A sequential approach for distributed set-membership estimation to exploit redundant information

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Abstract

This manuscript proposes a novel approach for the distributed set-membership estimation of linear time-invariant plants. The method is based on sequentially exploiting the information that the agents receive from their neighbourhood in an adequate manner, so that redundant information is not discarded, but used to further shrink the sets. To do so, they need to build the innovation-routing table, which integrates routing information and concepts of subspace observability. Then, sequential filtering steps are performed over the sets that contain the different modes in which the state vector is locally decomposed. The method is presented for sets described as zonotopes and takes advantage of the developments of well-known centralized zonotopic observers. Simulation results show that the use of redundant information reduces the uncertainty of the estimation, and the increment of computational cost is minor. Furthermore, the sensitivity of the method with respect to different parameters is analyzed with numerical simulations and compared to similar estimators in the literature.

Keywords: Distributed estimation, Set-membership observer, Sequential observers, Linear Time-Invariant plant, Zonotopes.

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1. Introduction

Set-membership observers are a particular kind of state estimators whose goal is to find a set that, in a deterministic way, contains the actual state vector. Unlike stochastic observers, such as Kalman filter, set-membership observers are built upon the assumption that the process and measurement noises are deterministically bounded.

The first set-membership observer dates back to the 60's, with the work of F. Schweppe [1], in which the state vector was confined in ellipsoidal sets. From that date, different mathematical descriptions of the sets have been proposed: intervals [2], ellipsoids [3], parallelotopes [4], zonotopes [5], constrained zonotopes [6], or the more general polytopes [7]. In general, the estimation is more accurate (the set is smaller) when the sets have mathematically more complex formulations. Of course, this usually incurs in an increment of the computational cost that needs to be considered.

This is where zonotopes show their strength, since they allow for relatively simple manipulations and can describe more complex regions that intervals, ellipsoids or parallelotopes [8]. A zonotope is a centrally symmetric convex set found by an affine transformation of the unit hypercube. Linear transformations or Minkowskii sums of zonotopes are straightforward operations. One of the most important limitations is that they are not closed under intersection, incurring in extra conservatism when finding the zonotope that contains the exact intersection of two zonotopes. To overcome this problem, constrained zonotopes [9] can be used if the increment of computational cost can be assumed.

When the plant is spread over a large region, the use of distributed formulations of the observer might come appealing. Now, a set of agents estimate the state of the same plant, but having only limited, and possibly insufficient, information of it. Therefore, communication between agents is needed to successfully estimate the state. The literature of distributed estimation is vast, and the reader might find some recent surveys in [10, 11, 12]. Some potential advantages mentioned in those documents are: reduced computational and memory burden, scalability, or robustness by avoiding a centralized unit.

Attending to the estimation objective and problem formulation, the available distributed set-membership observers in the literature might be categorised under three families:

Interconnected observers: The plant is modelled as a set of interconnected subsystems, coupled among them via state or input, and each observer aims to estimate the state vector of each local subsystem.

Decentralized + fusion observer: The plant is considered as a whole, but the estimation process is split in two phases: a decentralized one, in which each observer individually computes a local estimation; and a fusion phase, in which a centralized unit merges all the individual estimations made previously.

Fully distributed observers: The plant is considered as a whole, and the observers work in a complete distributed way, aiming at estimating the complete state vector, relying on local and received information.

Table 1 outlines the state of the art according to these categories and the mathematical description of the set.

Table 1: Distributed set-membership observers in the literature. In blue color, the works that are more closely related to this manuscript.

	Interval	Ellipsoid	Zonotope
Interconnected	[13]	[13, 14, 15]	[13, 16, 17, 18, 19, 20, 21]
Decentralized + fusion	[22, 23]	[24, 25]	[26]
Fully distributed		[27, 28, 29, 30, 31, 32, 33]	[34, 35, 36]

Fully distributed observers, which tackle the general problem, have been proposed using ellipsoids and zonotopes. For the former, many of the works have deepened in the effects of different communication schemes both in the design and in the performance of the observers: multi-rate sampling was considered in [27]; and event-based communication in [28, 29, 30]. The impact of attacks in the communications, and the design of resilient observers is proposed in [31]. Finally, two particular problems are addressed in [32, 33]: the estimation of the state of a system affected by an unknown dynamic bias, in the former; and the collaborative estimation of the position of an agent with bearing sensors that measure angles, in the latter.

Regarding those works that have used zonotopes, we can further organised them attending to the way in which each agent uses the information received from neighboring agents. Diffusion strategies, the most simple from a computational point of view, is proposed in [34]. The intersection between sets as a way to exploit neighboring information is proposed in [35]. Finally, the work in [36], based on the Zonotopic Kalman Filter of [37], makes use of a Kalman-inspired formulation to incorporate the received sets.

Although they are not included in the table because they are not purely distributed set-membership observers, it is worth mentioning that some authors have proposed formulations that partially remove full guarantees. This is the case of [16], that proposed a mixed Gaussian and zonotopic observer; of [38], that introduced an ellipsoidal distributed set-membership estimator with probabilistic guarantees; and of [39], in which a distributed fuzzy observer was used for fault detection.

This manuscript contributes with a novel formulation of a fully distributed set-membership zonotopic observer. The main contributions with respect to other fully distributed zonotopic observers (see Table 1) are:

- Similarly to [36], it makes use of the multi-hop subspace decomposition presented in [40] for linear time-invariant plants. This decomposition allows the observer to compute, instead of an unique zonotope that contains the full state vector, several zonotopes of smaller dimensions to contain each dynamical mode in which the state vector is decomposed. The benefit is a reduced computational effort than in [34, 35].

- The proposed observer is based on executing sequential filtering steps to exploit the information of neighbouring agents. The idea is to make subsequent corrections only to some of the sets that contain particular modes of the decomposition. To do so, the agents need to build the so-called innovation-routing table that, merging the information of a routing table with concepts of subspace observability, defines specific transformations needed to extract useful information from a received neighbouring set. This sequential approach allows to use redundant information in the network of agents, which was discarded in [36], without incurring in high computational requirements. The consequence is an improvement in the estimation performance compared to [36].

- The proposed formulation of the estimator allows for the sequential application of filtering steps designed for centralized zonotopic observers, either if they are based on diffusion, intersection of sets, or inspired in the Kalman filter. Then, it is a more general formulation than those in [34, 35, 36]. Furthermore, the proposed estimator is able to exploit the stability and convergence properties of the centralized observers to ensure theoretical guarantees for the distributed approach.

Numerical simulations are presented to illustrate the benefit of this distributed observer. In addition to the estimation performance and computational requirements, the sensibility of the method with respect to the maximum order of the zonotopes, the process noise, and the measurement noise, is analysed and compared with the other fully distributed zonotopic observers in the literature.

This paper is organized as follows. Section 2 presents some notation and preliminaries on zonotopes. The plant description and the assumptions are introduced in Section 3. The distributed set-membership estimation problem is defined in Section 4. The novel approach for the distributed observer can be found in Section 5, including the definition of the innovation-routing table. The simulations and numerical analysis appear in Section 6. Finally, conclusions and future works are drawn in Section 7.

2. Notation and preliminaries

Given a set of matrices A_i, for $i = 1, \ldots, n$, of appropriate dimensions, operator $\operatorname*{cat}_{i=1}^{n}\{A_i\}$ refers to the concatenation of the matrices, that is, $\operatorname*{cat}_{i=1}^{n}\{A_i\} = [A_1 \ A_2 \ \cdots \ A_n]$. The subspace generated by the columns of matrix A is denoted as $Im(A)$. When referred to subspaces, operations $\subseteq$ and $\cap$ refer to belonging and intersection, respectively.

Definition 1. *A zonotope $\mathcal{X}$, denoted with calligraphic, capital letters, is a centrally symmetric, convex set determined by its center $c \in \mathbb{R}^n$, and by a matrix $H \in \mathbb{R}^{n\times q}$: $\mathcal{X} = \langle c, H\rangle = \{c + \sum_{i=1}^{q} \varsigma_i h_i : \forall i,\ |\varsigma_i| \leq 1\}$, where $h_i \in \mathbb{R}^n$ (columns of H) are called generator vectors.*

The *order* of a zonotope is given by the number of generator vectors. The *F-radius* of zonotope $\mathcal{X}$ is the Frobenius norm of its generator matrix H, this is, $||\mathcal{X}||_F \equiv \sqrt{tr(H^T H)}$.

Let $\mathcal{X} = \langle c_x, H_x\rangle$ and $\mathcal{Y} = \langle c_y, H_y\rangle$, be two zonotopes, and let R be a matrix of appropriate dimensions. A linear transformation of a zonotope is given by $R\mathcal{X} = \langle Rc_x, RH_x\rangle$; the Minkowski sum of two zonotopes is obtained as $\mathcal{X} \oplus \mathcal{Y} = \langle c_x + c_y, [H_x, H_y]\rangle$. Given a matrix A and any vectors such that $x \in \mathcal{X}$ and $w \in \mathcal{W}$, it holds that $y := Ax + w \in A\mathcal{X} \oplus \mathcal{W}$.

