, X) \text{ and } f(\Omega) \text{ is } p\text{-compact}\}.$$

It follows from properties of p-compactness (see, e.g., [88]) that $\mathcal{C}_p(\Omega, X) \subset \mathcal{C}_q(\Omega, X)$ if $p \leq q$, and $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$.

Example 1.1.2. Let $f : \Omega \to X$ be a function of the following form:

$$f(\omega) = \sum_{n=1}^{\infty} \varphi_n(\omega) x_n, \ \ \omega \in \Omega,$$

where $(\varphi_n) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ and $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$). Then the series converges in $\mathcal{C}(\Omega, X)$ and $f \in \mathcal{C}_p(\Omega, X)$.

As we shall see from the main result of Section 1.3 (Theorem 1.3.7), all functions in $\mathcal{C}_p(\Omega, X)$ have the form as in Example 1.1.2. In order to obtain such a characterization, we use a result (Theorem 1.2.4) that is important by itself. We devote Section 1.2, after explaining its background, to prove it. In Section 1.3, we prove that $\mathcal{C}_p(\Omega, X)$ can be canonically identified with the Chevet–Saphar tensor product $\mathcal{C}(\Omega)\hat{\otimes}_{d_p} X$. In the final Section 1.4, we expand our method in a natural way from $\mathcal{C}_p(\Omega, X)$ to the space $\mathcal{UC}_p(\Omega, X)$ of unconditionally p-continuous vector-valued functions. They are continuous functions from Ω to X whose range is unconditionally p-compact. We prove that $\mathcal{UC}_p(\Omega, X)$ can be canonically identified with the Fourie–Swart tensor product $\mathcal{C}(\Omega)\hat{\otimes}_{w_{p'}} X$.

1.2 The main tool

Let X and Y be Banach spaces. A bounded linear operator $T \in \mathcal{L}(X, Y)$ is said to be *nuclear* if there exist $x_n^* \in X^*$ and $y_n \in Y$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \, \|y_n\| < \infty$ and $Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n$ for all $x \in X$. Denote by $\mathcal{N}(X, Y)$ the collection of all nuclear operators from X to Y.

Every operator $T \in \mathcal{L}(X, Y)$ may be viewed as an operator from X to Y^{**} considering the operator $j_Y T$, where $j_Y : Y \to Y^{**}$ denotes the canonical embedding. The following result goes back to Grothendieck's classics.

Theorem 1.2.1 (Grothendieck–Oja–Reinov)**.** *Assume that either X^* or Y^{***} has the approximation property. If $T \in \mathcal{L}(X, Y)$ and T is nuclear into Y^{**}, i.e., $j_Y T \in \mathcal{N}(X, Y^{**})$, then $T \in \mathcal{N}(X, Y)$.*

Theorem 1.2.1 was proved in [43, Chapter I, pp. 85–86] under the hypothesis on X (see, e.g., [84, Proposition 4.10]) and in [68] under the hypothesis on Y (announced in [67]); see [62] for a simpler proof in the both cases. From [35] and [67], we know that the assumptions about the approximation properties of X^* and Y^{***} are essential in Theorem 1.2.1 and cannot be weakened to the approximation properties of X or/and Y^{**} or even to the existence of bases. (Remark that Grothendieck affirmed (see [43, Chapter I, p. 86] and [42, p. 17] that Theorem 1.2.1 held if Y^{**} had the approximation property.) Thus, a non-nuclear operator may very well be nuclear for a larger range space: for the bidual of its range.

Let α be an arbitrary tensor norm. It is well known that α defines, as described below, the Banach operator ideal $\mathcal{N}_\alpha = (\mathcal{N}_\alpha, \|\cdot\|_{\mathcal{N}_\alpha})$ of *α-nuclear operators*. One starts by considering the identity embedding of $X^* \otimes_\alpha Y$ into $\mathcal{L}(X, Y)$. It has norm one and extends by continuity to $X^* \hat{\otimes}_\alpha Y$. Let the extension be denoted by J_α. One defines $\mathcal{N}_\alpha(X, Y) := \operatorname{ran} J_\alpha$ in $\mathcal{L}(X, Y)$ and equips $\mathcal{N}_\alpha(X, Y)$ with the quotient norm of $X^* \hat{\otimes}_\alpha Y / \ker J_\alpha$. The operator $J_\alpha : X^* \hat{\otimes}_\alpha Y \to \mathcal{N}_\alpha(X, Y)$ is called the *natural surjection*.

For example, concerning the (classical) nuclear operators, we have $\mathcal{N}(X, Y) = \mathcal{N}_\pi(X, Y)$, where π is the projective tensor norm.

If J_α is injective, then J_α is an isometric isomorphism between $X^* \hat{\otimes}_\alpha Y$ and $\mathcal{N}_\alpha(X, Y)$, and one writes $\mathcal{N}_\alpha(X, Y) = X^* \hat{\otimes}_\alpha Y$. Thanks to Grothendieck [42] (see, e.g., [24, Proposition 1.5.4] or [84, Proposition 8.7] for a proof), we know that this happens whenever X^* or Y has the approximation property.

Recall that X has the *approximation property* if the identity operator I_X on X can be uniformly approximated on compact subsets of X by bounded linear operators of finite rank, i.e., by members of $\mathcal{F}(X, X)$. From Grothendieck's classics (see [43, Chapter I, p. 165] or, e.g., [24, Theorem 1.4.18]), we know that X has the approximation property if and only if $\mathcal{N}(X, X) = X^* \hat{\otimes}_\pi X$ as Banach spaces. In this case, the trace functional is well defined on $\mathcal{N}(X, X)$.

In 2001, Theorem 1.2.1 was extended from nuclear operators (i.e., from π-nuclear operators) to α-nuclear operators by Kaijser and Reinov [45] (see [65] for a stronger result in the same vein). Recall that a Banach space Y is said to have the *α-approximation property* if for all Banach space X the natural map $X \hat{\otimes}_\alpha Y \to X \hat{\otimes}_\varepsilon Y$ is injective (see, e.g., [18, 21.7]).

Theorem 1.2.2 (Kaijser–Reinov)**.** *Let X and Y be Banach spaces and let α be a tensor norm. Assume that one of the following statements holds.*

(a) X^* *has the* α^t*-approximation property,*

(b) Y^{****} *has the* α*-approximation property,*

(c) Y^{**} *has the* α*-approximation property and* Y^{***} *has the* α^t*-approximation property.*

If $T \in \mathcal{L}(X, Y)$ *is such that* $T^{**} \in \mathcal{N}_\alpha(X^{**}, Y^{**})$*, then* $T \in \mathcal{N}_\alpha(X, Y)$*.*

On the other hand, in 2004, Oja [62] proved that in Theorem 1.2.1, Y^{**} can be replaced by any Banach space Z, where Y sits in as a closed subspace, such that there exists an *extension operator* $\Phi \in \mathcal{L}(Y^*, Z^*)$, meaning that Φ satisfies $(\Phi y^*)(y) = y^*(y)$ for all $y^* \in Y^*$ and all $y \in Y$. Remark that $\Phi = j_{Y^*} \in \mathcal{L}(Y^*, Y^{***})$ is clearly an extension operator.

Theorem 1.2.3 (Oja)**.** *Let* X *be a Banach space. Let* Y *be a closed subspace of a Banach space* Z *such that there exists an extension operator* $\Phi \in \mathcal{L}(Y^*, Z^*)$ *and let* $j : Y \to Z$ *denote the identity embedding. Assume that either* X^* *or* Z^* *has the approximation property. If* $T \in \mathcal{L}(X, Y)$ *and* T *is nuclear into* Z*, i.e.,* $jT \in \mathcal{N}(X, Z)$*, then* $T \in \mathcal{N}(X, Y)$*.*

Moreover,

$$\frac{1}{\|\Phi\|}\|T\|_{\mathcal{N}} \le \|jT\|_{\mathcal{N}} \le \|T\|_{\mathcal{N}}.$$

Remark that the existence of an extension operator Φ is equivalent to the annihilator of Y being complemented in Z^*. Pairs of Banach spaces Z and their closed subspaces Y for which there exists an extension operator $\Phi \in \mathcal{L}(Y^*, Z^*)$ were systematically studied by Fakhoury [33] and Kalton [46]. The existence of Φ with $\|\Phi\| = 1$ means, according to the terminology of Godefroy, Kalton, and Saphar [40], that Y is an *ideal* in Z. Different subclasses of ideals have been extensively studied by many authors (for references see [61, Section 4]).

The following theorem extends the above-mentioned results, and it is the basic result that will be used in the next section to characterize the space $\mathcal{C}_p(\Omega, X)$.

Theorem 1.2.4. *Let* X *be a Banach space. Let* Y *be a closed subspace of a Banach space* Z *such that there exists an extension operator* $\Phi \in \mathcal{L}(Y^*, Z^*)$ *and let* $j : Y \to Z$ *denote the identity embedding. Assume that either* X^* *or* Z^* *has the approximation property. Let* α *be a tensor norm. If* $T \in \mathcal{L}(X, Y)$ *and* T *is* α*-nuclear into* Z*, i.e.,* $jT \in \mathcal{N}_\alpha(X, Z)$*, then* $T \in \mathcal{N}_\alpha(X, Y)$*.*

Moreover,

$$\frac{1}{\|\Phi\|}\|T\|_{\mathcal{N}_\alpha} \le \|jT\|_{\mathcal{N}_\alpha} \le \|T\|_{\mathcal{N}_\alpha}. \tag{1.1}$$

Proof. It is a well-known Grothendieck's result [43] (see, e.g., [84, Corollary 4.7]) that if Z^* has the approximation property, then Z has too. Let us note that Y^*, being isomorphic to the complemented subspace $\operatorname{ran}\Phi$ of Z^*, has the approximation property. Hence also Y has the approximation property.

Therefore, under the assumptions of the theorem, the trace functional is well defined on $\mathcal{N}(X^*, X^*)$ (if X^* has the approximation property) or on $\mathcal{N}(Z^*, Z^*)$, $\mathcal{N}(Z, Z)$, and $\mathcal{N}(Y^*, Y^*)$ (if Z^* has the approximation property). Also

$$\mathcal{N}_\alpha(X, Z) = X^* \hat{\otimes}_\alpha Z$$

as Banach spaces, because X^* or Z has the approximation property, and

$$\mathcal{N}_\alpha(X, Y) = X^* \hat{\otimes}_\alpha Y$$

as Banach spaces, because X^* or Y has the approximation property.

By [66, Proposition 2.4], $X^* \hat{\otimes}_\alpha Y$ is naturally isomorphic to the closed subspace $\overline{X^* \otimes Y}$ of $X^* \hat{\otimes}_\alpha Z$ under the into isomorphism $I_{X^*} \otimes j : X^* \hat{\otimes}_\alpha Y \to X^* \hat{\otimes}_\alpha Z$ (note that this holds because of the existence of an extension operator and does not require any assumption about the approximation property).

Let $jT \in \mathcal{N}_\alpha(X, Z)$ be identified with $u \in X^* \hat{\otimes}_\alpha Z$. We have to prove that $u \in \overline{X^* \otimes Y}$, which would mean that $T \in \mathcal{N}_\alpha(X, Y)$.

(a) Assume that X^* has the approximation property. We shall use the canonical identification (see, e.g., [84, pp. 187–190])

$$(X^* \hat{\otimes}_\alpha Z)^* = \mathcal{A}(Z, X^{**}),$$

where $\mathcal{A}$ is the Banach operator ideal of the $(\alpha^t)'$-integral operators, called "the dual operator ideal of α" in [63] (here α^t and α', respectively, denote the transpose and dual norms of α).

Let $A \in \mathcal{A}(Z, X^{**})$ be arbitrary. Assume that A vanishes on $X^* \otimes Y$, i.e.,

$$\langle A, x^* \otimes y\rangle = (Ajy)(x^*) = 0 \text{ for all } x^* \in X^* \text{ and all } y \in Y,$$

meaning that $Aj = 0$.

It suffices to show that $\langle A, u\rangle = 0$. It is known (see, e.g., [65, Lemma 2.3]) that

$$A \otimes I_{X^*} \in \mathcal{L}(Z\hat{\otimes}_{\alpha^t}X^*, X^{**}\hat{\otimes}_\pi X^*)$$

and

$$\langle A, u\rangle = \operatorname{trace}(A \otimes I_{X^*})(u^t).$$

(Recall that if $v = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ (here X and Y are arbitrary Banach spaces), then v^t, the *transpose* of v, is defined as $v^t = \sum_{i=1}^n y_i \otimes x_i \in Y \otimes X$. In particular, if $S \in \mathcal{F}(X, Y) = X^* \otimes Y$, then $S^t = S^* \in Y \otimes X^* \subset \mathcal{F}(Y^*, X^*)$.)

Since X^* has the approximation property, $X^{**}\hat{\otimes}_\pi X^* = \mathcal{N}(X^*, X^*)$ as Banach spaces. We claim that, under the latter identification, $(A \otimes I_{X^*})(u^t)$ corresponds to the operator $T^*j^*A^*j_{X^*} \in \mathcal{N}(X^*, X^*)$. The argument is similar to the one given in [65, proof of Theorem 2.4].

Indeed, let $u = \lim T_n$ in $X^*\hat{\otimes}_\alpha Z$ for some $T_n \in X^* \otimes Z$. Then $(jT)^* = T^*j^* = \lim T_n^*$ in $\mathcal{N}_{\alpha^t}(Z^*, X^*)$ (see [65, Lemma 2.2]) and $(A \otimes I_{X^*})(u^t) = \lim(A \otimes I_{X^*})(T_n^*)$ in $X^{**}\hat{\otimes}_\pi X^*$. Every T_n is a finite sum of elements of the form $x^* \otimes z \in X^* \otimes Z$. Since for all $\bar{x}^* \in X^*$,

$$(x^* \otimes z)^*A^*j_{X^*}\bar{x}^* = (z \otimes x^*)(A^*j_{X^*}\bar{x}^*) = (A^*j_{X^*}\bar{x}^*)(z)x^*$$

$$= (j_{X^*}\bar{x}^*)(Az)x^* = (Az)(\bar{x}^*)x^*$$

and

$$((A \otimes I_{X^*})(x^* \otimes z)^*)\bar{x}^* = (Az \otimes x^*)\bar{x}^* = (Az)(\bar{x}^*)x^*,$$

we see that

$$T_n^*A^*j_{X^*}\bar{x}^* = ((A \otimes I_{X^*})(T_n^*))\bar{x}^*$$

for every T_n and all $\bar{x}^* \in X^*$. Hence, the limit operator $T^*j^*A^*j_{X^*}$ is identified with the limit tensor $(A \otimes I_{X^*})(u^t)$.

Recall that $Aj = 0$, so $T^*j^*A^*j_{X^*} = T^*(Aj)^*j_{X^*} = T^*0j_{X^*} = 0$. Therefore, $(A \otimes I_{X^*})(u^t) = 0$ and $\langle A, u\rangle = \operatorname{trace} 0 = 0$, as needed.

(b) Assume that Z^* has the approximation property. Then we shall use the canonical identification (see, e.g., [84, pp. 187–190])

$$(X^*\hat{\otimes}_\alpha Z)^* = \mathcal{A}(X^*, Z^*),$$

where $\mathcal{A}$ is the Banach operator ideal of the α'-integral operators.

Let $A \in \mathcal{A}(X^*, Z^*)$ be arbitrary. Assume that A vanishes on $X^* \otimes Y$, i.e.,

$$\langle A, x^* \otimes y\rangle = (Ax^*)(jy) = (j^*Ax^*)(y) = 0 \text{ for all } x^* \in X^* \text{ and all } y \in Y.$$

This means that $j^*A = 0$.

It suffices to show that $\langle A, u\rangle = 0$. We know (see, e.g., [65, Lemma 2.3]) that

$$A \otimes I_Z \in \mathcal{L}(X^*\hat{\otimes}_\alpha Z, Z^*\hat{\otimes}_\pi Z)$$

and

$$\langle A, u\rangle = \operatorname{trace}(A \otimes I_Z)u.$$

Since Z has the approximation property, $Z^*\hat{\otimes}_\pi Z = \mathcal{N}(Z, Z)$ as Banach spaces. Let an operator $S \in \mathcal{N}(Z, Z)$ be identified with $(A \otimes I_Z)u \in Z^*\hat{\otimes}_\pi Z$. We claim that $S^* = AT^*j^*$.

Our argument is essentially the same as in [65, proof of Theorem 2.4]. Indeed, let $u = \lim T_n$ in $X^*\hat{\otimes}_\alpha Z$ for some $T_n \in X^* \otimes Z$. Then $(A \otimes I_Z)u = \lim(A \otimes I_Z)T_n$ in $Z^*\hat{\otimes}_\pi Z$. Therefore $T^*j^* = \lim T_n^*$ in $\mathcal{N}_{\alpha^t}(Z^*, X^*)$ and $S^* = \lim((A \otimes I_Z)T_n)^*$ in $\mathcal{N}(Z^*, Z^*)$ (see [65, Lemma 2.2]). Now, every T_n is a finite sum of elements of the form $x^*\otimes z \in X^*\otimes Z$. Since for all $z^* \in Z^*$,

$$A(x^* \otimes z)^*z^* = A(z^*(z)x^*) = z^*(z)Ax^*$$

and

$$((A \otimes I_Z)(x^* \otimes z))^*z^* = (Ax^* \otimes z)^*z^* = z^*(z)Ax^*,$$

we see that

$$AT_n^*z^* = ((A \otimes I_Z)T_n)^*z^*$$

for every T_n and $z^* \in Z^*$. Hence, $AT^*j^*z^* = S^*z^*$ for all $z^* \in Z^*$.

Since the trace functional is defined not only on $\mathcal{N}(Z, Z)$ but also on $\mathcal{N}(Z^*, Z^*)$,

$$\langle A, u\rangle = \operatorname{trace} S = \operatorname{trace} S^* = \operatorname{trace} AT^*j^*.$$

The trace functional is also defined on $\mathcal{N}(Y^*, Y^*) = Y^{**}\hat{\otimes}_\pi Y^*$. It can be verified that $j^*AT^* \in \mathcal{N}(Y^*, Y^*)$ and

$$\operatorname{trace} AT^*j^* = \operatorname{trace} j^*AT^*.$$

Indeed, $AT^*j^* \in \mathcal{N}(Z^*, Z^*)$ implies that $AT^*j^*\Phi \in \mathcal{N}(Y^*, Z^*)$. Note that $j^*\Phi = I_{Y^*}$, therefore $AT^* \in \mathcal{N}(Y^*, Z^*)$. Hence $j^*AT^* \in \mathcal{N}(Y^*, Y^*)$. Let $AT^* = \sum_{n=1}^{\infty} y_n^{**} \otimes z_n^*$ be a nuclear representation of AT^*. Then

$$AT^*j^* = \sum_{n=1}^{\infty} (j^{**}y_n^{**}) \otimes z_n^* \in \mathcal{N}(Z^*, Z^*)$$

and

$$j^*AT^* = \sum_{n=1}^{\infty} y_n^{**} \otimes j^*z_n^* \in \mathcal{N}(Y^*, Y^*).$$

Their traces

$$\operatorname{trace} AT^*j^* = \sum_{n=1}^{\infty} (j^{**}y_n^{**})(z_n^*)$$

and

$$\operatorname{trace} j^*AT^* = \sum_{n=1}^{\infty} y_n^{**}(j^*z_n^*)$$

are trivially equal.

Finally, recalling that $j^*A = 0$, we get

$$\operatorname{trace} AT^*j^* = \operatorname{trace} j^*AT^* = \operatorname{trace} 0T^* = 0.$$

Hence, $\langle A, u\rangle = 0$, as desired.

(c) For the "moreover" part of the theorem, one has $\|jT\|_{\mathcal{N}_\alpha} \le \|j\|\,\|T\|_{\mathcal{N}_\alpha} = \|T\|_{\mathcal{N}_\alpha}$. The left-side inequality in (1.1) follows from [66, Proposition 2.4]. Indeed, by this result,

$$\frac{1}{\|\Phi\|}\alpha(v) \le \alpha((I_{X^*} \otimes j)v) \tag{1.2}$$

for all $v \in X^*\hat{\otimes}_\alpha Y$ (this inequality does not require any assumption about the approximation property). From the identifications $\mathcal{N}_\alpha(X, Y) = X^*\hat{\otimes}_\alpha Y$ and $\mathcal{N}_\alpha(X, Z) = X^*\hat{\otimes}_\alpha Z$ as Banach spaces (see the beginning of the proof of Theorem 1.2.4), identifying T with some $v \in X^*\hat{\otimes}_\alpha Y$, we see that jT identifies with $(I_{X^*} \otimes j)v \in X^*\hat{\otimes}_\alpha Z$. Hence, $\|T\|_{\mathcal{N}_\alpha} = \alpha(v)$ and $\|jT\|_{\mathcal{N}_\alpha} = \alpha((I_{X^*} \otimes j)v)$. Therefore (1.1) holds. In [66], (1.2) was established for a more general case. For completeness, let us present a proof of (1.2) as follows (the argument is the same as in [66, Proposition 2.4]).

Let $v = \sum_{n=1}^N x_n^* \otimes y_n \in X^* \otimes_\alpha Y$. Using the canonical identification (see, e.g., [84, pp. 187–190])

$$\mathcal{A}(X^*, Y^*) = (X^* \hat{\otimes}_\alpha Y)^*,$$

where $\mathcal{A}$ is the Banach operator ideal of the α'-integral operators, there exists an operator $A \in \mathcal{A}(X^*, Y^*)$ with $\|A\|_{\mathcal{A}} = 1$ such that

$$\alpha(v) = \langle A, v \rangle = \sum_{n=1}^N (Ax_n^*)(y_n) = \sum_{n=1}^N (\Phi A x_n^*)(jy_n),$$

where, in the very last equality, we use that $(\Phi y^*)(jy) = y^*(y)$ for all $y^* \in Y^*$ and all $y \in Y$. Note that $\Phi A \in \mathcal{A}(X^*, Z^*) = (X^* \hat{\otimes}_\alpha Z)^*$ and $(I_{X^*} \otimes j)v = \sum_{n=1}^N x_n^* \otimes jy_n \in X^* \otimes Z$. Hence

$$\alpha(v) = \sum_{n=1}^N (\Phi A x_n^*)(jy_n) = \langle \Phi A, (I_{X^*} \otimes j)v \rangle \le \|\Phi A\|_{\mathcal{A}} \alpha((I_{X^*} \otimes j)v)$$

$$\le \|\Phi\| \, \|A\|_{\mathcal{A}} \alpha((I_{X^*} \otimes j)v) = \|\Phi\| \, \alpha((I_{X^*} \otimes j)v). \qquad \square$$

1.3 The space $\mathcal{C}_p(\Omega, X)$ of p-continuous X-valued functions

In this section, we shall present an application of Theorem 1.2.4 in order to characterize the space $\mathcal{C}_p(\Omega, X)$ (Definition 1.1.1) of continuous functions from a compact Hausdorff space Ω to a Banach space X whose range is p-compact, $1 \le p \le \infty$. More specifically, we prove (Theorem 1.3.7) that $\mathcal{C}_p(\Omega, X)$ can be canonically identified with the Chevet–Saphar tensor product $\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$.

We start with an *algebraic identification* of $\mathcal{C}_p(\Omega, X)$ as a linear subspace of $\mathcal{K}_p(\ell_1(\Omega), X)$. For this, let us recall that the space $\ell_\infty(\Omega, X)$ of X-valued bounded functions defined on Ω may be identified with

$\mathcal{L}(\ell_1(\Omega), X)$ as Banach spaces. Under this identification, one associates with each $f \in \ell_\infty(\Omega, X)$ an operator $U_f \in \mathcal{L}(\ell_1(\Omega), X)$ defined by

$$U_f(e_\omega) = f(\omega), \ \ \omega \in \Omega,$$

where $(e_\omega)_{\omega\in\Omega}$ is the unit vector basis of $\ell_1(\Omega)$.

Keeping in mind this algebraic identification, $\mathcal{C}_p(\Omega, X)$ will be considered as a linear subspace of $\mathcal{L}(\ell_1(\Omega), X)$, i.e.,

$$\mathcal{C}_p(\Omega, X) \subset \mathcal{C}(\Omega, X) \subset \ell_\infty(\Omega, X) = \mathcal{L}(\ell_1(\Omega), X).$$

For describing $\mathcal{C}_p(\Omega, X)$ as a linear subspace of $\mathcal{L}(\ell_1(\Omega), X)$, it is useful to observe the following.

Proposition 1.3.1. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Suppose that $f \in \ell_\infty(\Omega, X)$ and $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$). Then $f(\Omega) \subset p\text{-co}(x_n)$ if and only if $U_f(B_{\ell_1(\Omega)}) \subset p\text{-co}(x_n)$.*

Proof. Assume that $f(\Omega) \subset p\text{-co}(x_n)$. We know that $p\text{-co}(x_n)$ is a closed absolutely convex set (this was observed in [21, p. 203] for the case $p > 1$; see [2, p. 230] for a proof of the general case). Hence, $\overline{\text{aco}}\{f(\omega) : \omega \in \Omega\} \subset p\text{-co}(x_n)$ (where $\overline{\text{aco}}A$ denotes the closed absolutely convex hull of A). Thus,

$$U_f(B_{\ell_1(\Omega)}) \subset \overline{\text{aco}}\{f(\omega) : \omega \in \Omega\} \subset p\text{-co}(x_n).$$

The "if" part is obvious because $\{f(\omega) : \omega \in \Omega\} \subset U_f(B_{\ell_1(\Omega)})$. □

The immediate corollaries follow.

Corollary 1.3.2. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Suppose that $U \in \mathcal{L}(\ell_1(\Omega), X)$ and $f \in \ell_\infty(\Omega, X)$ satisfy $U = U_f$. Then $U \in \mathcal{K}_p(\ell_1(\Omega), X)$ if and only if $f(\Omega)$ is p-compact.*

Corollary 1.3.3. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Then*

$$\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega, X) \cap \mathcal{K}_p(\ell_1(\Omega), X)$$

as linear spaces.

As it was mentioned in the Preliminaries, a suitable formula for the Banach operator ideal norm in $\mathcal{K}_p(Y, X)$ is

$$\|T\|_{\mathcal{K}_p} = \inf \|(x_n)\|_p \ , T \in \mathcal{K}_p(Y, X),$$

where the infimum is taken over all sequences $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$) such that $T(B_Y) \subset p\text{-co}(x_n)$.

Definition 1.3.4. Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. For every $f \in \mathcal{C}_p(\Omega, X)$, we define

$$\|f\|_{\mathcal{C}_p(\Omega,X)} = \inf \|(x_n)\|_p,$$

where the infimum is taken over all sequences $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$) such that $f(\Omega) \subset p\text{-co}(x_n)$.

By Proposition 1.3.1,

$$\|f\|_{\mathcal{C}_p(\Omega,X)} = \|U_f\|_{\mathcal{K}_p}$$

for all $f \in \mathcal{C}_p(\Omega, X)$, and therefore $\mathcal{C}_p(\Omega, X)$ is a subspace of the Banach space $\mathcal{K}_p(\ell_1(\Omega), X)$.

Proposition 1.3.5. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Then $\mathcal{C}_p(\Omega, X)$ is a closed subspace of $\mathcal{K}_p(\ell_1(\Omega), X)$. Hence $\mathcal{C}_p(\Omega, X)$ is a Banach space.*

Proof. By Corollary 1.3.3, $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega, X) \cap \mathcal{K}_p(\ell_1(\Omega), X)$. Let $\|U_{f_n} - U_f\|_{\mathcal{K}_p} \to 0$, where $f_n \in \mathcal{C}_p(\Omega, X)$ and $f \in \ell_\infty(\Omega, X)$ is such that $U_f \in \mathcal{K}_p(\ell_1(\Omega), X)$. Since

$$\|f_n - f\|_\infty = \|U_{f_n} - U_f\| \leq \|U_{f_n} - U_f\|_{\mathcal{K}_p} \to 0,$$

we have that $f_n \to f$ in $\ell_\infty(\Omega, X)$, where $\mathcal{C}(\Omega, X)$ sits as a closed subspace. Hence, $f \in \mathcal{C}(\Omega, X)$ as desired. □

We shall give now a finer description of $\mathcal{C}_p(\Omega, X)$ as a Banach space. Recall that $\mathcal{N}^p = \mathcal{N}_{d_p}$ and $\mathcal{N}_p = \mathcal{N}_{g_p}$, i.e., the Banach operator ideals of d_p- and g_p-nuclear operators coincide with the ideals of the *right p-nuclear operators* $\mathcal{N}^p$ and *p-nuclear operators* $\mathcal{N}_p$, respectively (see [71, 18.2.1] or, e.g., [84, pp. 139–140]). Since $\pi = d_1 = g_1$ (see [84, Proposition 6.6]), $\mathcal{N}_{d_1} = \mathcal{N}_{g_1} = \mathcal{N}$, the ideal of nuclear operators.

In [73], Piñeiro and Delgado introduced and studied the Banach space $c_{0,p}(X)$ of p-null sequences in a Banach space X (whose definition will be given below just before Corollary 1.3.9), which is a linear subspace of $c_0(X)$. In [65], Oja proved that

$$c_{0,p}(X) = c_0 \hat{\otimes}_{d_p} X$$

as Banach spaces. We shall prove that

$$\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X,$$

see Theorem 1.3.7 below. Note that the above description of $c_{0,p}(X)$ can be deduced from Theorem 1.3.7 (see Corollary 1.3.9).

The establishing of the description of $c_{0,p}(X)$ in [65] is an easy task that does not rely on any main result of [65], neither does it rely on a result of our Theorem 1.2.4 type. In contrast, the description of $\mathcal{C}_p(\Omega, X)$ will rely on Theorem 1.2.4 and also on a main result of [65]. Let us spell out, for completeness and easy reference, the relevant part of it.

Theorem 1.3.6 (cf. [65, Theorem 3.1])**.** *Let X and Y be Banach spaces and let $1 \le p \le \infty$. Assume that either X^{***} or Y has the approximation property. If $T \in \mathcal{N}_p(X^*, Y)$ and $T^*(Y^*) \subset X$, then T admits a representation $T = \sum_{n=1}^{\infty} x_n \otimes y_n$ with $(x_n) \in \ell_p(X)$ (or $c_0(X)$ when $p = \infty$) and $(y_n) \in \ell_{p'}^w(Y)$. In particular, the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ converges in $\mathcal{N}_p(X^*, Y)$.*

Theorem 1.3.7. *Let X be a Banach space and let $1 \le p \le \infty$. Then*

$$\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X = \mathcal{C}_p(\Omega, X)$$

under the canonical isometric isomorphism that associates with each elementary tensor $\varphi \otimes x \in \mathcal{C}(\Omega) \otimes X$ the function $\varphi(\omega)x, \omega \in \Omega$.

Moreover, every $f \in \mathcal{C}_p(\Omega, X)$ can be represented as

$$f(\omega) = \sum_{n=1}^{\infty} \varphi_n(\omega) x_n, \quad \omega \in \Omega,$$

where $(\varphi_n) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ and $(x_n) \in \ell_p(X)$ (or $(x_n) \in c_0(X)$ when $p = \infty$), where the series converges in $\mathcal{C}_p(\Omega, X)$.

Remark 1.3.8. The special case $p = \infty$ of Theorem 1.3.7, i.e., $\mathcal{C}(\Omega)\hat{\otimes}_{d_\infty}X = \mathcal{C}(\Omega, X)$, should be essentially known. Indeed, as observed by Saphar [87, p. 99], the tensor norm d_∞ coincides with the injective tensor norm ε on $\mathcal{C}(\Omega) \otimes X$, meaning that $\mathcal{C}(\Omega)\hat{\otimes}_{d_\infty}X = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X$. And, by Grothendieck's classics [43] (see, e.g., [84, pp. 49–50]), $\mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X = \mathcal{C}(\Omega, X)$ (under the mapping described in Theorem 1.3.7).

Proof of Theorem 1.3.7. Let us denote the mapping described in Theorem 1.3.7 by $\mathcal{J}$. By well-known Grothendieck's classics (see, e.g., [84, pp. 11, 49]), $\mathcal{J} : \mathcal{C}(\Omega) \otimes_{d_p} X \to \mathcal{C}(\Omega, X)$ is linear and injective. From Example 1.1.2, we know that $\operatorname{ran}\mathcal{J} \subset \mathcal{C}_p(\Omega, X)$. We need to prove that $\mathcal{J}$ is isometric and its extension by continuity (denoted also by $\mathcal{J}$) $\mathcal{J} : \mathcal{C}(\Omega)\hat{\otimes}_{d_p}X \to \mathcal{C}_p(\Omega, X)$ is surjective.

Let us introduce a mapping $\mathcal{U} : \mathcal{C}_p(\Omega, X) \to \ell_\infty(\Omega)\hat{\otimes}_{d_p}X$ as follows. We know that $\mathcal{C}_p(\Omega, X) \subset \mathcal{K}_p(\ell_1(\Omega), X)$ as a closed subspace. On the other hand, $\mathcal{K}_p = (\mathcal{N}^p)^{\mathrm{sur}}$, the surjective hull of $\mathcal{N}^p$, as Banach operator ideals. (This result is due to [21, Proposition 3.11] where, to prove this result, the authors use a roundabout approach, first describing $\mathcal{K}_p^{\mathrm{dual}}$, and rely on Reinov's recent study [83] on operators with p-nuclear adjoints. An easy straightforward proof was independently proposed in [1] (see Remark 3.8 in [1]) and [72, Theorem 1].) It is well known (and easy to see, because $\ell_1(\Omega)$ canonically embeds into $\ell_1(B_{\ell_1(\Omega)})$) that $\mathcal{A}^{\mathrm{sur}}(\ell_1(\Omega), X) = \mathcal{A}(\ell_1(\Omega), X)$ for any Banach operator ideal $\mathcal{A}$. Hence, $\mathcal{K}_p(\ell_1(\Omega), X) = \mathcal{N}^p(\ell_1(\Omega), X)$ as Banach spaces. Since, in turn, $\mathcal{N}^p(\ell_1(\Omega), X) = \ell_\infty(\Omega)\hat{\otimes}_{d_p}X$ (because $\mathcal{N}_{d_p} = \mathcal{N}^p$ and $\ell_1^*(\Omega) = \ell_\infty(\Omega)$ has the approximation property), we can define

$$\mathcal{U} : \mathcal{C}_p(\Omega, X) \to \ell_\infty(\Omega)\hat{\otimes}_{d_p}X$$

by $\mathcal{U}f = u_f$, $f \in \mathcal{C}_p(\Omega, X)$, where $u_f \in \ell_\infty(\Omega)\hat{\otimes}_{d_p}X$ canonically corresponds to $U_f \in \mathcal{N}^p(\ell_1(\Omega), X)$. Then, by its definition, $\mathcal{U}$ is an isometric isomorphism "into".

Let us look now at $\mathcal{U}\mathcal{J} : \mathcal{C}(\Omega) \otimes_{d_p} X \to \ell_\infty(\Omega)\hat{\otimes}_{d_p}X$. Since, clearly, $\mathcal{U}(\mathcal{J}(\varphi\otimes x)) = \varphi\otimes x$ if $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$, we have $\mathcal{U}\mathcal{J} = I_{\mathcal{C}(\Omega)\otimes_{d_p}X}$. It is a well-known application of Kakutani's representation theorem for abstract L-spaces (using, in the complex case, also a standard complexification argument (see, e.g., [55, 2.2])) that $\mathcal{C}(\Omega)^*$ is isometrically isomorphic to an $L_1(\mu)$-space for some measure μ, i.e., $\mathcal{C}(\Omega)$ is an L_1-*predual space*. Thanks to Fakhoury [33, Corollary 3.3] and Grothendieck [44, Theorem 1] (see, e.g., [24, pp. 76, 81]), L_1-predual spaces are ideals in their "superspaces" (for more details, see [51, p. 49]). In particular, $\mathcal{C}(\Omega)$ is an ideal in $\ell_\infty(\Omega)$. But then (see [66, Proposition 2.4]) $\mathcal{C}(\Omega)\otimes_{d_p}X$ is a subspace of $\ell_\infty(\Omega)\hat{\otimes}_{d_p}X$, and therefore $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X = \overline{\mathcal{C}(\Omega)\otimes_{d_p}X}$ is a closed subspace of $\ell_\infty(\Omega)\hat{\otimes}_{d_p}X$. Hence, for all $u \in \mathcal{C}(\Omega)\otimes_{d_p}X$,

$$\|u\|_{\mathcal{C}(\Omega)\otimes_{d_p}X} = \|\mathcal{U}(\mathcal{J}u)\|_{\mathcal{C}(\Omega)\otimes_{d_p}X} = \|\mathcal{U}(\mathcal{J}u)\|_{\ell_\infty(\Omega)\otimes_{d_p}X} = \|\mathcal{J}u\|_{\mathcal{C}_p(\Omega,X)},$$

giving that $\mathcal{J} : \mathcal{C}(\Omega)\otimes_{d_p}X \to \mathcal{C}_p(\Omega,X)$ is isometric.

We extend $\mathcal{J}$ by continuity to $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$ and we keep calling this map $\mathcal{J}$. We know that

$$\mathcal{J} : \mathcal{C}(\Omega)\hat{\otimes}_{d_p}X \to \mathcal{C}_p(\Omega,X)$$

is an isometric isomorphism "into" and $\mathcal{U}\mathcal{J} = I_{\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X}$. We shall prove that $\mathcal{J}$ is, in fact, surjective, meaning that $\mathcal{J}$ is an isometric isomorphism, as desired.

