Rational integrability of two-dimensional quasi-homogeneous polynomial differential systems

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1 Introduction and statement of main results.

In this paper, we deal with polynomial differential systems

$$(\dot{x}, \dot{y})^T = \mathbf{F}_r = (P, Q)^T, \tag{1}$$

where $\mathbf{F}_r$ is a polynomial vector field of degree $r \geq 0$ with respect to the type $\mathbf{t} = (t_1, t_2)$ fixed. In the particular case that $\mathbf{t} = (1, 1)$, (**??**) is a homogeneous polynomial differential system of degree $r + 1$. The purpose of our approach is to know when (**??**) is a rationally integrable system.

We recall that a function of two variables f is a quasi-homogeneous function of degree $k \in \mathbb{Z}$ with respect to the type $\mathbf{t} = (t_1, t_2)$ if $f(\varepsilon^{t_1} x, \varepsilon^{t_2} y) = \varepsilon^k f(x, y)$ (it will be said $\mathbf{t}$-function of degree $\deg_{\mathbf{t}}(f)$). We will denote $\mathcal{P}_k^{\mathbf{t}}$ the vector space of $\mathbf{t}$-polynomials of degree $k \geq 0$. A two-dimensional vector field $\mathbf{F} = (P, Q)^T$ is quasi-homogeneous of degree k with respect to the type $\mathbf{t}$ if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and

$Q \in \mathcal{P}^{\mathbf{t}}_{k+t_2}$ (it will be said $\mathbf{t}$-vector field of degree k). The vector space of polynomial $\mathbf{t}$-vector fields of degree k will be denoted by $\mathcal{Q}^{\mathbf{t}}_k$.

A function H is a first integral of system (**??**) in an open subset U of $\mathbb{R}^2$ if H is a nonconstant function in U which is constant on each solution curve of system (**??**). If there exists a rational first integral of (**??**) it is said that (**??**) is rationally integrable. Clearly, if $H = \frac{f}{g}$, with f, g polynomials, is a first integral of system (**??**) then the Lie derivative of H by $\mathbf{F}_r$ is zero in the open subset $\Omega_g = \{(x, y) \in \mathbb{R}^2 : g(x, y) \neq 0\}$, i.e. $L_{\mathbf{F}_r} H := \frac{\partial H}{\partial x} P + \frac{\partial H}{\partial y} Q \equiv 0$ in Ω_g.

The integrability problem consists in determining if the planar vector field has a first integral. In a general framework, the integrability is an important question because the existence of a first integral determines completely its phase portrait.

It is known that we can always calculate a first integral explicitly. It is enough to make the change of variables $(x, y) \to (u, v)$ according to $x = v^{t_1},\ y = v^{t_2} u$ which transforms the differential equation $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$ into a linear equation $\frac{v}{t_1}\frac{du}{dv} + \frac{t_2}{t_1} u = \frac{Q(1,u)}{P(1,u)}$ easy to integrate. But this first integral usually has a huge algebraic expression and, therefore, it is difficult to show whether it is rational or not.

As far as we know, the integrability problem of a planar quasi-homogeneous polynomial system has been only studied by Llibre & Zhang [**?**], Cairó & Llibre [**?**] and Tsygvintsev [**?**]. Llibre & Zhang in [**?**] solve the problem for planar polynomial systems (**??**) of degree one, which extends to the result given by Tsygvintsev [**?**] for the quadratic homogeneous polynomial differential system. Later, Cairó & Llibre [**?**] also find the two degree polynomially integrable systems.

For a higher dimension, Furta [**?**] and Goriely [**?**] proved, independently, the existence of a link between the Kowalevskaya exponents of quasi-homogeneous polynomial differential n-dimensional systems and the degree of their quasi-homogeneous polynomial first integrals. Later on, Llibre & Zhang [**?**] provided a new link that improves the above result given by [**?**,**?**].

To state our results, we need to recall the decomposition of a quasi-homogeneous vector field as a sum of two quasi-homogeneous fields, a conservative one (having zero-divergence) and a dissipative one that will be useful in what follows. Throughout this paper, it is denoted by $\mathbf{X}_f$ the Hamiltonian system associated to the $\mathcal{C}^1$ function f, i.e. $\mathbf{X}_f = (-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})^T$.

Lemma 1 *(Algaba et. al. [?]) Every vector field $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ can be expressed as*

$$\mathbf{F}_r = \frac{1}{r + |\mathbf{t}|}[\mathbf{X}_h + \operatorname{div}(\mathbf{F}_r)\mathbf{D}_0], \tag{2}$$

where $\mathbf{D}_0(x, y) = (t_1 x, t_2 y)^T$ (a dissipative $\mathbf{t}$-vector field of degree 0), $\operatorname{div}(\mathbf{F}_r) \in \mathcal{P}_r^{\mathbf{t}}$ (the divergence of $\mathbf{F}_r$), $h = t_1 x Q - t_2 y P \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ (the wedge product of $\mathbf{D}_0$ and $\mathbf{F}_r$) being $|\mathbf{t}| = t_1 + t_2$.
Furthermore, a such decomposition is unique.

Next, we state the main results of the paper, which characterize the rational integrability of $\mathbf{F}_r$. The first one gives a link between the existence of a $\mathbf{t}$-rational first integral of system (**??**) and the conservative and dissipative terms of the vector field $\mathbf{F}_r$.

Theorem 1 *An irreducible system (**??**) has got a rational first integral if and only if $\operatorname{div}(\mathbf{F}_r) \equiv 0$ or $h = \prod_{j=1}^k f_j$ where $f_1, ..., f_k$ are $\mathbf{t}$-polynomial irreducible on $\mathbb{K}[x, y]$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$), $k \geq 2$ and there exist k non-zero integer numbers, $n_1, n_2, ..., n_k$, such that $n_1 deg_{\mathbf{t}}(f_1) + \cdots + n_k deg_{\mathbf{t}}(f_k) \neq 0$ and*

$$(\sum_{j=1}^{k} n_j deg_{\mathbf{t}}(f_j))\operatorname{div}(\mathbf{F}_r) = h \sum_{j=1}^{k} \sum_{l=j+1}^{k} (n_l - n_j)\tfrac{1}{f_j f_l} L_{\mathbf{X}_{f_l}} f_j. \tag{3}$$

*Moreover, in such a case, $\prod_{j=1}^k f_j^{n_j}$ is a $\mathbf{t}$-rational first integral of (**??**).*

The second result simplifies the conditions of rational integrability of a $\mathbf{t}$-polynomial system. This provides an effective approach for computing rationally integrable systems, from a rational function η which is defined in section **??**.