3. Plant description

Consider a set $\mathcal{V} = \{1, 2, \ldots, p\}$ of agents able to measure some outputs from a linear time-invariant plant:

$$x(k+1) = Ax(k) + Bw(k), \tag{1}$$

$$y_i(k) = C_i x(k) + v_i(k),\ \forall i \in \mathcal{V}, \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ represents process noise, $y_i(k) \in \mathbb{R}^{m_i}$ represents the output vector for the i-th agent, and $v_i(k) \in \mathbb{R}^{n_v}$ are measurement noises, all of them defined at instant $k \in \{0, 1, 2, \ldots\}$. Matrices A, B, C_i are known matrices of appropriate dimensions.

Assumption 1. *Process and measurement noises are bounded signals, such that $w(k) \in \mathcal{W} = \langle 0, Q \rangle$, $v_i(k) \in \mathcal{V}_i = \langle 0, R_i \rangle$, being Q and R_i known matrices of adequate dimensions.*

Assumption 1 requires the noises to belong to some known bounded sets described as zonotopes. It is quite standard and a necessary assumption for set-membership estimation.

3.1. The multi-hop staircase decomposition and link to collective detectability

The multi-hop decomposition was introduced in [40]. Although some relevant details are described next, the reader is referred to that paper for a more complete description.

The set of agents $\mathcal{V}$ is connected by means of a given directional[1] graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with vertices $\mathcal{V} = \{1, 2, \ldots, p\}$, and directional edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ such that (i, j) means $i \leftarrow j$. The set of agents from which agent i receives information is named *the neighbourhood of* i, and is denoted by $N_i := \{j : (i, j) \in \mathcal{E}\}$.

Definition 2. *The ρ-hop output matrix of agent i, $C_{i,\rho}$, is a matrix that stacks the $(\rho - 1)$-hop output matrix of agent i and the $(\rho - 1)$-hop output matrices of its neighbourhood, N_i. That is:*

$$C_{i,\rho} := \begin{bmatrix} C_{i,\rho-1} \\ \mathrm{cat}\{C_{j,\rho-1}^\top\}_{j \in N_i}^\top \end{bmatrix}, \ \forall \rho \geq 1,$$

where $C_{i,0} := C_i$.

For any agent and hop, there exists a coordinate transformation matrix $T_{i,\rho} = \begin{bmatrix} \bar{V}_{i,\rho} \ V_{i,\rho} \end{bmatrix} \in \mathbb{R}^{n \times n}$ according to pair $(C_{i,\rho}, A)$ such that the change of variable $z_{i,\rho} \triangleq T_{i,\rho}^\top x = [\bar{V}_{i,\rho} \ V_{i,\rho}]^\top x \in \mathbb{R}^n$ transforms the original state-space representation into the standard observability staircase form.

[1]The original multi-hop decomposition was introduced for bidirectional graphs.

Definition 3. *The ρ-hop unobservable subspace from agent i, denoted $\bar{\mathcal{O}}_{i,\rho}$, is composed of all system modes that cannot be observed from the output locally measured by agent i and those measured by all the agents belonging to the s-hop reachable set of i, $\forall s \in \{0,\dots,\rho\}$. Equivalently, the ρ-hop unobservable subspace from agent i is the unobservable subspace related to pair $(C_{i,\rho}, A)$ using the above coordinate transformation, this is, $\bar{\mathcal{O}}_{i,\rho} := Im(\bar{V}_{i,\rho})$. With some abuse of notation, the orthogonal complement of $\bar{\mathcal{O}}_{i,\rho}$ is denoted ρ-hop observable subspace from agent i, $\mathcal{O}_{i,\rho} := Im(V_{i,\rho})$.*

According to previous definitions, it is clear that $\mathcal{O}_{i,\rho-1} \subseteq \mathcal{O}_{i,\rho}, \ \forall i \in \mathcal{V}, \ \rho \geq 0$, where we consider $\mathcal{O}_{i,-1} = \emptyset$. Then, the vectors of the "innovation" basis that generates $\mathcal{O}_{i,\rho} \cap (\mathcal{O}_{i,\rho-1})^{\perp}$ can be stacked into a matrix $W_{i,\rho}$ in such a way that:

$$Im(W_{i,\rho}) := \mathcal{O}_{i,\rho} \cap (\mathcal{O}_{i,\rho-1})^{\perp}, \quad \rho \geq 0. \tag{3}$$

Let us define $\ell_i \in \mathbb{Z}_{>0}$ as an arbitrary number of hops. From these definitions it is clear that for all $\rho \in \{0,\dots,\ell_i\}$ and all $i \in \mathcal{V}$, it holds that

$$Im(V_{i,\rho}) = Im\left([W_{i,\rho} \ \ V_{i,\rho-1}]\right), \tag{4}$$

$$Im(\bar{V}_{i,\rho-1}) = Im\left(\left[W_{i,\rho} \ \ \bar{V}_{i,\rho}\right]\right), \tag{5}$$

with $\bar{V}_{i,-1} := I_n$. It is worth pointing out that $Im(W_{i,\rho})$ corresponds to the innovation introduced by the ρ-hop reachable set of agent i, that is, the observable modes for agent i at hop ρ that are not observable at hop $\rho - 1$.

Now, one can link the multi-hop staircase decomposition to the required collective detectability assumption.

Definition 4. *System* (1)-(2) *is collectively detectable if for each agent $i \in \mathcal{V}$, there exists a finite number of hops $\ell_i \in \mathbb{N}$ such that pair (C_{i,ℓ_i}, A) is detectable.*

By definition, we see that a pair (C, A) is detectable if and only if the unobservable modes of the staircase decomposition are stable, namely if and only if there exists an observer ensuring the asymptotic stabilization of the estimation error. Similarly, system (1)-(2) is collectively detectable if for each agent, the complete information provided by the network (that is, the ρ-hop output matrix with ρ arbitrarily large) is sufficient to build such an observation law. Due to this fact, collective detectability is a necessary assumption to build a distributed observer.

Assumption 2. *System* (1)-(2) *is collectively detectable.*

Remark 1. *Please note that no assumptions are imposed to the topology of graph $\mathcal{G}$ per se. However, collective detectability in Assumption 2 demands some coupled constraints between graph topology and observability pairs (C_{i,ℓ_i}, A). This fact interestingly avoids the need to impose independent assumptions to the topology and to the dynamic matrices that could be more conservative, such as the need to have strongly-connected graphs.*

4. Problem statement

The observer has the mission of estimating the state of the perturbed plant (1), having limited and corrupted information about it (2). In particular, the distributed set-membership observer's goal is to find, for each agent $i \in \mathcal{V}$ and instant k, filtered and predicted sets, described as zonotopes, intended to contain the actual state of the plant:

- Filtered set: $\hat{\mathcal{X}}_i(k|k) = \langle c_i(k|k), H_i(k|k) \rangle$,
- Predicted set: $\hat{\mathcal{X}}_i(k+1|k) = \langle c_i(k+1|k), H_i(k+1|k) \rangle$,

being $c_i(\cdot|\cdot)$ and $H_i(\cdot|\cdot)$ the center and generator matrix of each zonotope as defined in Section 2.

Problem 1. *Consider a plant described by* (1)*, whose outputs* (2) *are measured by a set of agents $\mathcal{V}$ that communicate through a communication graph $\mathcal{G}$. Then, under Assumptions 1 and 2, the distributed set-membership observer must:*

a) *Find sets, by means of zonotopes, in which the actual state is continuously contained. In particular, filtered and predicted sets must be found such that $x(k) \in \hat{\mathcal{X}}_i(k|k)$ and $x(k+1) \in \hat{\mathcal{X}}_i(k+1|k)$, $\forall i \in \mathcal{V}$, $\forall k$.*

b) *Minimize the estimation uncertainty, measured through some metric related to the filtered zonotope $\hat{\mathcal{X}}_i(k|k)$, such as the F-radius (see [37]) or the volume (see [41]).*

5. Distributed observer formulation

5.1. Dynamics of the local modes

According to the multi-hop decomposition, the state of the plant (1) can be locally transformed as follows:

$$z_i(k) = T_i^\top x(k) = \begin{bmatrix} \bar{V}_{i,\ell_i}^\top \\ W_{i,\ell_i}^\top \\ \vdots \\ W_{i,\rho}^\top \\ \vdots \\ W_{i,0}^\top \end{bmatrix} x(k) = \begin{bmatrix} z_{i,\ell_i+1}(k) \\ z_{i,\ell_i}(k) \\ \vdots \\ z_{i,\rho}(k) \\ \vdots \\ z_{i,0}(k) \end{bmatrix}, \tag{6}$$

where $z_{i,0}(k)$ are the locally observable modes, $z_{i,\rho}(k)$, $\rho = 1,\ldots,\ell_i$ are the locally unobservable modes that are locally observable by the other agents in the corresponding ρ-hop reachable set of agent i, and $z_{i,\ell_i+1}(k)$ are the modes that are not locally observable by any of the agents in the ℓ_i-hop reachable set of agent i. By Assumption 2, the modes $z_{i,\ell_i+1}(k)$ must be stable.