Let $f \in \mathcal{C}_p(\Omega,X)$ be arbitrary. Then $u_f = \mathcal{U}f \in \ell_\infty(\Omega)\hat{\otimes}_{d_p}X$. If we could show that, in fact, $u_f \in \mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$, then $\mathcal{J}u_f = f$, because

$$\|\mathcal{J}u_f - f\|_{\mathcal{C}_p(\Omega,X)} = \|\mathcal{U}(\mathcal{J}u_f - f)\|_{\ell_\infty(\Omega)\hat{\otimes}_{d_p}X} = \|u_f - u_f\|_{\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X} = 0.$$

We know that $\ell_\infty(\Omega)\hat{\otimes}_{d_p}X = \mathcal{N}^p(\ell_1(\Omega),X)$ and u_f is identified with $U_f \in \mathcal{N}^p(\ell_1(\Omega),X)$. Thanks to Grothendieck [42] (see, e.g., [24, Corollary 1.4.9] or [84, Proposition 6.4]), we also know that $X\hat{\otimes}_{g_p}\ell_\infty(\Omega)$ is a closed subspace of $X^{**}\hat{\otimes}_{g_p}\ell_\infty(\Omega) = \mathcal{N}_p(X^*,\ell_\infty(\Omega))$. Since $g_p = d_p^t$, we see that $(u_f)^t \in X\hat{\otimes}_{g_p}\ell_\infty(\Omega)$ is identified with $(U_f)^* \in \mathcal{N}_p(X^*,\ell_\infty(\Omega))$. Observe that $\operatorname{ran}(U_f)^* \subset \mathcal{C}(\Omega)$, because $(U_f)^*x^* = x^*f$, $x^* \in X^*$. As $\mathcal{C}(\Omega)$ is an ideal in $\ell_\infty(\Omega)$ and $\ell_\infty(\Omega)^*$ has the approximation property, using Theorem 1.2.4, we can conclude that $(U_f)^* \in \mathcal{N}_p(X^*,\mathcal{C}(\Omega))$, meaning that $(u_f)^t \in X^{**}\hat{\otimes}_{g_p}\mathcal{C}(\Omega)$.

To avoid any ambiguity, let us define $V \in \mathcal{N}_p(X^*,\mathcal{C}(\Omega))$ to be the operator $(U_f)^*$ considered with values in $\mathcal{C}(\Omega)$, so that $(u_f)^t$ is identified with V. We can apply Theorem 1.3.6 to V, because $\mathcal{C}(\Omega)$ has the approximation property and $\operatorname{ran}V^* \subset X$. To see this inclusion, let $j : \mathcal{C}(\Omega) \to \ell_\infty(\Omega)$ be the identity embedding and let $\Phi : \mathcal{C}(\Omega)^* \to \ell_\infty(\Omega)^*$ be an extension operator (recall that $\mathcal{C}(\Omega)$ is an ideal in $\ell_\infty(\Omega)$). From $j^*\Phi = I_{\mathcal{C}(\Omega)^*}$ and $(U_f)^* = jV$, we get that $V^* = V^*j^*\Phi = (U_f)^{**}\Phi$. Hence,

$\operatorname{ran} V^* \subset \operatorname{ran}(U_f)^{**}$. But $\operatorname{ran}(U_f)^{**} \subset X$, because U_f is a compact operator.

By Theorem 1.3.6, $V = \sum_{n=1}^{\infty} x_n \otimes \varphi_n$, with $(x_n) \in \ell_p(X)$ (or $c_0(X)$ when $p = \infty$) and $(\varphi_n) \in \ell_{p'}^w(\mathcal{C}(\Omega))$. This means that $(u_f)^t = \sum_{n=1}^{\infty} x_n \otimes \varphi_n \in X \hat{\otimes}_{g_p} \mathcal{C}(\Omega)$. Hence, $u_f = (u_f)^{tt} \in \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$, completing the proof of the surjectivity of $\mathcal{J}$.

For the "moreover" part of the theorem, let $f \in \mathcal{C}_p(\Omega, X)$ be arbitrary. Then $\mathcal{J}^{-1} f \in \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ and, as was mentioned in the Preliminaries when we introduced $Y \hat{\otimes}_{d_p} X$, there exist sequences (φ_n) and (x_n) as needed such that $\mathcal{J}^{-1} f = \sum_{n=1}^{\infty} \varphi_n \otimes x_n$, where the series converges in $\mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$. Hence,

$$f = \mathcal{J}\left(\sum_{n=1}^{\infty} \varphi_n \otimes x_n\right) = \sum_{n=1}^{\infty} \mathcal{J}(\varphi_n \otimes x_n) = \sum_{n=1}^{\infty} \varphi_n(\omega) x_n, \; \omega \in \Omega,$$

where the series converges in $\mathcal{C}_p(\Omega, X)$. □

Recall (see [73]) that a sequence (x_n) in a Banach space X is *p-null* if for every $\varepsilon > 0$ there exist $(z_k) \in \varepsilon B_{\ell_p(X)}$ ($\varepsilon B_{c_0(X)}$ when $p = \infty$) and $N \in \mathbb{N}$ such that $x_n \in p\text{-co}(z_k)$ for all $n \geq N$. Clearly, $c_{0,\infty}(X) = c_0(X)$. For equivalent characterizations of p-null sequences, the reader is referred to [3, 50, 73]. Recall (see [73, p. 959]) that $c_{0,p}(X)$ is equipped with the norm from $\mathcal{K}_p(\ell_1, X)$, identifying $(x_n) \in c_{0,p}(X)$ with the operator $U_{(x_n)} \in \mathcal{K}_p(\ell_1, X)$ defined by $U_{(x_n)}(\alpha_n) = \sum_n \alpha_n x_n$, $(\alpha_n) \in \ell_1$.

Corollary 1.3.9 (see [65, Theorem 4.1])**.** *Let X be a Banach space and let $1 \leq p \leq \infty$. Then*

$$c_0 \hat{\otimes}_{d_p} X = c_{0,p}(X)$$

under the canonical isometric isomorphism that associates with each elementary tensor $(a_n) \otimes x \in c_0 \otimes X$ the sequence $(a_n x) \subset X$.

Proof. Let us denote by $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ the Alexandrov or one-point compactification of $\mathbb{N}$. From Theorem 1.3.7, we know that

$$\mathcal{C}(\mathbb{N}_\infty) \hat{\otimes}_{d_p} X = \mathcal{C}_p(\mathbb{N}_\infty, X)$$

under the canonical isometric isomorphism $\mathcal{J}$ that associates with each elementary tensor $\varphi \otimes x \in \mathcal{C}(\mathbb{N}_\infty) \otimes X$ the function (sequence) $(\varphi(n)x)_{n \in \mathbb{N}_\infty}$.

Let us identify c_0 with a subspace of $\mathcal{C}(\mathbb{N}_\infty)$ through the isometric embedding $(a_n) \mapsto \varphi$, where $\varphi(n) = a_n$ for $n \in \mathbb{N}$ and $\varphi(\infty) = 0$. Observe that c_0 is an ideal in $\mathcal{C}(\mathbb{N}_\infty)$ (because of the canonical embedding from $c_0^* = \ell_1$ into $\ell_1(\mathbb{N}_\infty) \subset \mathcal{M}(\mathbb{N}_\infty) = \mathcal{C}(\mathbb{N}_\infty)^*$, where $\mathcal{M}(\mathbb{N}_\infty)$ denotes the space of regular Borel measures on $\mathbb{N}_\infty$). Therefore, by [66, Proposition 2.4], $c_0 \hat{\otimes}_{d_p} X$ is canonically identified with a subspace of $\mathcal{C}(\mathbb{N}_\infty)\hat{\otimes}_{d_p} X$. Clearly, $\mathcal{J}((a_n) \otimes x) = (a_n x)_{n \in \mathbb{N}_\infty}$, where $a_\infty = 0$, for all $(a_n) \in c_0$ and $x \in X$.

We can identify $c_{0,p}(X)$ with a subspace of $\mathcal{C}_p(\mathbb{N}_\infty, X)$ through the isometric embedding $(x_n) \mapsto f$, where $f(n) = x_n$ for all $n \in \mathbb{N}$ and $f(\infty) = 0$. Indeed, every sequence $(x_n) \in c_{0,p}(X)$ is relatively p-compact in X (an easy straightforward observation (see [73, p. 959])), and the embedding is obviously linear. Moreover,

$$\|(x_n)\|_{c_{0,p}(X)} = \|U_{(x_n)}\|_{\mathcal{K}_p(\ell_1, X)} = \|U_f\|_{\mathcal{K}_p(\ell_1(\mathbb{N}_\infty), X)} = \|f\|_{\mathcal{C}_p(\mathbb{N}_\infty, X)}.$$

Finally, since $P : \varphi \mapsto (\varphi(n) - \varphi(\infty))_{n \in \mathbb{N}_\infty}$ is a projection from $\mathcal{C}(\mathbb{N}_\infty)$ onto c_0, it is clear that $P \otimes I_X$ is a projection from $\mathcal{C}(\mathbb{N}_\infty)\hat{\otimes}_{d_p} X$ onto $c_0 \hat{\otimes}_{d_p} X$. On the other hand, we define a projection Q from $\mathcal{C}_p(\mathbb{N}_\infty, X)$ onto $c_{0,p}(X)$ by $Q : f \mapsto (f(n) - f(\infty))_{n \in \mathbb{N}_\infty}$. Let $\varphi \in \mathcal{C}(\mathbb{N}_\infty)$ and $x \in X$. Since

$$Q\mathcal{J}(\varphi \otimes x) = Q((\varphi(n)x)_n) = (\varphi(n)x - \varphi(\infty)x)_n$$

and

$$\mathcal{J}(P \otimes I_X)(\varphi \otimes x) = \mathcal{J}((\varphi(n) - \varphi(\infty))_n \otimes x) = ((\varphi(n) - \varphi(\infty))x)_n,$$

we obtain that $Q\mathcal{J} = \mathcal{J}(P \otimes I_X)$. Thus, the restriction of $\mathcal{J}$ to $c_0 \hat{\otimes}_{d_p} X$ is the desired isometric isomorphism from $c_0 \hat{\otimes}_{d_p} X$ onto $c_{0,p}(X)$. □

1.4 The space $\mathcal{UC}_p(\Omega, X)$ of unconditionally p-continuous X-valued functions

Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$.

In this section, we show that, similarly to the previous section, the space $\mathcal{UC}_p(\Omega, X)$ of continuous functions from Ω to X whose range is unconditionally p-compact, can be canonically identified with the Fourie–Swart tensor product $\mathcal{C}(\Omega)\hat{\otimes}_{w_{p'}}X$, where $1/p + 1/p' = 1$. All relevant terminology and necessary preliminaries will be given below.

It is natural to introduce the concept of *unconditionally relatively p-compact set* in X by replacing $\ell_p(X)$ with $\ell_p^u(X)$ in the definition of relatively p-compact set (see [3] or [47]). A linear operator $T : Y \to X$ is *unconditionally p-compact* if $T(B_Y)$ is a relatively unconditionally p-compact subset of X (see [3] or [47]). Let $\mathcal{U}_p$ denote the class of all unconditionally p-compact operators acting between arbitrary Banach spaces. As observed in [3], $\mathcal{U}_p$ is a surjective operator ideal.

Definition 1.4.1. We define the space of *unconditionally p-continuous X-valued functions* $\mathcal{UC}_p(\Omega, X)$ as follows:

$$\mathcal{UC}_p(\Omega, X) = \{f : \Omega \to X \mid f \in \mathcal{C}(\Omega, X) \text{ and } f(\Omega) \text{ is unconditionally } p\text{-compact}\}.$$

The space $\ell_p^u(X)$ contains the space $\ell_p(X)$ as a linear subspace. This implies that every p-compact set is unconditionally p-compact. Therefore, $\mathcal{C}_p(\Omega, X) \subset \mathcal{UC}_p(\Omega, X)$. Since $\ell_\infty^u(X) = c_0(X)$, it is clear that ∞-compact sets (i.e., compact sets) coincide with unconditionally ∞-compact sets; in particular, $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X) = \mathcal{UC}_\infty(\Omega, X)$.

It is easy to check (as in Example 1.1.2) that all the functions $f : \Omega \to X$ of the form

$$f(\omega) = \sum_{n=1}^{\infty} \varphi_n(\omega)x_n, \quad \omega \in \Omega,$$

where $(\varphi_n) \in \ell_{p'}^u(\mathcal{C}(\Omega))$ and $(x_n) \in \ell_p^u(X)$, and the series converges in $\mathcal{C}(\Omega, X)$, belong to $\mathcal{UC}_p(\Omega, X)$. And as we shall see from the main result of this section (Theorem 1.4.6), all functions in $\mathcal{UC}_p(\Omega, X)$ have this form. Moreover, in the above representation, $\ell_{p'}^u(\mathcal{C}(\Omega))$ can be replaced by $\ell_{p'}^w(\mathcal{C}(\Omega))$ (the same space as in Example 1.1.2). (We just need to use that $(x_n) \in \ell_p^u(X)$ if and only if there exist $(\delta_n) \in c_0$ and $(y_n) \in \ell_p^w(X)$ such that $x_n = \delta_n y_n$, $n \in \mathbb{N}$ (see [37, p. 352]).)

Reasoning in a similar way as we did in Section 1.3, we obtain an algebraic identification of $\mathcal{UC}_p(\Omega, X)$ as a linear subspace of $\mathcal{U}_p(\ell_1(\Omega), X)$. Namely, for $f \in \ell_\infty(\Omega, X)$ and $(x_n) \in \ell_p^u(X)$, we can prove that

$f(\Omega) \subset p\text{-co}(x_n)$ if and only if $U_f(B_{\ell_1(\Omega)}) \subset p\text{-co}(x_n)$ (the analogous result to Proposition 1.3.1). This easily yields the identification

$$\mathcal{UC}_p(\Omega, X) = \mathcal{C}(\Omega, X) \cap \mathcal{U}_p(\ell_1(\Omega), X)$$

as linear spaces (the analogous result to Corollary 1.3.3).

In [3], it was observed that

$$\mathcal{U}_p = \mathcal{N}^{\text{sur}}_{(\infty,p',p)},$$

the surjective hull of the operator ideal of (∞, p', p)-*nuclear operators* (for the definition of $\mathcal{N}_{(t,u,v)}$, see [71, 18.1.1]). But, as *Banach* operator ideals,

$$\mathcal{N}_{(\infty,p',p)} = \mathcal{N}_{w_{p'}} = \mathrm{K}_{p'},$$

where K_p denotes the ideal of *classical* p-*compact operators* (see, e.g., [71, 18.3]). (Remark that $\mathcal{K}_p$ and K_p are different as operator ideals (see, [64] and [72]).) Since

$$\mathcal{U}_p = \mathrm{K}^{\text{sur}}_{p'}$$

as operator ideals, similarly to [1], a natural way to define a norm on $\mathcal{U}_p$ is as follows:

$$\|\cdot\|_{\mathcal{U}_p} = \|\cdot\|_{\mathrm{K}^{\text{sur}}_{p'}}.$$

Hence, $\mathcal{U}_p = (\mathcal{U}_p, \|\cdot\|_{\mathcal{U}_p})$ becomes a Banach operator ideal. The same technique used in [1, Theorem 3.4] gives the following explicit formula for $\|\cdot\|_{\mathcal{U}_p}$.

Proposition 1.4.2. *Let X and Y be Banach spaces. Let $1 \le p \le \infty$. Let $T \in \mathcal{U}_p(Y, X)$. Then*

$$\|T\|_{\mathcal{U}_p} = \inf \|(x_n)\|^w_p,$$

where the infimum is taken over all sequences $(x_n) \in \ell^u_p(X)$ such that $T(B_Y) \subset p\text{-co}(x_n)$.

Remark 1.4.3. In [47], $\mathcal{U}_p(Y, X)$ was equipped with the same norm, and then it was proved that $\mathcal{U}_p$ is a Banach operator ideal. From our approach, [47, Proposition 2.2], stating that $\mathcal{U}_2 = \mathrm{K}_2$ as Banach operator ideals, is obvious, because $\mathrm{K}^{\text{sur}}_2 = \mathrm{K}_2$ (see [71, 18.1.8]).

Definition 1.4.4. Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. For every $f \in \mathcal{UC}_p(\Omega, X)$, we define

$$\|f\|_{\mathcal{UC}_p(\Omega,X)} = \inf \|(x_n)\|_p^w,$$

where the infimum is taken over all sequences $(x_n) \in \ell_p^u(X)$ such that $f(\Omega) \subset p\text{-co}(x_n)$.

As in Section 1.3, by the equivalence between the inclusions $f(\Omega) \subset p\text{-co}(x_n)$ and $U_f(B_{\ell_1(\Omega)}) \subset p\text{-co}(x_n)$, we get

$$\|f\|_{\mathcal{UC}_p(\Omega,X)} = \|U_f\|_{\mathcal{U}_p}$$

for all $f \in \mathcal{UC}_p(\Omega, X)$, and therefore $\mathcal{UC}_p(\Omega, X)$ is a subspace of the Banach space $\mathcal{U}_p(\ell_1(\Omega), X)$. The next proposition is analogous to Proposition 1.3.5.

Proposition 1.4.5. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Then $\mathcal{UC}_p(\Omega, X)$ is a closed subspace of $\mathcal{U}_p(\ell_1(\Omega), X)$. Hence $\mathcal{UC}_p(\Omega, X)$ is a Banach space.*

Now, similarly to the description of $\mathcal{C}_p(\Omega, X)$ in terms of Chevet–Saphar tensor products (see Theorem 1.3.7), we shall give a description of $\mathcal{UC}_p(\Omega, X)$ using Fourie–Swart tensor products (see Theorem 1.4.6 below). First, let us recall the definition of Fourie–Swart tensor norm w_p (see [38] or, e.g., [18, 12.7]). Let $1 \le p \le \infty$. For every tensor $u \in Y \otimes X$, the *Fourie–Swart norm* is defined as follows:

$$w_p(u) = \inf \Big\{ \|(y_i)\|_p^w \|(x_i)\|_{p'}^w : u = \sum_{i=1}^n y_i \otimes x_i \Big\},$$

taking the infimum over all representations of $u \in Y \otimes X$.

Clearly, $w_p = w_{p'}^t$, $w_1 = d_\infty$, and $w_\infty = g_\infty$. One can describe the Fourie–Swart tensor product in the following way. If $u \in Y \hat{\otimes}_{w_p} X$, then there exist sequences $(y_n) \in \ell_p^u(Y)$ and $(x_n) \in \ell_{p'}^u(X)$ such that $u = \sum_{n=1}^\infty y_n \otimes x_n$ in $Y \hat{\otimes}_{w_p} X$ (see [38, Proposition 3.2] or, e.g., [18, Corollary, p. 153]).

Theorem 1.4.6. *Let X be a Banach space and let $1 \le p \le \infty$. Then*

$$\mathcal{C}(\Omega) \hat{\otimes}_{w_{p'}} X = \mathcal{UC}_p(\Omega, X)$$

under the canonical isometric isomorphism that associates with each elementary tensor $\varphi \otimes x \in \mathcal{C}(\Omega) \otimes X$ *the function* $\varphi(\omega)x, \omega \in \Omega$.

Moreover, every $f \in \mathcal{UC}_p(\Omega, X)$ *can be represented as*

$$f(\omega) = \sum_{n=1}^{\infty} \varphi_n(\omega)x_n, \omega \in \Omega,$$

where $(\varphi_n) \in \ell_{p'}^u(\mathcal{C}(\Omega))$ *and* $(x_n) \in \ell_p^u(X)$*, where the series converges in* $\mathcal{UC}_p(\Omega, X)$.

Proof. It is almost verbatim to the proof of Theorem 1.3.7. We only need to replace the Chevet–Saphar norm d_p by the Fourie–Swart norm $w_{p'}$, $\mathcal{K}_p$ by $\mathcal{U}_p$, $\mathcal{N}^p$ by $\mathrm{K}_{p'} = \mathcal{N}_{w_{p'}}$, and $\mathcal{N}_p$ by $\mathrm{K}_p = \mathcal{N}_{w_p}$. Then, in a similar situation, we have $V \in \mathrm{K}_p(X^*, \mathcal{C}(\Omega))$ and, instead of Theorem 1.3.6, we use the corresponding Theorem 1.4.7 below that can be easily deduced from [65, Theorem 2.4]. To obtain the sequences (φ_n) and (x_n) as needed, we use the description of the Fourie–Swart tensor product, recalled above. □

Theorem 1.4.7. *Under the assumptions of Theorem 1.3.6, if* $T \in \mathrm{K}_p(X^*, Y)$ *and* $T^*(Y^*) \subset X$*, then* T *admits a representation* $T = \sum_{n=1}^{\infty} x_n \otimes y_n$ *with* $(x_n) \in \ell_p^u(X)$ *and* $(y_n) \in \ell_{p'}^u(Y)$*. In particular, the series* $\sum_{n=1}^{\infty} x_n \otimes y_n$ *converges in* $\mathrm{K}_p(X^*, Y)$.

Proof. Assuming that either X^{***} or Y has the approximation property, [65, Theorem 2.4] asserts: if $T \in \mathcal{N}_\alpha(X^*, Y)$ and $T^*(Y^*) \subset X$, then $T \in \overline{X \otimes Y}$ in $\mathcal{N}_\alpha(X^*, Y)$, where α is a tensor norm. In our case, $\alpha = w_p$ and $\overline{X \otimes Y} = X \hat{\otimes}_{w_p} Y$, giving the desired representation of T. □

Remark 1.4.8. Theorem 1.4.7 was also observed in [47, Corollary 3.6].

Remark 1.4.9. Concerning the special case $p = \infty$ of Theorem 1.4.6, i.e., $\mathcal{C}(\Omega)\hat{\otimes}_{w_1} X = \mathcal{C}(\Omega, X)$, recall that $w_1 = d_\infty$ and see Remark 1.3.8.

In [3] and [47], the notion of p-null sequence was extended to the notion of *unconditionally* p*-null* sequence by replacing $\ell_p(X)$ with $\ell_p^u(X)$ in the corresponding definition. For equivalent characterizations of unconditionally p-null sequences, the reader is referred to [3].

Let us denote by $uc_{0,p}(X)$ the *space of unconditionally* p*-null sequences* in a Banach space X. Similarly to $c_{0,p}(X)$, the space $uc_{0,p}(X)$ is equipped with the norm from $\mathcal{U}_p(\ell_1, X)$. And we obtain a similar result to Corollary 1.3.9 which describes $uc_{0,p}(X)$ using tensor products. Its proof is almost verbatim to the proof of Corollary 1.3.9.

Corollary 1.4.10 (see [47, Theorem 1.1])**.** *Let* X *be a Banach space and let* $1 \leq p \leq \infty$. *Then* $c_0 \hat{\otimes}_{w_{p'}} X = uc_{0,p}(X)$ *under the canonical isometric isomorphism that associates with each elementary tensor* $(a_n) \otimes x \in c_0 \otimes X$ *the sequence* $(a_n x) \subset X$.

Remark 1.4.11. The proof of Corollary 1.4.10 in [47] follows the proof [65, Theorem 4.1], completing it with ideas from [21]. It does not use Theorem 1.4.7 (that was used in the proof of Theorem 1.4.6 whose consequence Corollary 1.4.10 is).

Chapter 2

Topology in $\mathcal{C}_p(\Omega, X)$

This chapter collects some topological properties of $\mathcal{C}_p(\Omega, X)$. Namely, we obtain some results related to density of simple vector-valued functions in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$ (where $\mathcal{B}(\Sigma)$ denotes the space of all bounded Borel-measurable scalar functions defined on Ω), complemented embeddings of $\mathcal{C}(\Omega)$ and X in $\mathcal{C}_p(\Omega, X)$, and sequences in $\mathcal{C}_p(\Omega, X)$. In this chapter, we also study the weak and weak* convergences of sequences in $\mathcal{C}_p(\Omega, X)$.

2.1 Introduction

Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. In this chapter, we shall show the first properties of the space $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$. Some of them are proved in the more general space $Z\hat{\otimes}_{d_p}X$, where Z is a Banach space.

In Section 2.2, we prove some results related to density, complemented subspaces, and sequences. We show that simple X-valued functions are dense in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$ (Lemma 2.2.1). Afterwards, we show that $\mathcal{C}(\Omega)$ and X are isometric to 1-complemented subspaces of $\mathcal{C}_p(\Omega, X)$ (Lemma 2.2.2). Finally, we provide some conditions for a sequence $(z_n \otimes x_n) \subset Z\hat{\otimes}_{d_p}X$ to be absolutely q-summable or weakly q-summable (Proposition 2.2.3).

In the last Section 2.3, we address ourselves to the study of the weak and weak* convergences of sequences in $Z\hat{\otimes}_{d_p}X$. We start with a characterization of the weak* convergence in $(Z\hat{\otimes}_{d_p}X)^*$ (Theorem 2.3.1). In relation with the weak convergence, four results are proved: a necessary condition valid for all sequences in $\mathcal{C}_p(\Omega, X)$ (Theorem 2.3.2) and three sufficient conditions valid for sequences of one-dimensional tensors in $Z\hat{\otimes}_{d_p}X$ (Lemmas 2.3.3, 2.3.4, and 2.3.5). This special kind of sequences will be very useful in Chapter 5.

2.2 Density, complemented subspaces, and sequences

Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. It is well known that $\mathcal{S}(\Sigma, X) = \mathcal{S}(\Sigma) \otimes X$ is dense in the Banach space $\mathcal{B}(\Sigma, X)$. The next lemma proves that $\mathcal{S}(\Sigma, X)$ is also dense in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$.

Lemma 2.2.1. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Then $\mathcal{S}(\Sigma, X)$ is dense in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$.*

Proof. Let $f \in \mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$. We shall prove that there exists a sequence (f_j) in $\mathcal{S}(\Sigma, X)$ such that $f_j \to f$ in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$ when j tends to infinity.

If $f \in \mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$, then there exist sequences $(\varphi_i) \in \ell^w_{p'}(\mathcal{B}(\Sigma))$ and $(x_i) \in \ell_p(X)$ (or $(x_i) \in c_0(X)$ when $p = \infty$) such that $f = \sum_{i=1}^{\infty} \varphi_i \otimes x_i$ in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$ (see, e.g., [84, Proposition 6.10]). For each $\varphi_i \in \mathcal{B}(\Sigma)$, $i \in \mathbb{N}$, there exists a sequence of simple functions $(\varphi^i_n)_n \in \mathcal{S}(\Sigma)$ such that $\varphi^i_n \to_n \varphi_i$ uniformly for the supremum norm. Therefore, given $\varepsilon > 0$, for each $i \in \mathbb{N}$, there exists $n_i \geq n_{i-1}$ (we take $n_0 = 1$) such that $\|\varphi^i_{n_i} - \varphi_i\|_\infty < \frac{\varepsilon}{2\sup_i \|x_i\| \, 2^{n_i}}$. Let $(\psi_k)_k := (\varphi^k_{n_k})_k \subset \mathcal{S}(\Sigma)$ and consider the functions $f_j = \sum_{i=1}^{j} \psi_i \otimes x_i \in \mathcal{S}(\Sigma, X)$, $j \in \mathbb{N}$.

Let us check that $f_j \to f$ in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$. We have that

$$f_j - f = \sum_{i=1}^{j} \psi_i \otimes x_i - \sum_{i=1}^{\infty} \varphi_i \otimes x_i = \sum_{i=1}^{j} (\psi_i - \varphi_i) \otimes x_i - \sum_{i=j+1}^{\infty} \varphi_i \otimes x_i.$$

Then

$$d_p(f_j - f) \leq d_p\Big(\sum_{i=1}^{j} (\psi_i - \varphi_i) \otimes x_i\Big) + d_p\Big(\sum_{i=j+1}^{\infty} \varphi_i \otimes x_i\Big).$$

Since $f = \sum_{i=1}^{\infty} \varphi_i \otimes x_i \in \mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X$, there exists $j \in \mathbb{N}$ such that $d_p(\sum_{i=j+1}^{\infty} \varphi_i \otimes x_i) < \frac{\varepsilon}{2}$. Therefore

$$d_p(f_j - f) < d_p\Big(\sum_{i=1}^{j}(\psi_i - \varphi_i) \otimes x_i\Big) + \frac{\varepsilon}{2} \leq \sum_{i=1}^{j} d_p\Big((\psi_i - \varphi_i) \otimes x_i\Big) + \frac{\varepsilon}{2}$$

$$= \sum_{i=1}^{j} \|\psi_i - \varphi_i\|_\infty \|x_i\| + \frac{\varepsilon}{2} < \sum_{i=1}^{j} \frac{\varepsilon}{2\sup_i \|x_i\| \, 2^{n_i}} \sup_i \|x_i\| + \frac{\varepsilon}{2} < \varepsilon. \quad \square$$

The next lemma shows that $\mathcal{C}(\Omega)$ and X both are isometric to 1-complemented subspaces of $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X$.

Lemma 2.2.2. *Let Z and X be Banach spaces and let α be a tensor norm. Then Z and X are isometric to 1-complemented subspaces of $Z\hat{\otimes}_\alpha X$.*

Proof. We shall only prove the lemma for Z, the same argument can be used for X.

Let $x_0 \in X$ and $x_0^* \in X^*$ be such that $\|x_0\| = \|x_0^*\| = x_0^*(x_0) = 1$. Consider the linear map $j : Z \to Z\hat{\otimes}_\alpha X$ defined by $jz = z \otimes x_0$ for all $z \in Z$. Then j is an isometric embedding. Namely, for all $z \in Z$,

$$\alpha(jz) = \alpha(z \otimes x_0) = \|z\|\|x_0\| = \|z\|.$$

Let us consider now the linear map $P := I_Z \otimes (x_0^* \otimes x_0)$. It is clear that $\|P\| = 1$ because

$$\|P\| = \|I_Z \otimes (x_0^* \otimes x_0)\| = \|I_Z\|\|x_0^* \otimes x_0\| = \|x_0^*\|\|x_0\| = 1.$$

In fact, P is a norm one projection in $Z\hat{\otimes}_\alpha X$ since

$$P(z \otimes x) = (I_Z \otimes (x_0^* \otimes x_0))(z \otimes x) = z \otimes x_0^*(x)x_0 \tag{2.1}$$

and

$$P(P(z \otimes x)) = P(z \otimes x_0^*(x)x_0) = (I_Z \otimes (x_0^* \otimes x_0))(z \otimes x_0^*(x)x_0)$$

$$= z \otimes x_0^*(x)x_0^*(x_0)x_0 = z \otimes x_0^*(x)x_0$$

for all $z \in Z$ and $x \in X$. From (2.1), we have that $\operatorname{ran} j = \operatorname{ran} P$. $\square$

We finish this section with a proposition that collects some results that prove special cases in which a sequence $(z_n \otimes x_n) \subset Z\hat{\otimes}_{d_p}X$ belongs to $\ell_q(Z\hat{\otimes}_{d_p}X)$ or $\ell_q^w(Z\hat{\otimes}_{d_p}X)$. Recall that $(Z\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Z, X^*)$ as Banach spaces.

Proposition 2.2.3. *Let X and Z be Banach spaces and let $1 \leq p \leq \infty$. Assume that (x_n) and (z_n) are sequences in X and Y, respectively. The following statements hold.*

(i) *For every $1 \leq q \leq \infty$, if $(x_n) \in \ell_q(X)$ and $(z_n) \in \ell_\infty(Z)$, then $(z_n \otimes x_n) \in \ell_q(Z\hat{\otimes}_{d_p}X)$.*

(ii) *For every $1 \leq q \leq \infty$, if $(x_n) \in \ell_\infty(X)$ and $(z_n) \in \ell_q(Z)$, then $(z_n \otimes x_n) \in \ell_q(Z\hat{\otimes}_{d_p}X)$.*

(iii) *For every $q \geq p'$, if $(x_n) \in \ell_\infty(X)$ and $(z_n) \in \ell_q^w(Z)$, then $(z_n \otimes x_n) \in \ell_q^w(Z\hat{\otimes}_{d_p}X)$.*

Proof. For (i), let $(x_n) \in \ell_q(X)$ and $(z_n) \in \ell_\infty(Z)$. Then

$$\|(z_n \otimes x_n)\|_q \leq \sup_n \|z_n\| \|(x_n)\|_q < \infty,$$

proving that (i) holds. The same argument can be used to prove (ii).

For (iii), let $(x_n) \in \ell_\infty(X)$ and $(z_n) \in \ell_q^w(Z)$. Fix an arbitrary $A \in (Z\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Z, X^*) \subset \mathcal{P}_q(Z, X^*)$, because $p' \leq q$ (see, e.g., [26, p. 39, Theorem 2.8]). Then

$$\sum_{n=1}^{\infty} |\langle z_n \otimes x_n, A\rangle|^q = \sum_{n=1}^{\infty} |\langle x_n, Az_n\rangle|^q \leq \sup_n \|x_n\|^q \|(Az_n)\|_q^q$$

$$\leq \sup_n \|x_n\|^q \|A\|_{\mathcal{P}_q}^q (\|(z_n)\|_q^w)^q < \infty.$$

This yields that statement (iii) holds. □

Remark 2.2.4. In the proof of Proposition 5.2.11, it can be seen another very particular case: let X and Z be $\mathcal{L}_\infty$-spaces, and let $1 \leq p \leq \infty$ and $2 \leq q \leq \infty$; if $(x_n) \in \ell_q^w(X)$ and $(z_n) \in \ell_\infty(Z)$, then $(z_n \otimes x_n) \in \ell_q^w(Z\hat{\otimes}_{d_p}X)$.

2.3 Some results of convergence in $\mathcal{C}_p(\Omega, X)$

Let X and Z be Banach spaces and let $1 \le p \le \infty$. We start this section with a characterization of the weak* convergence in $(Z\hat{\otimes}_{d_p}X)^*$.

Theorem 2.3.1. *Let X and Z be Banach spaces and let $1 \le p \le \infty$. Assume that (S_n) is a bounded sequence in $(Z\hat{\otimes}_{d_p}X)^*$. The following are equivalent.*

(i) $S_n \to 0$ *in the weak* topology of* $(Z\hat{\otimes}_{d_p}X)^*$.

(ii) $S_n z \to 0$ *in the weak* topology of* X^* *for all* $z \in Z$.

Proof. (i)⇒(ii). Assume that $S_n \to 0$ in the weak* topology of $(Z\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Z, X^*)$. Then $\langle u, S_n\rangle \to 0$ for all $u \in Z\hat{\otimes}_{d_p}X$. Thus $\langle z\otimes x, S_n\rangle \to 0$ for all $z \in Z$ and $x \in X$. Since $\langle z \otimes x, S_n\rangle = \langle x, S_n z\rangle$, we obtain that $\langle x, S_n z\rangle \to 0$ for all $x \in X$, proving (ii).

(ii)⇒(i). Let $u \in Z\hat{\otimes}_{d_p}X$. Then there exist sequences $(z_i) \in \ell^w_{p'}(Z)$ and $(x_i) \in \ell_p(X)$ (or $(x_i) \in c_0(X)$ when $p = \infty$) such that $u = \sum_{i=1}^{\infty} z_i \otimes x_i$ in $Z\hat{\otimes}_{d_p}X$ (see, e.g., [84, Proposition 6.10]). Let $\varepsilon > 0$. There exists $i_0 \in \mathbb{N}$ such that

$$\|(x_i)_{i>i_0}\|_p < \frac{\varepsilon}{2M\|(z_i)\|^w_{p'}},$$

where $M = \sup\{\|(S_n)\|_{\mathcal{P}_{p'}} : n \in \mathbb{N}\}$. Since $\langle z_i \otimes x_i, S_n\rangle = \langle x_i, S_n z_i\rangle$, by (ii), there also exists $n_0 \in \mathbb{N}$ such that

$$|\langle z_i \otimes x_i, S_n\rangle| < \frac{\varepsilon}{2i_0}$$

for all $n \ge n_0$ and $i \le i_0$. Then

$$|\langle u, S_n\rangle| = \Big|\sum_{i=1}^{\infty}\langle x_i, S_n z_i\rangle\Big| \le \sum_{i=1}^{i_0}|\langle x_i, S_n z_i\rangle| + \sum_{i>i_0}|\langle x_i, S_n z_i\rangle|$$

$$< \frac{\varepsilon}{2} + \|(S_n z_i)_{i>i_0})\|_{p'}\|(x_i)_{i>i_0}\|_p \le \frac{\varepsilon}{2} + \|S_n\|_{\mathcal{P}_{p'}}\|(z_i)\|^w_{p'}\|(x_i)_{i>i_0}\|_p$$

$$< \frac{\varepsilon}{2} + M\|(z_i)\|^w_{p'}\|(x_i)_{i>i_0}\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n \ge n_0$. This proves the statement (i). □

Let X be a Banach space and let Ω be a compact Hausdorff space. Related to the weak convergence in $\mathcal{C}(\Omega, X)$, in [31, Theorem 9] (see, e.g., [15, p. 36, Proposition 1.7.1]), the following characterization holds. For every bounded sequence (f_n) in $\mathcal{C}(\Omega, X)$ and $f \in \mathcal{C}(\Omega, X)$, $f_n \to f$ in the weak topology of $\mathcal{C}(\Omega, X)$ if and only if $f_n(\omega) \to f(\omega)$ in the weak topology of X for all $\omega \in \Omega$. The next theorem proves that the analogous necessary condition holds in $\mathcal{C}_p(\Omega, X)$.

Theorem 2.3.2. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that (f_n) is a bounded sequence in $\mathcal{C}_p(\Omega, X)$ and $f \in \mathcal{C}_p(\Omega, X)$. If $f_n \to f$ in the weak topology of $\mathcal{C}_p(\Omega, X)$, then $f_n(\omega) \to f(\omega)$ in the weak topology of X for all $\omega \in \Omega$.*

Proof. Since $f \in \mathcal{C}_p(\Omega, X)$, there exist sequences $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ and $(x_i) \in \ell_p(X)$ (or $(x_i) \in c_0(X)$ when $p = \infty$) such that $f = \sum_{i=1}^{\infty} \varphi_i \otimes x_i$ in $\mathcal{C}_p(\Omega, X)$ (see, e.g., [84, Proposition 6.10]). Let $\omega \in \Omega$ and $x^* \in X^*$. Consider $\delta_\omega \otimes x^* \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*) = \mathcal{C}_p(\Omega, X)^*$, where $\delta_\omega \in \mathcal{C}(\Omega)^*$ is defined by $\langle \varphi, \delta_\omega \rangle = \varphi(\omega)$ for all $\varphi \in \mathcal{C}(\Omega)$ (see, e.g., [17, Chapter V, Theorem 8.4]). Then

$$\begin{aligned}\langle f, \delta_\omega \otimes x^* \rangle &= \sum_{i=1}^{\infty} \langle x_i, (\delta_\omega \otimes x^*)\varphi_i \rangle = \sum_{i=1}^{\infty} \langle x_i, \varphi_i(\omega) x^* \rangle \\ &= \langle \sum_{i=1}^{\infty} \varphi_i(\omega) x_i, x^* \rangle = \langle f(\omega), x^* \rangle.\end{aligned} \tag{2.2}$$

The same argument proves that $\langle f_n, \delta_\omega \otimes x^* \rangle = \langle f_n(\omega), x^* \rangle$ for all $n \in \mathbb{N}$.