Theorem 2 *An irreducible system (**??**) has got a rational (polynomial, resp.) first integral if and only if* $\mathrm{div}(\mathbf{F}_r) \equiv 0$*, or the two following properties hold:*
i) *The* $\mathbf{t}$*-polynomial* h *has at least two irreducible factors on* $\mathbb{K}[x,y]$*, all of them are distinct, that is, it can be written as* $h = \prod_{j=1}^{m+2} f_j$*, where* $f_1 = x^{\delta_x},\ f_2 = y^{\delta_y},\ \delta_x, \delta_y \in \{0,1\},\ f_j = y^{t_1} - \lambda_j x^{t_2}$*, for* $j \geq 3$*, and* $m \geq 0$*,*
ii) *for any pole of* $\eta(1,y)$*, its residue is a rational number.*
Moreover, in such a case, by denoting the poles of $\eta(1,y)$ *by* $w_1 = \infty,\ w_2 = 0,\ w_j = \lambda_j,\ j = 3, \cdots, m+2$ *and the rational numbers* r_j *by*

$$r_j = \tfrac{1}{r+|\mathbf{t}|}(1 - \frac{t_1 t_2}{deg_{\mathbf{t}}(f_j)} Res[\eta(1,y), w_j]),\ j = 1, \cdots, m+2, \tag{4}$$

it has that $\prod_{j=1}^{m+2} f_j^{n_j}$ *is a* $\mathbf{t}$*-rational (polynomial, resp.) first integral of degree* M *being* M *such that* $n_j = M r_j \in \mathbb{Z}\,(\mathbb{N} \cup \{0\}$*, resp.).*

To state the third result, we define the Kowalevskaya exponent of a quasi-homogeneous vector field: $\rho \in \mathbb{C}$ is a Kowalevskaya exponent of $\mathbf{F}_r$ if ρ is an eigenvalue of the matrix $D\left(\mathbf{F}_r + \frac{1}{r}\mathbf{D}_0\right)(\mathbf{c})$ where $\mathbf{c} \in \mathbb{C}^2 \setminus \{O\}$ is a solution of the vectorial equation $\mathbf{F}_r(\mathbf{c}) + \frac{1}{r}\mathbf{D}_0(\mathbf{c}) = \mathbf{0}$.
Finally, the third result characterizes the rational (polynomial) integrability of system (**??**) through its Kowalevskaya exponents.

Theorem 3 *An irreducible system (**??**) has got a rational (polynomial, resp.) first integral if and only if* $\mathrm{div}(\mathbf{F}_r) \equiv 0$ *or the two following properties hold:*
i) *The* $\mathbf{t}$*-polynomial* h *has at least two irreducible factors on* $\mathbb{K}[x,y]$*, and all of them are distinct, that is, it can be written as* $h = \prod_{j=1}^{m+2} f_j$*, where* $f_1 = x^{\delta_x},\ f_2 = y^{\delta_y},\ \delta_x, \delta_y \in \{0,1\},\ f_j = y^{t_1} - \lambda_j x^{t_2}$*, for* $j \geq 3$*, and* $m \geq 0$*,*
ii) *for any Kowalevskaya exponents* $\rho_j \neq -1$*,* ρ_j *is a rational number.*
Moreover, in such a case, if it denotes the rational numbers r_j *by*

$$r_j = \tfrac{1}{r}\rho_j^{-1},\ j = 1, \cdots, m+2, \tag{5}$$

it has that $\prod_{j=1}^{m+2} f_j^{n_j}$ *is a* $\mathbf{t}$*-rational (polynomial, resp.) first integral of degree* M *with* M *such that* $n_j = M r_j \in \mathbb{Z}\,(\mathbb{N} \cup \{0\}$*, resp.).*

Theorem **??** extends to the results given by [?,?]. In fact, in these papers, the relationship between polynomial integrability of vector fields (**??**) with $r = 1$ and their Kowalevskaya exponents is proved.

The rest of the paper is organized as follows. In section **??**, we prove Theorems **??** and **??**. In section **??**, we give a link between the rational integrability of (**??**) and its Kowalevskaya exponents. As a consequence we obtain Theorem **??**. In section **??** we solve the problem of the rational and polynomial integrability of the $(1, 2)$-polynomial systems of degree 2.

2 Proof of Theorems ?? and ??.

First we shown a series of basic results which allow us to prove the main results of a organized way. The following lemma shows that if system (**??**) has a rational first integral, then the existence of a **t**-rational function first integral of (**??**) is necessary, that is, a first integral which is a quotient of two **t**-polynomials.

Lemma 2 *Let f, g polynomials with $f = \sum_{j=m_0}^{M} f_j$ and $g = \sum_{j=n_0}^{N} g_j$, and $f_j, g_j \in \mathcal{P}_j^{\mathbf{t}}$ (their expansions into* **t***-polynomials, respectively). If $\frac{f}{g}$ is a first integral of system (**??**) in Ω_g, then $\frac{f_{m_0}}{g_{n_0}}$ is a first integral of system (**??**) in $\Omega_{g_{n_0}}$.*

Proof. If $\frac{f}{g}$ is a first integral of system (**??**) in Ω_g it has that $L_{\mathbf{F}_r}\frac{f}{g} \equiv 0$ in Ω_g and therefore $0 \equiv gL_{\mathbf{F}_r}f - fL_{\mathbf{F}_r}g$. In particular its first **t**-term is also zero. So,

$$0 \equiv g_{n_0}L_{\mathbf{F}_r}f_{m_0} - f_{m_0}L_{\mathbf{F}_r}g_{n_0} = g_{n_0}^2 L_{\mathbf{F}_r}\frac{f_{m_0}}{g_{n_0}} \text{ in } \Omega_g \cap \Omega_{g_{n_0}}.$$

By continuity, it extends to $\Omega_{g_{n_0}}$. ∎

We will assume that system (**??**) is a irreducible system (i.e. P and Q are coprime and $PQ \not\equiv 0$) since, otherwise, if $PQ \equiv 0$ then x or y are first integrals of system (**??**); so system (**??**) is polynomially integrable. And if P, Q

are no coprime, it has that $P = fP'$, $Q = fQ'$ where $\mathbf{F}_{r'} = (P', Q')^T$ is a $\mathbf{t}$-polynomial vector field of degree $r' < r$. It is easy to prove that H is a first integral of (**??**) if and only if H is a first integral of $(\dot{x}, \dot{y})^T = \mathbf{F}_{r'}$. Therefore, it is sufficient to study the integrability of the second system.

From the decomposition conservative-dissipative, we note that if $h \equiv 0$, then system (**??**) is not irreducible. Moreover, in such a case, any $\mathbf{t}$-rational of the form $\frac{p}{q}$ with $p, q \in \mathcal{P}^{\mathbf{t}}_{m_0}$ is a first integral in Ω_q, since by Lemma **??**, it has that

$$L_{\mathbf{F}_r}\frac{p}{q} = \frac{1}{q^2}(qL_{\mathbf{F}_r}p - pL_{\mathbf{F}_r}q) = \frac{1}{(r+|\mathbf{t}|)q^2}\mathrm{div}(\mathbf{F}_r)(qL_{\mathbf{D}_0}p - pL_{\mathbf{D}_0}q),$$

in Ω_q, and from Euler Theorem for quasi-homogeneous polynomial, $L_{\mathbf{D}_0}p = m_0 p$, $L_{\mathbf{D}_0}q = m_0 q$, therefore $L_{\mathbf{F}_r}\frac{p}{q} \equiv 0$ in Ω_q.

The following properties characterize the quasi-homogeneous polynomial systems having a quasi-homogeneous rational first integral, and they give conditions on h.