Using some properties of the multi-hop decomposition (see Lemma 3 in [40]), the dynamics of all these modes can be written as:

$$z_{i,\rho}(k+1) = \sum_{r=0}^{\rho} W_{i,\rho}^\top A W_{i,r} z_{i,r}(k) + W_{i,\rho}^\top B w(k), \tag{7}$$

for $\rho = 0,\ldots,\ell_i+1$, where $W_{i,\ell_i+1} = \bar{V}_{i,\ell_i}$. Previous equation reveals a staircase or cascaded dynamics derived from an upper triangular transformed matrix $T_i^\top A T_i$. From the same transformation, the locally measured output (2) satisfies:

$$y_i(k) = C_i W_{i,0} z_{i,0}(k) + v_i(k). \tag{8}$$

Remark 2. *The multi-hop decomposition simply pursues the partition of the local state-space into a set of $\ell_i + 1$ subspaces. This partition can be made in several ways, but the method in [40] proposes to do it according to how the information measured from the plant (all the outputs in* (2) *collected by all agents) reaches the local agent through the graph. By doing so, a nice cascaded dynamics appear in* (7)*, which is exploited by the observer formulation. It is then, a static subspace division that has nothing to do with a multi-hop communication between agents.*

5.2. The innovation routing table

In [40], a distributed algorithm is proposed to build the multi-hop staircase decomposition. After the execution of that algorithm, each agent i knows, in addition to their local transformation matrices T_i, the innovation matrices of its neighbours, i.e. $W_{j,\rho}$, $\forall j \in N_i$, and constant ℓ_j, $\forall j \in N_i$.

To build the so-called *innovation-routing table*, each agent i requires to know the distance, in hops, to each other agent j in the network, and which neighbour $g(j) \in N_i$ serves as gateway for the information provided by that agent j. This is a sort of routing table, that can be found by executing any graph-based search algorithm, such as the Dijkstra's algorithm.

From this routing table and the multi-hop staircase decomposition, each agent can build the innovation-routing table.

Definition 5. *The innovation-routing table for an agent i comprises a set of tuples of the form:*

$$\left[j, g(j), W_{g,\rho(g)}, W_{i,\rho(i)}\right],$$

being j the agent that measures output $y_j(k)$ from the plant, $g(j) \in N_i$ the gateway neighbour for that agent j, and $W_{g,\rho(g)}, W_{i,\rho(i)}$ the innovation matrices that span the neighbour's and local's subspaces such that $Im(W_{j,0}) \cap Im(W_{g,\rho(g)}) \neq \emptyset$ and $Im(W_{g,\rho(g)}) \cap Im(W_{i,\rho(i)}) \neq \emptyset$.

Please note that $\rho(g)$ denotes the neighbour's hop in which the useful information contained in output $y_j(k)$ is included, while $\rho(i)$ states which local hop will be affected by this information. Therefore, each tuple indicates, for each output directly measured from the plant, the subspaces of the gateway neighbour containing such information, and the local subspaces that must be filtered using that information.

The innovation-routing table for each agent has to be computed only once and offline. From the information gathered to build the multi-hop staircase decomposition, each agent can, in a distributed way, build its own innovation-routing table, as it only requires the local's and the neighbour's innovation matrices.

Example 1. Consider a diagonal four-dimensional plant, and the network of agents in Figure 1. Vector e_i is the i-th canonical vector in four dimensions.

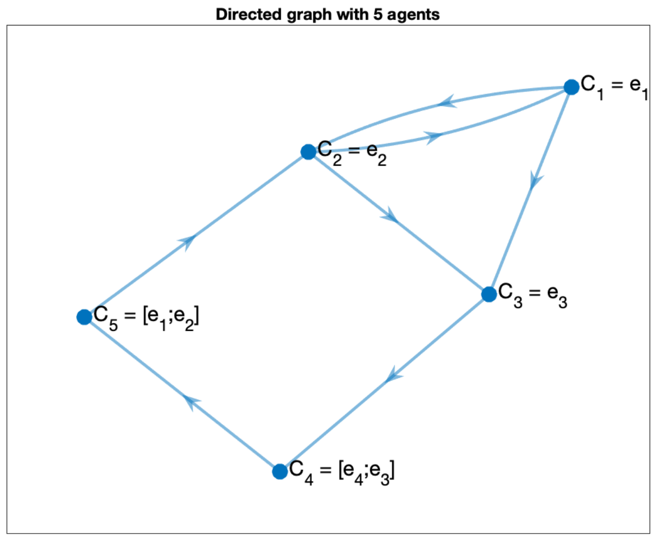

Figure 1: Directed graph with 5 agents. The labels indicate the output matrix for each agent.

The multi-hop staircase decomposition will produce the next innovation matrices[2]:

$$
\begin{aligned}
&W_{1,0} = \left[e_1^\top\right], \ W_{1,1} = \left[e_2^\top\right], \ W_{1,2} = \emptyset, \ W_{1,3} = \left[e_3^\top \ e_4^\top\right] \\
&W_{2,0} = \left[e_2^\top\right], \ W_{2,1} = \left[e_1^\top\right], \ W_{2,2} = \left[e_3^\top \ e_4^\top\right] \\
&W_{3,0} = \left[e_3^\top\right], \ W_{3,1} = \left[e_1^\top \ e_2^\top\right], \ W_{3,2} = \emptyset, \ W_{3,3} = \left[e_4^\top\right] \\
&W_{4,0} = \left[e_3^\top \ e_4^\top\right], \ W_{4,1} = \emptyset, \ W_{4,2} \left[e_1^\top \ e_2^\top\right] \\
&W_{5,0} = \left[e_1^\top \ e_2^\top\right], \ W_{5,1} \left[e_3^\top \ e_4^\top\right]
\end{aligned}
$$

The innovation-routing table for agent 3 for this decomposition and graph is shown in Table 2.

□

Remark 3. *It is possible that the information provided by an agent can reach*

[2] In general, the innovation matrices' columns do not correspond to the output matrices. This is true in this case, because we have intentionally chosen a diagonal plant to simplify the example.

Table 2: Innovation-routing table for agent 3

j	$g(j)$	$W_{g,\rho(g)}$	$W_{3,\rho(3)}$
1	1	$W_{1,0}$	$W_{3,1}$
2	2	$W_{2,0}$	$W_{3,1}$
4	2	$W_{2,2}$	$W_{3,0}$
4	2	$W_{2,2}$	$W_{3,3}$
5	2	$W_{2,1}$	$W_{3,1}$

an agent j through two or more different paths. In those cases, only the shortest path is included in the innovation-routing table. If there are several paths with the same length, one of them is randomly chosen. This occurs in the example with the information measured by agent 1, that can reach agent 3 *directly or through agent* 2.

Remark 4. *It is possible that, for two (or more) tuples in the innovation-routing table such that $j_1 \neq j_2$, the gateway neighbour is the same $g(j_1) = g(j_2)$, and the innovation matrices too. This defines a situation in which the gateway neighbour has included or fused in $Im(W_{g,\rho(g)})$ the information incoming from two different agents j_1 and j_2. In this case, the local agent will discard all the tuples with the same $g(j), W_{g,\rho(g)}, W_{i,\rho(i)}$ but one.*

Remark 5. *The innovation-routing table allows to use redundant information, which was one of the problem claimed by the authors in [40] and inherited in [36]. In the example, the tuple $[4, 2, W_{2,2}, W_{3,0}]$ introduces redundant information measured by agent* 4. *It is redundant in the sense that agent* 3 *would be able to reconstruct the subspace spanned by $e_3^\top$ with its local information. But, considering this extra information is helpful to further shrink the filtered set.*

5.3. Distributed set-membership observer

The proposed distributed set-membership observer intends to find guaranteed sets for every one of the modes (6) from the multi-hop decomposition. In summary, the observer relies on a set of sequential filtering steps. In each step, the observer exploits different sources of information, starting in the first step with the output measured from the plant, and continuing with the information received from neighbouring agents. Each filtering step aims to reduce the size of the sets of those modes of the multi-hop decomposition which are related to the particular source of information to be used. This relation between received set and mode can be extracted from the innovation-routing table.