By hypothesis, $\langle f_n, S \rangle \to \langle f, S \rangle$ for all $S \in \mathcal{C}_p(\Omega, X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$. In particular, this also holds for all $\delta_\omega \otimes x^* \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ with $\omega \in \Omega$ and $x^* \in X^*$. By (2.2), we obtain that $\langle f_n(\omega), x^* \rangle \to \langle f(\omega), x^* \rangle$ for all $\omega \in \Omega$ and $x^* \in X^*$, as desired. □

Related to the sufficient condition for the weak convergence of a sequence $(f_n) \subset Z \hat{\otimes}_{d_p} X$, the next three lemmas prove three different conditions which are sufficient in the special case when $f_n = z_n \otimes x_n$ where (z_n) and (x_n) are sequences in Z and X, respectively. This kind of functions f_n (and the corresponding sequences) is very simple, but it will be shown to be useful for some examples (see Chapter 5).

Let us recall that $(Z\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Z, X^*)$ as Banach spaces. In the proof of the next three lemmas, we shall use the well-known fact that absolutely q-summing operators, $1 \leq q < \infty$, are weakly compact and completely continuous (see, e.g., [26, p. 50, Theorem 2.17]).

Lemma 2.3.3. *Let X and Z be Banach spaces and let $1 < p \leq \infty$. If (z_n) is a weakly null sequence in Z and (x_n) is a bounded sequence in X, then $(z_n \otimes x_n)$ is a weakly null sequence in $Z\hat{\otimes}_{d_p}X$.*

Proof. Let $A \in (Z\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Z, X^*)$ arbitrary. Then

$$|\langle z_n \otimes x_n, A\rangle| = |\langle x_n, Az_n\rangle| \leq \|x_n\|\,\|Az_n\|.$$

As A is a completely continuous operator, the sequence (Az_n) is norm null convergent (because (z_n) is weakly null convergent). Then, given an arbitrary $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|Az_n\| < \frac{\varepsilon}{\|(x_n)\|_\infty}$ for all $n \geq n_0$. Therefore

$$|\langle z_n \otimes x_n, A\rangle| \leq \|x_n\|\,\|Az_n\| < \|x_n\|\frac{\varepsilon}{\|(x_n)\|_\infty} \leq \varepsilon$$

for all $n \geq n_0$. □

Lemma 2.3.4. *Let X and Z be Banach spaces and let $1 < p \leq \infty$. Assume that X enjoys the Dunford–Pettis property. If (z_n) is a bounded sequence in Z and (x_n) is a weakly null sequence in X, then the sequence $(z_n \otimes x_n)$ is weakly null in $Z\hat{\otimes}_{d_p}X$.*

Proof. Reasoning by contradiction, suppose that $\langle z_n \otimes x_n, A\rangle = \langle x_n, Az_n\rangle$ is not a null sequence for some $A \in \mathcal{P}_{p'}(Z, X^*) = (Z\hat{\otimes}_{d_p}X)^*$. Then passing to a subsequence, we may assume that $|\langle x_n, Az_n\rangle| > \varepsilon$ for some $\varepsilon > 0$ and for all $n \in \mathbb{N}$. As A is a weakly compact operator, the sequence (Az_n) admits a weakly convergent subsequence (Az_{n_k}) in X^*. Since X has the Dunford–Pettis property, $\lim_{k\to\infty}\langle x_{n_k}, Az_{n_k}\rangle = 0$, which is a contradiction. □

Lemma 2.3.5. *Let X and Z be Banach spaces and let $1 < p \leq \infty$. Assume that Z does not contain a subspace isomorphic to ℓ_1. If (z_n) is a bounded sequence in Z and (x_n) is a weakly null sequence in X, then $(z_n \otimes x_n)$ is weakly null in $Z\hat{\otimes}_{d_p}X$.*

Proof. Reasoning by contradiction, suppose that $\langle z_n \otimes x_n, A\rangle = \langle x_n, Az_n\rangle$ is not a null sequence for some $A \in \mathcal{P}_{p'}(Z, X^*) = (Z\hat{\otimes}_{d_p}X)^*$. Then passing to a subsequence, we may assume that $|\langle x_n, Az_n\rangle| > \varepsilon$ for some $\varepsilon > 0$ and for all $n \in \mathbb{N}$. Since $\ell_1 \not\subset Z$, by Rosenthal's ℓ_1 theorem, (z_n) admits a weak Cauchy subsequence (z_{n_k}). As A is a completely continuous operator, the sequence (Az_{n_k}) is norm convergent. If $x_0^* \in X^*$ denotes its limit, there exists $n_1 \in \mathbb{N}$ such that $\|Az_{n_k} - x_0^*\| < \frac{\varepsilon}{2\|(x_n)\|_\infty}$ for all $n \geq n_1$. There also exists $n_2 \in \mathbb{N}$ such that $|\langle x_{n_k}, x_0^*\rangle| < \frac{\varepsilon}{2}$ for all $n \geq n_2$. Thus, for $n \geq \max\{n_1, n_2\}$, we have that

$$\begin{aligned}|\langle x_{n_k}, Az_{n_k}\rangle| &\leq |\langle x_{n_k}, Az_{n_k} - x_0^*\rangle| + |\langle x_{n_k}, x_0^*\rangle| \\ &< \|x_{n_k}\|\|Az_{n_k} - x_0^*\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which is a contradiction. □

Remark 2.3.6. Notice that, in Lemma 2.3.3, where (z_n) is weakly null and (x_n) is bounded, no condition is imposed on the Banach spaces Z and X. In contrast, when (z_n) is bounded and (x_n) is weakly null, some conditions are imposed on the Banach space Z or X: in Lemma 2.3.4, no condition is imposed on Z but some condition is imposed on X; in Lemma 2.3.5, no condition is imposed on X but some condition is imposed on Z.

Remark 2.3.7. If a Banach space Z contains a subspace isomorphic to ℓ_1, then Lemma 2.3.5 may be untrue. We are going to show an example in $\mathcal{C}(\Delta)\hat{\otimes}_{d_p}L^2(\lambda)$, where $\Delta = \{-1, 1\}^{\mathbb{N}}$ is the Cantor group, λ is the Haar measure on Δ, (r_n) is the sequence of the Rademacher functions, and $p \leq 2$. Since Δ is not a dispersed compact Hausdorff space, $\ell_1 \subset \mathcal{C}(\Delta)$ (see, e.g., [70, Main Theorem]). The sequence $(r_n \otimes r_n)$ is not weakly null in $\mathcal{C}(\Delta)\hat{\otimes}_{d_p}L^2(\lambda)$. In fact, if $i_2 : \mathcal{C}(\Delta) \to L^2(\lambda)$ denotes the inclusion map, it is well known that i_2 is 2-summing (see, e.g., [26, p. 40]) and, therefore, p'-summing for all $p' \geq 2$ (see, e.g., [26, p. 39, Theorem 2.8]). Nevertheless, we have

$$\langle r_n \otimes r_n, i_2\rangle = \langle i_2 r_n, r_n\rangle = (r_n|r_n) = 1$$

for every $n \in \mathbb{N}$ (where $(\cdot|\cdot)$ denotes the inner product on $L^2(\lambda)$).

Chapter 3

Measures

This chapter focuses on a classical result of Analysis: integral representation of operators defined on continuous functions. In particular, we establish two integral representation theorems: one for operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ (which extends the classical Bartle–Dunford–Schwartz representation theorem (see, e.g., [27, p. 152, Theorem 1])) and another for operators $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ (which extends the classical Dinculeanu–Singer representation theorem (see, e.g., [27, p. 182])). We provide an alternative simpler proof of the latter result using the first one. We also build the needed integration theory. This chapter is based on [60].

3.1 Introduction

Let X and Y be a Banach spaces and let Ω be a compact Hausdorff space. Our main reference to the vector measure theory is the book [27] by Diestel and Uhl. In particular, a *vector measure* $m : \Sigma \to X$ is a finitely additive X-valued set function. The *semivariation of* m on Ω is denoted by $\|m\|(\Omega)$ and defined as

$$\|m\|(\Omega) = \sup \Big\| \sum_{E_i \in \Pi} \varepsilon_i m(E_i) \Big\|,$$

where the supremum is taken over all finite partitions $\Pi = (E_i)_{i=1}^n$ of Ω and all finite systems $(\varepsilon_i)_{i=1}^n$ with $|\varepsilon_i| \leq 1$, $1 \leq i \leq n$, $n \in \mathbb{N}$ (see, e.g., [27, p. 4, Proposition 11]). If $\|m\|(\Omega) < \infty$, then m is called a *measure of bounded semivariation*. A vector measure $m : \Sigma \to X$ is *bounded* if its range is bounded in X. This happens if and only if m is of bounded semivariation (see, e.g., [27, p. 4, Proposition 11]). Therefore, a vector measure of bounded semivariation is often called a *bounded vector measure* (see, e.g., [27, p. 5]), and we shall mainly use this term below.

Let $m : \Sigma \to Y$ be a vector measure of bounded semivariation. It is well known (see, e.g., [27, pp. 6, 56, 153]) that the (elementary Bartle) integral $\int_\Omega (\cdot)\, dm$ is defined on $\mathcal{B}(\Sigma)$. (The definition passes from characteristic functions to functions in $\mathcal{S}(\Sigma)$ by linearity and to functions in $\mathcal{B}(\Sigma)$ by density.) By the Bartle–Dunford–Schwartz representation theorem, for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ there exists a unique vector measure $m : \Sigma \to Y^{**}$ of bounded semivariation such that $S\varphi = \int_\Omega \varphi\, dm$ for all $\varphi \in \mathcal{C}(\Omega)$. The vector measure m is called the *representing measure* of S. Let us recall the statement of this theorem (see, e.g., [27, p. 152, Theorem 1]).

Theorem 3.1.1 (Bartle–Dunford–Schwartz)**.** *Let Y be a Banach space and let Ω be a compact Hausdorff space. For every $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ there exists a weak*-countably additive measure $m : \Sigma \to Y^{**}$ such that*

(i) *$\langle m(\cdot), y^* \rangle$ is a regular countably additive Borel measure for each $y^* \in Y^*$;*

(ii) *the map $Y^* \to \mathcal{C}(\Omega)^*$, $y^* \mapsto \langle m(\cdot), y^* \rangle$, is weak*-to-weak* continuous;*

(iii) *$\langle S\varphi, y^* \rangle = \int_\Omega \varphi\, d(\langle m(\cdot), y^* \rangle)$, for each $\varphi \in \mathcal{C}(\Omega)$ and each $y^* \in Y^*$; and*

(iv) *$\|S\| = \|m\|(\Omega)$.*

Conversely, any vector measure $m : \Sigma \to Y^{**}$ that satisfies (i) *and* (ii) *defines an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ by means of* (iii)*, and* (iv) *follows.*

In the next section, this representation is extended from $Y \cong \mathcal{L}(\mathbb{K}, Y)$ to $\mathcal{L}(X, Y)$. Namely, in the case when $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, a vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of bounded semivariation is built so that $S\varphi = \int_\Omega \varphi\, dm$ for all $\varphi \in \mathcal{C}(\Omega)$. We define a *representing measure* of $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ as a vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of bounded semivariation which satisfies

$$S\varphi = \int_\Omega \varphi\, dm \quad \text{for all } \varphi \in \mathcal{C}(\Omega).$$

In the next section, we also extend the Bartle–Dunford–Schwartz theorem, in all its aspects, to this general setting (see Theorem 3.2.5). We also find a formula connecting the measure m and the *classical* representing measure $\mu : \Sigma \to \mathcal{L}(X,Y)^{**}$ of S (as given by the Bartle–Dunford–Schwartz theorem) (see Corollary 3.2.10).

Results of Section 3.2 are applied in Section 3.4 to revisit the classical Dinculeanu–Singer representation theorem. By this theorem, for every operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, there exists a unique vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ such that

$$Uf = \int_\Omega f\, dm \quad \text{for all } f \in \mathcal{C}(\Omega, X),$$

where the existence of the above integral requires from the measure m that its Gowurin–Dinculeanu semivariation or, in our terminology (see Section 3.3), its 1-semivariation $\|m\|_1(\Omega)$ is bounded. Let us recall the statement of this theorem (see, e.g., [27, p. 182]).

Theorem 3.1.2 (Dinculeanu–Singer)**.** *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. For every $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ there exists a unique vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of bounded* 1-*semivariation such that*

(i) *for every $y^* \in Y^*$, the vector measure $m_{y^*} : \Sigma \to X^*$ defined by $\langle x, m_{y^*}(E)\rangle = \langle y^*, m(E)x\rangle$ for all $E \in \Sigma$ and $x \in X$, is regular;*

(ii) *the map $Y^* \to \mathcal{C}(\Omega, X)^*$, $y^* \mapsto m_{y^*}$, is weak*-to-weak* continuous;*

(iii) *$Uf = \int_\Omega f\, dm$ for all $f \in \mathcal{C}(\Omega, X)$;*

(iv) *$\|m\|_1(\Omega) = \|U\|$; and*

(v) *$U^*y^* = m_{y^*}$ for all $y^* \in Y^*$.*

Conversely, any vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of bounded 1-*semivariation that satisfies* (i) *and* (ii) *defines an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ by means of* (iii)*, and both* (iv) *and* (v) *follow.*

In Section 3.4, see Theorem 3.4.8, which is the main result of this chapter, we extend the Dinculeanu–Singer theorem, in all its aspects, from $\mathcal{C}(\Omega, X)$ to the Banach space $\mathcal{C}_p(\Omega, X)$ of p-continuous X-valued functions, where $1 \le p \le \infty$, the spaces $\mathcal{C}_p(\Omega, X)$ being contained in $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$. However, this is not a routine extension: we do not follow the traditional proofs of the Dinculeanu–Singer theorem (see Remark 3.4.11), but we provide a handy alternative to them.

The scheme of our proof is very simple: for $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, we consider the *associated operator* $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, defined by

$$(U^\# \varphi)x = U(\varphi x), \ \varphi \in \mathcal{C}(\Omega), \ x \in X.$$

By the above, we already have the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of $U^\#$. And we show (see Theorem 3.4.3) that $Uf = \int_\Omega f\, dm$ for all $f \in \mathcal{C}_p(\Omega, X)$, meaning that our m is also a representing measure of U.

On the other hand, one easily shows (see Proposition 3.4.4) that a representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ is also a representing measure of $U^\#$. Therefore, since the representing measure of $U^\#$ is unique, also the representing measure of U is unique. Hence, in the classical case when $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, we regain the classical representing measure from the Dinculeanu–Singer theorem. Moreover, for the first time in the literature, a general formula, connecting the representing measure m of U and the classical representing measure $\mu : \Sigma \to \mathcal{L}(X, Y)^{**}$ of $U^\#$, is given (see Corollary 3.4.6 and Remark 3.4.7).

In Section 3.3, since the integration on $\mathcal{C}_p(\Omega, X)$ requires from the measure more than just the boundedness of its semivariation (but less than the integration on $\mathcal{C}(\Omega, X)$), we build the needed theory. For this end, we introduce the concept of the *q-semivariation* of a vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of bounded semivariation. This enables us to define an integral on $\mathcal{C}_p(\Omega, X)$ with values in Y^{**}, provided that the p'-semivariation of m is bounded. In fact, the integral is defined on a larger space, where $\mathcal{C}_p(\Omega, X)$ sits as a closed subspace. This subspace contains $\mathcal{S}(\Sigma, X)$ and the integral extends the well-known "algebraic integral".

Finally, Section 3.5 is devoted to prove some qualitative complements to Theorem 3.4.8, our extension of the Dinculeanu–Singer theorem, it uses results from Chapter 4 and can be read just after Proposition 3.4.4.

3.2 Representing measure of $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. It is well known that, for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$, there exists a vector measure $m : \Sigma \to Y^{**}$ of bounded semivariation, called the *representing measure* of S, such that

$$S\varphi = \int_\Omega \varphi \, dm, \ \ \varphi \in \mathcal{C}(\Omega),$$

(see Theorem 3.1.1). We extend this result from $Y \cong \mathcal{L}(\mathbb{K}, Y)$ to $\mathcal{L}(X, Y)$: in the case when $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, we build a representing measure which takes its values in $\mathcal{L}(X, Y^{**})$ as follows.

So, let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. For every $x \in X$, we define an operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ by

$$S_x\varphi = (S\varphi)x, \varphi \in \mathcal{C}(\Omega).$$

Let $m_x : \Sigma \to Y^{**}$ be the representing measure of S_x given by the Bartle–Dunford–Schwartz theorem (see Theorem 3.1.1). Then the map $Y^* \to \mathcal{C}(\Omega)^*$, $y^* \mapsto \langle y^*, m_x(\cdot)\rangle$, is well defined and weak*-to-weak* continuous.

We define

$$m : \Sigma \to \mathcal{L}(X, Y^{**}),$$
$$E \mapsto m(E),$$

by

$$\langle y^*, m(E)x\rangle = \langle y^*, m_x(E)\rangle \tag{3.1}$$

for all $x \in X$ and $y^* \in Y^*$.

Proposition 3.2.1. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. The set function $m : \Sigma \to \mathcal{L}(X, Y^{**})$, defined by* (3.1)*, is a vector measure of bounded semivariation such that $\|m(E)\| \leq \|S\|$, $E \in \Sigma$.*

Proof. For all $x \in X$, clearly, $m(E)x \in Y^{**}$ and

$$|\langle y^*, m(E)x\rangle| \leq \|y^*\| \, \|m_x(E)\|.$$

We show that $m(E) \in \mathcal{L}(X, Y^{**})$ for all $E \in \Sigma$. For all $x_1, x_2 \in X$, $y^* \in Y^*$, and $\alpha \in \mathbb{K}$, we have that

$$\langle y^*, m(E)(x_1 + \alpha x_2)\rangle = \langle y^*, m_{x_1+\alpha x_2}(E)\rangle = \langle y^*, \int_\Omega \chi_E \, dm_{x_1+\alpha x_2}\rangle.$$

For each $\varphi \in \mathcal{C}(\Omega)$, it is clear that $S_{x_1+\alpha x_2}\varphi = S_{x_1}\varphi + \alpha S_{x_2}\varphi$. Or equivalently,

$$\int_\Omega \varphi\, dm_{x_1+\alpha x_2} = \int_\Omega \varphi\, dm_{x_1} + \alpha \int_\Omega \varphi\, dm_{x_2}.$$

For each $x \in X$, we can extend S_x to $\hat{S}_x \in \mathcal{L}(\mathcal{B}(\Sigma), Y^{**})$ by $\hat{S}_x\varphi = \int_\Omega \varphi\, dm_x$, $\varphi \in \mathcal{B}(\Sigma)$. Therefore,

$$\langle y^*, \int_\Omega \chi_E\, dm_{x_1+\alpha x_2}\rangle = \langle y^*, \hat{S}_{x_1+\alpha x_2}(\chi_E)\rangle = \langle y^*, \hat{S}_{x_1}(\chi_E) + \alpha \hat{S}_{x_2}(\chi_E)\rangle$$

$$= \langle y^*, m_{x_1}(E)\rangle + \langle y^*, \alpha m_{x_2}(E)\rangle = \langle y^*, m(E)x_1 + \alpha m(E)x_2\rangle.$$

Hence, $m(E)(x_1 + \alpha x_2) = m(E)x_1 + \alpha m(E)x_2$ showing that $m(E) : X \to Y^{**}$ is linear. Moreover, for all $x \in X$,

$$\|m(E)x\| = \sup_{\|y^*\|\le 1} |\langle y^*, m(E)x\rangle| \le \|m_x(E)\| \le \|m_x\|(E)$$

$$\le \|m_x\|(\Omega) = \|S_x\| \le \|S\|\,\|x\|$$

because $\|m_x\|(\Omega) = \|S_x\|$ (see Theorem 3.1.1). Therefore,

$$\|m(E)\| \le \|S\|,$$

showing that $m(E) \in \mathcal{L}(X, Y^{**})$ and the set function $m : \Sigma \to \mathcal{L}(X, Y^{**})$ has a bounded range.

Let $E_1, E_2 \in \Sigma$, $E_1 \cap E_2 = \emptyset$, $x \in X$, and $y^* \in Y^*$. Then

$$\langle y^*, m(E_1 \cup E_2)x\rangle = \langle y^*, m_x(E_1 \cup E_2)\rangle = \langle y^*, m_x(E_1) + m_x(E_2)\rangle$$

$$= \langle y^*, m_x(E_1)\rangle + \langle y^*, m_x(E_2)\rangle = \langle y^*, m(E_1)x\rangle + \langle y^*, m(E_2)x\rangle$$

$$= \langle y^*, (m(E_1) + m(E_2))x\rangle.$$

Thus, $m(E_1 \cup E_2) = m(E_1) + m(E_2)$, and m is finitely additive. Therefore, m is a vector measure.

Since m is a vector measure and its range is bounded, it follows that m is of bounded semivariation on Ω (see, e.g., [27, p. 4, Proposition 11]). $\square$

We can connect the integral with respect to m with the integral with respect to m_x as follows:

$$\langle y^*, \Big(\int_\Omega \varphi\, dm\Big)x\rangle = \langle y^*, \int_\Omega \varphi\, dm_x\rangle \tag{3.2}$$

for all $\varphi \in \mathcal{B}(\Sigma)$, $x \in X$, and $y^* \in Y^*$. (This equality follows easily from (3.1) using a standard argument which passes from characteristic functions to functions in $\mathcal{S}(\Sigma)$ by linearity, and finally to functions in $\mathcal{B}(\Sigma)$ by density.)

In particular, (3.2) is also true for $\varphi \in \mathcal{C}(\Omega)$. In this case, $\int_\Omega \varphi\, dm_x = S_x\varphi$, because m_x is the representing measure of S_x. Thus

$$\langle y^*, \Big(\int_\Omega \varphi\, dm\Big)x\rangle = \langle S_x\varphi, y^*\rangle = \langle (S\varphi)x, y^*\rangle$$

for all $\varphi \in \mathcal{C}(\Omega)$, $x \in X$, and $y^* \in Y^*$. Then,

$$S\varphi = \int_\Omega \varphi\, dm$$

for all $\varphi \in \mathcal{C}(\Omega)$, showing that m is a representing measure of S. The above integral is the restriction to $\mathcal{C}(\Omega)$ of the elementary Bartle integral $\int_\Omega (\cdot)dm$ defined on $\mathcal{B}(\Sigma)$.

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space.

Definition 3.2.2. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$. A *representing measure* of S is a bounded vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ which satisfies

$$S\varphi = \int_\Omega \varphi\, dm \quad \text{for all } \varphi \in \mathcal{C}(\Omega).$$

As it has been proved above, a representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ exists for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$. Since we are going to use such a measure, it would be good (but not crucial) to know that it is unique. We start by a general observation that will also be used in Section 3.3.

Let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Then, for every $x \in X$,

$$m_x := m(\cdot)x : \Sigma \to Y^{**}$$

is clearly a bounded vector measure (we use the same notation as above because, as we shall prove in Lemma 3.2.3 below, this is, in the case when $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ is given, the (classical) representing measure of the operator S_x). If $y^* \in Y^*$, then $x \otimes y^* \in \mathcal{L}(X, Y^{**})^*$ and for all $\varphi \in \mathcal{B}(\Sigma)$,

$$\langle \int_\Omega \varphi \, dm, x \otimes y^* \rangle = \int_\Omega \varphi \, d\mu_{x,y^*} = \langle y^*, \int_\Omega \varphi \, dm_x \rangle, \tag{3.3}$$

where $\mu_{x,y^*} := (x \otimes y^*)m$, because

$$\begin{aligned} \mu_{x,y^*}(E) &= ((x \otimes y^*)m)(E) = \langle m(E), x \otimes y^* \rangle \\ &= \langle y^*, m(E)x \rangle = \langle y^*, m_x(E) \rangle, \quad E \in \Sigma. \end{aligned}$$

Lemma 3.2.3. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be a representing measure of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$. Then $m_x : \Sigma \to Y^{**}$ is the (classical) representing measure of the operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ (given by the Bartle–Dunford–Schwartz theorem) and $\mu_{x,y^*} = S_x^* y^*$ for all $x \in X$ and $y^* \in Y^*$.*

Proof. For all $\varphi \in \mathcal{C}(\Omega)$, $x \in X$ and $y^* \in Y^*$, by (3.3),

$$\langle S_x \varphi, y^* \rangle = \langle (S\varphi)x, y^* \rangle = \langle S\varphi, x \otimes y^* \rangle = \langle y^*, \int_\Omega \varphi \, dm_x \rangle.$$

This shows that m_x is the representing measure of the operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$. Hence $S_x^* \in \mathcal{L}(Y^*, \mathcal{C}(\Omega)^*)$ and, by the Bartle–Dunford–Schwartz theorem, it is well known that $S_x^* y^* = \langle y^*, m_x(\cdot) \rangle = \mu_{x,y^*}$. □

Proposition 3.2.4. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Then the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ is unique.*

Proof. Let $m_1, m_2 : \Sigma \to \mathcal{L}(X, Y^{**})$ be two representing measures of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$, and let $\mu^i_{x,y^*} = (x \otimes y^*)m_i$, $i = 1, 2$. We know from Lemma 3.2.3 that $\mu^1_{x,y^*} = S_x^* y^* = \mu^2_{x,y^*}$, giving that $\langle y^*, m_1(E)x \rangle = \langle y^*, m_2(E)x \rangle$, for all $E \in \Sigma$, $x \in X$, and $y^* \in Y^*$. This means that $m_1 = m_2$. □

Recalling that $\mathcal{L}(\mathbb{K}, Y) \cong Y$, the following result extends the classical Bartle–Dunford–Schwartz theorem (see Theorem 3.1.1) in all its aspects.

Theorem 3.2.5. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space.*

(a) *Every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ has a unique representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$.*

(b) *Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ such that m is its representing measure if and only if for all $x \in X$,*

$$\mu_{x,y^*} = \langle y^*, m_x(\cdot)\rangle \in \mathcal{C}(\Omega)^*, \quad y^* \in Y^*,$$

and the map $Y^* \to \mathcal{C}(\Omega)^*$, $y^* \mapsto \langle y^*, m_x(\cdot)\rangle$, is linear, bounded, and weak-to-weak* continuous.*

In this case, $m_x : \Sigma \to Y^{**}$ is the representing measure of the operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ and $\mu_{x,y^*} = S_x^* y^*$ for all $x \in X$ and $y^* \in Y^*$, the equality $\|S\| = \|m\|(\Omega)$ holds, and the measure $m : \Sigma \to \mathcal{L}(X, Y^{**}) = (X \hat{\otimes}_\pi Y^*)^*$ is weak-countably additive.*

Proof. (a) The existence of a representing measure was proved in the beginning of this section. Its uniqueness comes from Proposition 3.2.4.

(b) By Lemma 3.2.3, we know that $m_x : \Sigma \to Y^{**}$ is the representing measure of $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$. Then $S_x^* \in \mathcal{L}(Y^*, \mathcal{C}(\Omega)^*)$ is weak*-to-weak* continuous. Also, by Lemma 3.2.3, $S_x^* y^* = \mu_{x,y^*} = \langle y^*, m_x(\cdot)\rangle$. This proves the "only if" part.

For the "if" part, let S be the restriction to $\mathcal{C}(\Omega)$ of the integration operator $\int_\Omega \varphi \, dm$, $\varphi \in \mathcal{B}(\Sigma)$. Then $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y^{**}))$. It remains to show that $(S\varphi)x \in Y$ for all $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. By (3.3),

$$\langle y^*, (S\varphi)x\rangle = \langle S\varphi, x \otimes y^*\rangle = \langle y^*, \int_\Omega \varphi \, dm_x\rangle, \quad y^* \in Y^*.$$

Hence, $(S\varphi)x \in Y^{**}$ is the composition of the weak*-to-weak* continuous map $y^* \mapsto \langle y^*, m_x(\cdot)\rangle$ from Y^* to $\mathcal{C}(\Omega)^*$ and the weak* continuous functional $\mu \mapsto \int_\Omega \varphi \, d\mu$ on $\mathcal{C}(\Omega)^*$. Therefore, $(S\varphi)x$ is a weak* continuous functional on Y^*, and $(S\varphi)x \in Y$ as desired.

For the "in this case" part, the first two claims come from Lemma 3.2.3.

Let us prove the third claim (for an alternative proof, see Corollary 3.2.8 below). Denoting by $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, Y^{**}))$ the integration operator $\hat{S}\varphi = \int_\Omega \varphi\, dm$, $\varphi \in \mathcal{B}(\Sigma)$, we see that $\hat{S}$ extends S (here, as usual, we identify Y with a subspace of Y^{**}). Hence,

$$\|S\| \leq \|\hat{S}\| = \|m\|(\Omega);$$

for the last equality see, e.g., [27, Theorem 13, p. 6].

On the other hand, when we look at $\mathcal{B}(\Sigma)$ as a closed subspace of $\mathcal{C}(\Omega)^{**}$, then $\hat{S}$ is the restriction to $\mathcal{B}(\Sigma)$ of $S^{**} \in \mathcal{L}(\mathcal{C}(\Omega)^{**}, \mathcal{L}(X, Y^{**})^{**})$ (this is a standard argument; see, e.g., [27, pp. 152–153]). Hence,

$$\|m\|(\Omega) = \|\hat{S}\| \leq \|S^{**}\| = \|S\|.$$

Finally, to show the weak*-countable additivity of m, let (E_n) be a sequence of pairwise disjoint members of Σ. Denote $f_k := \sum_{n=1}^{k} m(E_n)$, $k \in \mathbb{N}$, and $f := m(\bigcup_{n=1}^{\infty} E_n)$. By the countable additivity of μ_{x,y^*}, we have

$$\langle x \otimes y^*, f\rangle = \mu_{x,y^*}(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu_{x,y^*}(E_n) = \lim_{k\to\infty} \langle x \otimes y^*, f_k\rangle$$

for all $x \in X$ and $y^* \in Y^*$. Since also the sequence (f_k) is bounded (in fact, $\|f_k\| \leq \|m\|(\Omega)$), $f_k \to f$ pointwise on $X\hat{\otimes}_\pi Y^*$. This means that

$$\langle u, m(\bigcup_{n=1}^{\infty} E_n)\rangle = \sum_{n=1}^{\infty} \langle u, m(E_n)\rangle$$

for all $u \in X\hat{\otimes}_\pi Y^*$, as desired. □

In the classical case when $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ and $m : \Sigma \to Y^{**}$ is its representing measure, the integration operator $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), Y^{**})$, $\hat{S}\varphi = \int_\Omega \varphi\, dm$, $\varphi \in \mathcal{B}(\Sigma)$, extends the operator S from $\mathcal{C}(\Omega)$ to $\mathcal{B}(\Sigma)$, where $\mathcal{C}(\Omega)$ sits as a closed subspace. And, in turn, $S^{**} \in \mathcal{L}(\mathcal{C}(\Omega)^{**}, Y^{**})$ extends the operator $\hat{S}$ from $\mathcal{B}(\Sigma)$ to $\mathcal{C}(\Omega)^{**}$, where $\mathcal{B}(\Sigma)$ sits as a closed subspace.

In the proof of Theorem 3.2.5 (b), we pointed out that a similar phenomenon occurs also in the general case when $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. In the following, we shall make this precise.

Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. Then, as above, the integration operator $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, Y^{**}))$ extends the operator S. More precisely, let

$$J : \mathcal{L}(X, Y) \to \mathcal{L}(X, Y^{**}), \ J(A) = j_Y A, \ A \in \mathcal{L}(X, Y),$$

be the natural isometric embedding. Then

$$JS = \hat{S}|_{\mathcal{C}(\Omega)}.$$

To understand in which sense S^{**} "extends" $\hat{S}$, recall that $\mathcal{L}(X, Y^{**}) = (X \hat{\otimes}_\pi Y^*)^*$ as Banach spaces (π denotes the projective tensor norm, as usual), and put

$$P := (j_{X \hat{\otimes}_\pi Y^*})^*.$$

Then P is the (natural) projection from $\mathcal{L}(X, Y^{**})^{**} = (X \hat{\otimes}_\pi Y^*)^{***}$ onto $\mathcal{L}(X, Y^{**}) = (X \hat{\otimes}_\pi Y^*)^*$.

Theorem 3.2.6. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. Then, with the above notation,*

$$\hat{S} = PJ^{**}S^{**}|_{\mathcal{B}(\Sigma)}. \tag{3.4}$$

Proof. It suffices to verify that

$$\hat{S}\chi_E = PJ^{**}S^{**}\chi_E \quad \text{for all } E \in \Sigma.$$

Then by linearity, (3.4) holds on $\mathcal{S}(\Sigma)$, and by density, (3.4) holds on $\mathcal{B}(\Sigma)$.

For this end, in turn, it suffices to verify that

$$\langle x \otimes y^*, \hat{S}\chi_E \rangle = \langle x \otimes y^*, PJ^{**}S^{**}\chi_E \rangle, \quad x \in X, y^* \in Y^*. \tag{3.5}$$

For the left-hand side of (3.5), we have

$$\langle x \otimes y^*, \hat{S}\chi_E \rangle = \langle x \otimes y^*, m(E) \rangle = \langle y^*, m(E)x \rangle = \mu_{x,y^*}(E). \tag{3.6}$$

For the right-hand side of (3.5), we have, considering $\mathcal{C}(\Omega)^*$ embedded in $\mathcal{B}(\Sigma)^*$,

$$\langle x \otimes y^*, PJ^{**}S^{**}\chi_E \rangle = \langle j_{X \hat{\otimes}_\pi Y^*}(x \otimes y^*), J^{**}S^{**}\chi_E \rangle$$

$$= \langle \chi_E, S^* J^* j_{X\hat{\otimes}_\pi Y^*}(x \otimes y^*)\rangle = \langle \chi_E, S^*(x \otimes y^*)\rangle,$$

because

$$\langle A, J^* j_{X\hat{\otimes}_\pi Y^*}(x \otimes y^*)\rangle = \langle x \otimes y^*, J(A)\rangle = \langle y^*, j_Y Ax\rangle$$

$$= \langle Ax, y^*\rangle = \langle A, x \otimes y^*\rangle$$

for all $A \in \mathcal{L}(X, Y)$. But it is clear that

$$S^*(x \otimes y^*) = S_x^* y^*.$$

Indeed, for every $\varphi \in \mathcal{C}(\Omega)$, we have

$$\langle \varphi, S^*(x \otimes y^*)\rangle = \langle S\varphi, x \otimes y^*\rangle = \langle (S\varphi)x, y^*\rangle = \langle S_x \varphi, y^*\rangle = \langle \varphi, S_x^* y^*\rangle.$$

Therefore, using that $S_x^* y^* = \mu_{x,y^*}$ (see Lemma 3.2.3), we obtain

$$\langle x \otimes y^*, PJ^{**}S^{**}\chi_E\rangle = \langle \chi_E, \mu_{x,y^*}\rangle = \mu_{x,y^*}(E). \tag{3.7}$$

From (3.6) and (3.7), we get that (3.5) holds. □

Remark 3.2.7. The above proof does not require the uniqueness of the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ nor how the measure m is built.

Corollary 3.2.8 (see Theorem 3.2.5 and its proof)**.** *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. Then $\|m\|(\Omega) = \|S\|$.*

Proof. It is well known that $\|m\|(\Omega) = \|\hat{S}\|$ (see, e.g., [27, p. 6, Theorem 13]). But

$$\|S\| = \|JS\| = \|\hat{S}|_{\mathcal{C}(\Omega)}\| \le \|\hat{S}\| = \|PJ^{**}S^{**}|_{\mathcal{B}(\Sigma)}\| \le \|S\|. \qquad \square$$

Thanks to Theorem 3.2.6, we have an alternative proof for the uniqueness of the representing measure m.

Corollary 3.2.9 (see Proposition 3.2.4)**.** *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Then the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ is unique.*

Proof. Let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be a representing measure of $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then for all $E \in \Sigma$, we have

$$m(E) = \hat{S}\chi_E = PJ^{**}S^{**}\chi_E.$$

Hence, if $m_1, m_2 : \Sigma \to \mathcal{L}(X, Y^{**})$ are representing measures of S, then $m_1(E) = m_2(E)$ for all $E \in \Sigma$. □

For $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, together with its representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$, there also exists its classical representing measure, say $\mu : \Sigma \to \mathcal{L}(X, Y)^{**}$ (given by the Bartle–Dunford–Schwartz theorem). Let $\hat{\hat{S}} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, Y)^{**})$ denote the corresponding integration operator, i.e., $\hat{\hat{S}} = \int_\Omega \varphi \, d\mu$, $\varphi \in \mathcal{B}(\Sigma)$. As is well known (this was also mentioned above), $S^{**}|_{\mathcal{B}(\Sigma)} = \hat{\hat{S}}$. Hence, Theorem 3.2.6 tells us that

$$\hat{S} = PJ^{**}\hat{\hat{S}}.$$

On characteristic functions, this gives the following formula (3.8) which connects the measures m and μ.

Corollary 3.2.10. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ and $\mu : \Sigma \to \mathcal{L}(X, Y)^{**}$ be its representing measures. Then*

$$m(E) = PJ^{**}\mu(E) \quad \text{for all } E \in \Sigma. \tag{3.8}$$

Moreover, if S is weakly compact, then m takes its values in $\mathcal{L}(X, Y)$, and the measures m and μ coincide. In this case, the measure $m : \Sigma \to \mathcal{L}(X, Y)$ is countably additive and regular.

Proof. By the above, only the "moreover" part needs a proof. From the Bartle–Dunford–Schwartz theory [7] (see, e.g., [27, p. 153, Theorem 5]), it is well known that if $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ is weakly compact, then μ takes its values in $\mathcal{L}(X, Y)$ and $\mu : \Sigma \to \mathcal{L}(X, Y)$ is countably additive. It is also regular (see [27, p. 159, Corollary 14]). But for every $A \in \mathcal{L}(X, Y)$, considering $\mathcal{L}(X, Y)$ embedded in $\mathcal{L}(X, Y)^{**}$, we have

$$\langle x \otimes y^*, PJ^{**}(A) \rangle = \langle j_{X \hat{\otimes}_\pi Y^*}(x \otimes y^*), J^{**}(A) \rangle$$

$$= \langle A, J^* j_{X\hat{\otimes}_\pi Y^*}(x \otimes y^*)\rangle = \langle x \otimes y^*, j_Y A\rangle$$

for all $x \in X$ and $y^* \in Y^*$, implying that $PJ^{**}(A) = j_Y A$. Therefore, by (3.8),

$$m(E) = PJ^{**}\mu(E) = j_Y \mu(E)$$

for all $E \in \Sigma$. This means that m takes its values in $\mathcal{L}(X,Y)$ and considering $\mathcal{L}(X,Y)$ embedded in $\mathcal{L}(X,Y^{**})$, the measures $m : \Sigma \to \mathcal{L}(X,Y)$ and $\mu : \Sigma \to \mathcal{L}(X,Y)$ coincide. □

The next example shows that the fact that the representing measure $m : \Sigma \to \mathcal{L}(X,Y^{**})$ of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ takes its values in $\mathcal{L}(X,Y)$ does not imply the weak compactness of the operator S.