Lemma 3 *An irreducible system (**??**) has got a $\mathbf{t}$-rational first integral $\frac{p}{q}$, with $p \in \mathcal{P}^{\mathbf{t}}_{m_0}$ and $q \in \mathcal{P}^{\mathbf{t}}_{n_0}$, if and only if $L_{\mathbf{X}_h}\frac{p}{q} = (n_0 - m_0)\mathrm{div}(\mathbf{F}_r)\frac{p}{q}$, in Ω_q. Moreover, in such a case, m_0 must be different from n_0.*

Proof. If $\mathrm{div}(\mathbf{F}_r)$ is identically zero, then system (**??**) is a hamiltonian system and h is a polynomial first integral. In such a case, $p = h$, $q \equiv 1$, $n_0 = 0$, $m_0 = r + |\mathbf{t}|$.
If $\mathrm{div}(\mathbf{F}_r) \not\equiv 0$, by Lemma **??** and Euler theorem for quasi-homogeneous polynomial it arrives to

$$(r+|\mathbf{t}|)L_{\mathbf{F}_r}\frac{p}{q} = L_{\mathbf{X}_h}\frac{p}{q} + \mathrm{div}(\mathbf{F}_r)L_{\mathbf{D}_0}\frac{p}{q} = L_{\mathbf{X}_h}\frac{p}{q} + (m_0 - n_0)\mathrm{div}(\mathbf{F}_r)\frac{p}{q},$$

in Ω_q. We finish this proof, showing that $m_0 \neq n_0$. On the one hand, if $\frac{p}{q}$ is a first integral of $\mathbf{F}_r$, it holds $L_{\mathbf{F}_r}\frac{p}{q} = 0$ in Ω_q, i.e. $P(qp_x - pq_x) + Q(qp_y - pq_y) = 0$ in Ω_q, where p_x, p_y and q_x, q_y are the partial derivatives of p and q, respectively. On the other hand, as the components of $\mathbf{F}_r$ haven't got common factors, it has that there exists $k \in \mathcal{P}^{\mathbf{t}}_{m_0+n_0-r-|\mathbf{t}|}$, such that

$$(q_y p - p_y q) = kP, \qquad -(q_x p - p_x q) = kQ,$$

thus, it follows that $k\mathbf{F}_r = q\mathbf{X}_p - p\mathbf{X}_q = q^2\mathbf{X}_{\frac{p}{q}}$ in Ω_q ($k \not\equiv 0$ since $P.Q \not\equiv 0$). It has that

$$(r+|\mathbf{t}|)kh = \mathbf{D}_0 \wedge (k\mathbf{F}_r) = \mathbf{D}_0 \wedge (q^2\mathbf{X}_{\frac{p}{q}}) = \mathbf{D}_0 \wedge (q\mathbf{X}_p - p\mathbf{X}_q),$$

and from Euler theorem,

$$(r+|\mathbf{t}|)kh = (m_0 - n_0)pq, \text{ in } \Omega_q, \tag{6}$$

and this equality can be extend to $\mathbb{R}^2$. So, $m_0 \neq n_0$ and it follows the result.■

Lemma 4 *Let $\frac{p}{q}$ be an irreducible $\mathbf{t}$-rational first integral of a irreducible system (??). Then, any irreducible factor of p or q on $\mathbb{K}[x,y]$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) is a factor of h on $\mathbb{K}[x,y]$.*
Reciprocally, any irreducible factor of h on $\mathbb{K}[x,y]$ is either a factor of p or a factor of q.

Proof. From Lemma ??, $\frac{p(x,y)}{q(x,y)} = 0$ is an invariant rational curve of $\mathbf{X}_h$ in Ω_q and as $\frac{p}{q}$ is irreducible, then $\Omega_p \subset \mathbb{R}^2 \setminus \Omega_q$. Therefore, $p(x,y) = 0$ is a polynomial invariant curve of $\mathbf{X}_h$. On the other hand, the unique irreducible invariant curves of $\mathbf{X}_h$ are the irreducible factors of h. Thus, it follows that any irreducible factor of p is a factor of h.
Note that if $\frac{p}{q}$ is a first integral of $\mathbf{F}_r$, then $\frac{q}{p}$ also it is. Applying the same reasoning for the first integral $\frac{q}{p}$, it has that any irreducible factor of q is also a factor of h.
Finally, it follows by (??) that every irreducible factor of h must be either a factor of p or a factor of q. ■

Lemma 5 *If an irreducible system (??), with $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$, has got a $\mathbf{t}$-rational first integral, then h has at least two irreducible factors on $\mathbb{K}[x,y]$*

and all of them are distinct (i.e. the factors of h are simple).

Proof. On the one hand, if h had the form f^m with $m \geq 1$, then there would be a integer number non-zero n such that $\frac{p}{q} = f^n$, from Lemma **??**. Applying Lemma **??**, it has that $L_{\mathbf{X}_{f^m}} f^n = 0 = (n_0 - m_0)\text{div}(\mathbf{F}_r) f^n$, it would arrive at $\text{div}(\mathbf{F}_r) \equiv 0$. Thus, the assumption leads us to a contradiction.
On the other hand, if $h = \prod_{l=1}^{k} f_l^{m_l}$, with some $m_j > 1,\ 1 \leq j \leq k$, then f_j would be a factor of $\mathbf{X}_h$. From Lemma **??**, it has that $\frac{p}{q} = \prod_{l=1}^{k} f_l^{n_l}$, with n_j integers numbers non all zero, and in such a case, it is easy to check that the Lie derivative of $\frac{p}{q}$ by $\mathbf{X}_h$ is given by

$$L_{\mathbf{X}_h} \tfrac{p}{q} = \tfrac{p}{q} h \sum_{i=1}^{k} \sum_{j=i+1}^{k} (n_i m_j - n_j m_i) \frac{1}{f_i f_j} L_{\mathbf{X}_{f_j}} f_i. \qquad (7)$$

So, applying Lemma **??** and by cancelling, it would have that

$$h \sum_{i=1}^{k} \sum_{j=i+1}^{k} (n_i m_j - n_j m_i) \tfrac{1}{f_i f_j} L_{\mathbf{X}_{f_j}} f_i = (n_0 - m_0)\text{div}(\mathbf{F}_r).$$

Thus, f_j would be a factor of both, $\mathbf{X}_h$ and $\text{div}(\mathbf{F}_r)$, which would be in contradiction with the fact that the components of $\mathbf{F}_r$ are coprime. ∎

Proof of Theorem ??. If $\text{div}(\mathbf{F}_r) \equiv 0$, the system is rationally integrable from Lemma **??**, and if $h \equiv 0$, $\mathbf{F}_r$ is reducible.
We assume that $\text{div}(\mathbf{F}_r) \not\equiv 0$, $h \not\equiv 0$ and $\mathbf{F}_r = (P, Q)^T$ has got a rational first integral, $\frac{p}{q}$. Then, from Lemma **??**, $h = \prod_{j=1}^{k} f_j$ where $f_1, ..., f_k$ are $\mathbf{t}$-polynomial irreducible on $\mathbb{K}[x, y]$, $k \geq 2$ and, from Lemma **??**, there exist $n_1, n_2, \cdots, n_k$ non-zero integer numbers, such that $p = \prod_{n_i > 0} f_i^{n_i}$ and $q = \prod_{n_i < 0} f_i^{-n_i}$. Moreover, from Lemma **??**, $\deg_{\mathbf{t}}(p) - \deg_{\mathbf{t}}(q) = \sum_{i=1}^{k} n_i \deg_{\mathbf{t}}(f_i) \neq 0$.