Firstly, let us denote $\hat{\mathcal{X}}_{i,\rho}(k|k)|_l$ as the set obtained after the execution of the l-th filtering step applied to the predicted set $\hat{\mathcal{X}}_{i,\rho}(k|k-1)$ for the mode $z_{i,\rho}(k)$. The computation of $\hat{\mathcal{X}}_{i,\rho}(k|k)|_l$ from the previous set $\hat{\mathcal{X}}_{i,\rho}(k|k)|_{l-1}$ given a measured output can be made in different ways. Centralized set-membership estimators in the literature have proposed different mechanisms. For instance, in [42], an optimization problem is proposed to minimize the volume of the intersection between the previous set and the strip generated by the measured output. Another approach is presented in [37], in which a Zonotopic Kalman Filter formulation is proposed to minimise the *F-radius* of the resulting set. We can generalize all these filtering steps by introducing the next function:

$$\hat{\mathcal{X}}_{i,\rho}(k|k)|_l \equiv \langle c_{i,\rho}(k|k)|_l, H_{i,\rho}(k|k)|_l\rangle = f(\hat{\mathcal{X}}_{i,\rho}(k|k)|_{l-1}, y_i(k)|_l, C_i|_l, R_i|_l), \quad (9)$$

where $f(\cdot)$ is any of the filtering functions described in those documents, which depends on the previous set, the measured output $y_i(k)|_l$, the output matrix $C_i|_l$, and the noise affecting that output $R_i|_l$. The design of function $f(\cdot)$ is then out of the scope of this manuscript. However, this function will be asked to fulfill some constraints to be used for the distributed observer (see Asumption 3).

The estimation loop that locally runs in each agent is presented in Table 3. As previously mentioned, it carries out the execution of a first filtering step with the measured output (step 2), and sequential filtering steps with the received information from the neighbourhood (step 4). Let us go into more details next.

At the beginning of the loop, all the agents initialize their first sequential filtered set with the previous predicted set for all hops.

Later, each agent i makes use of the locally measured output $y_i(k)$ from the plant to find the first sequential filtered set for hop 0, denoted as $\hat{\mathcal{X}}_{i,0}(k|k)|_1$, as in (9). According to (8), the measured output $y_i(k)$ is only related to the locally observable modes. Therefore, the first filtering step only affects hop $\rho = 0$. Please notice that the output matrix $C_iW_{i,0}$ takes a vector in the coordinates of the mode $z_{i,0}(k)$, expands it to the plant's coordinates with $W_{i,0}$ and, then, projects it to the output's coordinates with C_i. The noise affecting the output is $v_i(k) \in \langle 0, R_i\rangle$.

Table 3: Estimation loop for agent i

1. Initial zonotopes	Hops $\rho = 0, \dots, \ell_i + 1$
1.1. Zonotope	$\hat{\mathcal{X}}_{i,\rho}(k\|k)\|_0 = \hat{\mathcal{X}}_{i,\rho}(k\|k-1)$
2. Sequential step	Only hop $\rho = 0$
2.1. Output,	$y_i(k)$
Matrix,	$C_i W_{i,0}$
and Noise (8)	$\langle 0, R_i \rangle$
2.2. Zonotope (9)	$\hat{\mathcal{X}}_{i,0}(k\|k)\|_1$
3. Communication	
3.1. Send (10)	$\tilde{\mathcal{X}}_{i,\rho}(k)$
3.2. Receive (10)	$\tilde{\mathcal{X}}_{g,\rho}(k),\ g \in N_i$
4. Sequential step	For each tuple
4.1. Output,	$\tilde{c}_{g,\rho(g)}(k)$
Matrix,	$W_{g,\rho(g)}^\top W_{i,\rho(i)}$
and Noise (11)	$\langle 0, \tilde{H}_{g,\rho(g)}(k) \rangle$
4.2. Zonotope (9)	$\hat{\mathcal{X}}_{i,\rho}(k\|k)\|_l$
5. Filtered set	$\hat{\mathcal{X}}_{i,\rho}(k\|k)$
6. Prediction step	For all hops
6.1. Zonotope (12)	$\hat{\mathcal{X}}_{i,\rho}(k+1\|k)$

Now, a single communication step between neighbouring agents takes place. All the agents send the previously computed filtered set for hop 0, i.e. $\hat{\mathcal{X}}_{i,0}(k|k)|_1$, and all the predicted sets for the rest of hops, i.e. $\hat{\mathcal{X}}_{i,\rho}(k|k)|_0,\ \forall \rho > 1$, to their neighbours. Let us define

$$\tilde{\mathcal{X}}_{i,\rho}(k) = \langle \tilde{c}_{i,\rho}(k), \tilde{H}_{i,\rho}(k) \rangle = \left\{ \begin{array}{ll} \hat{\mathcal{X}}_{i,0}(k|k)|_1, & \rho = 0 \\ \hat{\mathcal{X}}_{i,\rho}(k|k)|_0, & \rho > 0 \end{array} \right. , \qquad (10)$$

as the transmitted sets by agent i, and received by agents j such that $i \in N_j$. The received zonotopes contain the information that the agents might exploit to subsequently filter the sets. Note that, according to Definition 1, it is satisfied:

$$\tilde{c}_{i,\rho}(k) = z_{i,\rho}(k) + v_{i,\rho}(k),$$

with $v_{i,\rho}(k) \in \langle 0, \tilde{H}_{i,\rho}(k) \rangle$. Previous equation reveals that the center of a zonotope can play the role of an *output* that, instead of being measured directly from the plant, is transmitted/received. This special output is affected by a noise described by the generator matrix of the zonotope.

However, only some of the received zonotopes for a given agent i contain useful information, this is, information related to any of the measured outputs from the plant $y_j(k)$. Then, for every tuple $\left[j, g(j), W_{g,\rho(g)}, W_{i,\rho(i)}\right]$ in the innovation-routing table, agent i defines a received *output* as:

$$\tilde{c}_{g,\rho(g)}(k) = W_{g,\rho(g)}^{\top} W_{i,\rho(i)} z_{i,\rho(i)}(k) + v_{g,\rho(g)}(k), \tag{11}$$

with $v_{g,\rho(g)}(k) \in \langle 0, \tilde{H}_{g,\rho(g)}(k) \rangle$. The output matrix $W_{g,\rho(g)}^{\top} W_{i,\rho(i)}$ takes a vector in the local coordinates, expands it to the plant's coordinates with $W_{i,\rho(i)}$ and, then, projects it into the neighbour's coordinates with $W_{g,\rho(g)}^{\top}$. Example 2 below illustrates the double coordinate transformation with a graphical interpretation.

The information contained in this *output* (11) is used by the local agent with (9), in a similar way that it did with an output measured from the plant. After applying all the sequential steps for each row in the innovation-routing table, agent i will have found the final filtered set for each hop ρ, this is, $\hat{\mathcal{X}}_{i,\rho}(k|k)$. Please notice that, depending on the innovation-routing table, the sets containing the modes related to each hop will have been filtered, in general, a different amount of times. This is illustrated in Figure 2. To summarize, each local observer uses, as inputs, the measured output from the plant, and all the sets received from the neighbourhood.

Finally, according to (7), the predicted sets are computed as:

$$\hat{\mathcal{X}}_{i,\rho}(k+1|k) = \sum_{r=0}^{\rho} W_{i,\rho}^{\top} A W_{i,r} \hat{\mathcal{X}}_{i,\rho}(k|k) \oplus W_{i,\rho}^{\top} B \mathcal{W}, \tag{12}$$

for $\rho = 0, \ldots, \ell_i + 1$. An order reduction operation (see [43]) must be conducted over all the predicted zonotopes to keep the number of generators bounded.

Remark 6. *The proposed estimation loop can be also applied if the sets are described as constrained zonotopes as introduced in [9]. In that work, the authors describe the operations needed to intersect the strip of states consistent with a measurement with a constrained zonotope, this is, the implementation*

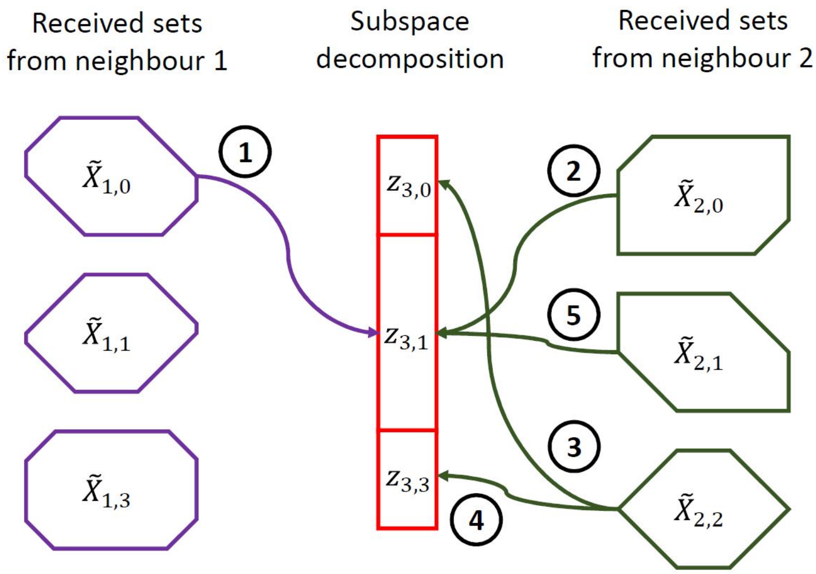

Figure 2: In red, the subspace decomposition for agent 3 for the situation referred in Example 1. In purple and green, some possible received sets from the two neighbours 1 and 2 in the graph depicted in Figure 1. The numbered arrows represent each of the sequential filtering step according to the innovation-routing table in Table 2. The subspace $Im(W_{3,1})$ is filtered three times, while the other ones are only filtered once.

of the function in (9). *The use of constrained zonotopes improves the estimation performance, as the filtering step is found by an exact intersection between the previous set and the strip of states consistent with the measurement. However, this incurs in an additional computational cost that needs to be considered.*

Example 2. Consider a two-dimensional plant observed by two bidirectionally connected agents. The multi-hop decomposition has generated the next innovation matrices:

$$W_{1,0} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ W_{1,1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, W_{2,0} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}.$$

Agent 1 locally observes one scalar mode, whereas agent 2 locally observes the complete state vector. The tuples for agent 1 are $[2, 2, W_{2,0}, W_{1,0}]$ and $[2, 2, W_{2,0}, W_{1,1}]$, whereas the unique tuple for agent 2 is $[1, 1, W_{1,0}, W_{2,0}]$.