Example 3.2.11. Denote by $\beta\mathbb{N}$ the Stone–Čech compactification of $\mathbb{N}$. As is well known, $\mathcal{C}(\beta\mathbb{N}) = \ell_\infty$. Consider the identity operator $I \in \mathcal{L}(\ell_\infty, \ell_\infty) = \mathcal{L}(\mathcal{C}(\beta\mathbb{N}), \mathcal{L}(\ell_1, \mathbb{K}))$. Since ℓ_∞ is not reflexive, I is a non-weakly compact operator. However, its representing measure $m : \Sigma \to \mathcal{L}(\ell_1, \mathbb{K}^{**})$ takes its values in $\mathcal{L}(\ell_1, \mathbb{K}) = \mathcal{L}(\ell_1, \mathbb{K}^{**})$.

Remark 3.2.12. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ and let $m : \Sigma \to \mathcal{L}(X,Y^{**})$ be the representing measure of S. By definition of the measures m_x, the measure m takes its values in $\mathcal{L}(X,Y)$ if and only if all $m_x : \Sigma \to Y^{**}$, $x \in X$, take their values in Y. Since m_x is the representing measure of S_x, by the Bartle–Dunford–Schwartz theory (see, e.g., [27, p. 153, Theorem 5]), this is equivalent to the fact that all operators $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$, $x \in X$, are weakly compact. This clearly happens when S is weakly compact.

Remark 3.2.13. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$. In the above, we only needed (and used) the fact (from the beginning of this section) that a representing measure $m : \Sigma \to \mathcal{L}(X,Y^{**})$ exists for S.

3.3 Integration of p-continuous vector-valued functions with respect to an operator-valued measure

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Let $m : \Sigma \to \mathcal{L}(X,Y)$ be a vector measure. It is well known that the "algebraic" integral $\int_\Omega (\cdot)\, dm$ is defined on $\mathcal{S}(\Sigma, X)$. (The definition passes from vector-valued characteristic functions $\chi_E x$, $E \in \Sigma$, $x \in X$, to functions in $\mathcal{S}(\Sigma, X)$ by linearity.)

The classical Dinculeanu–Singer representation theorem requires the integration on $\mathcal{C}(\Omega, X)$. The corresponding integral was built by Dinculeanu (see [30, II.7.1, II.9,1, and p. 398, Theorem 9]; an early idea of this integral can be found in [41] and [6]). In fact, the Dinculeanu integral was built on $\mathcal{B}(\Sigma, X)$, where $\mathcal{C}(\Omega, X)$ sits as a closed subspace, and then restricted to $\mathcal{C}(\Omega, X)$. On the other hand, the Dinculeanu integral restricted to $\mathcal{S}(\Sigma, X)$ coincides with the "algebraic" integral.

The existence of the Dinculeanu integral requires from m much more than does the existence of the elementary Bartle integral, where the semivariation $\|m\|(\Omega)$ was needed to be finite. Namely, a much bigger "semivariation" than $\|m\|(\Omega)$ must be finite. Let us call it the *Gowurin–Dinculeanu semivariation* (it was introduced by Gowurin [41] and deeply studied by Dinculeanu (see, e.g., [30, I.4])).

To be able to integrate on $\mathcal{C}_p(\Omega, X)$, we shall need an "intermediate semivariation", depending on p, which, in the "limit" cases for $\mathcal{C}_1(\Omega, X)$ and $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, coincides with the (usual) semivariation $\|m\|(\Omega)$ and the Gowurin–Dinculeanu semivariation, respectively (see Example 3.3.1 below).

Before introducing our "intermediate semivariation", we shall need the description of the dual space $\mathcal{C}_p(\Omega, X)^*$ as a space of operators from $\mathcal{C}(\Omega)$ to X^*. Recall that $(Z \hat{\otimes}_{d_p} X)^* = \mathcal{P}_{p'}(Z, X^*)$ as Banach spaces (here Z is an arbitrary Banach space). (Recall that $\mathcal{P}_q = (\mathcal{P}_q, \|\cdot\|_{\mathcal{P}_q})$, $1 \leq q \leq \infty$, denotes the Banach operator ideal of absolutely q-summing operators.) Since $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ as Banach spaces, we have

$$\mathcal{C}_p(\Omega, X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*),$$

as Banach spaces, under the duality

$$\langle \varphi x, T \rangle = \langle x, T\varphi \rangle, \quad \varphi \in \mathcal{C}(\Omega), x \in X, T \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*).$$

Let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Notice that this clearly encompasses the seemingly more general case when m takes its values in $\mathcal{L}(X, Y)$, because Y is canonically embedded in Y^{**}. Then, for every $y^* \in Y^*$,

$$m_{y^*} := (m(\cdot))^* y^* : \Sigma \to X^*$$

is clearly a bounded vector measure. From the beginning of Section 3.2, we know that

$$\langle x, m_{y^*}(E)\rangle = \langle x, (m(E))^* y^*\rangle = \langle y^*, m(E)x\rangle = \mu_{x,y^*}(E),$$

and therefore, for all $\varphi \in \mathcal{B}(\Sigma)$,

$$\langle \int_\Omega \varphi \, dm, x \otimes y^* \rangle = \int_\Omega \varphi \, d\mu_{x,y^*} = \langle x, \int_\Omega \varphi \, dm_{y^*}\rangle. \tag{3.9}$$

Denote by I_{y^*} the restriction of the latter integral from $\mathcal{B}(\Sigma)$ to $\mathcal{C}(\Omega)$, i.e., for every $y^* \in Y^*$,

$$I_{y^*}\varphi = \int_\Omega \varphi \, dm_{y^*}, \quad \varphi \in \mathcal{C}(\Omega).$$

Then $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ and $m_{y^*} : \Sigma \to X^*$ is its representing measure.

Let $1 \leq q \leq \infty$. We define the *q-semivariation* $\|m\|_q(\Omega)$ of a bounded vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ by

$$\|m\|_q(\Omega) = \sup_{y^* \in B_{Y^*}} \|I_{y^*}\|_{\mathcal{P}_q}.$$

We say that a bounded vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is of *bounded q-semivariation* if $\|m\|_q(\Omega) < \infty$. It follows from the inclusion theorem for absolutely q-summing operators (see, e.g., [26, p. 39, Theorem 2.8]) that

$$\|m\|_\infty(\Omega) \leq \|m\|_q(\Omega) \leq \|m\|_p(\Omega) \leq \|m\|_1(\Omega) \quad \text{if } 1 \leq p \leq q \leq \infty.$$

Example 3.3.1. Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Then $\|m\|_\infty(\Omega) = \|m\|(\Omega)$, the semivariation of m, and $\|m\|_1(\Omega)$ coincides with the Gowurin–Dinculeanu semivariation.

Proof. Let $y^* \in Y^*$. Since $(\mathcal{P}_\infty, \|\cdot\|_{\mathcal{P}_\infty}) = (\mathcal{L}, \|\cdot\|)$, we have that $\|I_{y^*}\|_{\mathcal{P}_\infty} = \|I_{y^*}\|$. And since m_{y^*} is the representing measure of $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$, we have that $\|I_{y^*}\| = \|m_{y^*}\|(\Omega)$, by the Bartle–Dunford–Schwartz theorem. Therefore

$$\|m\|_\infty(\Omega) = \sup_{y^* \in B_{Y^*}} \|m_{y^*}\|(\Omega)$$

$$= \sup\Big\{\Big\|\sum_{E_i\in\Pi}\varepsilon_i m_{y^*}(E_i)\Big\| \,:\, y^*\in B_{Y^*}, |\varepsilon_i|\le 1, \Pi\Big\}$$

$$= \sup\Big\{\Big|\langle x, \sum_{E_i\in\Pi}\varepsilon_i m_{y^*}(E_i)\rangle\Big| \,:\, x\in B_X, y^*\in B_{Y^*}, |\varepsilon_i|\le 1, \Pi\Big\}$$

$$= \sup\Big\{\Big|\langle y^*, \Big(\sum_{E_i\in\Pi}\varepsilon_i m(E_i)\Big)x\rangle\Big| \,:\, x\in B_X, y^*\in B_{Y^*}, |\varepsilon_i|\le 1, \Pi\Big\}$$

$$= \sup\Big\{\Big\|\Big(\sum_{E_i\in\Pi}\varepsilon_i m(E_i)\Big)x\Big\| \,:\, x\in B_X, |\varepsilon_i|\le 1, \Pi\Big\}$$

$$= \sup\Big\{\Big\|\sum_{E_i\in\Pi}\varepsilon_i m(E_i)\Big\| \,:\, |\varepsilon_i|\le 1, \Pi\Big\} = \|m\|(\Omega).$$

We know that $\mathcal{P}_1(\mathcal{C}(\Omega), X^*) = \mathcal{C}_\infty(\Omega, X)^* = \mathcal{C}(\Omega, X)^*$. We also know that $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ is absolutely summing, i.e., $I_{y^*} \in \mathcal{P}_1(\mathcal{C}(\Omega), X^*)$ if and only if its representing measure m_{y^*} is of bounded variation, and in this case, $\|I_{y^*}\|_{\mathcal{P}_1} = |m_{y^*}|(\Omega)$ (see, e.g., [27, p. 162, Theorem 3]). Hence

$$\|m\|_1(\Omega) = \sup_{y^*\in B_{Y^*}} |m_{y^*}|(\Omega), \tag{3.10}$$

which, thanks to [30, p. 55, Proposition 5], coincides with the Gowurin–Dinculeanu semivariation of m. Let us recall that in [27, p. 181], formula (3.10) is taken as the definition of the Gowurin–Dinculeanu semivariation of m. □

Below, we shall need the following result which, among others, may be used for calculating $\|m\|_q(\Omega)$. For $y^* \in Y^*$, let

$$\hat{I}_{y^*} := \int_\Omega (\cdot)\, dm_{y^*} \in \mathcal{L}(\mathcal{B}(\Sigma), X^*)$$

denote the integration operator with respect to m_{y^*}.

Proposition 3.3.2. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le q \le \infty$. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then*

$$\|\hat{I}_{y^*}\|_{\mathcal{P}_q} = \|I_{y^*}\|_{\mathcal{P}_q} \quad \textit{for all } y^* \in Y^*.$$

Proof. Since $\mathcal{C}(\Omega) \subset \mathcal{B}(\Sigma) \subset \mathcal{C}(\Omega)^{**}$ as closed subspaces, $\hat{I}_{y^*}$ is an extension of I_{y^*}, and $(I_{y^*})^{**}$ is an extension of $\hat{I}_{y^*}$, we have that

$$\|I_{y^*}\|_{\mathcal{P}_q} \leq \|\hat{I}_{y^*}\|_{\mathcal{P}_q} \leq \|(I_{y^*})^{**}\|_{\mathcal{P}_q}.$$

Hence, if $\|I_{y^*}\|_{\mathcal{P}_q} = \infty$, then also $\|\hat{I}_{y^*}\|_{\mathcal{P}_q} = \infty$. If $\|I_{y^*}\|_{\mathcal{P}_q} < \infty$, i.e., I_{y^*} is absolutely q-summing, then also $(I_{y^*})^{**}$ is, and in this case, $\|(I_{y^*})^{**}\|_{\mathcal{P}_q} = \|I_{y^*}\|_{\mathcal{P}_q}$ (see, e.g., [26, p. 50, Proposition 2.19]). Therefore $\|I_{y^*}\|_{\mathcal{P}_q} = \|\hat{I}_{y^*}\|_{\mathcal{P}_q}$, as desired. □

It is well known (see, e.g., [84, p. 11]) that $\mathcal{B}(\Sigma) \otimes X \subset \mathcal{B}(\Sigma, X)$ as a linear subspace, under the algebraic identification $\varphi \otimes x \leftrightarrow \varphi x$. This is used in the following result.

Theorem 3.3.3. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then the formula*

$$\int_\Omega (\varphi x)\, dm = \Big(\int_\Omega \varphi\, dm\Big) x, \quad \varphi \in \mathcal{B}(\Sigma), x \in X, \tag{3.11}$$

defines an integral on $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X$ with respect to m if and only if $\|m\|_{p'}(\Omega) < \infty$. In this case, the integration operator $\hat{U}$ belongs to $\mathcal{L}(\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X, Y^{**})$, $\|\hat{U}\| = \|m\|_{p'}(\Omega)$, the restriction of $\hat{U}$ to $\mathcal{S}(\Sigma, X) = \mathcal{S}(\Sigma) \otimes X$ concides with the "algebraic" integral, and $\hat{U}^* y^* = \hat{I}_{y^*}$ for all $y^* \in Y^*$.

Moreover, the measure m takes its values in $\mathcal{L}(X, Y)$ if and only if the integration operator $\hat{U}$ takes its values in Y.

Proof. First of all, notice that if the main part of the theorem holds true, then the "if" part of the "moreover" part is clear from (3.11). Indeed, assume that $\operatorname{ran} \hat{U} \subset Y$. Since

$$m(E) = \int_\Omega \chi_E\, dm \quad \text{for all } E \in \Sigma,$$

by (3.11), we have that

$$m(E)x = \int_\Omega (\chi_E x)\, dm = \hat{U}(\chi_E \otimes x) \in Y \quad \text{for all } E \in \Sigma \text{ and } x \in X.$$

This means that $\operatorname{ran} m \subset \mathcal{L}(X, Y)$.

To prove the theorem and to encompass also the "only if" part of the "moreover" part, let $W := Y^{**}$ or $W := Y$.

On the right-hand side of (3.11), the integral is just the elementary Bartle integral with respect to m. Denote by $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, W))$ this integration operator. Since, as is well known, $\mathcal{L}(Z, \mathcal{L}(X, W))$ is canonically isometrically isomorphic to $\mathcal{L}(Z \otimes_\pi X, W) = \mathcal{L}(Z \hat{\otimes}_\pi X, W)$ (for any Banach spaces X, W, and Z), there exists a unique linear operator $\hat{U} : \mathcal{B}(\Sigma) \otimes X \to W$ such that

$$\hat{U}(\varphi \otimes x) = (\hat{S}\varphi)x, \quad \varphi \in \mathcal{B}(\Sigma), x \in X,$$

and $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \otimes_\pi X, W)$. Hence, by (3.11),

$$\int_\Omega (\varphi x)\, dm = \hat{U}(\varphi \otimes x), \quad \varphi \in \mathcal{B}(\Sigma), x \in X,$$

giving that

$$\int_\Omega f\, dm = \hat{U} f, \quad f \in \mathcal{S}(\Sigma, X) = \mathcal{S}(\Sigma) \otimes X.$$

It remains to prove that

$$\nu := \sup\{\|\hat{U}v\| \,:\, v \in \mathcal{B}(\Sigma) \otimes X, \|v\|_{d_p} \le 1\} = \|m\|_{p'}(\Omega). \tag{3.12}$$

Then, in the case when $\|m\|_{p'}(\Omega) < \infty$ or, equivalently, $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \otimes_{d_p} X, W)$, by passing to the unique continuous linear extension of $\hat{U}$, we get that $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X, W)$ and $\|\hat{U}\| = \|m\|_{p'}(\Omega)$. Therefore, the integral $\int_\Omega (\cdot)\, dm$ is defined on $\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X$ by

$$\int_\Omega v\, dm = \hat{U} v, \quad v \in \mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X.$$

Let us now prove equality (3.12). Fix an arbitrary $y^* \in Y^*$. Then, for all $\varphi \in \mathcal{B}(\Sigma)$ and $x \in X$, by (3.9), we have

$$\begin{aligned} \langle x, \hat{I}_{y^*}\varphi\rangle =& \langle \hat{S}\varphi, x \otimes y^*\rangle = \langle y^*, (\hat{S}\varphi)x\rangle \\ =& \langle y^*, \hat{U}(\varphi \otimes x)\rangle = \langle \varphi \otimes x, \hat{U}^* y^*\rangle = \langle x, (\hat{U}^* y^*)\varphi\rangle; \end{aligned} \tag{3.13}$$

for the two last equalities, recall that we have

$$\hat{U}^* \in \mathcal{L}(W^*, (\mathcal{B}(\Sigma) \otimes_\pi X)^*),$$

so that $\hat{U}^* y^* \in (\mathcal{B}(\Sigma) \otimes_\pi X)^* = \mathcal{L}(\mathcal{B}(\Sigma), X^*)$. Therefore, $\hat{I}_{y^*} = \hat{U}^* y^*$ and thus

$$\|\hat{I}_{y^*}\|_{\mathcal{P}_{p'}} = \|\hat{U}^* y^*\|_{\mathcal{P}_{p'}} = \sup\{|\langle v, \hat{U}^* y^* \rangle| \,:\, v \in \mathcal{B}(\Sigma) \otimes X, \|v\|_{d_p} \leq 1\}$$

$$= \sup\{|\langle y^*, \hat{U} v \rangle| \,:\, v \in \mathcal{B}(\Sigma) \otimes X, \|v\|_{d_p} \leq 1\} \leq \nu \|y^*\|.$$

Hence,

$$\nu \geq \|m\|_{p'}(\Omega).$$

For the reverse inequality, let $v = \sum_{i=1}^n \varphi_i \otimes x_i \in \mathcal{B}(\Sigma) \otimes_{d_p} X$. For any $y^* \in Y^*$, by (3.13), we have

$$|\langle y^*, \hat{U} v \rangle| = \Big|\langle y^*, \sum_{i=1}^n \hat{U}(\varphi_i \otimes x_i) \rangle\Big| = \Big|\sum_{i=1}^n \langle y^*, \hat{U}(\varphi_i \otimes x_i) \rangle\Big|$$

$$= \Big|\sum_{i=1}^n \langle x_i, \hat{I}_{y^*} \varphi_i \rangle\Big| \leq \|(x_i)_{i=1}^n\|_p \|(\hat{I}_{y^*} \varphi_i)_{i=1}^n\|_{p'}$$

$$\leq \|(x_i)_{i=1}^n\|_p \|\hat{I}_{y^*}\|_{\mathcal{P}_{p'}} \|(\varphi_i)_{i=1}^n\|_{p'}^w.$$

Taking first the infimum over all the representations of $v \in \mathcal{B}(\Sigma) \otimes_{d_p} X$ and then the supremum over $y^* \in B_{Y^*}$, by Proposition 3.3.2, we obtain that

$$\|\hat{U} v\| \leq \|m\|_{p'}(\Omega) \|v\|_{d_p},$$

hence,

$$\nu \leq \|m\|_{p'}(\Omega),$$

and (3.12) holds.

Finally, if $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X, W)$, then we have

$$\hat{U}^* \in \mathcal{L}(W^*, (\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X)^*) = \mathcal{L}(W^*, \mathcal{P}_{p'}(\mathcal{B}(\Sigma), X^*)),$$

and equalities (3.13) hold true, giving that $\hat{I}_{y^*} = \hat{U}^* y^*$ for all $y^* \in Y^*$. $\square$

As we mentioned in the beginning of this section, $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$ as Banach spaces, under the identification $\varphi x \leftrightarrow \varphi \otimes x$. On the other hand, let us observe that *$\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$ is a closed subspace of $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$*. This statement is based on the same argument used in the proof of Theorem 1.3.7 to show that $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$ is a closed subspace of $\ell_\infty(\Omega)\hat{\otimes}_{d_p}X$, let us recall it. It is well known that $\mathcal{C}(\Omega)^*$ is isometrically isomorphic to an $L_1(\mu)$-space for some measure μ, i.e., $\mathcal{C}(\Omega)$ is an L_1-*predual space*. Thanks to Fakhoury [33, Corollary 3.3] and Grothendieck [44, Theorem 1] (see, e.g., [24, pp. 76, 81]), L_1-predual spaces are ideals in their "superspaces" (for more details, see [51, p. 49]). In particular, $\mathcal{C}(\Omega)$ is an ideal in $\mathcal{B}(\Sigma)$. But then (see [66, Proposition 2.4]) $\mathcal{C}(\Omega) \otimes_{d_p} X$ is a subspace of $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$, and therefore $\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X = \overline{\mathcal{C}(\Omega) \otimes_{d_p} X}$ is a closed subspace of $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$.

Therefore *$\mathcal{C}_p(\Omega, X)$ is a closed subspace of $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$*, and Theorem 3.3.3 almost immediately yields the integration result below (Theorem 3.3.4).

Let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be a vector measure of bounded p'-semivariation. Denote by

$$U := \hat{U}|_{\mathcal{C}_p(\Omega,X)} = \hat{U}|_{\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X}$$

the restriction to $\mathcal{C}_p(\Omega, X)$ of the integration operator $\hat{U}$ given by Theorem 3.3.3, i.e.,

$$Uf = \int_\Omega f\, dm, \quad f \in \mathcal{C}_p(\Omega, X).$$

Theorem 3.3.4. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a vector measure of bounded p'-semivariation. Then the formula* (3.11) *defines an integral on $\mathcal{C}_p(\Omega, X)$ with respect to m, the integration operator U belongs to $\mathcal{L}(\mathcal{C}_p(\Omega, X), Y^{**})$, $\|U\| = \|m\|_{p'}(\Omega)$, and $U^*y^* = I_{y^*}$ for all $y^* \in Y^*$.*

Moreover, if the integration operator U takes its values in Y, in particular, this is the case when the measure m takes its values in $\mathcal{L}(X, Y)$, then

$$U^{**}(\chi_E \otimes x) = m(E)x \quad \textit{for all } E \in \Sigma \textit{ and } x \in X,$$

where $\chi_E \otimes x \in \mathcal{C}_p(\Omega, X)^{**}$ is defined in the canonical way:

$$\langle A, \chi_E \otimes x\rangle = \langle A^*x, \chi_E\rangle, A \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*) = \mathcal{C}_p(\Omega, X)^*.$$

Proof. For the main part of the theorem, in view of Theorem 3.3.3, we only need to show that $\|U\| \geq \|m\|_{p'}(\Omega)$ (because $\|U\| \leq \|\hat{U}\| = \|m\|_{p'}(\Omega)$) and $U^*y^* = I_{y^*}$ for all $y^* \in Y^*$.

Let $y^* \in Y^*$. Using that $U \in \mathcal{L}(\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X, Y^{**})$ and $(\mathcal{C}(\Omega)\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$, so that $U^* \in \mathcal{L}(Y^{***}, \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*))$, we get from (3.13) that

$$\langle x, I_{y^*}\varphi\rangle = \langle y^*, U(\varphi \otimes x)\rangle = \langle \varphi \otimes x, U^*y^*\rangle = \langle x, (U^*y^*)\varphi\rangle$$

for all $x \in X$ and $\varphi \in \mathcal{C}(\Omega)$. Therefore $I_{y^*} = U^*y^*$ and

$$\|I_{y^*}\|_{\mathcal{P}_{p'}} = \|U^*y^*\|_{\mathcal{P}_{p'}} \leq \|U^*\|\|y^*\| = \|U\|\|y^*\|$$

for all $y^* \in Y^*$. This yields that

$$\|m\|_{p'}(\Omega) = \sup_{y^* \in B_{Y^*}} \|I_{y^*}\|_{\mathcal{P}_{p'}} \leq \|U\|.$$

Now, for the "moreover" part, assume that $\operatorname{ran} U \subset Y$. Then $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. Let $E \in \Sigma$, $x \in X$, and $y^* \in Y^*$. Then,

$$\langle y^*, U^{**}(\chi_E \otimes x)\rangle = \langle U^*y^*, \chi_E \otimes x\rangle = \langle I_{y^*}, \chi_E \otimes x\rangle = \langle (I_{y^*})^*x, \chi_E\rangle$$

$$= \langle x, (I_{y^*})^{**}\chi_E\rangle = \langle x, \hat{I}_{y^*}\chi_E\rangle = \langle y^*, \hat{U}(\chi_E \otimes x)\rangle,$$

where the last equality holds by (3.13). Therefore, $U^{**}(\chi_E \otimes x) = \hat{U}(\chi_E \otimes x)$ for all $E \in \Sigma$ and $x \in X$. But, by (3.11), we have that

$$\hat{U}(\chi_E \otimes x) = \Big(\int_\Omega \chi_E \, dm\Big)x = m(E)x,$$

proving that $U^{**}(\chi_E \otimes x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$.

Finally, let us recall from Theorem 3.3.3 that $\operatorname{ran} m \subset \mathcal{L}(X, Y)$ if and only if $\operatorname{ran} \hat{U} \subset Y$. Hence, in this case, $\operatorname{ran} U \subset Y$. $\square$

Remark 3.3.5. Form Example 3.3.1 and Theorem 3.3.4, it is clear that, in the special case when $p = \infty$, our integral coincides with the Dinculeanu integral from [30].

Remark 3.3.6. Our notion of the q-semivariation is different from the notion "q-semivariation" introduced in Dinculeanu's book [30, p. 246]. Let us call the latter "the Dinculeanu q-semivariation". Its definition is as follows.

Let $1 \le q \le \infty$ and let $\mu : \Sigma \to \mathbb{R}$ be a positive finite measure; we may assume that $\mu(\Omega) = 1$. For a vector measure $m : \Sigma \to \mathcal{L}(X, Y)$, the *Dinculeanu q-semivariation* on Ω (see [30, p. 246]) is defined by

$$\tilde{m}_q(\Omega) = \sup \Big\{ \Big\| \sum_{E_i \in \Pi} m(E_i) x_i \Big\| \Big\},$$

where the supremum is taken over all finite partitions $\Pi = (E_i)_{i=1}^n$ of Ω and all finite systems $(x_i)_{i=1}^n \subset X$ such that $\| \sum_{i=1}^n \chi_{E_i} x_i \|_{L_{q'}(\mu, X)} \le 1$, $n \in \mathbb{N}$. This notion is used in [30, II.13] to obtain the integral representation of an operator $U \in \mathcal{L}(L_p(\mu, X), Y)$, $1 \le p < \infty$, with respect to a vector measure $m : \Sigma \to \mathcal{L}(X, Y)$ such that $\tilde{m}_{p'}(\Omega) < \infty$.

It can be easily verified that $\|m\|_1(\Omega) \le \tilde{m}_1(\Omega)$ and $\|m\|_1(\Omega) = \tilde{m}_1(\Omega)$ if m is absolutely continuous with respect to μ (see [30, p. 246]). Since also $\tilde{m}_1(\Omega) \le \tilde{m}_q(\Omega)$ (see [30, p. 247]), we have that

$$\|m\|_q(\Omega) \le \|m\|_1(\Omega) \le \tilde{m}_1(\Omega) \le \tilde{m}_q(\Omega).$$

3.4 Representing measure of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Basing on Theorem 3.3.4, we may give the following definition whose special case when $p = \infty$, thanks to Example 3.3.1, coincides with the classical one, known from the Dinculeanu–Singer theorem.

Definition 3.4.1. Let $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. A *representing measure* of U is a vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of bounded p'-semivariation which satisfies

$$Uf = \int_\Omega f \, dm \quad \text{for all } f \in \mathcal{C}_p(\Omega, X). \tag{3.14}$$

Remark 3.4.2. In the classical case of $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$, Definition 3.4.1 differs from the definition of representing measure by Brooks and Lewis [12, Definition 2.9]. Namely, we do not require that the measures $m_{y^*} : \Sigma \to X^*$, $y^* \in Y^*$ (see Section 3.3) were regular. They have this regularity property thanks to Theorem 3.4.8 below. More precisely, the regularity holds whenever $p \neq 1$ (and this condition is essential by Example 3.4.9).

Theorem 3.4.3. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be the representing measure of the associated operator $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then m is a representing measure of U, $I_{y^*} = U^* y^* \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ for all $y^* \in Y^*$, $\|U\| = \|m\|_{p'}(\Omega)$, and*

$$U^{**}(\chi_E \otimes x) = m(E)x \quad \text{for all } E \in \Sigma \text{ and } x \in X.$$

Proof. We know that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. For all $\varphi \in \mathcal{C}(\Omega)$, $x \in X$, and $y^* \in Y^*$, by (3.9), we have that

$$\begin{aligned}\langle x, I_{y^*}\varphi\rangle &= \langle \int_\Omega \varphi\, dm, x \otimes y^*\rangle = \langle U^\# \varphi, x \otimes y^*\rangle = \langle y^*, (U^\#\varphi)x\rangle \\ &= \langle y^*, U(\varphi \otimes x)\rangle = \langle U^* y^*, \varphi \otimes x\rangle = \langle x, (U^* y^*)\varphi\rangle.\end{aligned}$$

Hence $I_{y^*} = U^* y^*$ for all $y^* \in Y^*$. Therefore $I_{y^*} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ and

$$\|m\|_{p'}(\Omega) = \sup_{y^* \in B_{Y^*}} \|U^* y^*\|_{\mathcal{P}_{p'}} = \|U^*\| = \|U\| < \infty.$$

Since m is of bounded p'-semivariation, the formula (3.11) defines

$$\int_\Omega (\cdot)\, dm \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y^{**})$$

(see Theorem 3.3.4). We only need to show (3.14), because then also the last claim holds true thanks to Theorem 3.3.4.

Let $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. Then

$$U(\varphi x) = (U^\#\varphi)x = \Big(\int_\Omega \varphi\, dm\Big)x = \int_\Omega (\varphi x)\, dm$$

by (3.11). It is well known (see, e.g., [84, p. 11]) that $\mathcal{C}(\Omega) \otimes X \subset \mathcal{C}_p(\Omega, X)$ as a linear subspace (under the algebraic identification $\varphi \otimes x \leftrightarrow \varphi x$ that was used in Section 3.3). Therefore, by linearity, (3.14) holds for every $f \in \mathcal{C}(\Omega) \otimes X$. If now $f \in \mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X$ is arbitrary, then $f = \lim_n f_n$ in $\mathcal{C}_p(\Omega, X)$ for some $f_n \in \mathcal{C}(\Omega) \otimes X$. Hence,

$$Uf = \lim_n Uf_n = \lim_n \int_\Omega f_n \, dm \text{ in } Y.$$

On the other hand, by the definition of the integral,

$$\int_\Omega f \, dm = \lim_n \int_\Omega f_n \, dm \text{ in } Y^{**}.$$

Consequently, (3.14) holds. □

Theorem 3.4.3 shows that a representing measure of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ may be defined as the representing measure of its associated operator $U^\#$. Now we see that this is, in fact, the unique way to define a representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ for U.

Proposition 3.4.4. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a representing measure of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. Then m is the representing measure of $U^\#$.*

Proof. Let $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. Then, using (3.11), we have

$$(U^\#\varphi)x = U(\varphi x) = \int_\Omega (\varphi x) \, dm = \Big(\int_\Omega \varphi \, dm\Big)x.$$

Thus

$$U^\#\varphi = \int_\Omega \varphi \, dm, \quad \varphi \in \mathcal{C}(\Omega).$$ □

Since the representing measure of $U^\#$ is unique (see Proposition 3.2.4), the following is immediate from Proposition 3.4.4.

Corollary 3.4.5. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Then the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ is unique.*

In view of Proposition 3.4.4 and Corollary 3.2.10, the next corollary is immediate; the operators J and P were introduced before Theorem 3.2.6.

Corollary 3.4.6. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure, and let $\mu : \Sigma \to \mathcal{L}(X, Y)^{**}$ be the (classical) representing measure of the associated operator $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then $m(E) = PJ^{**}\mu(E)$ for all $E \in \Sigma$.*

Moreover, if $U^{\#}$ is weakly compact, then m takes its values in $\mathcal{L}(X, Y)$, and the measures m and μ coincide. In this case, the measure $m : \Sigma \to \mathcal{L}(X, Y)$ is countably additive and regular.

Remark 3.4.7. Concerning the classical case of $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$, Corollary 3.4.6 provides, for the first time in the literature, a general formula connecting the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and the classical representing measure $\mu : \Sigma \to \mathcal{L}(X, Y)^{**}$ of $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. In this sense, let us point out the partial result due to Dinculeanu (see [29, Theorems 4 and 5], or, e.g., [30, p. 388, Theorem 4]): if $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ are *dominated* operators, then m takes its values in $\mathcal{L}(X, Y)$, and the measures m and μ coincide.

Recall that a vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is *weakly regular* if $m_{y^*} : \Sigma \to X^*$ is regular for all $y^* \in Y^*$ (see, e.g., [27, p. 181]). The following result extends the classical Dinculeanu–Singer theorem (see, e.g., [27, p. 182]) in all its aspects (see Corollary 3.4.10 and the paragraph preceding it).

Theorem 3.4.8. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$.*

(a) *Every operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ has a unique representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$. This measure coincides with the representing measure of its associated operator $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$.*

(b) *Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if for all $y^* \in Y^*$,*

$$I_{y^*} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*),$$

and the map $Y^* \to \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*) = \mathcal{C}_p(\Omega, X)^*$, $y^* \mapsto I_{y^*}$, is linear, bounded, and weak-to-weak* continuous.*

In this case, $I_{y^*} = U^*y^*$ for all $y^* \in Y^*$, $U^{**}(\chi_E \otimes x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$, $\|U\| = \|m\|_{p'}(\Omega)$, $\|U^\#\| = \|m\|(\Omega)$, and m is a weakly regular measure if $p > 1$.

Proof. (a) A representing measure for an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ (the associated operator $U^\#$ is of this type) always exists (see the beginning of Section 3.2). Theorem 3.4.3 and Proposition 3.4.4 show that the representing measures of U and $U^\#$ coincide. The measure is unique by Corollary 3.4.5.

(b) Let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be the representing measure of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$. Then, by (a) and Theorem 3.4.3, m is also the representing measure of $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, $I_{y^*} = U^*y^* \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ for all $y^* \in Y^*$, $U^{**}(\chi_E \otimes x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$, and $\|U\| = \|m\|_{p'}(\Omega)$. In particular, $U^* : y^* \mapsto I_{y^*}$ is linear, bounded, and weak*-to-weak* continuous. This shows the "only if" part.

For the "if" part, let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be a bounded vector measure. Denote by V the map given by the assumption, i.e.,

$$V : Y^* \to \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*) = \mathcal{C}_p(\Omega, X)^*, \quad y^* \mapsto I_{y^*}.$$

Since V is weak*-to-weak* continuous, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^* = V$.

We only need to show that

$$\int_\Omega \varphi \, dm = U^\# \varphi, \quad \varphi \in \mathcal{C}(\Omega),$$

because then $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is the representing measure of $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, hence also the representing measure of U (see Theorem 3.4.3).

For every $\varphi \in \mathcal{C}(\Omega)$, $x \in X$, and $y^* \in Y^*$, using (3.9), we have that

$$\begin{aligned} \langle y^*, \Big(\int_\Omega \varphi \, dm \Big) x \rangle &= \langle \int_\Omega \varphi \, dm, x \otimes y^* \rangle = \langle x, \int_\Omega \varphi \, dm_{y^*} \rangle \\ &= \langle x, I_{y^*} \varphi \rangle = \langle x, (V y^*) \varphi \rangle = \langle \varphi x, V y^* \rangle \\ &= \langle \varphi x, U^* y^* \rangle = \langle U(\varphi x), y^* \rangle = \langle (U^\# \varphi) x, y^* \rangle. \end{aligned}$$

This proves that m is the representing measure of $U^\#$, as desired.

For the "in this case" part, the first three claims were already observed above. Since m is also the representing measure of $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, by Corollary 3.2.8, we have that $\|U^{\#}\| = \|m\|(\Omega)$. Concerning the remaining claim about the weak regularity, recall that m_{y^*} is the representing measure of $I_{y^*} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ for every $y^* \in Y^*$. If $p > 1$, then $p' < \infty$, and I_{y^*} is a weakly compact operator (see, e.g., [26, p. 50, Theorem 2.17]). Therefore, m_{y^*} is regular (see, e.g., [27, p. 159, Corollary 14]) for all $y^* \in Y^*$. □

The next example shows that, for $p = 1$, the measure m in Theorem 3.4.8 is not weakly regular in general.

Example 3.4.9. Let X be a Banach space and let Ω be a compact Hausdorff space such that there exists a non-weakly compact operator $S \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), \mathbb{K})$ such that its representing measure $m : \Sigma \to \mathcal{L}(X, \mathbb{K}) = X^*$ is not weakly regular.

Proof. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ be a non-weakly compact operator. Then its representing measure $m : \Sigma \to X^{***}$ is not regular (see, e.g., [27, p. 159, Corollary 14]).

Since, as is well known, $\mathcal{L}(\mathcal{C}(\Omega), X^*)$ is canonically isometrically isomorphic to $(\mathcal{C}(\Omega) \otimes_\pi X)^* = (\mathcal{C}(\Omega)\hat{\otimes}_\pi X)^*$ and $\pi = d_1$, there exists a unique operator $U \in \mathcal{L}(\mathcal{C}(\Omega)\hat{\otimes}_{d_1} X, \mathbb{K}) = \mathcal{L}(\mathcal{C}_1(\Omega, X), \mathbb{K})$ such that $S = U^{\#}$. Then m is the representing measure of U because m is the representing measure of $U^{\#} = S$ (see Theorem 3.4.3). We know that $U^* \in \mathcal{L}(\mathbb{K}^*, \mathcal{C}_1(\Omega, X)^*) = \mathcal{L}(\mathbb{K}, \mathcal{L}(\mathcal{C}(\Omega), X^*))$ and, by Theorem 3.4.3, $I_1 = U^*1$. On the other hand, for every $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$, we have

$$((U^*1)\varphi)x = \langle \varphi x, U^*1 \rangle = 1\, U(\varphi x) = (U^{\#}\varphi)x = (S\varphi)x,$$

meaning that $U^*1 = S$. Therefore, $I_1 = S$, and its representing measure, which is m, is not regular. Hence, the representing measure of U is not weakly regular. □

Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$ and, hence, $\mathcal{P}_1(\mathcal{C}(\Omega), X^*) = \mathcal{C}_\infty(\Omega, X)^* = \mathcal{C}(\Omega, X)^*$, Theorem 3.4.8 immediately yields the classical Dinculeanu–Singer theorem. Notice that, for every $y^* \in Y^*$, we can identify $I_{y^*} \in \mathcal{P}_1(\mathcal{C}(\Omega), X^*) = \mathcal{C}(\Omega, X)^*$ with its (unique) representing measure

$m_{y^*} : \Sigma \to X^*$. Let us stress that below we do not need to know about the Riesz–Singer representation of $\mathcal{C}(\Omega, X)^*$ as $r\,ca\,bv(\Sigma, X^*)$. However, we get the regularity of the measures m_{y^*} from our general setting. We also obtain the countable additivity of m_{y^*} thanks to the Bartle–Dunford–Schwartz theorem (because I_{y^*} are weakly compact). Moreover, the measures m_{y^*} are of bounded variation (because they are the representing measures of absolutely summing operators I_{y^*} (see, e.g., [27, p. 162, Theorem 3])). So that, in the special case when $Y = \mathbb{K}$, also the Riesz–Singer theorem is contained in Corollary 3.4.10 below (recall that, for a vector measure $m : \Sigma \to X^*$, one has $\|m\|_1(\Omega) = |m|(\Omega)$, the variation of m on Ω (see, e.g., [30, p. 54, Proposition 4])).