In order to prove the sufficient condition, it is enough to apply Lemma **??** for $h = \prod_{j=1}^{k} f_j$ and $\frac{p}{q} = \prod_{j=1}^{k} f_j^{n_j}$. ∎

To define the function η that appears in Theorem **??**, it is convenient to choose a basis of $\mathcal{P}_k^{\mathbf{t}}$. It is easy to prove that if k can be expressed as $k = k_3t_1t_2+k_2t_2+k_1t_1$ with $0 \le k_1 < t_2,\ 0 \le k_2 < t_1$, then the set $B_k^{\mathbf{t}} = \{x^{t_2 i+k_1}y^{t_1(k_3-i)+k_2}, 0 \le i \le k_3\}$ is a basis of $\mathcal{P}_k^{\mathbf{t}}$. Otherwise, $\mathcal{P}_k^{\mathbf{t}} = \{0\}$. This allows us to write any non-vanishing $\mathbf{t}$-polynomial of degree k as $p(x,y) = x^{k_1}y^{k_2}p_0(x^{t_2}, y^{t_1})$ with

$$p_0(x,y) = \sum_{j=0}^{k_3} \alpha_j x^{k_3-j}y^j,$$

a homogeneous polynomial of degree k_3. Introducing the variable $v = \frac{y}{x}$ and denoting by s the higher index such that $\alpha_s \neq 0$ (i.e. $\alpha_{s+1} = \cdots = \alpha_{k_3} = 0$), we have that

$$p_0(x,y) = x^{k_3-s}\sum_{j=0}^{s} \alpha_j v^j.$$

If $\lambda_j \in \mathbb{C}$ are the distinct roots of the polynomial $\sum_{j=0}^{s} \alpha_j v^j$, by abusing the notation we can write any $\mathbf{t}$-polynomial in a compact form

$$p(x,y) = \alpha_s \prod_{j=1}^{d} f_j^{m_j}, \quad \text{where } f_j(x,y) = x,\ y \text{ or } y^{t_1} - \lambda_j x^{t_2}$$

with $\deg_{\mathbf{t}}(p) = \sum_{j=1}^{d} m_j \deg_{\mathbf{t}}(f_j)$.

Under the hypothesis of irreducibility of $\mathbf{F}_r$, the $\mathbf{t}$-polynomials h and $\text{div}(\mathbf{F}_r)$ have the following expressions, according to their degrees.

Lemma 6 *Let system (**??**) be an irreducible system with* $\text{div}(\mathbf{F}_r) \not\equiv 0$, *then* $r + |\mathbf{t}| = k_3t_1t_2 + \delta_y t_2 + \delta_x t_1$ *with* $\delta_x, \delta_y \in \{0,1\}$ *and* $k_3 \ge 0$.
As a consequence, it has that
i) $h(x,y) = x^{\delta_x}y^{\delta_y}h_0(x^{t_2}, y^{t_1})$ *with* $h_0(x,y)$ *homogeneous polynomial of degree* k_3.
ii) $\text{div}(\mathbf{F}_r)(x,y) = x^{(1-\delta_x)(t_2-1)}y^{(1-\delta_y)(t_1-1)}\mu_0(x^{t_2}, y^{t_1})$ *with* $\mu_0(x,y)$ *homogeneous polynomial of degree* $k_3 - (1-\delta_x) - (1-\delta_y)$.

Proof. We assume that there exist $k_1,\ k_2$ and k_3 integer numbers, with $0 \le k_1 < t_2,\ 0 \le k_2 < t_1,\ k_3 \ge 0$ such that $\deg_{\mathbf{t}}(h) = r + |\mathbf{t}| = k_1t_1 + k_2t_2 + k_3t_1t_2$, since otherwise, $h \equiv 0$. It follows easily that $k_1 = \delta_x$ and $k_2 = \delta_y$, with

$\delta_x, \delta_y \in \{0,1\}$, since if, for instance, $k_1 \geq 2$, then it will be

$$\deg_{\mathbf{t}}(P) = r + t_1 = k_1 t_1 + (k_2 - 1)t_2 + k_3 t_1 t_2,$$

$$\deg_{\mathbf{t}}(Q) = r + t_2 = (k_1 - 1)t_1 + k_2 t_2 + k_3 t_1 t_2,$$

that is, x would be a common factor of the components of $\mathbf{F}_r$. Also, if $t_1 = 1$ we will assume $\delta_y = 0$, and if $t_2 = 1$, $\delta_x = 0$. So, taking into account these considerations, it has that the degree of $\mathrm{div}(\mathbf{F}_r)$ is given by

$$r = [k_3 - (1-\delta_x) - (1-\delta_y)]t_1 t_2 + (1-\delta_x)(t_2 - 1)t_1 + (1-\delta_y)(t_1 - 1)t_2,$$

with $k_3 - (1-\delta_x) \geq 0$ and $k_3 - (1-\delta_y) \geq 0$, since otherwise if, for instance, $k_3 - (1-\delta_x) < 0$, then it will be $\deg_{\mathbf{t}}(P) = r + t_1 = (1-\delta_y)t_2(1-\delta_y - t_1) \leq 0$ that is P will be null.
From the expression of $r + |\mathbf{t}|$ and r it follows **i)** and **ii)**. ■
Next, we define the function $\eta(x,y) := \frac{\mu_0(x,y)}{x^{\delta_x} y^{\delta_y} h_0(x,y)}$ being $r + |\mathbf{t}| = \delta_x t_1 + \delta_y t_2 + k_3 t_1 t_2$. This function plays a important role in our research.

Proof of Theorem ??. First, we prove the necessary condition. If $\mathrm{div}(\mathbf{F}_r) \equiv 0$ or $h \equiv 0$, the system is rationally integrable or reducible.
We assume that $\mathrm{div}(\mathbf{F}_r) \not\equiv 0$, $h \not\equiv 0$ and system (**??**) has a rational first integral. By Theorem **??**, it has that h verifies **i)**.
Next, we prove the second property. By applying (**??**), with $M = \sum_{j=1}^{m+2} n_j \deg_{\mathbf{t}}(f_j)$ and h given by **i)**, it has that

$$\begin{aligned} \mathrm{div}(\mathbf{F}_r) = \tfrac{1}{M}\Big(&(n_2 - n_1)\delta_x\delta_y \tfrac{h}{f_1 f_2} L_{\mathbf{X}_{f_2}} f_1 + \sum_{j=3}^{m+2} (n_j - n_1)\delta_x \tfrac{h}{f_1 f_j} L_{\mathbf{X}_{f_j}} f_1 \\ &+ \sum_{j=3}^{m+2} (n_j - n_2)\delta_y \tfrac{h}{f_2 f_j} L_{\mathbf{X}_{f_j}} f_2 + \sum_{j=3}^{m+2} \sum_{l=j+1}^{m+2} (n_l - n_j) \tfrac{h}{f_l f_j} L_{\mathbf{X}_{f_l}} f_j \Big). \end{aligned} \tag{8}$$