Consider that, after the first sequential step, the filtered sets for the locally observable subspaces for both agents are given by

$$\begin{aligned} \mathcal{X}_{1,0}(k|k)|_1 &= \left\langle \begin{bmatrix} \sqrt{2} \end{bmatrix}, \begin{bmatrix} \sqrt{2} \end{bmatrix} \right\rangle, \\ \mathcal{X}_{2,0}(k|k)|_1 &= \left\langle \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \right\rangle. \end{aligned}$$

In order to understand the double coordinate transformation, we will make use of a graphical representation of the zonotopes.

First of all, for agent 1, the received zonotope $\mathcal{X}_{2,0}(k|k)|_1$ is transformed into the local coordinates through $W_{1,0}^{\top} W_{2,0}$. The double coordinate transformation generates, in this case, an interval that can be easily intersected with the local zonotope, which is another interval, as shown in Figure 3.

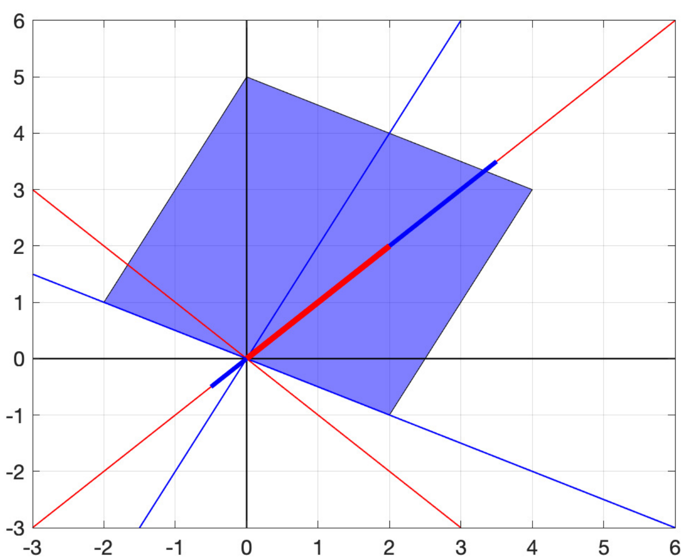

Figure 3: The received zonotope $\mathcal{X}_{2,0}(k|k)|_1$ is drawn in light blue. After the double transformation, the interval in dark blue is obtained. The local zonotope is drawn in thick red line.

Now, for agent 2, the received zonotope $\mathcal{X}_{1,0}(k|k)|_1$ is transformed into the local coordinates through $W_{2,0}^{\top} W_{1,0}$. The double coordinate transformation generates, in this case, a strip that can be easily intersected with the local zonotope, as shown in Figure 4. □

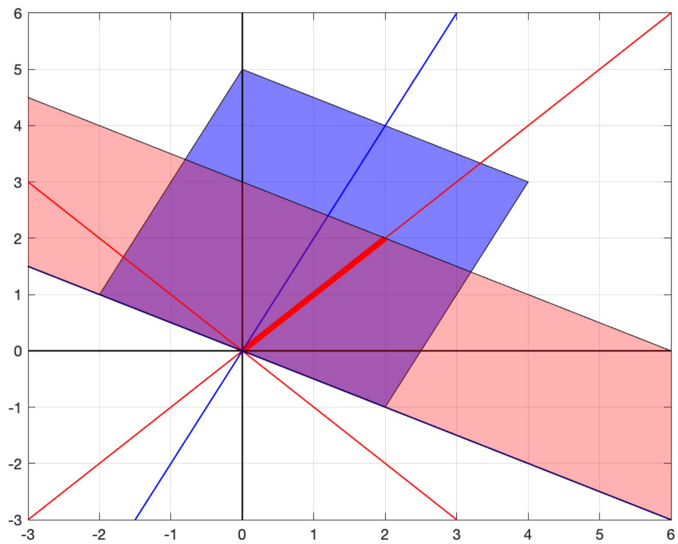

Figure 4: The received zonotope $\mathcal{X}_{1,0}(k|k)|_1$ is drawn with a thick red line. After the double transformation, the strip in light red is obtained. The local zonotope is drawn in light blue.

5.4. Guarantees of the proposed distributed observer

In order to present theoretical guarantees of the distributed set-membership observer, the filtering function (9) is required to satisfy the next assumption.

Assumption 3. *Operation* (9) *ensures that:*

- *If* $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k)|_{l-1}$ *then* $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k)|_l$.
- *The metric, F-radius or volume, that measures the estimation uncertainty of* $\hat{\mathcal{X}}_{i,\rho}(k|k)|_l$ *is minimized.*

Both the methods presented in [9], [37] and [42] satisfy Assumption 3. For instance, according to the method in [37], in order to guarantee that the state vector $z_{i,\rho}(k)$ belong to the set $\hat{\mathcal{X}}_{i,\rho}(k|k)|_l$, and to minimize the *F-radius* of $\hat{\mathcal{X}}_{i,\rho}(k|k)|_l$ one needs to perform the operation (9) as follows:

$$c_{i,\rho}(k|k)|_l = c_{i,\rho}(k|k)|_{l-1} + L_i(k)|_l\left(y_i(k)|_l - C_i|_l c_{i,\rho}(k|k)|_{l-1}\right) \tag{13}$$

$$H_{i,\rho}(k|k)|_l = \left[\,(I - L_i(k)|_l C_i|_l)\, H_{i,\rho}(k|k)|_{l-1}, \quad -L_i(k)|_l R_i|_l\right] \tag{14}$$

$$L_i(k)|_l = P_{i,\rho}(k)|_{l-1} C_i|_l^\top \left(C_i|_l P_{i,\rho}(k)|_{l-1} C_i|_l^\top + R_i|_l R_i|_l^\top\right)^{-1}, \tag{15}$$

with $P_{i,\rho}(k)|_{l-1} = H_{i,\rho}(k|k)|_{l-1} H_{i,\rho}(k|k)|_{l-1}^{\top}$.

Now, next result formally presents the guarantees of the proposed observer.

Theorem 1. *Let us assume that $z_{i,\rho}(k_o) \in \hat{\mathcal{X}}_{i,\rho}(k_o|k_o-1), \forall i \in \mathcal{V}, \rho = 0, \ldots, \ell_i + 1$ for some particular instant k_o. Then, if the estimation loop described in Table 3 under Assumption 3 is executed in a distributed manner by each agent $i \in \mathcal{V}$, for all modes $\rho = 0, \ldots, \ell_i + 1$, Problem 1 is solved for all $k > k_o$.*

Proof. The theorem is proved by induction. Thus, let us consider that $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k-1), \forall i, \rho$, for some instant k. Then, if $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k), \forall i, \rho$, $z_{i,\rho}(k+1) \in \hat{\mathcal{X}}_{i,\rho}(k+1|k), \forall i, \rho$, and the metric associated to the estimation uncertainty of set $\hat{\mathcal{X}}_{i,\rho}(k|k)$ is minimized, then the theorem holds true.

It is satisfied that $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k)|_0, \forall i, \rho$. Then, according to Assumption 3, it is also satisfied that $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k)|_l, \forall i, \rho$, for all sequential steps l implemented. Moreover, the metric that measures the estimation uncertainty of $\hat{\mathcal{X}}_{i,\rho}(k|k)|_l$ is minimized, for all sequential steps l.

Therefore, it is proven that $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k), \forall i, \rho$, and the metric is minimized for the final filtered set, since $\hat{\mathcal{X}}_{i,\rho}(k|k) = \hat{\mathcal{X}}_{i,\rho}(k|k)|_l$.