Corollary 3.4.10 (cf. the Dinculeanu–Singer theorem 3.1.2)**.** *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space.*

(a) *Every operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ has a unique representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$. This measure coincides with the representing measure of its associated operator $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$.*

(b) *Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ such that m is its representing measure if and only if for all $y^* \in Y^*$,*

$$m_{y^*} \in \mathcal{C}(\Omega, X)^*,$$

and the map $Y^* \to \mathcal{C}(\Omega, X)^*$, $y^* \mapsto m_{y^*}$, is linear, bounded, and weak-to-weak* continuous.*

In this case, $m_{y^*} : \Sigma \to X^*$ is countably additive and of bounded variation, $m_{y^*} = U^* y^*$ for all $y^* \in Y^*$, $\|U\| = \|m\|_1(\Omega)$, and m is weakly regular.

Remark 3.4.11. As we mentioned in the beginning of this chapter, in our general treatise, we did not follow any of the traditional proofs of the Dinculeanu–Singer theorem. The traditional proofs are of two types, although both extend methods of the classical proof of the Bartle–Dunford–Schwartz theorem in [7, Theorem 3.1] or [32, p. 492, Theorem 2]. The proofs, e.g., by Batt and König [9], Dinculeanu [28], [30, pp. 398–399, Theorem 9], Foiaş and Singer [36], Swong [91], Tucker [92], essentially rely on the Riesz–Singer representation theorem. The proofs, e.g., by Brooks and Lewis [12], and Diestel and Uhl [27, pp. 181–182] use “the device of embedding isometrically the simple functions in $\mathcal{C}(\Omega, X)^{**}$ and thus reducing the problem to utilizing the representing theorem for operators $L \in \mathcal{L}(\mathcal{B}(\Sigma, X), Y)$, which can be easily established". We quoted Brooks and Lewis [12, p. 139] here; the mentioned representing theorem can be found in Dinculeanu's book [30, p. 145, Theorem 1].

Remark 3.4.12. Batt and Berg [8] introduced the notion of the *weak extension* of an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, which is precisely the integration operator $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma, X), Y^{**})$ with respect to the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of U. They proved that $\|\hat{U}\| = \|U\|$, $\hat{U}(\chi_E x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$ (see [8, Theorem 1]), and that $\operatorname{ran} m \subset \mathcal{L}(X, Y)$ if and only if $\operatorname{ran} \hat{U} \subset Y$ (see [8, Theorem 2]). However, as our Theorems 3.3.3 and 3.3.4 clearly show, these are general properties of any integration operator $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma, X), Y^{**})$ and its restriction $U := \hat{U}|_{\mathcal{C}(\Omega,X)}$. Moreover, even $\|\hat{U}\| = \|U\| = \|m\|_1(\Omega)$ and (by (3.11) and the "moreover" part of Theorem 3.3.4) $\hat{U}(\chi_E x) = m(E)x = U^{**}(\chi_E \otimes x)$ for all $E \in \Sigma$ and $x \in X$ in this general case.

3.5 Complements to the Dinculeanu–Singer theorem

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. In Chapter 4, we shall study the problem when does there exist an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $S = U^{\#}$? In this section, we shall apply some result from Chapter 4 to prove some qualitative complements to Theorem 3.4.8, the extension of the Dinculeanu–Singer theorem.

The idea behind the results below is as follows: the existence of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that a given vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is its representing measure is equivalent to the existence of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ such that m is the representing measure of S and such that $S = U^{\#}$. Notice that we shall not need Theorem 3.4.8 at all. Besides Chapter 4, we shall rely on Theorem 3.2.5, our extension of the Bartle–Dunford–Schwartz theorem, together with Theorem 3.4.3 and Proposition 3.4.4.

The next theorem also contributes to the classical Dinculeanu–Singer case when $p = \infty$.

Theorem 3.5.1. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if*

(i) *for all $x \in X$,*

$$\langle y^*, m_x(\cdot)\rangle \in \mathcal{C}(\Omega)^*, \quad y^* \in Y^*,$$

and the map $Y^* \to \mathcal{C}(\Omega)^*$, $y^* \mapsto \langle y^*, m_x(\cdot)\rangle$, is linear, bounded and weak-to-weak* continuous, and*

(ii) *one of the following equivalent conditions holds:*

(a) *there exists a constant $c > 0$ such that, for all finite systems $(x_i)_{i=1}^n \subset X$ and $(\varphi_i)_{i=1}^n \subset \mathcal{C}(\Omega)$,*

$$\Big\|\Big(\int_\Omega \varphi_i\, dm_{x_i}\Big)\Big\|_{p'}^w \le c\,\|(x_i)\|_\infty \|(\varphi_i)\|_{p'}^w;$$

(b) *there exists a constant $c > 0$ such that, for all $(x_i) \in \ell_\infty(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$, and for all $n \in \mathbb{N}$,*

$$\Big\|\Big(\int_\Omega \varphi_i\, dm_{x_i}\Big)_{i=n}^\infty\Big\|_{p'}^w \le c\,\|(x_i)_{i=n}^\infty\|_\infty \|(\varphi_i)_{i=n}^\infty\|_{p'}^w;$$

(c) *if $(x_i) \in \ell_\infty(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$, then $(\int_\Omega \varphi_i\, dm_{x_i}) \in \ell_{p'}^w(Y)$;*

(d) *if $(x_i) \in c_0(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ (or $(x_i) \in \ell_\infty(X)$ and $(\varphi_i) \in \ell_{p'}^u(\mathcal{C}(\Omega))$), then $(\int_\Omega \varphi_i\, dm_{x_i}) \in \ell_{p'}^u(Y)$.*

Proof. We are going to use the following fact. Assume that m is the representing measure of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Since

$$(S\varphi)x = (\int_\Omega \varphi\, dm)x = \int_\Omega \varphi\, dm_x \quad \text{for all } \varphi \in \mathcal{C}(\Omega) \text{ and } x \in X,$$

by Corollary 4.3.4, every condition included in (ii) is equivalent to the existence of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^\# = S$.

For the "only if" part, let $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ be such that m is its representing measure. By Proposition 3.4.4, m is also the representing measure of its associated operator $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, and, by the above fact, (ii) holds; condition (i) is immediate from Theorem 3.2.5.

For the "if" part, condition (i) implies that there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ such that m is its representing measure (see Theorem 3.2.5). And, by the above fact, condition (ii) implies that there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$. Then, by Theorem 3.4.3, m is also the representing measure of U. □

In the next theorem, we use Corollary 4.2.5, which asserts that, for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), Y)$ such that $U^{\#} = S$.

Theorem 3.5.2. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), Y)$ such that m is its representing measure if and only if there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ such that m is its representing measure.*

Proof. The necessary condition is clear by taking $S = U^{\#}$ and applying Proposition 3.4.4. By Corollary 4.2.5, for a given operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), Y)$ such that $S = U^{\#}$. From Theorem 3.4.3, the sufficient condition is clear. □

The following results, which are similar to Theorem 3.5.2, can be obtained using Corollaries 4.2.6 and 4.2.7 (instead of Corollary 4.2.5).

Theorem 3.5.3. *Let X and Y be Banach spaces such that X^* is of cotype 2. Let Ω be a compact Hausdorff space. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, for every $p \leq 2$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ such that m is its representing measure.*

Theorem 3.5.4. *Let X and Y be Banach spaces such that X^* is of cotype q, where $2 \leq q < \infty$. Let Ω be a compact Hausdorff space. Assume that $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is a bounded vector measure. Then, for every $p \leq q'$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that m is its representing measure if and only if there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ such that m is its representing measure.*

Chapter 4

The problem of the associated operator

This chapter deals with the associated operator $U^{\#}$ defined in Chapter 3. Every operator $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ has an associated operator $U^{\#} \in \mathcal{L}(Z, \mathcal{L}(X,Y))$ defined in a natural way. In this chapter, we study the problem of the existence of an operator $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ such that $U^{\#} = S$ for a given operator $S \in \mathcal{L}(Z, \mathcal{L}(X,Y))$, solving a long-standing conjecture by Dinculeanu [30]. This chapter is based on [58].

4.1 Introduction

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. As in Chapter 3 (see Section 3.1), for every operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, we denote by $U^{\#}$ the *associated operator* from $\mathcal{C}(\Omega)$ to $\mathcal{L}(X,Y)$ defined by $(U^{\#}\varphi)x = U(\varphi x)$, $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. (The notation $U^{\#}$ is traditional; see, e.g., [56, 76, 77, 79, 81, 86, 90].) Then, clearly, $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$.

On the other hand, in a short remark (see [30, Remark, p. 379]), Dinculeanu pointed out that there exist operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ which are not associated to any $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, meaning that $S \neq U^{\#}$ for all operators $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$. (In [30], $U^{\#}$ is denoted by U'.) Professor Dinculeanu kindly informed us (personal communication, September 27, 2015) that his remark was just a conjecture based on Grothendieck's result quoted in Remark 4.2.4 below.

Many authors have studied the interplay between U and $U^\#$ for different classes of operators (see, e.g., the above references). However, it seems that nothing (apart from Dinculeanu's remark) has been said about the problem of the existence of an operator U such that $U^\# = S$ for a given operator S.

This chapter aims in studying this existence problem and, in particular, in proving Dinculeanu's conjecture. However, we shall study the problem in a more general context of operators defined on the Banach space $\mathcal{C}_p(\Omega, X)$ of p-continuous X-valued functions, $1 \leq p \leq \infty$. Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, this also encompasses the classical case of operators on $\mathcal{C}(\Omega, X)$.

By Grothendieck's classics [43] (see, e.g., [84, pp. 49–50]), we know that

$$\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X,$$

where ε denotes the injective tensor norm, under the canonical isometric isomorphism $\varphi x \leftrightarrow \varphi \otimes x$, $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. As is well known, this allows to extend the definition of $U^\#$ as follows.

Let Z be a Banach space and let α be a tensor norm. If $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$, then the operator $U^\# \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ *associated to* U is defined by $(U^\# z)x = U(z \otimes x)$, $z \in Z$ and $x \in X$. By Theorem 1.3.7,

$$\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X,$$

where d_p denotes the right Chevet–Saphar tensor norm (see [87] or, e.g., [84, Chapter 6]). Keeping this in mind, we shall study the existence problem in the general context of operators defined on tensor products of Banach spaces. In particular, we shall see that examples, proving Dinculeanu's conjecture, come out on the all three levels of the generality (see Remarks 4.2.2 and 4.2.4, Corollary 4.3.4, Proposition 4.4.3).

Let Z, X, and α be as above. In Section 4.2, we prove a general omnibus theorem (Theorem 4.2.1), which provides three equivalent conditions for the existence of $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ such that $S = U^\#$ for every Banach space Y and every operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$. The main applications (Theorem 4.2.3 and Corollaries 4.2.6 and 4.2.7) concern the case of p-continuous X-valued functions $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X$.

In Section 4.3, we fix Banach spaces Z, X, and Y, and a tensor norm d_p. We prove another omnibus theorem (Theorem 4.3.3), which provides four equivalent conditions for a given operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ to be the associated operator to an operator $U \in \mathcal{L}(Z\hat{\otimes}_{d_p}X, Y)$. Again, the main application (Corollary 4.3.4) concerns $\mathcal{C}_p(\Omega, X)$, yielding conditions that seem to be new even in the classical case $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$.

In Section 4.4, we are given an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. We present a necessary condition for the existence of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $S = U^{\#}$ (Proposition 4.4.1), which becomes also sufficient in the case $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$ (Proposition 4.4.3). This condition is expressed in terms of the representing measure of S, which was built in Chapter 3 (see Section 3.2).

Section 4.5 provides three examples (concerning Corollaries 4.2.7, 4.3.5, and Proposition 4.4.1) in order to show that our results are sharp in general.

4.2 Characterizing associated operators: "global" case

Let X, Y, and Z be Banach spaces, and let π be the projective tensor norm. It is well known that every operator $U \in \mathcal{L}(Z \otimes_\pi X, Y)$ induces an *associated operator* $U^{\#} \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ by

$$(U^{\#}z)x = U(z \otimes x), \; z \in Z \text{ and } x \in X.$$

It is also well known and easy to verify that the correspondence $U \mapsto U^{\#}$ is an isometric isomorphism between the Banach spaces $\mathcal{L}(Z \otimes_\pi X, Y) = \mathcal{L}(Z\hat{\otimes}_\pi X, Y)$ and $\mathcal{L}(Z, \mathcal{L}(X, Y))$. In particular, every $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ happens to be the associated operator to some $U \in \mathcal{L}(Z\hat{\otimes}_\pi X, Y)$.

In the special case when $Y = \mathbb{K}$, the operators U and $U^{\#}$ are canonically identified, and the corresponding identification

$$(Z\hat{\otimes}_\pi X)^* = \mathcal{L}(Z, X^*)$$

(as Banach spaces) uses the duality

$$\langle S, \sum_{i=1}^{n} z_i \otimes x_i \rangle = \sum_{i=1}^{n} (Sz_i)x_i,$$

i.e., $S = U^{\#}$ is identified with U.

The same phenomenon occurs for any tensor norm α: thanks to Grothendieck [42] (see, e.g., [84, pp. 187–190]), one has the canonical identification

$$(Z\hat{\otimes}_\alpha X)^* = \mathcal{A}(Z, X^*)$$

(as Banach spaces), where $\mathcal{A}$ is the Banach operator ideal of the α'-integral operators. Following [66], let us say that $\mathcal{A}$ is the *dual space operator ideal* of α. (Note that in [63], the dual space operator ideal was defined differently, but in a symmetric way.)

Let α be a tensor norm. Since $\alpha(u) \leq \pi(u)$, $u \in Z \otimes X$, we have $\mathcal{L}(Z\hat{\otimes}_\alpha X, Y) \subset \mathcal{L}(Z\hat{\otimes}_\pi X, Y)$. Therefore the associated operator $U^{\#} \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ is defined for any $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$, and, in particular, for any $U \in \mathcal{A}(Z\hat{\otimes}_\alpha X, Y)$, where $\mathcal{A} = (\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an *arbitrary* Banach operator ideal. However, in this case, in general, not all $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ enjoy the "privilege" of being associated to some $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$. Theorem 4.2.1 below will give three equivalent conditions for the existence of such $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ for every Banach space Y and every operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$.

Theorem 4.2.1. *Let X and Z be Banach spaces. Let α be a tensor norm and let $\mathcal{A}$ be the dual space operator ideal of α. The following statements are equivalent.*

(a) *For every Banach space Y and for every operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$, there exists $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ such that $U^{\#} = S$.*

(b) *There exists a Banach space $Y \neq \{0\}$ such that for every operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$, there exists $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ such that $U^{\#} = S$.*

(c) $\mathcal{L}(Z, X^*) = \mathcal{A}(Z, X^*)$ *as sets.*

(d) *The tensor norms α and π are equivalent on $Z \otimes X$.*

Proof. (a)⇒(b). This is trivial.

(b)⇒(c). Fix $y_0 \in Y$ and $y_0^* \in Y^*$ satisfying $\|y_0\| = \|y_0^*\| = y_0^*(y_0) = 1$. Let $A \in \mathcal{L}(Z, X^*)$. Define $S : Z \to \mathcal{L}(X, Y)$ by $Sz = Az \otimes y_0 \in \mathcal{F}(X, Y)$. Then $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ and there exists $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ such that $U^{\#} = S$. Let us consider

$$y_0^* U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, \mathbb{K}) = (Z\hat{\otimes}_\alpha X)^* = \mathcal{A}(Z, X^*).$$

Since

$$\begin{aligned}(y_0^*U)(z \otimes x) &= y_0^*(U(z \otimes x)) = y_0^*((Sz)x) \\ &= y_0^*((Az)(x)y_0) = (Az)(x),\ x \in X, z \in Z,\end{aligned}$$

$y_0^*U = A$; hence $A \in \mathcal{A}(Z, X^*)$.

(c)⇒(d). We know that

$$(Z \otimes_\pi X)^* = (\mathcal{L}(Z, X^*), \|\cdot\|) \text{ and } (Z \otimes_\alpha X)^* = (\mathcal{A}(Z, X^*), \|\cdot\|_{\mathcal{A}})$$

as Banach spaces. By (c), the Banach space $(\mathcal{L}(Z, X^*), \|\cdot\|)$ also carries another complete norm $\|\cdot\|_{\mathcal{A}}$. Since, as is well known, $\|\cdot\| \le \|\cdot\|_{\mathcal{A}}$, the norms $\|\cdot\|$ and $\|\cdot\|_{\mathcal{A}}$ are equivalent. Hence also π and α are equivalent.

(d)⇒(a). Let Y be a Banach space. By the canonical isometric isomorphism between $\mathcal{L}(Z, \mathcal{L}(X, Y))$ and $\mathcal{L}(Z \hat{\otimes}_\pi X, Y)$, for every $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$, there exists $U \in \mathcal{L}(Z \hat{\otimes}_\pi X, Y)$ such that $U^{\#} = S$. But $\mathcal{L}(Z \hat{\otimes}_\pi X, Y) = \mathcal{L}(Z \hat{\otimes}_\alpha X, Y)$ as sets (and isomorphic as Banach spaces), because π and α are equivalent on $Z \otimes X$. □

Remark 4.2.2. The particular case of $\alpha = \varepsilon$ concerns one of the most famous long-standing conjectures in functional analysis. In [43, p. 153] (see also [42, Section 4.6]), Grothendieck conjectured: if the injective tensor norm ε and the projective tensor norm π are equivalent on $Z \otimes X$, then Z or X must be finite dimensional. In 1981, Pisier [74] constructed an infinite-dimensional separable Banach space P such that ε and π are equivalent on $P \otimes P$. Since $\varepsilon(u) \le \alpha(u) \le \pi(u)$, $u \in Z \otimes X$, all tensor norms are equivalent on $P \otimes P$. By Theorem 4.2.1, for every tensor norm α and every Banach space Y, every operator $S \in \mathcal{L}(P, \mathcal{L}(P, Y))$ is associated to some $U \in \mathcal{L}(P \hat{\otimes}_\alpha P, Y)$.

In [43, p. 153, Corollary 2] (see [63, Corollary 3.1] for a different proof), it is proved that if ε and π are equivalent on $Z \otimes Z^*$, then Z is finite dimensional. Again, by Theorem 4.2.1, for every infinite-dimensional Banach space Z and every Banach space $Y \neq \{0\}$, there exists an operator $S \in \mathcal{L}(Z, \mathcal{L}(Z^*, Y))$ which is not associated to any operator $U \in \mathcal{L}(Z \hat{\otimes}_\varepsilon Z^*, Y)$. With $Z = \mathcal{C}(\Omega)$, this clearly can be used to prove Dinculeanu's conjecture (see Section 4.1).

Let $1 \le p \le \infty$. Recall (see Theorem 1.3.7) that $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$ as Banach spaces. It is known (see, e.g., [84, p. 142]) that the dual space operator ideal of the Chevet–Saphar tensor norm d_p coincides with $\mathcal{P}_{p'}$, where $\mathcal{P}_q$, $1 \le q \le \infty$, denotes the Banach operator ideal of absolutely q-summing operators. This leads us to the following immediate application of Theorem 4.2.1.

Theorem 4.2.3. *Let X be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. The following statements are equivalent.*

(a) *For every Banach space Y and for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, there exists $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^\# = S$.*

(b) *There exists a Banach space $Y \neq \{0\}$ such that for every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, there exists $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^\# = S$.*

(c) $\mathcal{L}(\mathcal{C}(\Omega), X^*) = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ *as sets.*

(d) *The tensor norms d_p and π are equivalent on $\mathcal{C}(\Omega) \otimes X$.*

Remark 4.2.4. As was mentioned, $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$. Saphar [87, p. 99] has shown that d_∞ coincides with ε on $\mathcal{C}(\Omega) \otimes X$. But, thanks to Grothendieck [43, p. 152, Proposition 33], ε and π cannot be equivalent on $\mathcal{C}(\Omega) \otimes X$ when X is an infinite-dimensional Banach space (assuming, of course, that Ω is infinite). Thus, in this special case, Theorem 4.2.3 says that, whenever $Y \neq \{0\}$, there exists an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ which is not associated to any $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$. This refines Dinculeanu's remark and proves his conjecture (see Section 4.1).

Since $\mathcal{P}_\infty = \mathcal{L}$, the following is immediate from Theorem 4.2.3.

Corollary 4.2.5. *Let X be a Banach space and let Ω be a compact Hausdorff space. For every Banach space Y and every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, there exists $U \in \mathcal{L}(\mathcal{C}_1(\Omega, X), Y)$ such that $U^\# = S$.*

Equality (c) of Theorem 4.2.3, which is well studied in the literature (see, e.g., [26, Chapter 11] for results and references), enables us to move from $\mathcal{C}_1(\Omega, X)$ to larger domain spaces $\mathcal{C}_p(\Omega, X)$ for special cases of X.

Corollary 4.2.6. *Let X and Y be Banach spaces such that X^* is of cotype 2. Let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. Then, for each $p \le 2$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^\# = S$.*

Proof. Let X^* be of cotype 2. Then $\mathcal{L}(\mathcal{C}(\Omega), X^*) = \mathcal{P}_2(\mathcal{C}(\Omega), X^*)$ (see, e.g., [26, Theorem 11.14]). Since $p' \geq 2$, we have $\mathcal{P}_2(\mathcal{C}(\Omega), X^*) \subset \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ (see, e.g., [26, Theorem 2.8]). Hence, $\mathcal{L}(\mathcal{C}(\Omega), X^*) = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ and we only need to apply Theorem 4.2.3 to finish the proof. □

In Section 4.5, we shall show that Corollary 4.2.6 does not hold for $p > 2$ (see Example 4.5.1).

Corollary 4.2.7. *Let X and Y be Banach spaces such that X^* is of cotype q, where $2 \leq q < \infty$. Let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$. Then, for each $p < q'$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$.*

Proof. Let X^* be of cotype q, $2 \leq q < \infty$. Then $\mathcal{L}(\mathcal{C}(\Omega), X^*) = \mathcal{P}_r(\mathcal{C}(\Omega), X^*)$, for all $r > q$ (see, e.g., [26, Theorem 11.14]). Since $p' > q$, $\mathcal{L}(\mathcal{C}(\Omega), X^*) = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ and the proof finishes using again Theorem 4.2.3. □

4.3 Characterizing associated operators: "local" case

In this section, let an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ be given, where X and Y are Banach spaces. Let $1 \leq p \leq \infty$. We are interested in equivalent conditions for the existence of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$. These conditions will be presented in Corollary 4.3.4 below. They immediately follow from the more general case when $S \in \mathcal{L}(Z, \mathcal{L}(X,Y))$ and $U \in \mathcal{L}(Z \hat{\otimes}_{d_p} X, Y)$, where Z is a Banach space, see Theorem 4.3.3. Theorem 4.3.3 in turn will be deduced from Theorem 4.3.1, which characterizes operators that take Z to the space $\mathcal{P}_{(r,q)}(X,Y)$ of absolutely (r,q)-summing operators, and is the main result of this section.

Let $1 \leq q \leq r \leq \infty$. Recall that an operator $U \in \mathcal{L}(X,Y)$ is *absolutely (r,q)-summing* if there is a constant $C \geq 0$ such that

$$\|(Ux_i)_{i=1}^n\|_r \leq C \,\|(x_i)_{i=1}^n\|_q^w$$

for all finite systems $(x_i)_{i=1}^n \subset X$, $n \in \mathbb{N}$. The least constant C for which the previous inequality holds is denoted by $\|U\|_{\mathcal{P}_{(r,q)}}$. All absolutely (r,q)-summing operators between arbitrary Banach spaces form a Banach operator ideal, denoted by $\mathcal{P}_{(r,q)}$. Recall also that the Banach operator ideal $\mathcal{P}_q$ of *absolutely q-summing* operators is defined as $\mathcal{P}_q = \mathcal{P}_{(q,q)}$.

Theorem 4.3.1. *Let X, Y, and Z be Banach spaces. Let $1 \le q \le r \le \infty$. Assume that $T \in \mathcal{L}(Z, \mathcal{L}(X,Y))$. The following statements are equivalent.*

(a) *$T \in \mathcal{L}(Z, \mathcal{P}_{(r,q)}(X,Y))$.*

(b) *There exists a constant $c > 0$ such that, for all finite systems $(y_i^*)_{i=1}^n \subset Y^*$ and $(x_i)_{i=1}^n \subset X$,*

$$\|(T^*(x_i \otimes y_i^*))\|_r^w \le c\,\|(y_i^*)\|_\infty \|(x_i)\|_q^w.$$

(b′) *There exists a constant $c > 0$ such that, for all $(y_i^*) \in \ell_\infty(Y^*)$ and $(x_i) \in \ell_q^w(X)$, and for all $n \in \mathbb{N}$,*

$$\|(T^*(x_i \otimes y_i^*))_{i=n}^\infty\|_r^w \le c\,\|(y_i^*)_{i=n}^\infty\|_\infty \|(x_i)_{i=n}^\infty\|_q^w.$$

(c) *If $(y_i^*) \in \ell_\infty(Y^*)$ and $(x_i) \in \ell_q^w(X)$, then $(T^*(x_i \otimes y_i^*)) \in \ell_r^w(Z^*)$.*

(d) *If $(y_i^*) \in c_0(Y^*)$ and $(x_i) \in \ell_q^w(X)$ (or $(y_i^*) \in \ell_\infty(Y^*)$ and $(x_i) \in \ell_q^u(X)$), then $(T^*(x_i \otimes y_i^*)) \in \ell_r^u(Z^*)$.*

Proof. (a)⇔(b). Condition (a) is equivalent to the existence of a constant $c > 0$ such that

$$\|Tz\|_{\mathcal{P}_{(r,q)}} \le c\,\|z\|$$

for all $z \in Z$. This means that, for all $z \in B_Z$ and finite systems $(x_i)_{i=1}^n \subset X$,

$$\|((Tz)x_i)\|_r \le c\,\|(x_i)\|_q^w.$$

Consider $y_i^* \in B_{Y^*}$ such that

$$\|(Tz)x_i\|_Y = \sup_{y_i^* \in B_{Y^*}} |y_i^*((Tz)x_i)|$$

for $i = 1, \ldots, n$. We may write

$$\|((Tz)x_i)\|_r^r = \sum_{i=1}^n \sup_{y_i^* \in B_{Y^*}} |y_i^*((Tz)x_i)|^r = \sup_{\|(y_i^*)_{i=1}^n\|_\infty \le 1} \sum_{i=1}^n |y_i^*((Tz)x_i)|^r.$$

Hence, (a) is equivalent to the existence of $c > 0$ such that, for all $(x_i)_{i=1}^n \subset X$,

$$\sup_{\|(y_i^*)_{i=1}^n\|_\infty \leq 1} \sup_{z \in B_Z} \Big(\sum_{i=1}^n |y_i^*((Tz)x_i)|^r \Big)^{1/r} \leq c \, \|(x_i)\|_q^w.$$

Since

$$\sup_{z \in B_Z} \Big(\sum_{i=1}^n |y_i^*((Tz)x_i)|^r \Big)^{1/r} = \|(T^*(y_i^* \otimes x_i))\|_r^w,$$

the above inequality reads as

$$\sup_{\|(y_i^*)_{i=1}^n\|_\infty \leq 1} \|(T^*(y_i^* \otimes x_i))\|_r^w \leq c \, \|(x_i)\|_q^w.$$

Therefore, (a) is clearly equivalent to (b).

For the equivalences of the remaining conditions, see Proposition 4.3.2 below, where it is shown that conditions (b), (b′), (c), and (d) are equivalent in a more general context with T^* replaced by an arbitrary continuous bilinear map. □

Recall that a separately continuous bilinear map is continuous (see, e.g., [18, Theorem 1.2, p. 8]) and this is equivalent to be bounded (see, e.g., [18, Proposition 1.1, p. 8]).

Proposition 4.3.2. *Let X, Y, and Z be Banach spaces. Let $1 \leq q \leq r \leq \infty$. Assume that $A : X \times Y \to Z$ is a continuous bilinear map. The following statements are equivalent.*

(i) *There exists a constant $c > 0$ such that, for all finite systems $(x_i)_{i=1}^n \subset X$ and $(y_i)_{i=1}^n \subset Y$,*

$$\|(A(x_i, y_i))\|_r^w \leq c \, \|(y_i)\|_\infty \|(x_i)\|_q^w.$$

(i′) *There exists a constant $c > 0$ such that, for all $(x_i) \in \ell_q^w(X)$ and $(y_i) \in \ell_\infty(Y)$, and for all $n \in \mathbb{N}$,*

$$\|(A(x_i, y_i))_{i=n}^\infty\|_r^w \leq c \, \|(y_i)_{i=n}^\infty\|_\infty \|(x_i)_{i=n}^\infty\|_q^w.$$

(ii) *If $(x_i) \in \ell_q^w(X)$ and $(y_i) \in \ell_\infty(Y)$, then $(A(x_i, y_i)) \in \ell_r^w(Z)$.*

(iii) *If $(x_i) \in \ell_q^w(X)$ and $(y_i) \in c_0(Y)$ (or $(x_i) \in \ell_q^u(X)$ and $(y_i) \in \ell_\infty(Y)$), then $(A(x_i, y_i)) \in \ell_r^u(Z)$.*

Proof. (i)⇒(i′). Let $(y_i) \in \ell_\infty(Y)$ and $(x_i) \in \ell_q^w(X)$. For all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$ with $k > n$, we have

$$\|(A(x_i, y_i))_{i=n}^k\|_r^w \leq c\,\|(y_i)_{i=n}^k\|_\infty \|(x_i)_{i=n}^k\|_q^w$$
$$\leq c\,\|(y_i)_{i=n}^\infty\|_\infty \|(x_i)_{i=n}^\infty\|_q^w.$$

As this inequality holds for all $k \in \mathbb{N}$ with $k > n$, it is clear from the definition of the norm $\|\cdot\|_r^w$ that

$$\|(A(x_i, y_i))_{i=n}^\infty\|_r^w \leq c\,\|(y_i)_{i=n}^\infty\|_\infty \|(x_i)_{i=n}^\infty\|_q^w.$$

(i′)⇒(i). This is trivial.

(ii)⇒(i). Consider the bilinear map

$$B : \ell_q^w(X) \times \ell_\infty(Y) \to \ell_r^w(Z),$$
$$(x_i, y_i)_i := ((x_i), (y_i)) \mapsto (A(x_i, y_i)).$$

We shall prove that the graph of B is closed. Let $u_k = (x_i^k, y_i^k)_i$ be a sequence in $\ell_q^w(X) \times \ell_\infty(Y)$ such that $u_k \to_k u = (x_i, y_i)_i$. Looking at the norms of $\ell_q^w(X)$ and $\ell_\infty(Y)$, we get that $x_i^k \to_k x_i$ and $y_i^k \to_k y_i$ for all $i \in \mathbb{N}$. Let $v = (z_i) \in \ell_r^w(Z)$ be such that $Bu_k \to_k v$, i.e., $(A(x_i^k, y_i^k)) \to_k (z_i)$. As before, looking at the norm of $\ell_r^w(Z)$, we get that $A(x_i^k, y_i^k) \to_k z_i$ for all $i \in \mathbb{N}$. On the other hand, by the continuity of A, $A(x_i^k, y_i^k) \to_k A(x_i, y_i)$ for all $i \in \mathbb{N}$. Hence, $z_i = A(x_i, y_i)$ for all $i \in \mathbb{N}$, i.e., $v = Bu$.

So B has a closed graph. By the closed graph theorem for bilinear maps (see, e.g., [18, Exercise 1.11, p. 14] or [34]), B is continuous, hence bounded, and therefore (i) clearly holds.

(iii)⇒(i). We can use the same argument as in (ii)⇒(i).

(i′)⇒(ii). Let $(y_i) \in \ell_\infty(Y)$ and $(x_i) \in \ell_q^w(X)$. Then $(A(x_i, y_i)) \in \ell_r^w(Z)$, because

$$\|(A(x_i, y_i))_{i=1}^\infty\|_r^w \leq c\,\|(y_i)_{i=1}^\infty\|_\infty \|(x_i)_{i=1}^\infty\|_q^w < \infty.$$

(i′)⇒(iii). Let $(y_i) \in c_0(Y)$ and $(x_i) \in \ell_q^w(X)$. Then $\|(y_i)_{i=n}^\infty\|_\infty \to 0$ when n tends to infinity and $\|(x_i)_{i=n}^\infty\|_q^w \leq \|(x_i)_{i=1}^\infty\|_q^w$. Hence,

$$\|(A(x_i, y_i))_{i=n}^\infty\|_r^w \leq c\,\|(y_i)_{i=n}^\infty\|_\infty \|(x_i)_{i=1}^\infty\|_q^w \underset{n}{\to} 0,$$

showing that $(A(x_i, y_i)) \in \ell_r^u(Z)$.

Let now $(y_i) \in \ell_\infty(Y)$ and $(x_i) \in \ell_q^u(X)$. Then $\|(y_i)_{i=n}^\infty\|_\infty \le \|(y_i)_{i=1}^\infty\|_\infty$ and $\|(x_i)_{i=n}^\infty\|_q^w \to 0$ when n tends to infinity. Hence,

$$\|(A(x_i, y_i))_{i=n}^\infty\|_r^w \le c\,\|(y_i)_{i=1}^\infty\|_\infty\|(x_i)_{i=n}^\infty\|_q^w \underset{n}{\to} 0,$$

showing that $(A(x_i, y_i)) \in \ell_r^u(Z)$. □

Let α be a tensor norm and let $\mathcal{A}$ be the dual space operator ideal of α. As was said in Section 4.2, given an operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$, there always exists an operator $U \in \mathcal{L}(Z\hat{\otimes}_\pi X, Y)$ such that $U^{\#} = S$. Being interested in the case when $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$, we are going to use that

$$U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y) \;\Leftrightarrow\; U^* \in \mathcal{L}(Y^*, \mathcal{A}(Z, X^*)),$$

which is straightforward to verify. We shall now apply Theorem 4.3.1 and the above observation to the special case $\alpha = d_p$. Then, as was recalled above, $\mathcal{A} = \mathcal{P}_{p'} = \mathcal{P}_{(p',p')}$.

Theorem 4.3.3. *Let X, Y, and Z be Banach spaces. Let $1 \le p \le \infty$. Assume that $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$. The following statements are equivalent.*

(a) *There exists an operator $U \in \mathcal{L}(Z\hat{\otimes}_{d_p} X, Y)$ such that $U^{\#} = S$.*

(b) *There exists a constant $c > 0$ such that, for all finite systems $(x_i)_{i=1}^n \subset X$ and $(z_i)_{i=1}^n \subset Z$,*

$$\|((Sz_i)x_i)\|_{p'}^w \le c\,\|(x_i)\|_\infty\|(z_i)\|_{p'}^w.$$

(b′) *There exists a constant $c > 0$ such that, for all $(x_i) \in \ell_\infty(X)$ and $(z_i) \in \ell_{p'}^w(Z)$, and for all $n \in \mathbb{N}$,*

$$\|((Sz_i)x_i)_{i=n}^\infty\|_{p'}^w \le c\,\|(x_i)_{i=n}^\infty\|_\infty\|(z_i)_{i=n}^\infty\|_{p'}^w.$$

(c) *If $(x_i) \in \ell_\infty(X)$ and $(z_i) \in \ell_{p'}^w(Z)$, then $((Sz_i)x_i) \in \ell_{p'}^w(Y)$.*

(d) *If $(x_i) \in c_0(X)$ and $(z_i) \in \ell_{p'}^w(Z)$ (or $(x_i) \in \ell_\infty(X)$ and $(z_i) \in \ell_{p'}^u(Z)$), then $((Sz_i)x_i) \in \ell_{p'}^u(Y)$.*

Proof. Recall that S is associated to some $U \in \mathcal{L}(Z\hat{\otimes}_\pi X, Y)$, meaning that $U(z \otimes x) = (Sz)x$ for all $z \in Z$ and $x \in X$.

By the above observation, condition (a) is equivalent to the fact that $U^* \in \mathcal{L}(Y^*, \mathcal{P}_{p'}(Z, X^*))$. Therefore, viewing U^* in the role of T in Theorem 4.3.1, condition (a) is equivalent to the existence of a constant $c > 0$ such that, for all finite systems $(x_i^{**})_{i=1}^n \subset X^{**}$ and $(z_i)_{i=1}^n \subset Z$,

$$\|(U^{**}(z_i \otimes x_i^{**}))\|_{p'}^w \le c\,\|(x_i^{**})\|_\infty \|(z_i)\|_{p'}^w. \tag{4.1}$$

Now, an easy application of the principle of local reflexivity yields that the above condition is equivalent to the existence of a constant $C > 0$ such that, for all finite systems $(x_i)_{i=1}^n \subset X$ and $(z_i)_{i=1}^n \subset Z$,

$$\|(U(z_i \otimes x_i))\|_{p'}^w \le C\,\|(x_i)\|_\infty \|(z_i)\|_{p'}^w, \tag{4.2}$$

which in turn is clearly equivalent to (b).