The above Lie derivatives is given by

$$L_{\mathbf{X}_{f_2}} f_1 = -1,$$

$$L_{\mathbf{X}_{f_j}} f_1 = -t_1 y^{t_1-1},\ j=3,\cdots,m+2$$

$$L_{\mathbf{X}_{f_j}} f_2 = -t_2\lambda_2 x^{t_2-1},\ j=3,\cdots,m+2$$

$$L_{\mathbf{X}_{f_l}} f_j = (\lambda_j-\lambda_l) t_1 t_2 x^{t_2-1} y^{t_1-1},\ j,l=3,\cdots,m+2.$$

By Lemma **??**, it has

$$\frac{\operatorname{div}(\mathbf{F}_r)(x,y)}{h(x,y)} = \frac{x^{t_2-1}y^{t_1-1}\mu_0(x^{t_2},y^{t_1})}{x^{t_2\delta_x}y^{t_1\delta_y}h_0(x^{t_2},y^{t_1})} = x^{t_2-1}y^{t_1-1}\eta(x^{t_2},y^{t_1}), \tag{9}$$

thus, dividing by h in (**??**) and changing (x^{t_2}, y^{t_1}) by (x,y), it has the following expression of η,

$$\begin{aligned}\eta(x,y) = \tfrac{1}{M}\Big(&(n_1-n_2)\delta_x\delta_y\tfrac{1}{xy} + \sum_{j=3}^{m+2} t_1(n_1-n_j)\delta_x \tfrac{1}{x(y-\lambda_j x)} \\ &+ \sum_{j=3}^{m+2} t_2\lambda_j(n_2-n_j)\delta_y \tfrac{\lambda_j}{y(y-\lambda_j x)} \\ &+ \sum_{j=3}^{m+2}\sum_{l=j+1}^{m+2} t_1t_2(n_l-n_j)(\lambda_j-\lambda_l)\tfrac{1}{(y-\lambda_j x)(y-\lambda_l x)}\Big).\end{aligned}$$

Now, we prove that it holds

$$\operatorname{Res}[\eta(1,y),\infty] = -\delta_x \operatorname{Res}[\eta(x,1),0], \tag{10}$$

where, by definition, $\operatorname{Res}[\eta(1,y),\infty] = \frac{1}{2\pi i}\oint_{\gamma^-}\eta(1,y)dy$ being γ^- any closed curve negatively oriented which contains in its interior all the poles of $\eta(1,y)$. Actually, if $\delta_x = 0$, by Lemma **??**,

$$\deg(\eta(1,y)) = \deg(\mu_0(1,y)) - \deg(y^{\delta_y}h_0(1,y)) = 2,$$

therefore $\operatorname{Res}[\eta(1,y),\infty]=0$.
If $\delta_x = 1$, the difference of both degrees is one, therefore $\operatorname{Res}[\eta(1,y),\infty] =$

$-\lim_{y\to\infty} y\eta(1,y)$, and expressing η in the form

$$\eta(x,y) = \frac{\sum_{j=0}^{m+1+\delta_y} d_j x^{m+1+\delta_y-j} y^j}{cxy^{\delta_y}\prod_{j=3}^{m+2}(y-\lambda_j x)},$$

then

$$\mathrm{Res}[\eta(x,1),0] = \lim_{x\to 0} x\eta(x,1) = \frac{d_{m+1+\delta_y}}{c} = \lim_{y\to\infty} y\eta(1,y) = -\mathrm{Res}[\eta(1,y),\infty],$$

thus, (**??**) holds. Now, we prove that

$$\mathrm{Res}[\eta(1,y),w_i] = \tfrac{1}{t_1t_2}(1-\tfrac{n_i(r+|\mathbf{t}|)}{M})\mathrm{deg}_{\mathbf{t}}(f_i),\ i=1,\cdots,m+2. \tag{11}$$

For $w_1=\infty$ (case $\delta_x=1$), we have that

$$\begin{aligned}\mathrm{Res}[\eta(1,y),w_1] &= -\mathrm{Res}[\eta(x,1),0] = -\lim_{x\to 0} x\eta(x,1)\\ &= -\tfrac{1}{M}[(n_1-n_2)\delta_y + \textstyle\sum_{j=3}^{m+2} t_1(n_1-n_j)] = -\tfrac{n_1(r+|\mathbf{t}|)-M}{t_2M}.\end{aligned}$$

Analogously, for $w_2=0$ ($\delta_y=1$), it holds

$$\mathrm{Res}[\eta(1,y),w_2] = \tfrac{1}{M}[(n_1-n_2)\delta_x - \sum_{j=3}^{m+2} t_2(n_2-n_j)] = -\tfrac{n_2(r+|\mathbf{t}|)-M}{t_1M}.$$

And for each $w_i=\lambda_i,\ i=3,\cdots,m+2$, it has that

$$\begin{aligned}\mathrm{Res}[\eta(1,y),w_i] &= \lim_{y\to\lambda_i}(y-\lambda_i)\eta(1,y)\\ &= \tfrac{1}{M}\Big(t_1(n_1-n_i)\delta_x + t_2(n_2-n_i)\delta_y + t_1t_2\textstyle\sum_{j=3}^{m+2}(n_j-n_i)\Big)\\ &= \tfrac{-n_i(r+|\mathbf{t}|)+M}{M}.\end{aligned}$$

Therefore, (**??**) is proved. As a consequence, it has **ii)**.

Also, defining $r_j=\frac{n_j}{M}\in\mathbb{Q}$ and by solving r_j in (**??**) it has (**??**). By Theorem **??**, $\prod_{j=1}^{m+2} f_j^{n_j}$ with $n_j=Mr_j\in\mathbb{Z}$, is a rational first integral whose degree is $\sum_{j=1}^{m+2} n_j\mathrm{deg}_{\mathbf{t}}(f_j) = M\sum_{j=1}^{m+2} r_j\mathrm{deg}_{\mathbf{t}}(f_j)$. By (**??**), it has that

$$\sum_{j=1}^{m+2} r_j\mathrm{deg}_{\mathbf{t}}(f_j) = \sum_{j=1}^{m+2}\frac{\mathrm{deg}_{\mathbf{t}}(f_j)}{r+|\mathbf{t}|} - t_1t_2\sum_{j=1}^{m+2}\mathrm{Res}[\eta(1,y),w_j].$$

The first summand on the right-hand is 1 and the second is 0, from Residues Theorem. Thus, the degree of the first integral is M.