Regarding the prediction steps, taking into account the evolution of the system in (7), and using $z_{i,\rho}(k) \in \hat{\mathcal{X}}_{i,\rho}(k|k), \forall \rho$ and $w(k) \in \mathcal{W}$ by Assumption 1, it is possible to find a set containing $z_{i,\rho}(k+1)$:

$$z_{i,\rho}(k+1) \in \bigoplus_{r=0}^{\rho} \left(W_{i,\rho}^{\top} A W_{i,r} \hat{\mathcal{X}}_{i,r}(k|k) \right) \oplus W_{i,\rho}^{\top} B \mathcal{W},$$

where $\oplus$ stands for the Minkowski sum of sets. The set on the right-hand size of the previous equation is exactly the same that in (12), proving that $z_{i,\rho}(k+1) \in \hat{\mathcal{X}}_{i,\rho}(k+1|k), \forall i, \rho$. □

Remark 7. *The proposed set-membership estimator depends on a set of parameters that must be tuned:*

- *The initial zonotopes must be chosen such $z_{i,\rho}(k_o) \in \hat{\mathcal{X}}_{i,\rho}(k_o|k_o-1), \forall i \in \mathcal{V}, \rho = 0, \ldots, \ell_i+1$. One simple way to fulfill this is to choose sufficiently large generator vectors. Equivalently to the initial covariance matrix for the Kalman filter, the impact of this choice is negligible in the steady-state operation of the observer, but only affects the transient.*

- *The maximum order of the zonotopes allows to trade off computational burden and memory usage with estimation performance. The choice of this value might be constrained by the computational or memory capabilities of the device that will implement the algorithm. Anyway, the analysis conducted in the next section will help the designer to pick an adequate value for this parameter.*

Finally, it is worth noting that the multi-hop decomposition and the innovation-routing table are fixed for a given plant, output matrices and graph topology, so no additional parameter has to be tuned.

6. Results

6.1. Description of the experiment

The proposed distributed zonotopic observer is tested over the LTI plant presented in [36]:

$$x(k+1) = \begin{bmatrix} 1.005 & 0 & 0 & 0 \\ 0 & 0.9954 & -0.08757 & 0 \\ 0 & 0.1248 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.9775 \end{bmatrix} x(k) + w(k).$$

Four agents measure different outputs from the plant:

$$\begin{aligned} y_1(k) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(k) + v_1(k),\ y_2(k) = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \end{bmatrix} x(k) + v_2(k), \\ y_3(k) &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(k) + v_3(k),\ y_4(k) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(k) + v_4(k). \end{aligned}$$

The communication graph comprises the edges $\mathcal{E} = \{(1,2),(2,1),(1,3),(3,1),(2,4),(4,2)\}$, so collective detectability in Assumption 2 is fulfilled. Noises and disturbances are bounded according to Assumption 1, with $R_i = R_{\max}, \forall i$, and $Q = Q_{\max} I$, where $R_{\max}, Q_{\max}$ are some scalars to be chosen later. Table 4 presents the innovation-routing tables for agents 1 and 2:

To illustrate the performance of the proposed observer, it will be compared with the observer in [36] based also on the multi-hop decomposition but descarding redundant information, and with the observer in [35] based in intersections of neighbouring sets trying to minimize the *F-radius* as well. The method proposed in [34], based on a distributed diffusion of zonotopic sets, cannot be easily compared with these solutions. The reason is that they proposed to share the measurements $y_i(k)$ between neighbours in a

Table 4: Innovation-routing table for agents 1 (left) and 2 (right)

j	$g(j)$	$W_{g,\rho(g)}$	$W_{1,\rho(1)}$
2	2	$W_{2,0}$	$W_{1,1}$
3	3	$W_{3,0}$	$W_{1,1}$
4	2	$W_{2,1}$	$W_{1,2}$

j	$g(j)$	$W_{g,\rho(g)}$	$W_{2,\rho(2)}$
1	1	$W_{1,0}$	$W_{2,1}$
3	1	$W_{1,1}$	$W_{2,0}$
4	4	$W_{4,0}$	$W_{2,1}$

prior communication phase. The comparison of the method in [34], assuming that this communication in the measurement update is not allowed, was made in [36], showing that the performance degrades very fast without this additional information.

For the rest of the section, operation (9) is performed as in (13)-(15), since all the methods under comparison intends to minimize the *F-radius* of the filtered sets. Then, Assumption 3 is also fulfilled.

In order to present a numerical comparison of the performance of the observers, we use the next index (an extension to the one defined in [36]) for a set of N_{sim} simulation of N_{ins} instants:

$$I_i = \frac{1}{N_{sim}} \sum_{j=1}^{N_s} \left(\frac{1}{N_{ins}} \sum_{k=1}^{N_{ins}} \sqrt{\sum_{\rho=0}^{\ell_{i+1}} ||\hat{\mathcal{X}}_{i,\rho}(k|k)||_F^2} \right),$$

where, with some abuse of notation, $\ell_i = 0, \forall i$, for the method in [35], in which only one hop is computed. This index tries to capture the average value of the *F-radius* of the sets computed for all the hops, time instants, and simulations.

The initial condition for each of the N_{sim} simulations is chosen randomly such that $x(0) = [x(0)_i]$, with $x(0)_i \sim U(-1, 1)$. Noises and disturbances are also generated randomly meeting Assumption 1. The same initial condition, noises and disturbances sequences are applied to all the methods for each simulation $j = 1, \ldots, N_{sim}$.

Finally, the order reduction algorithm described in [37] is used for all the methods. The order reduction is performed after the prediction step in all cases.

6.2. Benefits from using redundant information

For $R_{\max} = Q_{\max} = 0.02$, $N_{ins} = 100$ and $N_{sim} = 20$, Table 5 includes the value of index I_i for three different values of the maximum order of the zonotopes.

As expected, the performance index reduces for smaller orders. It can be seen that the methods in [35, 36] behaves quite similarly, and are more sensitive to this reduction than the sequential method in Table 3.

Table 5: Numerical results at average

Order	Algorithm	I_1	I_2	I_3	I_4
10	[35]	0.0454	0.3452	0.2464	0.2789
	[36]	0.0487	0.3458	0.2473	0.2795
	Table 3	0.0462	0.0627	0.0696	0.0797
100	[35]	0.0380	0.0798	0.0826	0.0645
	[36]	0.0393	0.0820	0.0830	0.0653
	Table 3	0.0382	0.0441	0.0529	0.0536
500	[35]	0.0361	0.0798	0.0826	0.0550
	[36]	0.0374	0.0819	0.0828	0.0560
	Table 3	0.0367	0.0402	0.0493	0.0486

The cases of order $q = 100$ or $q = 500$ illustrate the potential of the proposed sequential observer. Regarding agent 2, the algorithm in [36] was designed to discard redundant information at different hops, so that agent 2 does not use the information measured by agent 3 (see second row in Table 4 right). However, the proposed algorithm is able to incorporate this information via one additional sequential filtering step. This is why the performance is highly increased. The same reason applies for agent 3, which also receives redundant information.

However, according to Table 4, agent 1 does not receive redundant information at different hops. Note that agent 2 and 3 are located at the same distance to agent 1, so the redundant information is received at the same hop, and then used for the method in [36]. Therefore, the estimation performance of this agent is quite similar for both methods and all the orders analysed.

The steady-state value of the index, i.e. for $k = N_{ins}$, in contrast to the average value obtained from $k = 1$ to $k = N_{ins}$ as computed before, is another interesting way to compare the observers. Table 6 includes the value of the index for $q = 100$ and $k = N_{ins}$, showing again the benefit of using redundant information.

To evaluate the transient behaviour of the algorithms, the same experiment is performed, now with $q = 100$, $N_{sim} = 30$ and $N_{ins} = 5$, $N_{ins} = 10$, $N_{ins} = 15$. The results are included in Table 7. The same conclusions can

Table 6: Numerical results in the steady-state

Order	Algorithm	I_1	I_2	I_3	I_4
100	[35]	0.0344	0.0626	0.0656	0.0531
	[36]	0.0354	0.0653	0.0659	0.0538
	Table 3	0.0340	0.0383	0.0416	0.0432

be observed from this analysis, showing that the speed of convergence to the steady-state set is similar for all the methods.

Table 7: Numerical results in the transient

N_{ins}	Algorithm	I_1	I_2	I_3	I_4
5	[35]	0.1362	0.3317	0.3599	0.2682
	[36]	0.1391	0.3323	0.3609	0.2697
	Table 3	0.1385	0.1444	0.2597	0.2304
10	[35]	0.0852	0.2180	0.2286	0.1689
	[36]	0.0875	0.2189	0.2293	0.1702
	Table 3	0.0869	0.0949	0.1577	0.1431
15	[35]	0.0669	0.1710	0.1775	0.1312
	[36]	0.0690	0.1721	0.1781	0.1324
	Table 3	0.0684	0.0764	0.1201	0.1113

6.3. Sensitivity to q, $Q_{\max}$ and $R_{\max}$

Figures 5-7 illustrate the sensitivity of the methods to the maximum order of the zonotope q, to the maximum energy of the process noise $Q_{\max}$, and to the maximum energy of the measurement noises $R_{\max}$. To simplify the plots, it includes the value of the average index $I = (I_1 + I_2 + I_3 + I_4)/4$ for each of the methods.

Figure 5 complements the first conclusion of Table 5. It can be observed that the proposed observer is less sensitive to the variation of the order, and the performance is always better than that of the two other formulations for this particular example.