For completeness, let us include the proof that the conditions concerning (4.1) and (4.2) are equivalent. Since U^{**} is an extension of U, (4.1) implies (4.2). For the converse, let $(x_i^{**})_{i=1}^n \subset X^{**}$ and $(z_i)_{i=1}^n \subset Z$. Fix an arbitrary $y^* \in B_{Y^*}$. Considering $\operatorname{span}\{x_i^{**}\}_{i=1}^n$ in X^{**} and $\operatorname{span}\{(U^*y^*)z_i\}_{i=1}^n$ in X^*, the principle of local reflexivity yields an operator $V : \operatorname{span}\{x_i^{**}\}_{i=1}^n \to X$, with $\|V\| \le 2$, such that $\langle Vx_i^{**}, (U^*y^*)z_i\rangle = \langle (U^*y^*)z_i, x_i^{**}\rangle$ for all $i = 1, \dots, n$. But then $y^*(U(z_i \otimes Vx_i^{**})) = \langle U^*y^*, z_i \otimes Vx_i^{**}\rangle = \langle Vx_i^{**}, (U^*y^*)z_i\rangle = \langle (U^*y^*)z_i, x_i^{**}\rangle$ for all $i = 1, \dots, n$. Therefore,

$$\Big(\sum_{i=1}^n |\langle y^*, U^{**}(z_i \otimes x_i^{**})\rangle|^{p'}\Big)^{1/p'} = \Big(\sum_{i=1}^n |\langle (U^*y^*)z_i, x_i^{**}\rangle|^{p'}\Big)^{1/p'}$$

$$= \Big(\sum_{i=1}^n |y^*(U(z_i \otimes Vx_i^{**}))|^{p'}\Big)^{1/p'} \le \|(U(z_i \otimes Vx_i^{**}))\|_{p'}^w$$

$$\le 2\,C\,\|(x_i^{**})\|_\infty \|(z_i)\|_{p'}^w.$$

Taking the supremum over $y^* \in B_{Y^*}$ gives us (4.1) (with $c = 2C$).

The equivalences of (b), (b′), (c), and (d) are clear from Proposition 4.3.2. □

Recalling that $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X$, let us spell out the desired (immediate) consequence of Theorem 4.3.3.

Corollary 4.3.4. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$. The following statements are equivalent.*

(a) *There exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$.*

(b) *There exists a constant $c > 0$ such that, for all finite systems $(x_i)_{i=1}^n \subset X$ and $(\varphi_i)_{i=1}^n \subset \mathcal{C}(\Omega)$,*

$$\|((S\varphi_i)x_i)\|_{p'}^w \le c\,\|(x_i)\|_\infty \|(\varphi_i)\|_{p'}^w.$$

(b$'$) *There exists a constant $c > 0$ such that, for all $(x_i) \in \ell_\infty(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$, and for all $n \in \mathbb{N}$,*

$$\|((S\varphi_i)x_i)_{i=n}^\infty\|_{p'}^w \le c\,\|(x_i)_{i=n}^\infty\|_\infty \|(\varphi_i)_{i=n}^\infty\|_{p'}^w.$$

(c) *If $(x_i) \in \ell_\infty(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$, then $((S\varphi_i)x_i) \in \ell_{p'}^w(Y)$.*

(d) *If $(x_i) \in c_0(X)$ and $(\varphi_i) \in \ell_{p'}^w(\mathcal{C}(\Omega))$ (or $(x_i) \in \ell_\infty(X)$ and $(\varphi_i) \in \ell_{p'}^u(\mathcal{C}(\Omega))$), then $((S\varphi_i)x_i) \in \ell_{p'}^u(Y)$.*

For the case $p = \infty$, and hence $p' = 1$, recall from Section 4.2 that $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$ as Banach spaces. Thus, the above corollary also characterizes those operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ which are associated to an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$. This develops further Dinculeanu's remark (see Section 4.1) that such an operator U does not necessarily exist for a given arbitrary S.

Another consequence of Theorem 4.3.3 is the next result.

Corollary 4.3.5. *Let X, Y, and Z be Banach spaces. Let $1 \le p \le \infty$. If $S \in \mathcal{P}_{p'}(Z, \mathcal{L}(X,Y))$, then there exists an operator $U \in \mathcal{L}(Z \hat{\otimes}_{d_p} X, Y)$ such that $U^{\#} = S$.*

Proof. Given $(x_i) \in \ell_\infty(X)$ and $(z_i) \in \ell_{p'}^w(Z)$, we have

$$\begin{aligned}\|((Sz_i)x_i)\|_{p'}^w &\le \|((Sz_i)x_i)\|_{p'} \le \|(x_i)\|_\infty \|(Sz_i)\|_{p'} \\ &\le \|S\|_{\mathcal{P}_{p'}} \|(x_i)\|_\infty \|(z_i)\|_{p'}^w.\end{aligned}$$ □

In Section 4.5, we shall show that the converse of this result, in general, does not hold (see Example 4.5.2).

4.4 Characterizing associated operators: classical case

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$. In Sections 4.2 and 4.3, for a given operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$, we obtained conditions for the existence of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$. This was done in a rather general framework involving tensor products of Banach spaces. In this section, we are interested in specific conditions involving a representing measure of S, which was built in Chapter 3 (see Section 3.2). In terms of the representing measure of S, we establish a necessary condition for the existence of such an operator U (Proposition 4.4.1) that becomes also sufficient in the classical case of $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$ (Proposition 4.4.3).

Let us recall how m was built in Chapter 3 (see Section 3.2). Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$. Let $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$, $x \in X$, be defined by $S_x\varphi = (S\varphi)x$, $\varphi \in \mathcal{C}(\Omega)$, and let $m_x : \Sigma \to Y^{**}$ be its representing measure (given by the Bartle–Dunford–Schwartz theorem). Then $m : \Sigma \to \mathcal{L}(X, Y^{**})$ is defined by

$$\langle y^*, m(E)x\rangle = \langle y^*, m_x(E)\rangle, \quad E \in \Sigma, \tag{4.3}$$

for all $x \in X$ and $y^* \in Y^*$. We can connect the integral with respect to m with the integral with respect to m_x as follows:

$$\langle y^*, \Big(\int_\Omega \varphi\, dm\Big)x\rangle = \langle y^*, \int_\Omega \varphi\, dm_x\rangle \tag{4.4}$$

for all $\varphi \in \mathcal{B}(\Sigma)$, $x \in X$, and $y^* \in Y^*$. (This equality follows easily from (4.3) using a standard argument which passes from characteristic functions to functions in $\mathcal{S}(\Sigma)$ by linearity, and finally to functions in $\mathcal{B}(\Sigma)$ by density.).

Let us present now the promised necessary condition announced at the beginning of this section.

Proposition 4.4.1. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. Let $1 \le p \le \infty$. If there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$, then there exists a constant $c > 0$ such that, for every sequence (E_i) of* *pairwise disjoint sets in* Σ *and every sequence* (x_i) *in* B_X*, the inequality* $\|(m(E_i)x_i)\|_{p'}^w \le c$ *holds.*

Proof. Let $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. As in the proof of Theorem 3.2.5, we define $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, Y^{**}))$ by $\hat{S}\varphi = \int_\Omega \varphi\, dm$, $\varphi \in \mathcal{B}(\Sigma)$. Then $\hat{S}|_{\mathcal{C}(\Omega)} = S$ and, by (4.4), recalling that $\int_\Omega \varphi\, dm_x = S_x^{**}\varphi$, $\varphi \in \mathcal{B}(\Sigma)$, we have

$$\langle y^*, (\hat{S}\varphi)x\rangle = \langle S_x^* y^*, \varphi\rangle \tag{4.5}$$

for all $\varphi \in \mathcal{B}(\Sigma) \subset \mathcal{C}(\Omega)^{**}$, $x \in X$, and $y^* \in Y^*$.

Let $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ be such that $U^{\#} = S$. We can define $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X, Y^{**})$ by $\hat{U}(\varphi \otimes x) = (\hat{S}\varphi)x$. Indeed, we already know that $\hat{U} : \mathcal{B}(\Sigma) \otimes_{d_p} X \to Y^{**}$ is a linear operator. It suffices to show that $\hat{U}$ is bounded on $\mathcal{B}(\Sigma) \otimes_{d_p} X$. Recall that $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X$. Let $\varphi \in \mathcal{C}(\Omega)$, $x \in X$, and $y^* \in Y^*$. Using that $U(\varphi \otimes x) = S_x\varphi$ in Y, hence $\langle \varphi \otimes x, U^*y^*\rangle = \langle \varphi, S_x^* y^*\rangle$, and using also that $(\mathcal{C}(\Omega)\hat{\otimes}_{d_p} X)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*)$ as Banach spaces, we get that $(U^*y^*)^*x = S_x^* y^*$ in $\mathcal{C}(\Omega)^*$. Since $\mathcal{B}(\Sigma) \subset \mathcal{C}(\Omega)^{**}$, for all $\varphi \in \mathcal{B}(\Sigma)$, $x \in X$, and $y^* \in Y^*$, we have

$$\langle S_x^* y^*, \varphi\rangle = \langle x, (U^*y^*)^{**}\varphi\rangle = \langle \varphi \otimes x, U_{y^*}\rangle, \tag{4.6}$$

where $U_{y^*} := (U^*y^*)^{**}|_{\mathcal{B}(\Sigma)}$. It is well known (see, e.g., [26, Theorem 2.17 and Proposition 2.19]) that $(U^*y^*)^{**} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega)^{**}, X^*)$ and $\|(U^*y^*)^{**}\|_{\mathcal{P}_{p'}} = \|U^*y^*\|_{\mathcal{P}_{p'}}$. It follows that $U_{y^*} \in \mathcal{P}_{p'}(\mathcal{B}(\Sigma), X^*)$ and $\|U_{y^*}\|_{\mathcal{P}_{p'}} = \|U^*y^*\|_{\mathcal{P}_{p'}}$, because $\mathcal{C}(\Omega) \subset \mathcal{B}(\Sigma) \subset \mathcal{C}(\Omega)^{**}$ as closed subspaces. Recalling that $\mathcal{P}_{p'}(\mathcal{B}(\Sigma), X^*) = (\mathcal{B}(\Sigma) \otimes_{d_p} X)^*$ and using (4.5) and (4.6), we may write

$$\langle y^*, \hat{U}(\varphi \otimes x)\rangle = \langle \varphi \otimes x, U_{y^*}\rangle, \quad \varphi \in \mathcal{B}(\Sigma), x \in X, y^* \in Y^*.$$

Hence, for any $v = \sum_{i=1}^n \varphi_i \otimes x_i \in \mathcal{B}(\Sigma) \otimes_{d_p} X$ and $y^* \in Y^*$, we get that

$$|\langle y^*, \hat{U}v\rangle| = \Big|\sum_{i=1}^n \langle \varphi_i \otimes x_i, U_{y^*}\rangle\Big| = |\langle v, U_{y^*}\rangle| \le \|U_{y^*}\|_{\mathcal{P}_{p'}} \|v\|_{d_p}.$$

But since

$$\|U_{y^*}\|_{\mathcal{P}_{p'}} = \|U^*y^*\|_{\mathcal{P}_{p'}} = \sup\Big\{|\langle v, U^*y^*\rangle| \,:\, v \in B_{\mathcal{C}(\Omega)\otimes_{d_p} X}\Big\}$$

$$= \sup\Big\{|\langle Uv, y^*\rangle| \,:\, v \in B_{\mathcal{C}(\Omega)\otimes_{d_p} X}\Big\} \le \|U\|\,\|y^*\|,$$

$\hat{U}$ is bounded on $\mathcal{B}(\Sigma) \otimes_{d_p} X$, as desired.

We have defined $\hat{U}$ so that $\hat{U}^{\#} = \hat{S}$. Hence, by Theorem 4.3.3, there exists a constant $c > 0$ such that, for all $(x_i) \in \ell_\infty(X)$ and $(\varphi_i) \in \ell^w_{p'}(\mathcal{B}(\Sigma))$,

$$\|((\hat{S}\varphi_i)x_i)\|^w_{p'} \le c\,\|(x_i)\|_\infty \|(\varphi_i)\|^w_{p'}. \tag{4.7}$$

Let (E_i) be a sequence of pairwise disjoint sets in Σ. Then $(\chi_{E_i}) \in \ell^w_{p'}(\mathcal{B}(\Sigma))$ and $\|(\chi_{E_i})\|^w_{p'} \le 1$. Indeed, let us show that even $\|(\chi_{E_i})\|^w_1 \le 1$. By definition,

$$\|(\chi_{E_i})\|^w_1 = \sup\Big\{\|(\langle \chi_{E_i}, \mu\rangle)_i\|_1 \,:\, \mu \in B_{\mathcal{B}(\Sigma)^*}\Big\}.$$

Also, it is well known that $\mathcal{B}(\Sigma)^* = ba(\Sigma)$ as Banach spaces, where $ba(\Sigma)$ denotes the space of all bounded additive measures defined on Σ with the variation norm, and the duality works as follows:

$$\langle \varphi, \mu\rangle = \int_\Omega \varphi\, d\mu \ , \varphi \in \mathcal{B}(\Sigma), \mu \in ba(\Sigma)$$

(see, e.g., [32, Theorem 1, p. 258]). Since, in particular,

$$\langle \chi_E, \mu\rangle = \mu(E), \text{ for all } E \in \Sigma,$$

we have

$$\|(\chi_{E_i})\|^w_1 = \sup\Big\{\|(\mu(E_i))_i\|_1 \,:\, \mu \in B_{ba(\Sigma)}\Big\}.$$

But

$$\|(\mu(E_i))_i\|_1 \le |\mu|(\bigcup_{i=1}^\infty E_i) \le |\mu|(\Omega),$$

giving that $\|(\chi_{E_i})\|^w_1 \le 1$.

Therefore, since $\hat{S}\chi_{E_i} = m(E_i)$, we get from (4.7) the desired inequality for $(x_i) \subset B_X$. □

Remark 4.4.2. Using the general inequality $\|z_i\| \le \|(z_i)_i\|^w_1$ for all i, it is clear that $\|(\chi_{E_i})\|^w_1 = 1$ if there exists $E_{i_0} \neq \emptyset$. An alternative proof of the inequality $\|(\chi_{E_i})\|_1^w \le 1$ can be done using extreme points. It is well known that we only need to use any norming subset of $\mathcal{B}(\Sigma)^*$ for evaluating $\|(\chi_{E_i})\|_1^w$ and a particularly useful example of this kind is the set of all extreme points of $B_{\mathcal{B}(\Sigma)^*}$ (see, e.g., [26, p. 36]). This set coincides with $\{\alpha\delta_\omega : \alpha \in \mathbb{K}, |\alpha| = 1, \omega \in \Omega\}$, where $\delta_\omega \in \mathcal{B}(\Sigma)^*$ is defined by $\langle \varphi, \delta_\omega\rangle = \varphi(\omega)$, $\omega \in \Omega$ (see, e.g., [17, Chapter V, Theorem 8.4]). Then

$$\|(\chi_{E_i})\|_1^w = \sup\Big\{\|(\langle \chi_{E_i}, \alpha\delta_\omega\rangle)_i\|_1 : \alpha \in \mathbb{K}, |\alpha| = 1, \omega \in \Omega\Big\}$$

$$= \sup\Big\{\|(\alpha\chi_{E_i}(\omega))_i\|_1 : \alpha \in \mathbb{K}, |\alpha| = 1, \omega \in \Omega\Big\}$$

$$= \sup\Big\{\|(\chi_{E_i}(\omega))_i\|_1 : \omega \in \Omega\Big\} \le 1,$$

where the last inequality uses that (E_i) is a sequence of pairwise disjoint sets in Σ (this inequality becomes an equality if there exists $E_{i_0} \neq \emptyset$).

In Section 4.5, we shall see that the necessary condition from Proposition 4.4.1, in general, is not sufficient (see Example 4.5.3). But it is so for the case $p = \infty$, as the next proposition shows.

Proposition 4.4.3. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. The following statements are equivalent.*

(a) *There exists an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ such that $U^{\#} = S$.*

(b) *There exists a constant $c > 0$ such that, for every sequence (E_i) of pairwise disjoint sets in Σ and every sequence (x_i) in B_X, the inequality $\|(m(E_i)x_i)\|_1^w \le c$ holds.*

Proof. We only need to prove (b)⇒(a). Similarly to the construction of the elementary Bartle integral (the case when $X = \mathbb{K}$) (see, e.g., [27, pp. 5–6]), we have a linear operator $V : \mathcal{S}(\Sigma, X) \to Y^{**}$ defined by $Vf = \sum_{i=1}^n m(E_i)x_i$ for $f = \sum_{i=1}^n \chi_{E_i}x_i$, where $E_1, \ldots, E_n$ are pairwise disjoint sets in Σ and $x_i \in X \setminus \{0\}$. Using that $\|f\| = \max_i \|x_i\|$, we have that

$$\|Vf\| = \sup\Big\{\Big|\langle y^*, \sum_{i=1}^n m(E_i)x_i\rangle\Big| \;:\; \|y^*\| \le 1\Big\}$$

$$\le \sup\Big\{\sum_{i=1}^{n}\Big|\langle y^*, m(E_i)\frac{x_i}{\|x_i\|}\rangle\Big|\,\|x_i\| \,:\, \|y^*\|\le 1\Big\}$$

$$\le \sup\Big\{\sum_{i=1}^{n}\Big|\langle y^*, m(E_i)\frac{x_i}{\|x_i\|}\rangle\Big| \,:\, \|y^*\|\le 1\Big\}\max_i \|x_i\|$$

$$= \Big\|\Big(m(E_i)\Big(\frac{x_i}{\|x_i\|}\Big)\Big)\Big\|_1^w \|f\| \le c\,\|f\|.$$

Therefore, V is continuous and it can be extended by continuity to $\mathcal{B}(\Sigma, X)$. We keep calling its extension V. We shall show that $U = V|_{\mathcal{C}(\Omega,X)}$ is the desired operator.

As in the proof of Proposition 4.4.1, we have $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X, Y^{**}))$ defined by $\hat{S}\varphi = \int_\Omega \varphi\, dm$, $\varphi \in \mathcal{B}(\Sigma)$. Let $x \in X$. Since

$$(\hat{S}\chi_E)x = m(E)x = V(\chi_E x),\ E \in \Sigma,$$

and $\mathcal{B}(\Sigma) = \overline{\mathcal{S}(\Sigma)}$,

$$(\hat{S}\varphi)x = V(\varphi x),\ \varphi \in \mathcal{B}(\Sigma),\ x \in X.$$

In particular,

$$(S\varphi)x = U(\varphi x),\ \varphi \in \mathcal{C}(\Omega),\ x \in X,$$

meaning that $U^\# = S$. It remains to show that $\operatorname{ran} U \subset Y$. From the last equality, we see that $U(\varphi x) \in Y$ for all $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. But, as is well known,

$$\mathcal{C}(\Omega, X) = \overline{\operatorname{span}}\{\varphi x \,:\, \varphi \in \mathcal{C}(\Omega), x \in X\}$$

(see, e.g., [84, p. 49]; this is, in fact, the main argument in the proof of Grothendieck's description $\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X$). □

4.5 Examples

In Corollary 4.2.6, we proved that, for all Banach spaces X and Y, with X^* being of cotype 2, and for every $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ and $p \le 2$, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^\# = S$. The next example proves that this result does not hold for $p > 2$.

Example 4.5.1. There exists an operator $S \in \mathcal{L}(\mathcal{C}([0,1]), \mathcal{L}(\ell_2, \ell_2))$ such that, for each $p > 2$, there does not exist any operator $U \in \mathcal{L}(\mathcal{C}_p([0,1], \ell_2), \ell_2)$ such that $U^{\#} = S$.

Proof. Consider an operator $S \in \mathcal{L}(\mathcal{C}([0,1]), \mathcal{L}(\ell_2, \ell_2))$ defined by $(S\varphi)x = (x(n)\varphi(\delta_n))_n$, for $x = (x(n))_n \in \ell_2$, where $(\delta_n)_{n=1}^{\infty}$ is the following sequence in $[0,1]$:

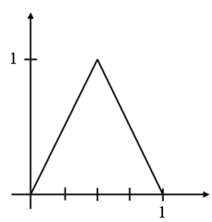
$$\frac{1}{2}; \frac{1}{2^2}, \frac{3}{2^2}; \frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}; \cdots ; \frac{1}{2^n}, \frac{3}{2^n}, \cdots , \frac{2^n - 1}{2^n}; \cdots$$

By contradiction, let us suppose that there exist $p > 2$ and $U \in \mathcal{L}(\mathcal{C}_p([0,1], \ell_2), \ell_2)$ such that $U^{\#} = S$. By Theorem 4.3.3, there exists a constant $c > 0$ such that $\|((S\varphi_n)x_n)\|_{p'}^{w} \leq c$, for all $(x_n) \subset B_{\ell_2}$ and $(\varphi_n) \subset \mathcal{C}([0,1])$ with $\|(\varphi_n)\|_{p'}^{w} \leq 1$. Taking $(x_n) = (e_n)$, the unit vector basis in ℓ_2, we get that $(S\varphi_n)e_n = \varphi_n(\delta_n)e_n$, $n \in \mathbb{N}$. Therefore, $\sum_{n=1}^{\infty} |\langle \varphi_n(\delta_n)e_n, y\rangle|^{p'} \leq c^{p'}$, for all $y = (y(n)) \in B_{\ell_2}$, meaning that

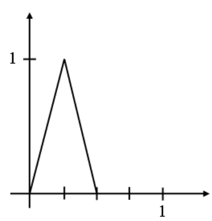
$$\sum_{n=1}^{\infty} |\varphi_n(\delta_n)y(n)|^{p'} \leq c^{p'} \tag{4.8}$$

for all $y = (y(n)) \in B_{\ell_2}$ and $(\varphi_n) \subset \mathcal{C}([0,1])$ with $\|(\varphi_n)\|_{p'}^{w} \leq 1$.

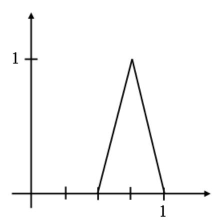
Let us consider the functions of the classical Faber–Schauder basis, except the two first ones. Here are the first few of them:



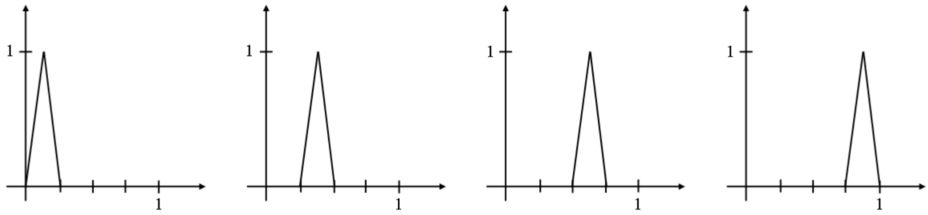
φ_1^1



φ_1^2



φ_2^2



φ_1^3 φ_2^3 φ_3^3 φ_4^3

In general, given $k \in \mathbb{N}$, we take φ_n^k with $n = 1, 2, \ldots, 2^{k-1}$ defined by

$$\varphi_n^k(t) = \begin{cases} 1 - |2^k t - 2n + 1| & \text{if } \frac{2n-2}{2^k} \leq t \leq \frac{2n}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\varphi_1^1(\delta_1) = 1, \varphi_1^2(\delta_2) = \varphi_2^2(\delta_3) = 1, \varphi_1^3(\delta_4) = \ldots = \varphi_4^3(\delta_7) = 1$, and so on, and consider the sequences $(\psi_n^k)_n$ defined as

$$(\psi_n^1) = (\varphi_1^1, 0, 0, \ldots),$$
$$(\psi_n^2) = (0, \varphi_1^2, \varphi_2^2, 0, \ldots),$$
$$(\psi_n^3) = (0, 0, 0, \varphi_1^3, \varphi_2^3, \varphi_3^3, \varphi_4^3, 0, \ldots),$$

and so on. In general,

$$(\psi_n^k)_n = (0, \ldots, 0, \varphi_1^k, \varphi_2^k, \ldots, \varphi_{2^{k-1}}^k, 0, \ldots),$$

where φ_1^k is in the (2^{k-1})-th position. It is clear that $\|(\psi_n^k)_n\|_1 \leq 1$ and therefore $\|(\psi_n^k)_n\|_{p'}^w \leq 1$ for all $k \in \mathbb{N}$. Using (4.8) for $(\psi_n^k)_n$, we obtain

$$\sum_{n=2^{k-1}}^{2^k-1} |y(n)|^{p'} \leq c^{p'}, \text{ for all } k \in \mathbb{N} \text{ and } y = (y(n)) \in B_{\ell_2}.$$

This is, however, impossible. Indeed, take

$$y = (0, \ldots, 0, \frac{1}{\sqrt{2^n}}, \ldots, \frac{1}{\sqrt{2^n}}, 0, \ldots, 0),$$

where y is null except for the positions from 2^{k-1} to $(2^k - 1)$. Obviously, $y \in B_{\ell_2}$ and

$$\sum_{n=2^{k-1}}^{2^k-1} |y(n)|^{p'} = 2^{k-1}\Big(\frac{1}{\sqrt{2^{k-1}}}\Big)^{p'} = 2^{(k-1)(1-\frac{p'}{2})},$$

which is certainly (much) bigger than some $c^{p'}$ for sufficiently large k. $\square$

In Corollary 4.3.5, we proved that, for all Banach spaces X, Y, and Z, every p with $1 \le p \le \infty$, and every operator $S \in \mathcal{P}_{p'}(Z, \mathcal{L}(X,Y))$, there exists an operator $U \in \mathcal{L}(Z\hat{\otimes}_{d_p}X, Y)$ such that $U^{\#} = S$. The next example proves that the converse of this result, in general, does not hold. We use notation from Section 4.4 for $\Omega = [0,1]$: Σ is the σ-algebra of Borel subsets of $[0,1]$ and $\mathcal{B}(\Sigma)$ is the Banach space of all bounded Σ-measurable functions on $[0,1]$.

Example 4.5.2. There exists an operator $S \in \mathcal{L}(c_0, \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{B}(\Sigma)))$ which is not absolutely r-summing for $1 \le r < \infty$ and such that, whenever $1 \le p \le \infty$, there exists an operator $U \in \mathcal{L}(c_0\hat{\otimes}_{d_p}\mathcal{B}(\Sigma), \mathcal{B}(\Sigma))$ satisfying $U^{\#} = S$.

Proof. Denote by (e_n) the unit vector basis in c_0. Consider a linear operator $S : c_0 \to \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{B}(\Sigma))$ defined by $Se_n = T_n$ for all $n \in \mathbb{N}$, where $T_n \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{B}(\Sigma))$ is defined by $T_n\varphi = \varphi\chi_{(1/(n+1),1/n]}$, $\varphi \in \mathcal{B}(\Sigma)$.

Let $(\alpha_n) \in c_0$. Then

$$\|S(\alpha_n)\| = \Big\|\sum_{n=1}^{\infty} \alpha_n T_n\Big\| = \sup_{\|\varphi\|\le 1} \Big\|\sum_{n=1}^{\infty} \alpha_n \varphi\chi_{(1/(n+1),1/n]}\Big\|.$$

Let $\varphi \in \mathcal{B}(\Sigma)$ with $\|\varphi\| \le 1$. For every $t \in (0,1]$, there exists a unique $n_0 \in \mathbb{N}$ such that $t \in (1/(n_0+1), 1/n_0]$. Then

$$\Big|\sum_{n=1}^{\infty} \alpha_n \varphi(t)\chi_{(1/(n+1),1/n]}(t)\Big| = |\alpha_{n_0}\varphi(t)| \le \|(\alpha_n)\|\,\|\varphi\| \le \|(\alpha_n)\|$$

(this inequality is trivially true for $t = 0$). Thus S is bounded.

In order to show that there exists an operator $U \in \mathcal{L}(c_0\hat{\otimes}_{d_p}\mathcal{B}(\Sigma), \mathcal{B}(\Sigma))$ such that $U^{\#} = S$, we check that S satisfies condition (b) of Theorem 4.3.3. Let $(\tilde{\alpha}_n)_{n=1}^N \subset c_0$, with $\tilde{\alpha}_n = (\alpha_n(m))_m$, and $(\varphi_n)_{n=1}^N \subset \mathcal{B}(\Sigma)$ be finite systems. Then (see, e.g., [26, p. 35] for the formula of $\|(\cdot)\|_{p'}^w$ below)

$$\|((S\tilde{\alpha}_n)\varphi_n)\|_{p'}^w = \sup\Big\{\Big\|\sum_{n=1}^{N} \gamma_n (S\tilde{\alpha}_n)\varphi_n\Big\| \,:\, (\gamma_n) \in B_{\ell_p^N}\Big\}.$$

For $(\gamma_n) \in B_{\ell_p^N}$, notice that

$$\Big\|\sum_{n=1}^N \gamma_n (S\tilde{\alpha}_n)\varphi_n\Big\| = \Big\|\sum_{n=1}^N \gamma_n \sum_{m=1}^\infty \alpha_n(m) T_m \varphi_n\Big\|$$

$$= \Big\|\sum_{n=1}^N \gamma_n \sum_{m=1}^\infty \alpha_n(m)\varphi_n \chi_{(1/(m+1),1/m]}\Big\|.$$

But, again, for every $t \in (0,1]$, there exists a unique $m_0 \in \mathbb{N}$ such that $t \in (1/(m_0+1), 1/m_0]$ (for $t = 0$, the chain of inequalities below is trivially true), and we obtain

$$\Big|\sum_{n=1}^N \gamma_n \sum_{m=1}^\infty \alpha_n(m)\varphi_n(t)\chi_{(1/(m+1),1/m]}(t)\Big| = \Big|\sum_{n=1}^N \gamma_n \alpha_n(m_0)\varphi_n(t)\Big|$$

$$\leq \|(\varphi_n)\|_\infty \Big|\sum_{n=1}^N \gamma_n \alpha_n(m_0)\Big| \leq \|(\varphi_n)\|_\infty \Big\|\sum_{n=1}^N \gamma_n \tilde{\alpha}_n\Big\| \leq \|(\varphi_n)\|_\infty \|(\tilde{\alpha}_n)\|^w_{p'},$$

yielding

$$\|((S\tilde{\alpha}_n)\varphi_n)\|^w_{p'} \leq \|(\varphi_n)\|_\infty \|(\tilde{\alpha}_n)\|^w_{p'},$$

meaning that condition (b) of Theorem 4.3.3 holds.

Finally, let $1 \leq r < \infty$. Since $(e_n) \in \ell_1^w(c_0)$, also $(e_n) \in \ell_r^w(c_0)$. If now S were an absolutely r-summing operator, then the sequence $(Se_n) = (T_n)$ would be absolutely r-summable (see, e.g., [26, Proposition 2.1, p. 34]). This is however impossible, because $\|T_n\| = 1$ for all $n \in \mathbb{N}$. Thus S is not absolutely r-summing. □

In Proposition 4.4.1, for a given operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$, we obtained a necessary condition, expressed in terms of the representing measure of S, for the existence of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^\# = S$. The next example proves that this condition, in general, is not sufficient.

Example 4.5.3. Let $1 < p < 2$. There exist a compact Hausdorff space Ω and an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(\ell_p, \mathbb{K}))$ such that its representing measure $m : \Sigma \to \mathcal{L}(\ell_p, \mathbb{K})$ verifies that

$$\|(m(E_i)x_i)\|_{p'} \leq c, \text{ hence also } \|(m(E_i)x_i)\|^w_{p'} \leq c,$$

for some constant $c > 0$ and for every sequence (E_i) of pairwise disjoint sets in Σ and every sequence (x_i) in B_{ℓ_p}. However, there does not exist any operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, \ell_p), \mathbb{K})$ such that $U^{\#} = S$.

Proof. Kwapień showed (see [48, Theorem 7] or, e.g., [26, p. 208]) that there exist bounded linear operators S from ℓ_∞ to $\ell_{p'} = \mathcal{L}(\ell_p, \mathbb{K})$, $2 < p' < \infty$, which are not absolutely p'-summing. Since ℓ_∞ can be identified with $\mathcal{C}(\Omega)$, where Ω is the Stone–Čech compactification of $\mathbb{N}$, we have an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \ell_{p'}) \setminus \mathcal{P}_{p'}(\mathcal{C}(\Omega), \ell_{p'})$.

The canonical identification $\mathcal{C}_p(\Omega, \ell_p)^* = \mathcal{P}_{p'}(\mathcal{C}(\Omega), \ell_p^*)$ identifies $U \in \mathcal{C}_p(\Omega, \ell_p)^*$ with $U^{\#} \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \ell_{p'})$ (see Section 4.2). Since our S is not absolutely p'-summing, there does not exist $U \in \mathcal{C}_p(\Omega, \ell_p)^* = \mathcal{L}(\mathcal{C}_p(\Omega, \ell_p), \mathbb{K})$ such that $U^{\#} = S$.

On the other hand, thanks to Lindenstrauss and Pełczyński [52] and Maurey [54] (see, e.g., [26, Theorems 10.6 and 10.9]), we know that $\mathcal{L}(\mathcal{C}(\Omega), \ell_{p'}) = \mathcal{P}_{(p',1)}(\mathcal{C}(\Omega), \ell_{p'})$ (here we use again that $2 < p' < \infty$). Using that S is an absolutely $(p', 1)$-summing operator, we shall show that its representing measure $m : \Sigma \to \ell_{p'}$ verifies the above condition. Let (E_i) be a sequence of pairwise disjoint sets in Σ and let $(x_i) \subset B_{\ell_p}$. Then, the sequence $(\chi_{E_i}) \in \ell_1^w(\mathcal{B}(\Sigma))$ and $\|(\chi_{E_i})\|_1^w \le 1$ (see the proof of Proposition 4.4.1 or Remark 4.4.2). As $m : \Sigma \to \ell_{p'} = (\ell_{p'})^{**}$ is the (classical) representing measure of S, the integration operator $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \ell_{p'})$ coincides with $S^{**}|_{\mathcal{B}(\Sigma)}$. Since S is absolutely $(p', 1)$-summing, also S^{**} is (see [4, Theorem 3.4] or, e.g., [71, 17.1.5]), and hence $\hat{S}$ is, giving that

$$\|(\hat{S}\chi_{E_i})\|_{p'} \le \|\hat{S}\|_{\mathcal{P}_{(p',1)}} \|(\chi_{E_i})\|_1^w \le \|\hat{S}\|_{\mathcal{P}_{(p',1)}} =: c.$$

Therefore

$$\|(m(E_i)x_i)\|_{p'} = \|((\hat{S}\chi_{E_i})x_i)\|_{p'} \le \|(\hat{S}\chi_{E_i})\|_{p'} \le c. \qquad \square$$

Chapter 5

Absolutely (r, q)-summing operators

This chapter is devoted to the study of absolutely (r, q)-summing operators from $\mathcal{C}_p(\Omega, X)$ to Y. We study the interplay between U, its associated operator $U^\#$, and its representing measure (built in Chapter 3). Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, this encompasses not only the classical Swartz theorem about absolutely summing operators from $\mathcal{C}(\Omega, X)$ to Y [90] but also its existing extensions, providing an improvement even to the Swartz theorem. Counterexamples are exhibited to indicate sharpness of our results. This chapter is based on [59].

5.1 Introduction

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. As in Chapters 3 and 4, for every operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, we denote by $U^\#$ the *associated operator* from $\mathcal{C}(\Omega)$ to $\mathcal{L}(X, Y)$ defined by $(U^\#\varphi)x = U(\varphi x)$, $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. (The notation $U^\#$ is traditional; see, e.g., [56, 75–77, 79–81, 86, 90]; in the book [30] by Dinculeanu, $U^\#$ is denoted by U'; in [10, 13], $U^\#$ is denoted by U_1.) Then, clearly, $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$.

Many authors have studied the interplay between U and $U^\#$ for different classes of operators with the aim to characterize properties of U

in terms of its associated operator $U^{\#}$; see, e.g., the above references. On the other hand, in the above-mentioned papers and, e.g., in [78, 82, 85], the authors also studied the interplay between the operator U and its representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$, where Σ denotes the σ-algebra of Borel subsets of Ω.

The starting point for the present paper, as well as for the above-mentioned papers, is the following by-now classical theorem from 1973, due to Swartz [90, Theorem 12]. Below, we denote by $\mathcal{P}_1 = (\mathcal{P}_1, \|\cdot\|_{\mathcal{P}_1})$ the Banach operator ideal of absolutely 1-summing operators (called also absolutely summing operators).

Theorem 5.1.1 (Swartz). *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. The following statements are equivalents.*

(i) *$U \in \mathcal{P}_1(\mathcal{C}(\Omega, X), Y)$.*

(ii) *$U^{\#} \in \mathcal{P}_1(\mathcal{C}(\Omega), \mathcal{P}_1(X, Y))$.*

(iii) *$m(E) \in \mathcal{P}_1(X, Y)$ for all $E \in \Sigma$, and $m : \Sigma \to \mathcal{P}_1(X, Y)$ is of bounded variation.*

In this case, $\|U\|_{\mathcal{P}_1} = \|U^{\#}\|_{\mathcal{P}_1} = |m|(\Omega)$, where $|m|(\Omega)$ denotes the variation of $m : \Sigma \to \mathcal{P}_1(X, Y)$ on Ω.

As we recalled in Section 4.1, from Grothendieck's classics [43] (see, e.g., [84, pp. 49–50]), we know that $\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X$, where ε denotes the injective tensor norm, under the canonical isometric isomorphism $\varphi x \leftrightarrow \varphi \otimes x$, $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. As is well known, this allows to extend the definition of $U^{\#}$ as follows. Let Z be a Banach space and let α be a tensor norm. If $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$, then the operator $U^{\#} \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ *associated to* U is defined by $(U^{\#}z)x = U(z \otimes x)$, $z \in Z$ and $x \in X$.

Keeping this in mind, the most far-reaching extension of the Swartz theorem – after the works, for instance, by Bilyeu and Lewis [10, Proposition 2.2 and Theorem 2.5], Montgomery-Smith and Saab [56, Theorems 2.3 and 3.2] – is due to Popa [77, Theorems 1, 2, 5] in 2001. We state Popa's extension in two theorems, the first one (Theorem 5.1.10) concerning the interplay between U and $U^{\#}$, the second one (Theorem 5.1.11) concerning the interplay between U and its representing measure. Let us recall a bit the history from Swartz's theorem to Popa's theorem. Below, we denote by $\mathcal{P}_{(r,q)} = (\mathcal{P}_{(r,q)}, \|\cdot\|_{\mathcal{P}_{(r,q)}})$ the Banach operator ideal of absolutely (r,q)-summing operators. Recall that the Banach operator ideal of absolutely q-summing operators is defined as $\mathcal{P}_q = \mathcal{P}_{(q,q)}$.

In 1976, Bilyeu and Lewis [10] extended the Swartz theorem to a more general context. Since $\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X$, they considered operators $U \in \mathcal{L}(Z\hat{\otimes}_\varepsilon X, Y)$ and they replaced the condition of being absolutely summing by the condition of being absolutely p-summing, $1 \le p < \infty$ (see [10, Proposition 2.2 (ii)]).