We now see the sufficient condition. We assume that h verifies **i)** and $\eta(1, y)$ satisfies **ii)**. First, we check that η is univocally determined. It is easily followed because, making $v = \frac{y}{x}$, η can be expressed in the form

$$\eta(x, xv) = \frac{\mu_0(x, xv)}{x^{\delta_x} y^{\delta_y} v^{\delta_y} h_0(1, v)} = \frac{\mu_0(1, v)}{x^2 v^{\delta_y} h_0(1, v)} == \tfrac{1}{x^2}(\tfrac{A_2\delta_y}{v} + \sum_{j=3}^{m+2} \tfrac{A_j}{v-\lambda_j})$$

and $A_2, A_3, \cdots A_{m+2}$ are given by

$$A_j = \mathrm{Res}[\eta(1, v), w_j] = \tfrac{1}{t_1 t_2}(1 - \tfrac{n_j(r+|\mathbf{t}|)}{M})\deg_{\mathbf{t}}(f_j), j = 2, \cdots, m + 2.$$

Finally, we prove that $\mathbf{F}_r$ verifies (**??**). Let $\tilde{\mathbf{F}}_r = \frac{1}{r+|\mathbf{t}|}[\mathbf{X}_h + \mu\mathbf{D}_0]$, where μ is the $\mathbf{t}$-polynomial such that $(\sum_{j=1}^{m+2} n_j \deg_{\mathbf{t}}(f_j))\mu$ is given by the right-side of (**??**). Trivially, $\tilde{\mathbf{F}}_r$ verifies (**??**), thus $\tilde{\mathbf{F}}_r$ has got a rational first integral. From the necessary condition, $\tilde{\eta}$ associated to $\tilde{\mathbf{F}}_r$ holds **ii)** and therefore, $\eta = \tilde{\eta}$ and $\mu = \mathrm{div}(\mathbf{F}_r)$. So, system (**??**) is rationally integrable. ■

3 Kowalevskaya exponents and rational integrability

The Kowalevskaya exponents arose from the study of the existence of particular solutions of the form $(x(t), y(t)) = (c_1 t^{-t_1}, c_2 t^{-t_2})$ of system (**??**), where the coefficients $\mathbf{c} = (c_1, c_2) \in \mathbb{C}^2 \setminus \{O\}$ are given by the vectorial equation

$$\mathbf{F}_r(\mathbf{c}) + \tfrac{1}{r}\mathbf{D}_0(\mathbf{c}) = \mathbf{0}. \qquad (12)$$

For a given type $\mathbf{t}$, there may exist different $\mathbf{c}$-s called system *balances.*

Now, for each balance $\mathbf{c}$, it defines the differential of $\mathbf{F}_r + \frac{1}{r}\mathbf{D}_0$ evaluated at $\mathbf{c}$, that is $K(\mathbf{c}) = D\left(\mathbf{F}_r + \frac{1}{r}\mathbf{D}_0\right)(\mathbf{c})$. The eigenvalues of $K(\mathbf{c})$ are called the *Kowalevskaya exponents* of the balance $\mathbf{c}$, see Kowalevskaya [**?**]. It can be shown that there always exists a Kowalevskaya exponent equal to -1, see [**?**,**?**].

Several authors have studied the integrability of quasi-homogeneous systems and its relation with both n-dimensional and planar systems, see, Furta [?], Goriely [?], Llibre & Zhang [?], Tsygvintsev [?] and Cairó & Llibre [?], among others.

Next, we calculate the balances of system (**??**), by showing the relation among them and the irreducible factors of h over $\mathbb{C}[x,y]$.

Proposition 4 *If* $\mathbf{c}$ *is a balance of (**??**), then* $h(\mathbf{c})=0$.
Furthermore, if P,Q *are coprime and* $PQ\not\equiv 0$, *it holds:*
i) *if* x *is a factor of* h, *then* $(0,c_2)$, *with* $c_2^{r/t_2}=-\frac{t_2}{rQ(0,1)}$, *is a balance of (**??**),*
ii) *if* y *is a factor of* h, *then* $(c_1,0)$, *with* $c_1^{r/t_1}=-\frac{t_1}{rP(1,0)}$, *is a balance of (**??**),*
iii) *if* $y^{t_1}-\lambda x^{t_2}$ *is a factor of* h, *with* $\lambda\in\mathbb{C}\setminus\{0\}$, *then* (c_1,c_2) *with* $c_1=u^{t_1}$, $c_2=u^{t_2}\lambda^{1/t_1}$ *and* $u^r=-\frac{t_1}{rP(1,\lambda^{1/t_1})}$, *is a balance of (**??**).*

Proof. Every balance $\mathbf{c}=(c_1,c_2)$ is a solution of an irreducible factor of h over $\mathbb{C}[x,y]$, since $h(\mathbf{c})=(\mathbf{D}_0\wedge\mathbf{F}_r)(\mathbf{c})=\mathbf{D}_0(\mathbf{c})\wedge[\mathbf{F}_r(\mathbf{c})+\frac{1}{r}\mathbf{D}_0(\mathbf{c})]=0$.

Let us assume that P,Q are coprime and $PQ\not\equiv 0$. From Lemma **??**, we can assume that $r+|\mathbf{t}|=k_3t_1t_2+\delta_yt_2+\delta_xt_1$ with $\delta_x,\delta_y\in\{0,1\}$ and $k_3\geq 0$. Also, it follows easily that P and Q evaluated at $\mathbf{c}$ may be written as

$$P(\mathbf{c})=c_1^{\delta_x}c_2^{(t_1-1)(1-\delta_y)}P_0(c_1^{t_2},c_2^{t_1}),\quad Q(\mathbf{c})=c_1^{(t_2-1)(1-\delta_x)}c_2^{\delta_y}Q_0(c_1^{t_2},c_2^{t_1}),$$

where P_0 and Q_0 are homogeneous polynomials of degree $k_3+\delta_y-1$ and $k_3+\delta_x-1$, respectively.
The factors of h can be of three types: x, y or $y^{t_1}-\lambda x^{t_2}$ with $\lambda\in\mathbb{C}\setminus\{0\}$.
If x is a factor of $h=t_1xQ-t_2yP$ then x is a factor of P (that is, $P(0,y)\equiv 0$) and $Q(0,1)\neq 0$, since otherwise x would be a factor of both components of $\mathbf{F}_r$. We compute the balances $\mathbf{c}=(0,c_2)$ of system (**??**) associated to x. As $P(0,c_2)=0$, the first equation of (**??**) evaluated at $(0,c_2)$ holds. On the other hand, we note that $(t_2-1)(1-\delta_x)=0$, since if $\delta_x=0$ and $t_2>1$, x would be factor of P and Q. Therefore, the second equation of (**??**) becomes

$$0=c_2^{\delta_y}Q(0,1)c_2^{(k_3+\delta_x-1)t_1}+\frac{t_2c_2}{r}=c_2\left[c_2^{\delta_y-1+(k_3+\delta_x-1)t_1}Q(0,1)+\frac{t_2}{r}\right],$$

and as $r = t_2[t_1(k_3 - 1 + \delta_x) - (1 - \delta_y)]$, it has $c_2^{r/t_2} = -\frac{t_2}{rQ(0,1)}$, item **i)**.
If y is a factor of h, by the same reasoning we arrive at **ii)**.
If $y^{t_1} - \lambda x^{t_2}$ is a factor of h with $\lambda \in \mathbb{C} \setminus \{0\}$, there exist balances of the form $(c_1, c_2) = \left(u^{t_1}, u^{t_2}\lambda^{1/t_1}\right)$ with $u \neq 0$, since $h\left(u^{t_1}, u^{t_2}\lambda^{1/t_1}\right) = u^{r+|\mathbf{t}|}h\left(1, \lambda^{1/t_1}\right) = 0$.
In this case, the first equation of (**??**) is

$$0 = P(u^{t_1}, u^{t_2}\lambda^{1/t_1}) + \tfrac{t_1 u^{t_1}}{r} = u^{t_1}\left[u^r P(1, \lambda^{1/t_1}) + \tfrac{t_1}{r}\right].$$