Figures 6-7 highlight that the proposed observer is less sensitive to an increment of the energy of the process noise (the slope is smaller), but more sensitive to an increment of the energy of the measurement noise. In particular, the method in [35], which is not based on any subspace decomposition is the less sensitive against measurement noise.

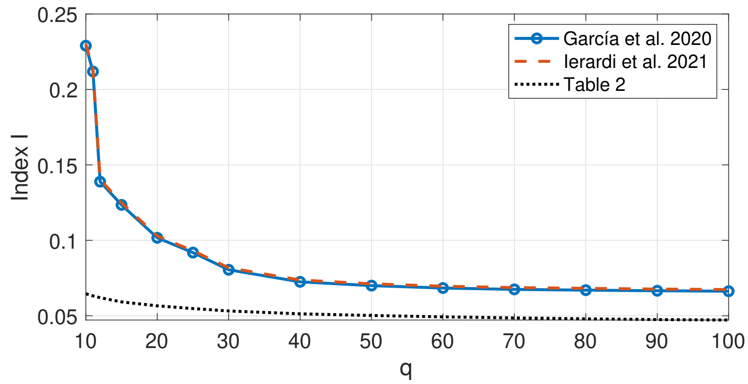

Figure 5: Values of index I for different values of the order and $R_{\max} = Q_{\max} = 0.02$

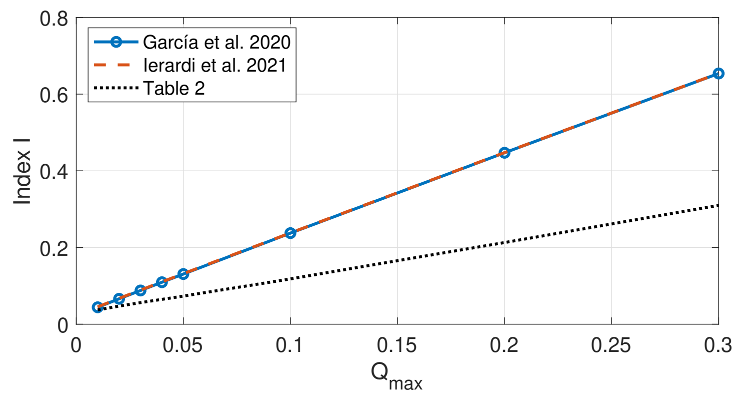

Figure 6: Values of index I for different values of the $Q_{\max}$ and $q = 100, R_{\max} = 0.02$

6.4. Computational requirements

The computational requirements of all the estimators is analysed here, which is related to the complexity of the formulations. Providing a general computational analysis for all the methods is rather complex, since it would depend on the particular graph topology, and system and output matrices. Instead of that, the analysis is conducted for this particular example. Furthermore, and due to fact that the prediction and local filtering steps are equivalent for all the compared observers, only the computational costs of the filtering step with the received neighbouring information is considered. In Table 8, the notation $\mathcal{O}(n, m)$ defines the computational cost of a filtering step with a zonotope of dimension n and an *output* of dimension m. Please

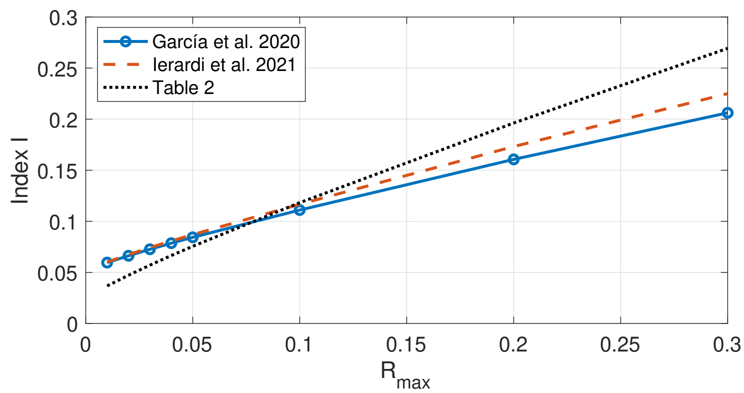

Figure 7: Values of index I for different values of $R_{\max}$ and $q = 100, Q_{\max} = 0.02$

remember that the filtering step is computed as in (13)-(15) for the three algorithms.

Table 8: Computational requirements of the filtering step

Agent	[35]	[36]	Table 3
1	$2 \times \mathcal{O}(4,4)$	$\mathcal{O}(2,1) + 2 \times \mathcal{O}(1,1)$	$\mathcal{O}(2,1) + 2 \times \mathcal{O}(1,1)$
2	$2 \times \mathcal{O}(4,4)$	$2 \times \mathcal{O}(1,1)$	$\mathcal{O}(2,1) + 2 \times \mathcal{O}(1,1)$
3	$\mathcal{O}(4,4)$	$2 \times \mathcal{O}(1,1)$	$\mathcal{O}(2,1) + 2 \times \mathcal{O}(1,1)$
4	$\mathcal{O}(4,4)$	$\mathcal{O}(2,1) + 2 \times \mathcal{O}(1,1)$	$\mathcal{O}(2,1) + 2 \times \mathcal{O}(1,1)$

Due to the multi-hop decomposition, the observer in [36] and the one presented here require a number of filtering step whose joint computational cost is smaller than the one in [35] for all agents. The difference between the algorithm in [36] and the new one appears only in those agents that are able to use redundant information (agents 2 and 3 for this particular example). Those two agents introduce an extra filtering step of cost $\mathcal{O}(2,1)$, and the benefit of doing so is a reduction in the *F-radius* of the sets, as previously observed in Tables 5-7.

6.5. Graphical performance

Finally, Figure 8 plots the performance of the zonotopic observer in a graphical way. In red, the actual value of the transformed states is drawn for agent 4. We can observe, as expected from matrix A, a stable mode, an unstable mode, and a pair of oscillatory modes. In solid blue lines, the extreme bounds for each mode are drawn. Those minimum and maximum bounds can be obtained from the filtered zonotope $\hat{\mathcal{X}}_{4,\rho}(k|k)$ of each hop ρ. Although it is not drawn for the sake of clarity, the center of the filtered set $c_{4,\rho}(k|k)$ is located in the middle of the blue solid band. Tighter bounds are seen for the stable mode, which is the one directly measured by agent 4.

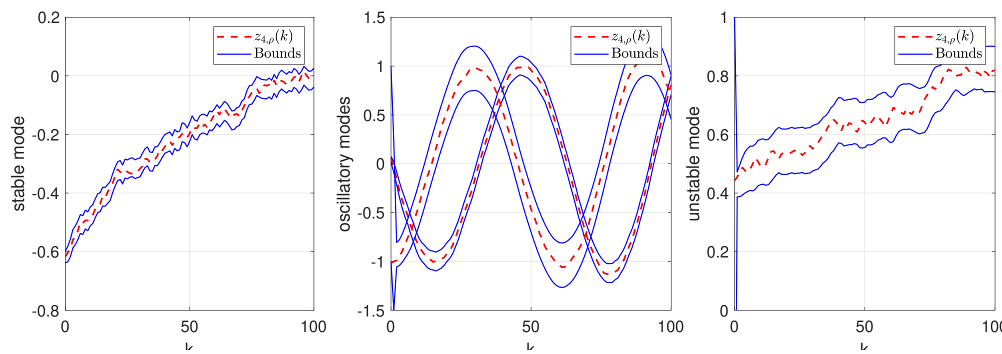

Figure 8: The evolution of the real states in the local coordinates of agent 4 (red dashed line) and the bounds computed with the proposed observer (blue solid lines).

6.6. Discussion

After some numerical, computational, sensitivity, and even graphical analysis carried out in the last subsections, a more general discussion about the performance of the proposed distributed observer is presented here.

First of all, and compared to the observer in [36] that made use of the same multi-hop decomposition, it is shown that the control engineer has to make a decision between computational requirements and estimation performance. The proposed observer requires to make more filtering steps to exploit the redundant information, and the benefits are clear from Table 5-6-7. There is another difference revealed in the sensitivity analysis which is quite unexpected. And it is the fact that the proposed observer is more sensitive to the energy of the measurement noise, than to the energy of the process noise. Therefore, attending to the particular noise specifications of the real problem, the practitioner might choose one approach or the other.

With respect to the method in [35], it has been shown that the proposed observer obtains better performance with less computational effort. This is, clearly, thanks to the multi-hop decomposition. Again, the higher sensitivity to measurement noise is something to be considered.

7. Conclusions

A novel approach for the distributed set-membership estimation problem is reported. By performing sequential filtering steps with the information that travels through the graph, the agents are able to use redundant information, improving the results from previously published observers.

Future work will explore different subspace decomposition, other than the multi-hop one, such as the Jordan form. The application of distributed zonotopic observers for linear time-variant plants will be also explored. Considering observability quality measurements, such as those derived from the gramian, can be exploited to eliminate rows of the innovation routing table that does not introduce useful information. Another line of future research will consist of mixing set-membership observers with probabilistic ones or based on fuzzy sets aiming to reduce the conservatism while keeping robustness.