Theorem 5.1.2 (Bilyeu and Lewis)**.** *Let X, Y, and Z be Banach spaces. Let $1 \le p < \infty$. Assume that $U \in \mathcal{P}_p(Z\hat{\otimes}_\varepsilon X, Y)$. Then $U^\# \in \mathcal{P}_p(Z, \mathcal{L}(X, Y))$.*

Remark 5.1.3. This result also appears in [13, Proposition 3.1].

Let $1 \le r < \infty$. For a vector measure $m : \Sigma \to Y$, we define $|m|_r(\Omega)$, the *r-variation* of m on Ω, by

$$|m|_r(\Omega) = \sup \Big(\sum_{E_i \in \Pi} \|m(E_i)\|^r \Big)^{1/r},$$

where the supremum is taken over all finite partitions $\Pi = (E_i)_{i=1}^n$ of Ω, $n \in \mathbb{N}$. If $|m|_r(\Omega) < \infty$, then m is a *measure of bounded r-variation.* For $r = 1$, we have $|m|_1(\Omega) = |m|(\Omega)$, the *variation* of m on Ω (see, e.g. [27, p. 2, Definition 4]). It is clear that $|m|_r(\Omega) \le |m|_s(\Omega) \le |m|(\Omega)$ if $1 \le s \le r < \infty$. (See Remark 5.3.4 for a different notion of r-variation.)

Bilyeu and Lewis also obtained the next result about the interplay between the operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and its representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ (see [10, Theorem 2.5])

Theorem 5.1.4 (Bilyeu and Lewis)**.** *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p < \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and denote by $m : \Sigma \to \mathcal{L}(X, Y^{**})$ its representing measure.*

(i) *If $U \in \mathcal{P}_p(\mathcal{C}(\Omega, X), Y)$, then $m(E) \in \mathcal{P}_p(X, Y)$ for all $E \in \Sigma$, and $|m|_p(\Omega) \le \|U\|_{\mathcal{P}_p}$.*

(ii) *If $m(E) \in \mathcal{P}_p(X, Y)$ for all $E \in \Sigma$, and $m : \Sigma \to \mathcal{P}_p(X, Y)$ is of bounded variation, then $U \in \mathcal{P}_p(\mathcal{C}(\Omega, X), Y)$ and $\|U\|_{\mathcal{P}_p} \le |m|(\Omega)$.*

Remark 5.1.5. A result similar to part (ii) was proved in [82, Proposition 4] but in the framework of p-lattice summing operators.

As we can see, in Theorem 5.1.2, only one implication is proved. However, nothing is said about the relation between the norms of U and $U^{\#}$, or about the converse implication. The next step was walked by Montgomery-Smith and Saab [56] in 1992. They observed (see [56, Theorem 3.1]) that the statement in Theorem 5.1.2 can be expressed in a more precise way.

Theorem 5.1.6 (Montgomery-Smith and Saab)**.** *Let X, Y, and Z be Banach spaces. Let $1 \leq p < \infty$. Assume that $U \in \mathcal{P}_p(Z\hat{\otimes}_\varepsilon X, Y)$. Then $U^{\#} \in \mathcal{P}_p(Z, \mathcal{P}_p(X, Y))$.*

They noticed that, if $U \in \mathcal{P}_p(Z\hat{\otimes}_\varepsilon X, Y)$, then $U^{\#}z \in \mathcal{P}_p(X, Y)$ for all $z \in Z$. (For a more general fact, see Proposition 5.2.1.) In [56, Theorem 3.4], they also showed the first example that the converse of Theorem 5.1.2 is not true in general.

Theorem 5.1.7 (Montgomery-Smith and Saab)**.** *For each $1 < p < \infty$, there exists $U \in \mathcal{L}(\mathcal{C}([0,1], \ell_2), \ell_2)$ which is not absolutely p-summing, but such that $U^{\#} \in \mathcal{P}_p(\mathcal{C}[0,1], \mathcal{P}_p(\ell_2, \ell_2))$.*

Montgomery-Smith and Saab also extended the Swartz Theorem 5.1.1 to the following one (see [56, Theorem 2.3]).

Theorem 5.1.8 (Montgomery-Smith and Saab)**.** *Let X and Y be Banach spaces and let Z be an $\mathcal{L}_\infty$-space. The following statements are equivalent.*

(i) $U \in \mathcal{P}_1(Z\hat{\otimes}_\varepsilon X, Y)$.

(ii) $U^{\#} \in \mathcal{P}_1(Z, \mathcal{P}_1(X, Y))$.

They showed that the condition Z being an $\mathcal{L}_\infty$-space is essential (see [56, Theorem 3.2]).

Theorem 5.1.9 (Montgomery-Smith and Saab)**.** *There exists $U \in \mathcal{L}(\ell_2\hat{\otimes}_\varepsilon \ell_2, \ell_2)$ such that U is not absolutely summing, yet $U^{\#} \in \mathcal{P}_1(\ell_2, \mathcal{P}_1(\ell_2, \ell_2))$.*

Montgomery-Smith and Saab's work (similarly to Bilyeu and Lewis' work) did not mention anything about the relation between the norms of U and $U^{\#}$, neither about the relation between these operators and the representing measure of $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$.

As it was mentioned above, the latest extension of the Swartz theorem is due to Popa (see [77, Theorems 1, 2, 5]) in 2001.

Theorem 5.1.10 (Popa). *Let X, Y, and Z be Banach spaces. Let $1 \le q \le r < \infty$. If $U \in \mathcal{P}_{(r,q)}(Z \hat{\otimes}_\varepsilon X, Y)$, then $U^{\#} \in \mathcal{P}_{(r,q)}(Z, \mathcal{P}_{(r,q)}(X,Y))$ and $\|U^{\#}\|_{\mathcal{P}_{(r,q)}} \le \|U\|_{\mathcal{P}_{(r,q)}}$.*

Theorem 5.1.11 (Popa). *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le q \le r < \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure.*

(i) *If $U \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega, X), Y)$, then $m(E) \in \mathcal{P}_{(r,q)}(X,Y)$ for all $E \in \Sigma$, and $|m|_r(\Omega) \le \|U\|_{\mathcal{P}_{(r,q)}}$.*

(ii) *If $m(E) \in \mathcal{P}_{(r,q)}(X,Y)$ for all $E \in \Sigma$, and $m : \Sigma \to \mathcal{P}_{(r,q)}(X,Y)$ is of bounded variation, then $U \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega, X), Y)$ and $\|U\|_{\mathcal{P}_{(r,q)}} \le |m|(\Omega)$.*

This chapter aims in studying the interplay between U, $U^{\#}$, and the representing measure of U in a more general context of operators defined on the Banach space $\mathcal{C}_p(\Omega, X)$ of p-continuous X-valued functions (see Section 5.2 for the definition and references). Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, this also encompasses the classical case of operators on $\mathcal{C}(\Omega, X)$. Recall (see Theorem 1.3.7) that $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega) \hat{\otimes}_{d_p} X$, where d_p denotes the right Chevet–Saphar tensor norm (see [87] or, e.g., [84, Chapter 6]).

In Section 5.2, we extend Popa's Theorem 5.1.10 from the injective tensor norm ε to the norms d_p, $1 \le p \le \infty$ (see Theorem 5.2.3), providing an improvement of it in the special case when $p = \infty$ (see Corollary 5.2.5). We also exhibit counterexamples (see Proposition 5.2.8 and 5.2.12), demonstrating that $U^{\#} \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), \mathcal{P}_{(r,q)}(X,Y))$ does not imply that $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega, X), Y)$.

Measures enter into play in Section 5.3. We extend Popa's Theorem 5.1.11 from $\mathcal{C}(\Omega, X)$ to $\mathcal{C}_p(\Omega, X)$, $1 \le p \le \infty$ (see Theorem 5.3.5). Then we introduce a notion of r-dominated vector measures, giving a sufficient condition under which $U^{\#} \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), \mathcal{P}_{(r,q)}(X,Y))$ implies that $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega, X), Y)$ (see Theorem 5.3.6). This, in turn, leads us to an improvement even in Swartz's Theorem 5.1.1 (see Corollary 5.3.8).

5.2 Interplay between U and $U^{\#}$ for absolutely (r,q)-summing operators

Let X, Y, and Z be Banach spaces, and let α be a tensor norm. As we already mentioned above (see also Section 4.2), every operator

$U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ induces an associated operator $U^\# \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ by $(U^\# z)x = U(z \otimes x)$, $z \in Z$, $x \in X$. It is well known and easy to verify that the correspondence $U \mapsto U^\#$ is an isometric isomorphism between the Banach spaces $\mathcal{L}(Z \otimes_\pi X, Y) = \mathcal{L}(Z\hat{\otimes}_\pi X, Y)$ and $\mathcal{L}(Z, \mathcal{L}(X, Y))$, where π denotes the projective tensor norm. In particular, every $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ happens to be the associated operator to some (unique) $U \in \mathcal{L}(Z\hat{\otimes}_\pi X, Y)$. However, in the general case, not all $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ enjoy the "privilege" of being associated to some $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$.

The "existence problem" to characterize these S for which there exists U such that $U^\# = S$ was systematically (and probably the first time in the literature) studied in the paper [58] by Muñoz, Oja, and Piñeiro (see Chapter 4). Among others, we proved Dinculeanu's conjecture from 1967 that even in the most important classical case, when $Z = \mathcal{C}(\Omega)$ and $\alpha = \varepsilon$, the injective tensor norm, such operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ do exist, meaning that $S \neq U^\#$ for all operators $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ (see Section 4.1 for an overview and references). Let us recall an "existence" result, Corollary 4.3.5, which asserts that if $S \in \mathcal{P}_{p'}(Z, \mathcal{L}(X, Y))$, $1 \le p \le \infty$, then there exists an operator $U \in \mathcal{L}(Z\hat{\otimes}_{d_p} X, Y)$ such that $U^\# = S$. This result will be used below several times.

Before passing to the Banach operator ideals $\mathcal{P}_{(r,q)}$ and $\mathcal{P}_q = \mathcal{P}_{(q,q)}$ of the absolutely (r, q)- and absolutely q-summing operators, let us make a useful general observation concerning an arbitrary Banach operator ideal $\mathcal{A}$ (see Proposition 5.2.1). Since $U^\# \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ is defined for any $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$, it is, in particular, defined for any $U \in \mathcal{A}(Z\hat{\otimes}_\alpha X, Y)$.

Proposition 5.2.1. *Let X, Y, and Z be Banach spaces. Let α be a tensor norm and let $\mathcal{A}$ be a Banach operator ideal. If $U \in \mathcal{A}(Z\hat{\otimes}_\alpha X, Y)$, then $U^\# z \in \mathcal{A}(X, Y)$ for all $z \in Z$, and $\|U^\# z\|_\mathcal{A} \le \|U\|_\mathcal{A} \|z\|$.*

Proof. The idea of the proof comes from [56] (see Remark 5.2.2). Let $z \in Z$ and consider $\hat{z} \in \mathcal{L}(X, Z\hat{\otimes}_\alpha X)$, defined by $\hat{z}(x) = z \otimes x$. Since $(U^\# z)x = U(z \otimes x) = (U\hat{z})x$, $x \in X$, and $U \in \mathcal{A}(Z\hat{\otimes}_\alpha X, Y)$, we get that $U^\# z \in \mathcal{A}(X, Y)$ and

$$\|U^\# z\|_\mathcal{A} = \|U\hat{z}\|_\mathcal{A} \le \|U\|_\mathcal{A} \|\hat{z}\| = \|U\|_\mathcal{A} \|z\|. \qquad \square$$

Remark 5.2.2. The special cases of Proposition 5.2.1 when $\alpha = \varepsilon$ are contained in [86, Theorem 3] for $\mathcal{A} = \mathcal{N}$, the Banach operator ideal of nuclear operators, in [56, proofs of Theorems 2.1 and 3.1 (see Theorem 5.1.6)] for $\mathcal{A} = \mathcal{I}$, the Banach operator ideal of integral operators, and $\mathcal{A} = \mathcal{P}_q$, and in [77, Theorem 1 (see Theorem 5.1.10)] for $\mathcal{A} = \mathcal{P}_{(r,q)}$.

The next result extends Popa's Theorem 5.1.10 from the injective tensor norm ε to the norms d_p, $1 \le p \le \infty$, providing an improvement of it in the special case when $p = \infty$ (see Corollary 5.2.5).

Theorem 5.2.3. *Let X, Y, and Z be Banach spaces. Let $1 \le p \le \infty$ and $p' \le q \le r \le \infty$. If $U \in \mathcal{P}_{(r,q)}(Z\hat{\otimes}_{d_p}X, Y)$, then $U^{\#} \in \mathcal{P}_{(r,q)}(Z, \mathcal{P}_{(r,q)}(X, Y))$ and $\|U^{\#}\|_{\mathcal{P}_{(r,q)}} \le \|U\|_{\mathcal{P}_{(r,q)}}$.*

Proof. In the case when $r = \infty$, the claim is immediate from Proposition 5.2.1, because $(\mathcal{P}_{(\infty,q)}, \|\cdot\|_{\mathcal{P}_{(\infty,q)}}) = (\mathcal{L}, \|\cdot\|)$. Let us consider the case when $r < \infty$.

We already know that $U^{\#}z \in \mathcal{P}_{(r,q)}(X, Y)$ for all $z \in Z$ (see Proposition 5.2.1). Let us fix $z_1, \ldots, z_n$ in Z. We need to prove that

$$\sum_{i=1}^{n} \|U^{\#}z_i\|^r_{\mathcal{P}_{(r,q)}} \le \|U\|^r_{\mathcal{P}_{(r,q)}} \big(\|(z_i)\|^w_q\big)^r. \tag{5.1}$$

Fix $\varepsilon > 0$. For every $i = 1, \ldots, n$, we have by definition,

$$\|U^{\#}z_i\|^r_{\mathcal{P}_{(r,q)}} = \sup\Big\{ \sum_{j=1}^{n} \|(U^{\#}z_i)x_{ij}\|^r \ :\ \|(x_{ij})\|^w_q \le 1\,,\ n \in \mathbb{N}\Big\}.$$

Therefore, for each $i = 1, \ldots, n$, there is a finite system $(x_{ij})_{j=1}^{n_i}$ in X with $\|(x_{ij})\|^w_q \le 1$ such that

$$\|U^{\#}z_i\|^r_{\mathcal{P}_{(r,q)}} \le \sum_{j=1}^{n_i} \|(U^{\#}z_i)x_{ij}\|^r + \frac{\varepsilon}{n}.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n} \|U^{\#}z_i\|^r_{\mathcal{P}_{(r,q)}} &\le \sum_{i=1}^{n}\sum_{j=1}^{n_i} \|U(z_i \otimes x_{ij})\|^r + \varepsilon \\ &\le \|U\|^r_{\mathcal{P}_{(r,q)}} \big(\|(z_i \otimes x_{ij})\|^w_q\big)^r + \varepsilon,\end{aligned} \tag{5.2}$$

where $(z_i \otimes x_{ij}) = (z_i \otimes x_{ij})_{i=1,j=1}^{n,n_i} \in \ell^w_q(Z\hat{\otimes}_{d_p}X)$.

Using that $(Z\hat{\otimes}_{d_p}X)^* = \mathcal{P}_{p'}(Z,X^*)$ as Banach spaces, calculate

$$\|(z_i\otimes x_{ij})\|_q^w = \sup\Big\{\Big(\sum_{i,j=1}^{n,n_i}|(Az_i)(x_{ij})|^q\Big)^{1/q} : A\in\mathcal{P}_{p'}(Z,X^*), \|A\|_{\mathcal{P}_{p'}}\le 1\Big\}$$

$$= \sup\Big\{\Big(\sum_{i=1}^{n}\|Az_i\|^q\sum_{j=1}^{n_i}\Big|\frac{Az_i}{\|Az_i\|}(x_{ij})\Big|^q\Big)^{1/q} : A\in\mathcal{P}_{p'}(Z,X^*), \|A\|_{\mathcal{P}_{p'}}\le 1\Big\}.$$

Here

$$\sum_{j=1}^{n_i}\Big|\frac{Az_i}{\|Az_i\|}(x_{ij})\Big|^q \le \big(\|(x_{ij})_{j=1}^{n_i}\|_q^w\big)^q \le 1.$$

Since $p'\le q$, we have $(\mathcal{P}_{p'},\|\cdot\|_{\mathcal{P}_{p'}})\subset(\mathcal{P}_q,\|\cdot\|_{\mathcal{P}_q})$. In particular, $\|A\|_{\mathcal{P}_{p'}}\le 1$ implies $\|A\|_{\mathcal{P}_q}\le 1$. Hence,

$$\|(z_i\otimes x_{ij})\|_q^w \le \sup\big\{\|A\|_{\mathcal{P}_q}\|(z_i)\|_q^w \;:\; \|A\|_{\mathcal{P}_q}\le 1\big\} \le \|(z_i)\|_q^w. \tag{5.3}$$

As $\varepsilon>0$ is arbitrary, (5.2) together with (5.3) imply (5.1). □

It is well known that

$$\varepsilon(v)\le d_\infty(v)\le d_{p_2}(v)\le d_{p_1}(v),\ v\in Z\otimes X,$$

whenever $1\le p_1\le p_2\le\infty$ (see, e.g., [84, p. 135, Proposition 6.6] or [18, p. 152, 12.7, and p. 150, 12.5]). Therefore, concerning possible applications of Theorem 5.2.3, the following observation is useful.

Proposition 5.2.4. *Let X, Y, and Z be Banach spaces, and let α and β be tensor norms. Assume that $1\le q\le r\le\infty$. If $\beta(v)\le\alpha(v)$ for all $v\in Z\otimes X$, then every operator $U\in\mathcal{P}_{(r,q)}(Z\hat{\otimes}_\beta X,Y)$ induces an operator $\overline{U}\in\mathcal{P}_{(r,q)}(Z\hat{\otimes}_\alpha X,Y)$ by*

$$\overline{U}v = Uv,\ v\in Z\otimes X,$$

satisfying $\|\overline{U}\|_{\mathcal{P}_{(r,q)}}\le\|U\|_{\mathcal{P}_{(r,q)}}$.

Proof. Denote $V=Z\otimes X$. The claim of the proposition holds in a general case when V is just a linear space with two norms α and β on it, such that $\beta(v)\le\alpha(v)$ for all $v\in V$. Indeed, denote by V_α and V_β the corresponding completions of V, and let $(v_i)_{i=1}^n$ be an arbitrary finite system in V. Since $B_{V_\beta^*} \subset B_{V_\alpha^*}$, from the definition of $\|\cdot\|_q^w$, it is clear that

$$\|(v_i)\|_{\ell_q^w(V_\beta)} \leq \|(v_i)\|_{\ell_q^w(V_\alpha)}.$$

Now the claim follows, because

$$\|(\overline{U}v_i)\|_r = \|(Uv_i)\|_r \leq \|U\|_{\mathcal{P}_{(r,q)}}\|(v_i)\|_{\ell_q^w(V_\beta)} \leq \|U\|_{\mathcal{P}_{(r,q)}}\|(v_i)\|_{\ell_q^w(V_\alpha)}. \quad \square$$

An immediate application of Theorem 5.2.3 gives the following result. We spell it out, because, thanks to Proposition 5.2.4, it represents a slight improvement of Popa's Theorem 5.1.10. Note that the tensor norms ε and d_∞ are not equivalent (because d_∞ is right-projective (see, e.g., [18, p. 254, 20.4] or [84, p. 137, Proposition 6.7]) but ε is not (see, e.g., [18, p. 49, 4.3] or [84, p. 47])).

Corollary 5.2.5. *Let X, Y, and Z be Banach spaces. Let $1 \leq q \leq r < \infty$. If $U \in \mathcal{P}_{(r,q)}(Z\hat{\otimes}_{d_\infty}X, Y)$, then $U^{\#} \in \mathcal{P}_{(r,q)}(Z, \mathcal{P}_{(r,q)}(X,Y))$ and $\|U^{\#}\|_{\mathcal{P}_{(r,q)}} \leq \|U\|_{\mathcal{P}_{(r,q)}}$.*

Recall that, by Theorem 1.3.7, $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$, where X is a Banach space and Ω is a compact Hausdorff space. Hence, by Theorem 5.2.3, the following holds.

Corollary 5.2.6. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$ and $p' \leq q \leq r \leq \infty$. If $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega, X), Y)$, then $U^{\#} \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), \mathcal{P}_{(r,q)}(X,Y))$ and $\|U^{\#}\|_{\mathcal{P}_{(r,q)}} \leq \|U\|_{\mathcal{P}_{(r,q)}}$.*

The converse of Corollary 5.2.6 and therefore also the converse of Theorem 5.2.3 are not true. In the classical case, when one assumes that $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, there exist quite a lot of counterexamples demonstrating that $U^{\#} \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ does not imply that $U \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega, X), Y)$. The first examples of such kind were constructed by Montgomery-Smith and Saab [56] in 1991 (see, e.g., Theorem 5.1.7). A systematic method for constructing counterexamples with $r = q = 2$ was developed by Popa [76] in 1999.

The relevant examples by Montgomery-Smith and Saab (see [56, Theorems 3.4 (see Theorem 5.1.7) and 3.5] rely on the essence of the

ε-norm on $\mathcal{C}[0,1] \otimes X$ (with $X = \ell_2$ and $X = \ell_1$), and they are not easy to modify for the d_p-norms, $1 < p < \infty$. From the four examples of Popa (see [76, Propositions 3–6]), his example below (see Proposition 5.2.7) is rather appropriate for modifying to give a couple of desired counterexamples for the d_p-norms. This will be done in Propositions 5.2.8 and 5.2.12 below.

We shall work, like Popa [76], on $\mathcal{C}(\Delta)$, where $\Delta = \{-1,1\}^{\mathbb{N}}$ is the Cantor group with Haar measure λ on it. We denote by (r_n) the sequence of the Rademacher functions in $\mathcal{C}(\Delta)$ (recall that r_n is defined as the nth coordinate functional, i.e., $r_n(\delta) = \delta_n$ for $\delta = (\delta_n) \in \Delta$).

Proposition 5.2.7 (see [76, Proposition 4])**.** *Let X and Y be Banach spaces. Assume that X contains a subspace isomorphic to c_0 and Y does not have the Orlicz property. Then there exists an operator $U \in \mathcal{L}(\mathcal{C}(\Delta, X), Y)$ such that $U^\# \in \mathcal{P}_2(\mathcal{C}(\Delta), \mathcal{P}_2(X, Y))$, but U is not absolutely 2-summing.*

Recall (see, e.g., [18, p. 103]) that a Banach space Y has the *Orlicz property* if $\ell_1^w(Y) \subset \ell_2(Y)$. In other words, the identity map on Y is absolutely $(2,1)$-summing.

Proposition 5.2.8. *Let X and Y be Banach spaces and let $1 < p \leq 2$. Assume that X enjoys the Dunford–Pettis property, but not the Schur property. Assume further that Y contains a sequence $(y_n) \in \ell_p^w(Y)$ which is not a null sequence. Then there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Delta, X), Y)$ such that $U^\# \in \mathcal{P}_{p'}(\mathcal{C}(\Delta), \mathcal{P}_{p'}(X, Y))$, but U is not completely continuous. In particular, U is not absolutely q-summing for any q, $1 \leq q < \infty$.*

Remark 5.2.9. It is well known that $L_1(\mu)$- and $\mathcal{C}(\Omega)$-spaces have the Dunford–Pettis property (see, e.g., [23, pp. 19, 20]). Therefore, concerning X, Proposition 5.2.8 applies to all infinite-dimensional $\mathcal{C}(\Omega)$-spaces, including $L_\infty(\mu)$-, $c_0(\Gamma)$-, and $\ell_\infty(\Gamma)$-spaces, and more generally, to their complemented infinite-dimensional subspaces (these contain subspaces isomorphic to c_0 (see, e.g., [23, p. 38])). And it also applies to all those $L_1(\mu)$-spaces that are not $\ell_1(\Gamma)$-spaces (these enjoy the Schur property; recall that $L_1(\mu)$ has the Schur property if and only if μ is an atomic measure (see [14, Theorem 6.5])). Concerning Y, Proposition 5.2.8 applies, for instance, to all Banach spaces containing a subspace isomorphic to ℓ_q for some $q \geq p'$ or to c_0. Indeed, if $(y_n) \subset Y$ is equivalent to the unit vector basis of ℓ_q for some $q \geq p'$ (or of c_0 when $q = \infty$), then $(y_n) \in \ell_{q'}^w(Y) \subset \ell_p^w(Y)$ and (y_n) is not a null sequence. This includes Hilbert spaces, the Banach space c of convergent sequences, and ℓ_∞.

Proof of Proposition 5.2.8. Let (x_n) be a weakly null sequence in X, which is not a null sequence. We clearly may assume that $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Choose $x_n^* \in X^*$ satisfying $\|x_n^*\| = x_n^*(x_n) = 1$, $n \in \mathbb{N}$.

We define an operator S by a general formula that Popa [76] uses in all his counterexamples. Namely, let

$$S\varphi = \sum_{n=1}^{\infty} \Big(\int_\Delta \varphi r_n \, d\lambda \Big) x_n^* \otimes y_n, \quad \varphi \in \mathcal{C}(\Delta). \tag{5.4}$$

Then $S\varphi \in \mathcal{N}_{p'}(X,Y)$ for all $\varphi \in \mathcal{C}(\Delta)$, where $\mathcal{N}_{p'}$ denotes the Banach operator ideal of p'-nuclear operators. Indeed, $(y_n) \in \ell_p^w(Y)$ by assumption, and for

$$a_n^\varphi := \Big(\int_\Delta \varphi r_n \, d\lambda \Big) x_n^*, \quad n \in \mathbb{N},$$

we have, recalling that $(r_n)_{n\in\mathbb{N}}$ is an orthonormal system in $L_2(\lambda)$ and using Bessel's inequality, that

$$\|(a_n^\varphi)_{n=1}^\infty\|_{p'} \leq \|(a_n^\varphi)\|_2 \leq \Big(\sum_{n=1}^{\infty} \Big| \int_\Delta \varphi r_n \, d\lambda \Big|^2 \Big)^{1/2} \leq \|i_2 \varphi\|_{L_2(\lambda)},$$

where $i_2 : \mathcal{C}(\Delta) \to L_2(\lambda)$ is the canonical inclusion map.

It is well known that $(\mathcal{N}_{p'}, \|\cdot\|_{\mathcal{N}_{p'}}) \subset (\mathcal{P}_{p'}, \|\cdot\|_{\mathcal{P}_{p'}})$ (see, e.g., [26, p. 113, Corollary 5.24, and p. 97, Proposition 5.5]). Therefore, $S\varphi \in \mathcal{P}_{p'}(X,Y)$ and

$$\|S\varphi\|_{\mathcal{P}_{p'}} \leq \|S\varphi\|_{\mathcal{N}_{p'}} \leq \|(a_n^\varphi)\|_{p'} \|(y_n)\|_p^w \leq \|i_2\varphi\|_{L_2(\lambda)} \|(y_n)\|_p^w$$

for all $\varphi \in \mathcal{C}(\Delta)$.

Thus, we have a map $S : \mathcal{C}(\Delta) \to \mathcal{P}_{p'}(X,Y)$ which is clearly linear. To show that $S \in \mathcal{P}_{p'}(\mathcal{C}(\Delta), \mathcal{P}_{p'}(X,Y))$, we use the well-known fact that i_2 is absolutely 2-summing (see, e.g., [26, p. 40]), in particular, i_2 is absolutely

p'-summing (see, e.g., [26, p. 39, Theorem 2.8]). If now $(\varphi_k) \in \ell_{p'}^w(\mathcal{C}(\Delta))$, then

$$\sum_{k=1}^{\infty} \|S\varphi_k\|_{\mathcal{P}_{p'}}^{p'} \le \big(\|(y_n)\|_p^w\big)^{p'} \sum_{k=1}^{\infty} \|i_2\varphi_k\|_{L_2(\lambda)} < \infty.$$

Since $S \in \mathcal{P}_{p'}(\mathcal{C}(\Delta), \mathcal{P}_{p'}(X,Y))$, applying Corollary 4.3.5, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Delta, X), Y)$ such that $U^{\#} = S$. Reasoning by contradiction, suppose that U is completely continuous. Thanks to Lemma 2.3.4, the sequence $(r_n \otimes x_n)$ is weakly null in $\mathcal{C}_p(\Delta, X) = \mathcal{C}(\Delta)\hat{\otimes}_{d_p} X$, so $(U(r_n \otimes x_n))_{n=1}^{\infty}$ converges to 0 in Y. Since

$$U(r_n \otimes x_n) = (U^{\#} r_n)x_n = (Sr_n)x_n = \sum_{m=1}^{\infty} \Big(\int_{\Delta} r_n r_m \, d\lambda\Big) x_m^*(x_n) y_m$$

$$= x_n^*(x_n) y_n = y_n,$$

we see that also (y_n) converges to 0. This is a contradiction.

Finally, let us recall that the absolutely q-summing operators, $1 \le q < \infty$, are completely continuous (see, e.g., [26, p. 50, Theorem 2.17]). □

Our final aim in this section is to show that the Popa example (Proposition 5.2.7), in the special case when X is an $\mathcal{L}_\infty$-space, can be extended to yield desired counterexamples for all d_p, $1 \le p \le \infty$ (see Proposition 5.2.12 below). For this end, let us introduce a property of operators $S \in \mathcal{L}(Z, \mathcal{L}(X,Y))$, which we call "property $\mathbf{S}_q$" because it concerns the absolute q-summability.

Let $1 \le q \le \infty$. We say that an operator $S \in \mathcal{L}(Z, \mathcal{L}(X,Y))$ has *property* $\mathbf{S}_q$ if for all sequences $(z_n) \in \ell_\infty(Z)$ and $(x_n) \in \ell_q^w(X)$, it holds that $((Sz_n)x_n) \in \ell_q(Y)$.

In Proposition 5.2.10, we shall characterize property $\mathbf{S}_q$ using the notion of "uniformly dominated" set. Let $1 \le q < \infty$. Recall that a set $M \subset \mathcal{P}_q(X,Y)$ is *uniformly dominated* (see, e.g., [20, p. 307]) if there exists a positive Radon measure μ defined on the compact space B_{X^*}, endowed with the restriction of the weak* topology of X^*, such that

$$\|Tx\|^q \le \int_{B_{X^*}} |\langle x, x^* \rangle|^q \, d\mu$$

for all $x \in X$ and $T \in M$. In [20, Proposition 2.1], it was proved that $M \subset \mathcal{P}_q(X,Y)$ is uniformly dominated if and only if there exists a constant $C > 0$ such that

$$\|(T_i x_i)_{i=1}^n\|_q \leq C\|(x_i)_{i=1}^n\|_q^w$$

for all finite systems $(x_i)_{i=1}^n \subset X$ and $(T_i)_{i=1}^n \subset M$. Using this notion, we have obtained the following characterization of property $\mathbf{S}_q$.

Proposition 5.2.10. *Let X, Y, and Z be Banach spaces and let $1 \leq q < \infty$. Assume that $S \in \mathcal{L}(Z, \mathcal{L}(X,Y))$. The following statements are equivalent.*

(a) *S has property $\mathbf{S}_q$.*

(b) *$M = \{Sz : z \in B_Z\} \subset \mathcal{P}_q(X,Y)$ is uniformly dominated.*

Proof. By the above observation, condition (b) is equivalent to the existence of a constant $C > 0$ such that $\|((Sz_n)x_n)\|_q \leq C\|(x_n)\|_q^w$ for all finite systems $(z_n) \subset B_Z$ and $(x_n) \in \ell_q^w(X)$, and this easily yields that S has property $\mathbf{S}_q$. In view of this, we only need to prove that (a) implies (b).

By contradiction, assume that $\sup\{\|((Sz_n)x_n)\|_q : \|(z_n)\|_\infty \leq 1, \|(x_n)\|_q^w \leq 1\} = \infty$. Then, inductively, we can choose a strictly increasing sequence (n_k) of integers so that, for each $k \in \mathbb{N}$, there exist sequences $(x_n^k)_n \in \ell_q^w(X)$ with $\|(x_n^k)_n\|_q^w \leq 1$ and $(z_n^k)_n \subset Z$ with $\|(z_n^k)_n\|_\infty \leq 1$ satisfying $\sum_{n_{k-1}+1}^{n_k} \|((Sz_n^k)x_n^k)\|^q > k^{3q}$. Now, we consider the sequences

$$z_1^1, \ldots, z_{n_1}^1, z_{n_1+1}^2, \ldots, z_{n_2}^2, \ldots, z_{n_{k-1}+1}^k, \ldots, z_{n_k}^k, \ldots$$

and

$$x_1^1, \ldots, x_{n_1}^1, \frac{x_{n_1+1}^2}{2^2}, \ldots, \frac{x_{n_2}^2}{2^2}, \ldots, \frac{x_{n_{k-1}+1}^k}{k^2}, \ldots, \frac{x_{n_k}^k}{k^2}, \ldots.$$

We denote these sequences by (z_m) and (x_m), respectively.

Obviously, $(z_m) \subset B_Z$. We show that $(x_m) \in \ell_q^w(X)$. Given $x^* \in X^*$, we have

$$\sum_{m=1}^{\infty} |\langle x_m, x^* \rangle|^q \leq \sum_{k=1}^{\infty} \sum_{m=n_{k-1}+1}^{n_k} |\langle \frac{x_m^k}{k^2}, x^* \rangle|^q \leq \sum_{k=1}^{\infty} \frac{1}{k^{2q}} \sum_{m=n_{k-1}+1}^{n_k} |\langle x_m^k, x^* \rangle|^q$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k^{2q}} \|(x_n^k)_n\|_q^w \leq \sum_{k=1}^{\infty} \frac{1}{k^{2q}} < \infty.$$

So (z_m) and (x_m) satisfy the hypothesis for (a) and thus $\|((Sz_m)x_m)\|_q < \infty$. On the other hand, we can obtain

$$\sum_{m=1}^{\infty} \|((Sz_m)x_m)\|^q = \sum_{k=1}^{\infty} \sum_{m=n_{k-1}+1}^{n_k} \|(Sz_m^k)\frac{x_m^k}{k^2})\|^q$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=n_{k-1}+1}^{n_k} \|(Sz_m^k)x_m^k)\|^q \geq \sum_{k=1}^{\infty} \frac{1}{k^2} k^{3q} = \sum_{k=1}^{\infty} k^q = \infty,$$

showing that $\|((Sz_m)x_m)\|_q = \infty$, which is a contradiction. □

Proposition 5.2.11. *Let X, Y, and Z be Banach spaces. Let $1 \leq p \leq \infty$ and $2 \leq q \leq \infty$. Assume that X and Z are $\mathcal{L}_\infty$-spaces. If $U \in \mathcal{P}_q(Z\hat{\otimes}_{d_p}X, Y)$, then $U^\#$ has property $\mathbf{S}_q$.*

Proof. Take arbitrary $(z_n) \in \ell_\infty(Z)$ and $(x_n) \in \ell_q^w(X)$. It suffices to show that $(z_n \otimes x_n) \in \ell_q^w(Z\hat{\otimes}_{d_p}X)$, because then

$$((U^\# z_n)x_n) = (U(z_n \otimes x_n))_{n=1}^{\infty} \in \ell_q(Y)$$

by assumption, and $U^\#$ has property $\mathbf{S}_q$ as desired.

To show that $(z_n \otimes x_n) \in \ell_q^w(Z\hat{\otimes}_{d_p}X)$, fix an arbitrary $A \in \mathcal{P}_{p'}(Z, X^*) = (Z\hat{\otimes}_{d_p}X)^*$. Since

$$\langle z_n \otimes x_n, A\rangle = \langle x_n, Az_n\rangle = \langle z_n, (A^* j_X)x_n\rangle$$

and

$$A^* j_X \in \mathcal{L}(X, Z^*) = \mathcal{P}_2(X, Z^*) = \mathcal{P}_q(X, Z^*)$$

(the operator ideal equalities (with equivalent norms) are well known; see, e.g., [26, Theorem 3.7] for the first equality, which holds because X is an $\mathcal{L}_\infty$-space and Z^* is an $\mathcal{L}_1$-space, and recall that $\mathcal{P}_2 \subset \mathcal{P}_q$ because $q \geq 2$), we have

$$\sum_{n=1}^{\infty} |\langle z_n \otimes x_n, A\rangle|^q \leq \sup_n \|z_n\|^q \sum_{n=1}^{\infty} \|(A^* j_X)x_n\|^q < \infty.$$

Hence, $(z_n \otimes x_n) \in \ell_q^w(Z\hat{\otimes}_{d_p}X)$. This completes the proof. □

The claim of Proposition 5.2.11 is used in the statement of the promised counterexample.

Proposition 5.2.12. *Let X and Y be Banach spaces and let $1 \leq p \leq \infty$. Assume that X is an $\mathcal{L}_\infty$-space, containing a subspace isomorphic to c_0, and Y does not have the Orlicz property. Then there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Delta, X), Y)$ such that $U^{\#} \in \mathcal{P}_2(\mathcal{C}(\Delta), \mathcal{P}_2(X, Y))$, but $U^{\#}$ does not have property $\mathbf{S}_2$ and, therefore, U is not absolutely 2-summing.*

Proof. Let a sequence (x_n) in X be equivalent to the unit vector basis of c_0. Then $(x_n) \in \ell_1^w(X)$ and $\|x_n\| \geq c$, $n \in \mathbb{N}$, for some $c > 0$.

Take $U \in \mathcal{L}(\mathcal{C}(\Delta, X), Y)$ given by Proposition 5.2.7. From its proof in [76], we know that $S := U^{\#}$ is given by formula (5.4) (see the proof of Proposition 5.2.8), where $x_n^* \in X^*$ with $\|x_n^*\| = 1$, $n \in \mathbb{N}$, are chosen so that $x_n^*(x_n) = \|x_n\|$, $n \in \mathbb{N}$, and $(y_n) \in \ell_1^w(Y)$ satisfies $\sum_{n=1}^{\infty} \|y_n\|^2 = \infty$ (the existence of such a sequence (y_n) means, by definition, that Y does not have the Orlicz property).