Moreover $P(1, \lambda^{1/t_1}) \neq 0$, since otherwise $Q(1, \lambda^{1/t_1})$ would be zero and therefore $y^{t_1} - \lambda x^{t_2}$ would be a common factor of P and Q. Thus, it holds **iii)**. The second equation of (**??**), by replacing Q, becomes $h(c_1, c_2) + t_2 c_2\left(P(c_1, c_2) + \frac{t_1 c_1}{r}\right) = 0$ which is true. ■
Next, we obtain the Kowalevskaya exponents associated to the balances of (**??**) through the rational function η defined in the previous section.

Proposition 5 *Let system (**??**) be an irreducible system with* $h = \prod_{j=1}^{m+2} f_j^{m_j} \not\equiv 0$. *If* $w_1 = \infty,\ w_2 = 0,\ w_j = \lambda_j,\ j = 3, \cdots, m+2$ *are the poles of* $\eta(1, y)$, *then* $\rho_i = 0$ *if* $m_i > 1$, *otherwise,*

$$\rho_i^{-1} = \tfrac{r}{r+|\mathbf{t}|}\left(1 - \frac{t_1 t_2}{deg_{\mathbf{t}}(f_i)} Res[\eta(1, y), w_i]\right),$$

where every $(-1, \rho_i)$ *is the Kowalevskaya exponents associated to the factor* f_i *of* h, *for* $i = 1, \cdots, m+2$.

Proof. Throughout the demonstration, we will denote $\mu = \operatorname{div}(\mathbf{F}_r)$ and $h_x,\ h_y$ and $\mu_x,\ \mu_y$ the partial derivatives of h and μ respect to the variables x and y, respectively.
From Lemma **??**,

$$\mathbf{F}_r + \tfrac{1}{r}\mathbf{D}_0 = \tfrac{1}{r+|\mathbf{t}|}\left(\mathbf{X}_h + (\mu + \tfrac{r+|\mathbf{t}|}{r})\mathbf{D}_0\right).$$

The trace of its differential is

$$\begin{aligned}
&\tfrac{1}{r+|\mathbf{t}|}\left(\mu_x t_1 x + \mu t_1 + \tfrac{r+|\mathbf{t}|}{r} t_1 + \mu_y t_2 y + \mu t_2 + \tfrac{r+|\mathbf{t}|}{r} t_2\right)\\
&= \tfrac{1}{r+|\mathbf{t}|}\left(\nabla\mu\cdot\mathbf{D}_0 + (\mu + \tfrac{r+|\mathbf{t}|}{r})|\mathbf{t}|\right)\\
&= \tfrac{1}{r+|\mathbf{t}|}\left(r\mu + |\mathbf{t}|\mu + \tfrac{r+|\mathbf{t}|}{r}|\mathbf{t}|\right) = \mu + \tfrac{|\mathbf{t}|}{r}.
\end{aligned}$$

Then, for each balance $\mathbf{c}$, if $\rho(\mathbf{c})$ is the eigenvalue different from -1, it has that $\rho(\mathbf{c}) - 1 = \mathrm{Trace}(K(\mathbf{c})) = \mu(\mathbf{c}) + \frac{|\mathbf{t}|}{r}$, that is,

$$\rho(\mathbf{c}) = \mu(\mathbf{c}) + \tfrac{r+|\mathbf{t}|}{r}. \tag{13}$$

We compute the Kowalevskaya exponents different from -1 associated to the factors of h.

We suppose that x is a factor of h ($\delta_x = 1$), that is, there exist balances of the form $(0, c_2)$. From Lemma **??**, $h_x = (r+|\mathbf{t}|)Q - yt_2\mu$. As μ is a $\mathbf{t}$-polynomial of degree r with respect to $\mathbf{t}$, it has that $\mu(0, c_2) = c_2^{r/t_2}\mu(0,1)$, and by applying (**??**) and Proposition **??**, we have that

$$\begin{aligned}
h_x(0,1) &= (r + |\mathbf{t}|)Q(0,1) - t_2\mu(0,1)\\
&= (r + |\mathbf{t}| + r\mu(0,c_2))Q(0,1) = r\rho_1 Q(0,1),
\end{aligned}$$

with $Q(0,1) \neq 0$. Therefore, all the balances $(0, c_2)$ have the same eigenvalues. Moreover, $\rho_1 = 0$ if and only if $h_x(0,1) = 0$, i.e. $m_x > 1$. Otherwise,

$$\begin{aligned}
\tfrac{1}{\rho_1} &= \tfrac{r}{r+|\mathbf{t}|}\left(\tfrac{h_x(0,1)+t_2\mu(0,1)}{h_x(0,1)}\right)\\
&= \tfrac{r}{r+|\mathbf{t}|}\left(1 + t_2\tfrac{\mu(0,1)}{h_x(0,1)}\right) = \tfrac{r}{r+|\mathbf{t}|}\left(1 + t_2\mathrm{Res}[\eta(x,1),0]\right),
\end{aligned}$$

and by (**??**) it follows the result. If y is a factor of h, the reasoning is analogous. Finally, we compute the exponents ρ_i associated to the factors $y^{t_1} - \lambda_i x^{t_2}$, where $\lambda_i \in \mathbb{C}\setminus\{0\},\ i = 3, \cdots, m+2$. From Lemma **??**, $h_y = -(r+|\mathbf{t}|)P + xt_1\mu$.