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References

[1] F. Schweppe, Recursive state estimation: Unknown but bounded errors and system inputs, IEEE Transactions on Automatic Control 13 (1) (1968) 22–28.

[2] S. Chebotarev, D. Efimov, T. Raïssi, A. Zolghadri, Interval observers for continuous-time LPV systems with L1/L2 performance, Automatica 58 (2015) 82–89.

[3] Z. Wang, X. Shen, Y. Zhu, Ellipsoidal fusion estimation for multisensor dynamic systems with bounded noises, IEEE Transactions on Automatic Control 64 (11) (2019) 4725–4732.

[4] D. Qu, Z. Huang, Y. Zhao, G. Song, K. Yi, X. Zhao, Nonlinear state estimation by extended parallelotope set-membership filter, ISA transactions (2021).

[5] V. T. H. Le, C. Stoica, T. Alamo, E. F. Camacho, D. Dumur, Zonotopic guaranteed state estimation for uncertain systems, Automatica 49 (11) (2013) 3418–3424.

[6] B. S. Rego, G. V. Raffo, J. K. Scott, D. M. Raimondo, Guaranteed methods based on constrained zonotopes for set-valued state estimation of nonlinear discrete-time systems, Automatica 111 (2020) 108614.

[7] J. Wan, S. Sharma, R. Sutton, Guaranteed state estimation for nonlinear discrete-time systems via indirectly implemented polytopic set computation, IEEE Transactions on Automatic Control 63 (12) (2018) 4317–4322.

[8] W. Tang, Z. Wang, Y. Wang, T. Raïssi, Y. Shen, Interval estimation methods for discrete-time linear time-invariant systems, IEEE Transactions on Automatic Control 64 (11) (2019) 4717–4724.

[9] J. K. Scott, D. M. Raimondo, G. R. Marseglia, R. D. Braatz, Constrained zonotopes: A new tool for set-based estimation and fault detection, Automatica 69 (2016) 126–136.

[10] X. Ge, Q.-L. Han, X.-M. Zhang, L. Ding, F. Yang, Distributed event-triggered estimation over sensor networks: A survey, IEEE Transactions on Cybernetics 50 (3) (2019) 1306–1320.

[11] C. Ierardi, L. Orihuela, I. Jurado, Distributed Estimation Techniques for Cyber-Physical Systems: A Systematic Review, Sensors 19 (21) (2019) 4720.

[12] L. Zou, Z. Wang, J. Hu, Q.-L. Han, Moving horizon estimation meets multi-sensor information fusion: Development, opportunities and challenges, Information Fusion 60 (2020) 1 – 10.

[13] Y. Ding, Y. Cong, X. Wang, Distributed set-membership filtering frameworks for multi-agent systems with absolute and relative measurements, arXiv preprint arXiv:2305.15797 (2023).

[14] L. P. Rodriguez, J. M. Maestre, E. F. Camacho, M. C. Sánchez, Decentralized ellipsoidal state estimation for linear model predictive control of an irrigation canal, Journal of Hydroinformatics 22 (3) (2020) 593–605.

[15] C. Hu, S. Ding, X. Xie, Event-based distributed set-membership estimation for complex networks under deception attacks, IEEE Transactions on Automation Science and Engineering (Early Access) (2023) 1–10.

[16] C. Combastel, A. Zolghadri, FDI in cyber physical systems: A distributed zonotopic and gaussian Kalman filter with bit-level reduction, IFAC-PapersOnLine 51 (24) (2018) 776–783.

[17] L. Orihuela, P. Millán, S. Roshany-Yamchi, R. A. García, Negotiated distributed estimation with guaranteed performance for bandwidth-limited situations, Automatica 87 (2018) 94–102.

[18] L. Orihuela, S. Roshany-Yamchi, R. García, P. Millán, Distributed set-membership observers for interconnected multi-rate systems, Automatica 85 (2018) 221–226.

[19] Y. Wang, T. Alamo, V. Puig, G. Cembrano, Distributed zonotopic set-membership state estimation based on optimization methods with partial projection, IFAC-PapersOnLine 50 (1) (2017) 4039 – 4044, 20th IFAC World Congress.

[20] Y. Wang, T. Alamo, V. Puig, G. Cembrano, A Distributed Set-membership Approach based on Zonotopes for Interconnected Systems, in: 2018 IEEE Conference on Decision and Control, 2018, pp. 668–673.

[21] W. Chen, Z. Wang, J. Hu, H. Dong, G.-P. Liu, Distributed resilient state estimation for cyber-physical systems against bit errors: A zonotopic set-membership approach, IEEE Transactions on Network Science and Engineering (2023).

[22] Y. Zhang, J. Wang, Z. Li, Interval State Estimation with Uncertainty of Distributed Generation and Line Parameters in Unbalanced Distribution Systems, IEEE Transactions on Power Systems 35 (1) (2020) 762–772.

[23] T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, J.-P. Richard, Interval observer design and control of uncertain non-homogeneous heat equations, Automatica 111 (2020) 108595.

[24] N. Xia, F. Yang, Q. Han, Distributed networked set-membership filtering with ellipsoidal state estimations, Information Sciences 432 (2018) 52–62.

[25] P. Liu, S. Zhou, P. Zhang, M. Li, Distributed state fusion estimation of multi-source localization nonlinear systems, Sensors 23 (2) (2023) 698.

[26] P. Segovia, V. Puig, E. Duviella, Set-membership-based distributed moving horizon estimation of large-scale systems, ISA transactions 128 (2022) 402–413.

[27] S. Liu, Z. Wang, G. Wei, M. Li, Distributed Set-Membership Filtering for Multirate Systems Under the Round-Robin Scheduling Over Sensor Networks, IEEE Transactions on Cybernetics 50 (5) (2020) 1910–1920.

[28] L. Ma, Z. Wang, H. Lam, N. Kyriakoulis, Distributed event-based set-membership filtering for a class of nonlinear systems with sensor saturations over sensor networks, IEEE Transactions on Cybernetics 47 (11) (2016) 3772–3783.

[29] D. Ding, Z. Wang, Q.-L. Han, A set-membership approach to event-triggered filtering for general nonlinear systems over sensor networks, IEEE Transactions on Automatic Control 65 (4) (2019) 1792–1799.

[30] A. Basit, M. Tufail, M. Rehan, H. U. Rashid, A non-uniform event-triggered distributed filtering scheme for discrete-time nonlinear systems over wireless sensor networks, Transactions of the Institute of Measurement and Control 0 (0) (2022).

[31] H. Wu, H. Zhu, X. Li, M. J. V. Amuri, Trust-based distributed set-membership filtering for target tracking under network attacks, IEEE Access 11 (2023) 84468–84474.

[32] X. Li, J. Song, N. Hou, D. Dai, F. Yang, Finite-horizon distributed set-membership filtering with dynamical bias and dos attacks under binary encoding schemes, Information Sciences 641 (2023) 119084.

[33] M. Zamani, J. Trumpf, C. Manzie, Collaborative bearing estimation using set membership methods, arXiv preprint arXiv:2304.07996 (2023).

[34] A. Alanwar, J. Rath, H. Said, M. Althoff, Distributed Set-Based Observers Using Diffusion Strategy, arXiv preprint arXiv:2003.10347 (2020).

[35] R. A. García, L. Orihuela, P. Millán, F. R. Rubio, M. G. Ortega, Guaranteed estimation and distributed control of vehicle formations, International Journal of Control 93 (11) (2020) 2729–2742.

[36] C. Ierardi, L. Orihuela, I. Jurado, A distributed set-membership estimator for linear systems with reduced computational requirements, Automatica 132 (2021) 109802.

[37] C. Combastel, Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence, Automatica 55 (1) (2015) 265–273.

[38] Y. Xie, S. Ding, F. Yang, L. Wang, X. Xie, Probabilistic-constrained distributed set-membership estimation over sensor networks: A dynamic periodic event-triggered approach, IEEE Transactions on Network Science and Engineering 9 (6) (2022) 4444–4457.

[39] Y. Gao, F. Xiao, J. Liu, R. Wang, Distributed soft fault detection for interval type-2 fuzzy-model-based stochastic systems with wireless sensor networks, IEEE Transactions on Industrial Informatics 15 (1) (2018) 334–347.

[40] A. R. del Nozal, P. Millán, L. Orihuela, A. Seuret, L. Zaccarian, Distributed estimation based on multi-hop subspace decomposition, Automatica 99 (2019) 213–220.

[41] H. Montgomery, Computing the volume of a zonotope, American Mathematical Monthly 96 (1989) 431.

[42] T. Alamo, J. M. Bravo, E. F. Camacho, Guaranteed state estimation by zonotopes, Automatica 41 (6) (2005) 1035–1043.

[43] A. K. Kopetzki, B. Schürmann, M. Althoff, Methods for order reduction of zonotopes, in: 2017 IEEE 56th Annual Conference on Decision and Control, IEEE, 2017, pp. 5626–5633.