Now, we have a fixed p and, without loss of generality, we may assume that $U \in \mathcal{L}(\mathcal{C}(\Delta, X), Y)$ (see Proposition 5.2.4, the case when $q = r = \infty$, $\beta = \varepsilon$, and $\alpha = d_p$). Moreover, from the proof of Proposition 5.2.8, we know that

$$(Sr_n)x_n = x_n^*(x_n)y_n = \|x_n\|y_n, \quad n \in \mathbb{N}.$$

We only have to show that S does not have property $\mathbf{S}_2$. This is so, because $(r_n) \in \ell_\infty(\mathcal{C}(\Delta))$ and $(x_n) \in \ell_2^w(X)$, but

$$\sum_{n=1}^{\infty} \|(Sz_n)x_n\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 \|y_n\|^2 = \infty$$

(recall that $\|x_n\| \geq c$, $n \in \mathbb{N}$). □

Remark 5.2.13. Concerning the assumption of Proposition 5.2.12, recall that (thanks to Bourgain and Delbaen [11]) there exist infinite-dimensional $\mathcal{L}_\infty$-spaces which do not contain subspaces isomorphic to c_0. It was an open question for more than ten years (asked by Lindenstrauss and Pełczyński [52] and Lindenstrauss and Rosenthal [53]) whether every infinite-dimensional $\mathcal{L}_\infty$-space contains a subspace isomorphic to c_0.

Let X, Y, and Z be Banach spaces. Let $1 \leq p \leq \infty$. We know (see Corollary 4.3.5) that there exists an operator $U \in \mathcal{L}(Z \hat{\otimes}_{d_p} X, Y)$

such that $U^{\#} = S$ whenever $S \in \mathcal{P}_{p'}(Z, \mathcal{L}(X, Y))$. But U is not absolutely p'-summing in general (as Propositions 5.2.7, 5.2.8, and 5.2.12 show). Let Ω be a compact Hausdorff space. The next example provides a rather general case of operators $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ for which $U \in \mathcal{P}_{p'}(\mathcal{C}_p(\Omega, X), Y)$.

Example 5.2.14. Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 < p \leq \infty$. There exists an operator $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X, Y))$ such that there exists an operator $U \in \mathcal{P}_{p'}(\mathcal{C}_p(\Omega, X), Y)$ with $U^{\#} = S$.

Proof. Let $(y_n) \in \ell_p^w(Y)$, $(x_n^*) \in B_{X^*}$, and $(\mu_n) \in \ell_{p'}(\mathcal{C}(\Omega)^*)$. We consider the operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X, Y))$ defined by

$$S\varphi = \sum_{n=1}^{\infty} \big(\mu_n(\varphi) x_n^*\big) \otimes y_n$$

for all $\varphi \in \mathcal{C}(\Omega)$. By definition, $S\varphi \in \mathcal{N}_{p'}(X, Y)$, and therefore $S\varphi \in \mathcal{P}_{p'}(X, Y)$ (recall that $(\mathcal{N}_{p'}, \|\cdot\|_{\mathcal{N}_{p'}}) \subset (\mathcal{P}_{p'}, \|\cdot\|_{\mathcal{P}_{p'}})$ (see, e.g., [26, p. 113, Corollary 5.24, and p. 97, Proposition 5.5])). Moreover,

$$\|S\varphi\|_{\mathcal{P}_{p'}} \leq \|S\varphi\|_{\mathcal{N}_{p'}} \leq \|(y_n)\|_p^w \|(\mu_n(\varphi) x_n^*)\|_{p'} \leq \|(y_n)\|_p^w \|(\mu_n(\varphi))\|_{p'}$$
$$\leq \|(y_n)\|_p^w \|(\mu_n)\|_{p'} \|\varphi\|.$$

In fact, we have that

$$S = \sum_{n=1}^{\infty} \mu_n \otimes (x_n^* \otimes y_n).$$

To show that $S \in \mathcal{N}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X, Y))$ (and therefore $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X, Y))$), we only need to prove that $(x_n^* \otimes y_n) \in \ell_p^w(\mathcal{P}_{p'}(X, Y))$. For this, we use the fact that $\|(z_n)\|_p^w = \sup\{\| \sum_{k=1}^n \alpha_k z_k\| : n \in \mathbb{N}, (\alpha_n) \in B_{\ell_{p'}}\}$ for all sequences (z_n) in an arbitrary Banach space Z (see, e.g., [18, p. 91] or [26, pp. 35–36]). Let $(\alpha_n) \in B_{\ell_{p'}}$ be arbitrary. Then

$$\Big\| \sum_{k=1}^{n} \alpha_k (x_k^* \otimes y_k) \Big\|_{\mathcal{P}_{p'}} \leq \Big\| \sum_{k=1}^{n} \alpha_k (x_k^* \otimes y_k) \Big\|_{\mathcal{N}_{p'}} = \Big\| \sum_{k=1}^{n} (\alpha_k x_k^*) \otimes y_k \Big\|_{\mathcal{N}_{p'}}$$

$$\leq \|(\alpha_k)\|_{p'}\|(y_k)\|_p^w \leq \|(y_k)\|_p^w$$

for all $n \in \mathbb{N}$. Therefore, $(x_n^* \otimes y_n) \in \ell_p^w(\mathcal{P}_{p'}(X,Y))$.

Since $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X,Y))$, by Corollary 4.3.5, there exists an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$. Let us prove now that U is absolutely p'-summing. To do this, consider the representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of U (see Section 3.4), which coincides with the representing measure of $U^{\#} = S$ (see Theorem 3.4.3 and Proposition 3.4.4). In fact, since $U^{\#} = S$ is weakly compact, m takes its values in $\mathcal{L}(X,Y)$ (see Corollary 3.2.10). To prove that U is absolutely p'-summing, it suffices to show that the integration operator $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X, Y)$, defined by $\hat{U}(\varphi \otimes x) = (\int_\Omega \varphi\, dm)x$, $\varphi \in \mathcal{B}(\Sigma)$, $x \in X$ (see Section 3.3), is absolutely p'-summing on $\mathcal{S}(\Sigma, X)$ (which is dense in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X$, see Lemma 2.2.1).

Let $(f_j)_{j=1}^{\nu} \subset \mathcal{S}(\Sigma, X)$, $\nu \in \mathbb{N}$, with $\|(f_j)\|_{p'}^w \leq 1$. We may assume without loss of generality that, for some $n \in \mathbb{N}$,

$$f_j = \sum_{i=1}^{n} \chi_{E_i} x_i^j, \quad 1 \leq j \leq \nu,$$

where (E_i) are pairwise disjoint members of Σ and $(x_i^j)_{i=1,j=1}^{n,\nu} \subset X$. Therefore

$$\hat{U} f_j = \hat{U}\Big(\sum_{i=1}^{n} \chi_{E_i} x_i^j\Big) = \sum_{i=1}^{n} \hat{U}(\chi_{E_i} x_i^j) = \sum_{i=1}^{n} \Big(\int_\Omega \chi_{E_i}\, dm\Big) x_i^j. \tag{5.5}$$

Consider the integration operator $\hat{S} \in \mathcal{L}(\mathcal{B}(\Sigma), \mathcal{L}(X,Y))$, defined by $\hat{S} = \int_\Omega \varphi\, dm$, $\varphi \in \mathcal{B}(\Sigma)$ (see Section 3.2), which is an extension of S. Then

$$\begin{aligned} \sum_{i=1}^{n} \Big(\int_\Omega \chi_{E_i}\, dm\Big) x_i^j &= \sum_{i=1}^{n} (\hat{S}\chi_{E_i}) x_i^j \\ = \sum_{i=1}^{n} \Big(\sum_{k=1}^{\infty} (\mu_k(\chi_{E_i}) x_k^* \otimes y_k) x_i^j\Big) &= \sum_{i=1}^{n} \Big(\sum_{k=1}^{\infty} (\mu_k(E_i) x_k^* \otimes y_k) x_i^j\Big) \\ = \sum_{i=1}^{n} \Big(\sum_{k=1}^{\infty} \mu_k(E_i) x_k^*(x_i^j) y_k\Big) &= \sum_{k=1}^{\infty} y_k \sum_{i=1}^{n} \mu_k(E_i) x_k^*(x_i^j) \\ &= \Big(\sum_{k=1}^{\infty} (\mu_k \otimes x_k^*) \otimes y_k\Big) f_j. \end{aligned} \tag{5.6}$$

Thus, (5.5) and (5.6) yield that U has the following representation

$$U = \sum_{n=1}^{\infty} (\mu_n \otimes x_n^*) \otimes y_n$$

A similar reasoning as used above permits us to check that $(\mu_n \otimes x_n^*) \in \ell_{p'}(\mathcal{C}_p(\Omega, X)^*) = \ell_{p'}(\mathcal{P}_{p'}(\mathcal{C}(\Omega), X^*))$. Then, $U \in \mathcal{N}_{p'}(\mathcal{C}_p(\Omega, X), Y) \subset \mathcal{P}_{p'}(\mathcal{C}_p(\Omega, X), Y)$, as desired. □

5.3 Interplay between U and its representing measure m for absolutely (r,q)-summing operators

Let X and Y be Banach spaces, and let Ω be a compact Hausdorff space. By the classical Dinculeanu–Singer theorem, to each operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ there corresponds a unique representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$. The same holds in the general case when $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, $1 \leq p \leq \infty$. This extension of the Dinculeanu–Singer theorem, together with a needed integration theory, was established in Chapter 3.

Let us summarize our method. First, we built a (unique) representing measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ for an arbitrary operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ (see Section 3.2). Then (see Theorem 3.4.3 and Proposition 3.4.4), we proved that *the representing measure* $m : \Sigma \to \mathcal{L}(X, Y^{**})$ *of the associated operator* $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ *is also a representing measure of* U *and vice versa.* (Among others, our approach in Chapter 3 allowed us to give an alternative simpler proof to the Dinculeanu–Singer theorem.) In this section, we shall formulate our results using the representing measure of U, so the equality of the measures is useful to keep in mind below.

Each operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ has also its classical representing measure, say $\mu : \Sigma \to \mathcal{L}(X, Y)^{**}$ (given by the Bartle–Dunford–Schwartz theorem). The formula connecting m and μ, found in Corollary 3.2.10, shows that *m and μ coincide whenever S is weakly compact*. Summarizing, the common representing measure m of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ and of its associated operator $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$

coincides with the classical representing measure μ of $U^{\#}$ whenever $U^{\#}$ is weakly compact.

But if $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ happens to be weakly compact, then its representing measure $\mu : \Sigma \to Y^{**}$ *takes its values in* Y and $\mu : \Sigma \to Y$ is *countably additive* and *regular*. This important fact holds by a well-known theory of Bartle–Dunford–Schwartz [7] (see, e.g., [27, pp. 153–154, Theorem 5]) and by [27, p. 159, Corollary 14].

It is well known that absolutely q-summing operators ($1 \le q < \infty$) are weakly compact (see, e.g., [26, p. 50, Theorem 2.17]), but, in general, absolutely (r, q)-summing operators are not (see, e.g., [26, pp. 209–210]). However, absolutely (r, q)-summing operators defined on $\mathcal{C}(\Omega)$-spaces are weakly compact. This fact was observed by Popa (see [77, proof of Theorem 2]). We shall provide a proof for the sake of completeness.

Lemma 5.3.1 (Popa)**.** *Let Y be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le q \le r < \infty$. If $S \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), Y)$, then S is a weakly compact operator.*

Proof. By contradiction, suppose that $S \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), Y)$ is not weakly compact. Then, by a theorem of Pełczyński [69, Theorem 5] (see, e.g., [27, p. 159, Theorem 15]), $\mathcal{C}(\Omega)$ contains a subspace Z which is isometric to c_0 and on which S acts as an isomorphism. Let $(e_i) \subset Z$ correspond to the unit vector basis in c_0. Since $\|(e_i)\|_r = \infty$, also $\|(Se_i)\|_r = \infty$. On the other hand,

$$\|(Se_i)\|_r \le \|S\|_{\mathcal{P}_{(r,q)}} \|(e_i)\|_q^w \le \|S\|_{\mathcal{P}_{(r,q)}} \|(e_i)\|_1^w < \infty,$$

which is a contradiction. $\square$

One of the fundamental results about absolutely summing operators on $\mathcal{C}(\Omega)$ (essentially due to Grothendieck [43]) asserts that a bounded linear operator $S \in \mathcal{P}_1(\mathcal{C}(\Omega), Y)$ if and only if its representing measure $\mu : \Sigma \to Y^{**}$ is of bounded variation, and, in this case, $\|S\|_{\mathcal{P}_1} = |\mu|(\Omega)$ (see, e.g., [27, p. 162, Theorem 3]). The next lemma may be considered as an extension of the "only if" part. As usual, the characteristic function of $E \in \Sigma$ is denoted by χ_E.

Lemma 5.3.2. *Let Y be a Banach space and let Ω be a compact Hausdorff space. Let $1 \le q \le r < \infty$. If $S \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), Y)$, then its representing* *measure* $\mu : \Sigma \to Y^{**}$ *takes its values in* Y*,* $\mu : \Sigma \to Y$ *is countably additive and regular, and* $|\mu|_r(\Omega) \leq \|S\|_{\mathcal{P}_{(r,q)}}$.

Proof. Let $S \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), Y)$. By Proposition 5.3.1, S is weakly compact and, thus, μ enjoys the well-known properties pointed out above. To conclude with the proof, let $(E_i)_{i=1}^n \subset \Sigma$ be a finite partition of Ω. Then

$$\Big(\sum_{i=1}^n \|\mu(E_i)\|_{\mathcal{P}_{(r,q)}}^r\Big)^{1/r} = \Big(\sum_{i=1}^n \|S^{**}\chi_{E_i}\|_{\mathcal{P}_{(r,q)}}^r\Big)^{1/r} \leq \|S^{**}\|_{\mathcal{P}_{(r,q)}} \|(\chi_{E_i})\|_q^w$$
$$\leq \|S^{**}\|_{\mathcal{P}_{(r,q)}} = \|S\|_{\mathcal{P}_{(r,q)}};$$

a proof of the fact that $\|(\chi_{E_i})\|_q^w \leq 1$ can be seen in the proof of Proposition 4.4.1 or Remark 4.4.2, and the last equality holds by [4, Theorem 3.4] (see, e.g., [71, p. 228, 17.1.3, and p. 77, 4.9.11]). Now, taking the supremum over all finite partitions $(E_i)_{i=1}^n$ of Ω, $n \in \mathbb{N}$, we obtain that $|\mu|_r(\Omega) \leq \|S\|_{\mathcal{P}_{(r,q)}}$. □

As was said in the beginning of Section 5.3, the representing measures $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of an operator $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ $(1 \leq p \leq \infty)$ and its associated operator $U^{\#} \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ coincide. Even more, if $U^{\#}$ is weakly compact, then both measures coincide with the (classical) measure μ of $U^{\#}$. And this is the case when $U^{\#}$ is absolutely (r,q)-summing (see Lemma 5.3.1). This together with Lemma 5.3.2 yield the next result.

Proposition 5.3.3. *Let* X *and* Y *be Banach spaces and let* Ω *be a compact Hausdorff space. Let* $1 \leq p \leq \infty$ *and* $1 \leq q \leq r < \infty$*. Assume that* $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ *and let* $m : \Sigma \to \mathcal{L}(X, Y^{**})$ *be its representing measure. If* $U^{\#} \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), \mathcal{P}_{(r,q)}(X,Y))$*, then* $m(E) \in \mathcal{P}_{(r,q)}(X,Y)$ *for all* $E \in \Sigma$*,* $m : \Sigma \to \mathcal{P}_{(r,q)}(X,Y)$ *is countably additive and regular, and* $|m|_r(\Omega) \leq \|U^{\#}\|_{\mathcal{P}_{(r,q)}}$.

Remark 5.3.4. A bigger r-variation was introduced in Dinculeanu's book [30]. We call it "the Dinculeanu r-variation". Let $1 \leq r \leq \infty$ and let $\mu : \Sigma \to \mathbb{R}$ be a positive finite measure; we may assume that $\mu(\Omega) = 1$. For a vector measure $m : \Sigma \to \mathcal{L}(X,Y)$, the *Dinculeanu* r*-variation* on Ω (see [30, p. 241]) is defined by

$$\overline{m}_r(\Omega) = \sup \sum_{E_i \in \Pi} \|m(E_i)\| \|x_i\|,$$

where the supremum is taken over all finite partitions $\Pi = (E_i)_{i=1}^n$ of Ω and all finite systems $(x_i)_{i=1}^n \subset X$ such that $\|\sum_{i=1}^n \chi_{E_i} x_i\|_{L_{r'}(\mu,X)} \leq 1$, $n \in \mathbb{N}$. This notion is used in [30, II.13] to obtain the integral representation of an operator $U \in \mathcal{L}(L_p(\mu,X),Y)$, $1 \leq p < \infty$, with respect to a vector measure $m : \Sigma \to \mathcal{L}(X,Y)$ such that $\overline{m}_{p'}(\Omega) < \infty$.

It can be easily verified that $|m|_1(\Omega) \leq \overline{m}_1(\Omega)$ and $|m|_1(\Omega) = \overline{m}_1(\Omega)$ if m is absolutely continuous with respect to μ (see [30, p. 242]). Since also $\overline{m}_1(\Omega) \leq \overline{m}_q(\Omega)$ (see [30, p. 243]), we have that

$$|m|_q(\Omega) \leq |m|_1(\Omega) = |m|(\Omega) \leq \overline{m}_1(\Omega) \leq \overline{m}_q(\Omega).$$

The next result extends Popa's Theorem 5.1.11 from $\mathcal{C}(\Omega,X) = \mathcal{C}_\infty(\Omega,X)$ to $\mathcal{C}_p(\Omega,X)$, $1 \leq p \leq \infty$.

Theorem 5.3.5. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \leq p \leq \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}_p(\Omega,X),Y)$ and let $m : \Sigma \to \mathcal{L}(X,Y^{**})$ be its representing measure.*

(i) *Let $p' \leq q \leq r < \infty$. If $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega,X),Y)$, then $m(E) \in \mathcal{P}_{(r,q)}(X,Y)$ for all $E \in \Sigma$, $m : \Sigma \to \mathcal{P}_{(r,q)}(X,Y)$ is countably additive and regular, and $|m|_r(\Omega) \leq \|U\|_{\mathcal{P}_{(r,q)}}$.*

(ii) *Let $1 \leq q \leq r < \infty$. If $m(E) \in \mathcal{P}_{(r,q)}(X,Y)$ for all $E \in \Sigma$, and $m : \Sigma \to \mathcal{P}_{(r,q)}(X,Y)$ is of bounded variation, then $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega,X),Y)$ and $\|U\|_{\mathcal{P}_{(r,q)}} \leq |m|(\Omega)$.*

Proof. (i) Let $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega,X),Y)$. By Corollary 5.2.6, $U^\# \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega),\mathcal{P}_{(r,q)}(X,Y))$ and $\|U^\#\|_{\mathcal{P}_{(r,q)}} \leq \|U\|_{\mathcal{P}_{(r,q)}}$. Hence, by Proposition 5.3.3, the claim (i) is immediate.

(ii) From Section 3.3, we know that $\mathcal{C}_p(\Omega,X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p}X$ is a closed subspace of $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X$ and U is the restriction to $\mathcal{C}_p(\Omega,X)$ of the integration operator $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma)\hat{\otimes}_{d_p}X,Y)$ with respect to m. Remark that $\hat{U}$ takes its values in Y (and not in Y^{**}) because m takes its values in $\mathcal{L}(X,Y)$ (see Theorem 3.3.3). It clearly suffices to prove that $\hat{U}$ is absolutely (r,q)-summing and $\|\hat{U}\|_{\mathcal{P}_{(r,q)}} \leq |m|(\Omega)$.

The integration operator $\hat{U}$ is defined by the formula

$$\hat{U}(\varphi \otimes x) = \Big(\int_\Omega \varphi\, dm\Big)x, \quad \varphi \in \mathcal{B}(\Sigma), x \in X \tag{5.7}$$

(see Theorem 3.3.3). On the right-hand side, we have just the elementary Bartle integral. Hence, $\int_\Omega \varphi\, dm \in \mathcal{P}_{(r,q)}(X,Y)$ for all $\varphi \in \mathcal{B}(\Sigma)$. Recall that $\mathcal{S}(\Sigma) \otimes X = \mathcal{S}(\Sigma, X)$ under the algebraic identification $\varphi \otimes x = \varphi x$. Using that $\mathcal{S}(\Sigma)$ is dense in $\mathcal{B}(\Sigma)$ and relying on the definition of the tensor norm d_p, $d_p(v) = \inf\{\|(\varphi_i)\|_{p'}^w \|(x_i)\|_p : v = \sum_{i=1}^n \varphi_i \otimes x_i\}$, where the infimum is taken over all representations of $v \in \mathcal{B}(\Sigma) \otimes X$ (see, e.g., [84, p. 135]), it is straightforward to show that $\mathcal{S}(\Sigma, X)$ is dense in $\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X$ (see Lemma 2.2.1).

Let $(f_i)_{i=1}^n$ be an arbitrary finite system in $\mathcal{S}(\Sigma, X)$. Since the restriction of $\hat{U}$ to $\mathcal{S}(\Sigma, X)$ is just the "algebraic" integral (i.e., the integral whose definition passes from the X-valued characteristic functions $\chi_E x$, $E \in \Sigma$, $x \in X$, to functions in $\mathcal{S}(\Sigma, X)$ by linearity). But for the "algebraic" integral on $\mathcal{S}(\Sigma, X)$, we know from [77, proof of Theorem 5, inequality (11)] that

$$\|(\hat{U} f_i)\|_r \le |m|(\Omega) \|(f_i)\|_{\ell_q^w(\mathcal{B}(\Sigma, X))}.$$

Since

$$\|f\|_{\mathcal{B}(\Sigma,X)} = \sup_{\omega \in \Omega} \|f(\omega)\| = \|f\|_{\mathcal{S}(\Sigma)\otimes_\varepsilon X} \le \|f\|_{\mathcal{S}(\Sigma)\otimes_{d_p} X}$$

for all $f \in \mathcal{S}(\Sigma, X) = \mathcal{S}(\Sigma) \otimes X$, as in the proof of Proposition 5.2.4, we see that

$$\|(f_i)\|_{\ell_q^w(\mathcal{B}(\Sigma,X))} \le \|(f_i)\|_{\ell_q^w(\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X)}.$$

Hence,

$$\|(\hat{U} f_i)\|_r \le |m|(\Omega) \|(f_i)\|_{\ell_q^w(\mathcal{B}(\Sigma)\hat{\otimes}_{d_p} X)},$$

as desired. □

Let us now turn to Swartz's classical Theorem 5.1.1. There is only one implication, namely (ii) ⇒ (i), which cannot be extended, in general, from $\mathcal{P}_1$ to $\mathcal{P}_{(r,q)}$ (relevant counterexamples were discussed in Section 5.2). In our next Theorem 5.3.6, we show that the implication (ii) ⇒ (i) can be extended from $\mathcal{P}_1$ to $\mathcal{P}_{(r,q)}$ under a reasonable supplementary assumption. Moreover (see Theorem 5.3.7), an exceptional feature is that our extension from $\mathcal{P}_1$ to $\mathcal{P}_q = \mathcal{P}_{(q,q)}$ does not require a priori the existence of an operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$. This, in turn, leads us to an improvement even in the Swartz theorem (see Corollary 5.3.8).

We say that a vector measure $m : \Sigma \to \mathcal{L}(X, Y)$ is r-*dominated*, where $1 \le r < \infty$, if

$$\Big\| \sum_{E_i \in \Pi} m(E_i) x_i \Big\| \le \Big(\sum_{E_i \in \Pi} \| m(E_i) x_i \|^r \Big)^{1/r}$$

for all finite systems $\Pi = (E_i)_{i=1}^n$ of pairwise disjoint members of Σ and $(x_i)_{i=1}^n \subset X$, $n \in \mathbb{N}$.

Theorem 5.3.6. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le p \le \infty$ and $1 \le q \le r < \infty$. Assume that $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ and let $m : \Sigma \to \mathcal{L}(X, Y^{**})$ be its representing measure. If m is r-dominated and $U^\# \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), \mathcal{P}_{(r,q)}(X, Y))$, then $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega, X), Y)$ and $\|U\|_{\mathcal{P}_{(r,q)}} \le |m|_r(\Omega) \le \|U^\#\|_{\mathcal{P}_{(r,q)}}$. Moreover, if $p' \le q$, then $\|U\|_{\mathcal{P}_{(r,q)}} = |m|_r(\Omega) = \|U^\#\|_{\mathcal{P}_{(r,q)}}$.*

Proof. Since $U^\# \in \mathcal{P}_{(r,q)}(\mathcal{C}(\Omega), \mathcal{P}_{(r,q)}(X, Y))$, by Proposition 5.3.3, the vector measure m takes its values in $\mathcal{P}_{(r,q)}(X, Y)$, i.e., $m : \Sigma \to \mathcal{P}_{(r,q)}(X, Y)$, and $|m|_r(\Omega) \le \|U^\#\|_{\mathcal{P}_{(r,q)}}$. To prove that $U \in \mathcal{P}_{(r,q)}(\mathcal{C}_p(\Omega, X), Y)$ and $\|U\|_{\mathcal{P}_{(r,q)}} \le |m|_r(\Omega)$, as in the proof of Theorem 5.3.5 (ii), it suffices to show that the integration operator $\hat{U} \in \mathcal{L}(\mathcal{B}(\Sigma) \hat{\otimes}_{d_p} X, Y)$ satisfies the inequality

$$\|(\hat{U} f_j)\|_r \le |m|_r(\Omega) \|(f_j)\|_{\ell_q^w(\mathcal{B}(\Sigma, X))} \tag{5.8}$$

for every finite system (f_j) in $\mathcal{S}(\Sigma, X)$.

Let $(f_j)_{j=1}^\nu \subset \mathcal{S}(\Sigma, X)$, $\nu \in \mathbb{N}$, with $\|(f_j)\|_q^w \le 1$. We may assume without loss of generality that, for some $n \in \mathbb{N}$,

$$f_j = \sum_{i=1}^n \chi_{E_i} x_i^j, \quad 1 \le j \le \nu,$$

where (E_i) are pairwise disjoint members of Σ and $(x_i^j)_{i=1,j=1}^{n,\nu} \subset X$. As it was proved in [77, the proof of Theorem 5], $\|(x_i^j)_j\|_q^w \le 1$ for $1 \le i \le n$ whenever $\|(f_j)\|_q^w \le 1$. Moreover, by the identification $\chi_E x = \chi_E \otimes x$ and formula (5.7), we have $\hat{U}(\chi_E x) = m(E)x$ for all $E \in \Sigma$ and $x \in X$. Therefore, since m is r-dominated,

$$\sum_{j=1}^\nu \|\hat{U}(f_j)\|^r = \sum_{j=1}^\nu \Big\| \sum_{i=1}^n m(E_i) x_i^j \Big\|^r \le \sum_{j=1}^\nu \sum_{i=1}^n \|m(E_i) x_i^j\|^r$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{\nu} \|m(E_i)x_i^j\|^r.$$

Using that $m : \Sigma \to \mathcal{P}_{(r,q)}(X,Y)$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{\nu} \|m(E_i)x_i^j\|^r \leq \sum_{i=1}^{n} \|m(E_i)\|_{\mathcal{P}_{(r,q)}}^r (\|(x_i^j)_j\|_q^w)^r$$

$$\leq \sum_{i=1}^{n} \|m(E_i)\|_{\mathcal{P}_{(r,q)}}^r \leq (|m|_r(\Omega))^r.$$

This proves condition (5.8).

For the "moreover" part, in the case when $p' \leq q$, by Theorem 5.3.5 (i), we have that $|m|_r(\Omega) \leq \|U\|_{\mathcal{P}_{(r,q)}}$. This together with $\|U^{\#}\|_{\mathcal{P}_{(r,q)}} \leq \|U\|_{\mathcal{P}_{(r,q)}}$ (see Corollary 5.2.6) yield $\|U\|_{\mathcal{P}_{(r,q)}} = |m|_r(\Omega) = \|U^{\#}\|_{\mathcal{P}_{(r,q)}}$. □

As it was recalled above, absolutely q-summing operators $(1 \leq q < \infty)$ are weakly compact.

Theorem 5.3.7. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 < p \leq \infty$. Assume that $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X,Y))$ and its representing measure μ is p'-dominated. Then there exists an operator $U \in \mathcal{P}_{p'}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$ and $\|U\|_{\mathcal{P}_{p'}} = \|S\|_{\mathcal{P}_{p'}} = |\mu|_{p'}(\Omega) = |m|_{p'}(\Omega)$, where m is the representing measure of U.*

Proof. Since $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X,Y))$, there exists $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $U^{\#} = S$ (see Corollary 4.3.5). Keeping in mind that the representing measures m and μ coincide because $U^{\#} = S$ is a weakly compact operator (see the introduction of Section 5.3), by Theorem 5.3.6 (for the particular case when $r = q = p'$), the operator U is absolutely p'-summing and $\|U\|_{\mathcal{P}_{p'}} = |\mu|_{p'}(\Omega) = \|S\|_{\mathcal{P}_{p'}}$. □

Bilyeu and Lewis [10] prove that, for a given operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$, its associated operator $U^{\#}$ is absolutely q-summing if U is absolutely q-summing (see Theorem 5.1.2). Theorem 5.3.7 provides a strong converse to this result which does not need the assumption of the existence of $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ to imply that U is absolutely q-summing, but only the existence of some $S \in \mathcal{P}_q(\mathcal{C}(\Omega), \mathcal{P}_q(X,Y))$ whose representing measure is q-dominated.

Notice that a vector measure $m : \Sigma \to \mathcal{L}(X, Y)$ is always 1-dominated (this condition is just the triangle inequality). Let us spell out this particular case of Theorem 5.3.7 because it represents an improvement to Swartz's Theorem 5.1.1.

Corollary 5.3.8. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Assume that $S \in \mathcal{P}_1(\mathcal{C}(\Omega), \mathcal{P}_1(X, Y))$. Then there exists $U \in \mathcal{P}_1(\mathcal{C}(\Omega, X), Y)$ such that $U^{\#} = S$ and $\|U\|_{\mathcal{P}_1} = \|S\|_{\mathcal{P}_1} = |m|(\Omega)$, where m is the representing measure of U.*

Our final aim in this section is to provide a "good" condition in terms of an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, which would imply that its representing measure $\mu : \Sigma \to \mathcal{L}(X, Y)$ is r-dominated (see Proposition 5.3.9 below). This allows us to exhibit, for all $p > 1$, examples of operators $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(X, Y))$ to which Theorem 5.3.7 applies (see Example 5.3.10). For this end, let us introduce a property of operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, which we call "property $\mathbf{D}_r$" because it concerns some r-domination.

We say that an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ has *property* $\mathbf{D}_r$, where $1 \le r < \infty$, if

$$\Big\| \sum_{i=1}^{n} (S\varphi_i)x_i \Big\| \le \Big(\sum_{i=1}^{n} \|(S\varphi_i)x_i\|^r \Big)^{1/r}$$

for all finite systems $(\varphi_i)_{i=1}^n \subset \mathcal{C}(\Omega)$ and $(x_i)_{i=1}^n \subset X$, $n \in \mathbb{N}$, whenever the supports of the functions $(\varphi_i)_{i=1}^n$ are pairwise disjoint.

Proposition 5.3.9. *Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Let $1 \le r < \infty$. Assume that $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ is a weakly compact operator. If S has property $\mathbf{D}_r$, then the representing measure μ of S is r-dominated.*

Proof. Since S is weakly compact, μ is $\mathcal{L}(X, Y)$-valued, countably additive, and regular (see the introduction of Section 5.3). Thus, given $(x_i)_{i=1}^n \subset X$, pairwise disjoint sets $(E_i)_{i=1}^n \subset \Sigma$, $n \in \mathbb{N}$, and $\varepsilon > 0$, for each $1 \le i \le n$, there exist an open subset O_i of Ω and a compact subset K_i of Ω such that

$$K_i \subset E_i \subset O_i \quad \text{and} \quad \|\mu\|(O_i / K_i) < \frac{\varepsilon}{4n \max_i \|x_i\|}, \tag{5.9}$$

where $\|\mu\|(\cdot)$ denotes the semivariation of μ.

Since the compact sets $(K_i)_{i=1}^n$ are pairwise disjoint in the Hausdorff compact space Ω, we can inductively choose open and pairwise disjoint sets $(U_i)_{i=1}^n$ such that $K_i \subset U_i$, for $1 \leq i \leq n$. Now put $A_i = O_i \cap U_i$ and notice that the sets $(A_i)_{i=1}^n$ are open, pairwise disjoint, and $K_i \subset A_i$ for $1 \leq i \leq n$. By Urysohn's lemma, for each $1 \leq i \leq n$, there exists a continuous function $\varphi_i : \Omega \to [0,1]$ satisfying

$$\text{supp } \varphi_i \subset A_i \text{ and } \varphi_i(\omega) = 1 \text{ for all } \omega \in K_i$$

(where supp φ_i denotes the support of φ_i). Observe that $\|\varphi_i - \chi_{E_i}\| \leq 2$ and $\varphi_i(\omega) - \chi_{E_i}(\omega) = 0$ for every $\omega \notin O_i/K_i$. By (5.9), we have

$$\|S\varphi_i - \mu(E_i)\| = \Big\| \int_\Omega \varphi_i \, d\mu - \int_\Omega \chi_{E_i} \, d\mu \Big\| \leq 2\|\mu\|(O_i/K_i)$$

for every $1 \leq i \leq n$.

Now we can prove that μ is r-dominated. Using that S has property $\mathbf{D}_r$, we obtain

$$\begin{aligned}
\Big\| \sum_{i=1}^n \mu(E_i)x_i \Big\| &\leq \Big\| \sum_{i=1}^n (S\varphi_i)x_i \Big\| + \Big\| \sum_{i=1}^n \Big(\mu(E_i) - S\varphi_i\Big)x_i \Big\| \\
&\leq \Big(\sum_{i=1}^n \|(S\varphi_i)x_i\|^r \Big)^{1/r} + \sum_{i=1}^n 2\|\mu\|(O_i/K_i)\|x_i\| \\
&< \Big(\sum_{i=1}^n \|(S\varphi_i - \mu(E_i))x_i\|^r \Big)^{1/r} + \Big(\sum_{i=1}^n \|\mu(E_i)x_i\|^r \Big)^{1/r} + \frac{\varepsilon}{2} \\
&< \Big(\sum_{i=1}^n (2\|\mu\|(O_i/K_i)\|x_i\|)^r \Big)^{1/r} + \Big(\sum_{i=1}^n \|\mu(E_i)x_i\|^r \Big)^{1/r} + \frac{\varepsilon}{2} \\
&\leq \Big(\sum_{i=1}^n \|\mu(E_i)x_i\|^r \Big)^{1/r} + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ concludes the proof. □

Example 5.3.10. Let Ω be a compact Hausdorff space and let λ be a positive regular Borel measure on Σ. Assume that $1 < p \leq \infty$. Then there exists an operator $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(\mathcal{C}(\Omega), L_{p'}(\lambda)))$, with $\|S\|_{\mathcal{P}_{p'}} = \lambda(\Omega)^{1/p'}$, whose representing measure μ is p'-dominated. Hence, by Theorem 5.3.7, there exists an operator $U \in \mathcal{P}_{p'}(\mathcal{C}_p(\Omega, \mathcal{C}(\Omega)), L_{p'}(\lambda))$ such that $U^{\#} = S$ and $\|U\|_{\mathcal{P}_{p'}} = \|S\|_{\mathcal{P}_{p'}} = |\mu|_{p'}(\Omega) = |m|_{p'}(\Omega)$, where m is the representing measure of U.

Proof. Consider the linear operator $S : \mathcal{C}(\Omega) \to \mathcal{L}(\mathcal{C}(\Omega), L_{p'}(\lambda))$ defined by $(S\varphi)\psi = \varphi\psi$ for all $\varphi, \psi \in \mathcal{C}(\Omega)$. We have to show that $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(\mathcal{C}(\Omega), L_{p'}(\lambda)))$, $\|S\|_{\mathcal{P}_{p'}} = \lambda(\Omega)^{1/p'}$, and S satisfies the assumptions of Proposition 5.3.9 with $r = p'$.

For every $\varphi \in \mathcal{C}(\Omega)$, the operator $S\varphi$ is just the multiplication operator from $\mathcal{C}(\Omega)$ to $L_{p'}(\lambda)$, denoted by M_φ in [26]. Therefore, as is well known (see, e.g., [26, p. 40]),

$$S\varphi \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), L_{p'}(\lambda)) \text{ and } \|S\varphi\|_{\mathcal{P}_{p'}} = \|i_{p'}\varphi\|_{L_{p'}(\lambda)} \text{ for all } \varphi \in \mathcal{C}(\Omega),$$

where $i_{p'} : \mathcal{C}(\Omega) \to L_{p'}(\lambda)$ is the canonical inclusion map. Both S and $i_{p'}$ are linear operators, which are defined on the same space. Therefore, by the definition of absolutely p'-summing operators, the last equality clearly implies that S is absolutely p'-summing if and only if $i_{p'}$ is, and in this case, $\|S\|_{\mathcal{P}_{p'}} = \|i_{p'}\|_{\mathcal{P}_{p'}}$. But this is indeed the case: $i_{p'}$ is absolutely p'-summing and, moreover $\|i_{p'}\|_{\mathcal{P}_{p'}} = \lambda(\Omega)^{1/p'}$ (see, e.g., [26, p. 40]). This proves that $S \in \mathcal{P}_{p'}(\mathcal{C}(\Omega), \mathcal{P}_{p'}(\mathcal{C}(\Omega), L_{p'}(\lambda)))$ and $\|S\|_{\mathcal{P}_{p'}} = \lambda(\Omega)^{1/p'}$.

Finally, since S is also a weakly compact operator, we are left to show that S has property $\mathbf{D}_{p'}$. Consider two arbitrary finite systems $(\varphi_i)_{i=1}^n, (\psi_i)_{i=1}^n \subset \mathcal{C}(\Omega)$, where the supports $\operatorname{supp} \varphi_i$ of the functions $(\varphi_i)_{i=1}^n$ are pairwise disjoint. Then

$$\Big\| \sum_{i=1}^n (S\varphi_i)\psi_i \Big\|_{L_{p'}(\lambda)}^{p'} = \int_\Omega \Big| \sum_{i=1}^n \varphi_i(\omega)\psi_i(\omega) \Big|^{p'} d\lambda(\omega)$$

$$= \sum_{i=1}^n \int_{\operatorname{supp} \varphi_i} |\varphi_i(\omega)\psi_i(\omega)|^{p'} d\lambda(\omega) = \sum_{i=1}^n \|(S\varphi_i)\psi_i\|_{L_{p'}(\lambda)}^{p'},$$

showing that S has property $\mathbf{D}_{p'}$. □

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