Also $\mu(c_1, c_2) = \mu(u^{t_1}, u^{t_2}\lambda_i^{1/t_i}) = u^r\mu(1, \lambda_i^{1/t_i})$. So, from Proposition **??**,

$$h_y(1, \lambda_i^{1/t_i}) = -(r + |\mathbf{t}|)P(1, \lambda_i^{1/t_i}) + t_1\mu(1, \lambda_i^{1/t_i})$$
$$= -(r + |\mathbf{t}| + r\mu(c_1, c_2))P(1, \lambda_i^{1/t_i}) = -r\rho_i P(1, \lambda_i^{1/t_i}),$$

with $P(1, \lambda_i^{1/t_i}) \neq 0$. Thus, $\rho_i = 0$ if and only if $h_y(1, \lambda_i^{1/t_i}) = 0$, i.e. $m_i > 1$. Otherwise,

$$\frac{1}{\rho_i} = -\frac{r}{r+|\mathbf{t}|}\left(\frac{-h_y(1,\lambda_i^{1/t_i})+t_1\mu(1,\lambda_i^{1/t_i})}{h_y(1,\lambda_i^{1/t_i})}\right)$$
$$= \frac{r}{r+|\mathbf{t}|}\left(1 - t_1\frac{\mu(1,\lambda_i^{1/t_i})}{h_y(1,\lambda_i^{1/t_i})}\right) = \frac{r}{r+|\mathbf{t}|}\left(1 - \lim_{y\to\lambda_i^{1/t_1}} t_1\frac{(y - \lambda_i^{1/t_1})\mu(1, y)}{h(1, y)}\right).$$

To prove the result, it is enough to check that the above limit is $\mathrm{Res}[\eta(1, y), \lambda_i]$. This is followed from $lim_{y\to\lambda_i^{1/t_1}} \frac{y-\lambda_i^{1/t_1}}{y^{t_1}-\lambda_i} = t_1$ and $\frac{\mu(1,y)}{h(1,y)} = y^{t_1-1}\eta(1, y^{t_1})$. ∎

As a consequence of Theorem **??** and Proposition **??**, it has Theorem **??**.

4 Application.

We illustrate our method by studying the integrability of the $(1, 2)$-polynomial systems of degree 2, i.e.

$$\begin{aligned} \dot{x} &= a_1x^3 + a_2xy, \\ \dot{y} &= b_1x^4 + b_2x^2y + b_3y^2, \end{aligned} \tag{14}$$

with a_1, a_2, b_1, b_2 and b_3 real parameters with $b_3 \neq 0$ and $b_1a_2^2 - b_2a_1a_2 + b_3a_2^2 \neq 0$ (irreducibility of system (**??**)). The function h associated to (**??**) is $h(x, y) = \frac{1}{5}x[(b_3 - 2a_2)y^2 + (b_2 - 2a_1)x^2y + b_1x^4]$. If $b_3 - 2a_2 = 0$, then x is a multiple factor of h and therefore systems (**??**) are not integrable. Otherwise, we can write h in the form $h(x, y) = \frac{1}{5}(b_3 - 2a_2)x[(y + Bx^2)^2 + Ax^4]$ with $A = \frac{4b_1(b_3-2a_2)-(b_2-2a_1)^2}{4(b_3-2a_2)^2}$, $B = \frac{b_2-2a_1}{2(b_3-2a_2)}$. If $A = 0$, systems (**??**) are not integrable, since h would have multiple factors. So, under the assumption of integrability, systems (**??**) can be transformed, by means of the change $u = \sqrt[4]{\mathrm{sig}(A)Ax},\ v = y + Bx^2$ into $(\dot{u}, \dot{v})^T = \tilde{\mathbf{F}}_2(u, v)$, with

$$\tilde{h}(u, v) = \frac{1}{5}cu(v^2 + \sigma u^4), \qquad \mathrm{div}(\tilde{\mathbf{F}}_2)(u, v) = \frac{1}{5}(d_1 v + d_2 u^2),$$

where $c \neq 0$ and $\sigma = \pm 1$. That is, the systems become

$$\begin{aligned} \dot{u} &= (-2c + d_1)uv + d_2 u^3, \\ \dot{v} &= (c + 2d_1)v^2 + 2d_2 u^2 v + 5c\sigma u^4, \end{aligned} \tag{15}$$

with c, d_1 and d_2 real parameters and $c \neq 0,\ \sigma = \pm 1$.
The following result characterizes both rationally and polynomially integrable systems of the family (**??**).

Theorem 6 *A system (**??**) with $\sigma = -1$ is rationally integrable if and only if $\frac{d_1}{c}$, $\frac{d_1+d_2}{2c}$ and $\frac{d_2-d_1}{2c}$ are integers numbers.*
*A such system (**??**) is polynomially integrable if and only if there is a natural number M such that $\frac{M}{5}(1 + \frac{2d_1}{c})$, $\frac{M}{5}(1 - \frac{d_1+d_2}{2c})$, $\frac{M}{5}(1 - \frac{d_1-d_2}{2c})$ are natural numbers. In such a case*

$$u^{\frac{M}{5}(1+\frac{2d_1}{c})}(v - u^2)^{\frac{M}{5}(1-\frac{d_1+d_2}{2c})}(v + u^2)^{\frac{M}{5}(1-\frac{d_1-d_2}{2c})},$$

*is a $(1, 2)$-polynomial first integral of degree M of the system (**??**).*

Proof. In this case, $\eta(u, v) = \frac{d_1 v + d_2 u}{cu(v-u)(v+u)}$. By Theorem **??**, system (**??**) is rationally integrable if and only if

$$\begin{aligned} \mathrm{Res}[\eta(1, v), \infty] &= -\mathrm{Res}[\eta(u, 1), 0] = -\tfrac{d_1}{c}, \\ \mathrm{Res}[\eta(1, v), 1] &= \tfrac{d_1+d_2}{2c}, \\ \mathrm{Res}[\eta(1, v), -1] &= \tfrac{d_1-d_2}{2c} \end{aligned}$$

are integer numbers. System (**??**) has a $(1,2)$-polynomial first integral of degree $M>0$ if and only if

$$\frac{M}{5}(1+2\frac{d_1}{c}),\ \frac{M}{5}(1-\frac{d_1+d_2}{2c}),\ \frac{M}{5}(1+\frac{d_2-d_1}{2c})$$

are non-negative integer numbers. This completes the proof. ■

Theorem 7 *A system (**??**) with $\sigma=1$ is rationally integrable if and only if $d_2=0$ and $\frac{d_1}{2c}$ is an integer number.*
*Such system (**??**) is polynomially integrable if and only if $d_2=0$ and there is a natural number M such that $\frac{M}{5}(1+\frac{2d_1}{c}),\ \frac{M}{5}(1-\frac{d_1}{2c})$ are natural numbers. In such a case*

$$u^{\frac{M}{5}(1+\frac{2d_1}{c})}(v^2+u^4)^{\frac{M}{5}(1-\frac{d_1}{2c})},$$

*is a $(1,2)$-polynomial first integral of degree M of the system (**??**).*

Proof. In this case, $\eta(u,v)=\frac{d_1v+d_2u}{cu(v-Iu)(v+Iu)}$. So,

$$\begin{aligned}\mathrm{Res}[\eta(1,v),\infty] &= -\mathrm{Res}[\eta(u,1),0]=-\tfrac{d_1}{c},\\ \mathrm{Res}[\eta(1,v),I] &= \tfrac{d_1I+d_2}{2Ic}=\tfrac{d_1}{2c}-\tfrac{d_2}{2c}I,\\ \mathrm{Res}[\eta(1,v),-I] &= \tfrac{d_1I-d_2}{2Ic}=\tfrac{d_1}{2c}+\tfrac{d_2}{2c}I.\end{aligned}$$

Therefore, system (**??**) is rationally integrable if and only if $d_2=0$ and $\frac{d_1}{2c}$ is an integer number. A system of (**??**) is polynomially integrable if and only if

$$\frac{M}{5}(1+2\frac{d_1}{c}),\ \frac{M}{5}(1-\frac{d_1}{2c})$$

are non-negative integer numbers. ■

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