Characterizing isochronous points and computing isochronous sections.

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We consider two-dimensional autonomous systems of differential equations

$$\dot{x} = -y + \lambda x + P(x, y), \qquad \dot{y} = x + \lambda y + Q(x, y),$$

where λ is a real constant and P and Q are smooth functions of order greater than or equal to two. These systems, so-called centre-focus type systems, have either a centre or a focus at the origin. We characterize the systems with a weak isochronous focus at the origin by means of their radial and azimuthal coefficients. We prove, in this case, the existence of a normalized vector field and an isochronous section which arrives at the origin with defined direction. We also provide algorithms that compute the radial and azimuthal coefficients, terms of normalized vector field and of isochronous section of a system. As applications, we analyze the weak isochronous foci for quadratic systems and for systems with cubic non-linearities, and we give a three-parameter family of Rayleigh equations with four local critical periods.

Centres, foci, isochronous sections, commutators, normalized vector fields.

1. INTRODUCTION AND MAIN RESULTS.

Let us consider the vector field

$$\mathcal{X} = (-y + \lambda x + P)\partial_x + (x + \lambda y + Q)\partial_y, \tag{1.1}$$

where $\lambda \in \mathbf{R}$ and P and Q are smooth functions of order greater than or equal to two. The origin O is an isolated singular point of (1.1). It is said to be a centre of (1.1) if it has a punctured neighbourhood filled with periodic orbits, and it is a focus if there is a neighbourhood in which all the orbits are spirals forward or backward in time.

If $\lambda \neq 0$, O is a strong focus of (1.1). Otherwise, O can be either a centre or a weak focus.

The problem of determining whether or not O is an isochronous centre (all the closed orbits neighbouring O have the same period) has been studied by several authors. However, it is far from being completely solved, even for specific families

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of vector fields (see [4] and the references therein). Algaba et al. [3] proved that if there is an analytic vector field $\mathcal{W}$ with linear part $x\partial_x + y\partial_y$ commuting with the analytic vector field $\mathcal{X}$ for $\lambda = 0$ ($[\mathcal{X}, \mathcal{W}] \equiv 0$), then the origin of (1.1) is an isochronous centre. Sabatini [22] proved the same result assuming that (1.1) has a centre at origin.

We now take the following into consideration in order to understand the concept of isochronous focus. The smooth vector field (1.1) in polar coordinates has the form $\mathcal{X} = f(r,\theta)\partial_x + g(r,\theta)\partial_y$ with $g(r,\theta) = 1 + \frac{1}{r}(cos\theta Q(r\cos\theta, r\sin\theta) - sin\theta P(r\cos\theta, r\sin\theta))$, that is, $g(r,\theta) = 1 + \sum_{i\geq 1} r^i g_i(\theta) + G(r,\theta)$, where G is a smooth function in a neighbourhood of $r = 0$ and flat in $r = 0$. Giné and Grau [15] define O as an isochronous point of $\mathcal{X}$ if it can be transformed by means of an analytic near-identity change of variables into an analytic vector field with $g(r,\theta) = g(\theta)$. This implies that $g_i(\theta) = 0$, for any $i \geq 1$ and for every $\theta \in [0, 2\pi)$. In such a case, the return time of the orbits of $\mathcal{X}$ is constant on every ray of the origin; concretely, it is 2π. Thus, in cartesian coordinates, the vector field will have the form $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a smooth scalar function. This leads us to the following definition, which is less restrictive than that given in [15], since we do not require the analyticity of the change of variables.

Definition 1.1. *The origin of (1.1) is said to be an isochronous point if there exists a smooth near-identity change of variables which transforms $\mathcal{X}$ into $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a smooth scalar function with $H(O) = 0$, i.e. $\mathcal{X}$ is smoothly conjugated to $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$.*

On the one hand, if O is an isochronous centre of an analytic system, the diffeomorphism can be chosen analytic since the normal form of (1.1) verifies the conditions "A" and "ω" (see [7]) and also $H(x, y) = 0$. The vector field (1.1) is analytically linearizable.
On the other hand, if O is a focus of an analytic system, in general, we cannot guarantee the existence of a convergent transformation (see [7, 28]), i.e. it can exist an analytic vector fields whose O is an isochronous focus, according to Definition 1.1, but do not verify the definition provided in [15].

The Poincaré-Dulac normal form for a critical point of a plane vector field is useful to study the problems of centre, isochronous centre and isochronous focus.

Let $\mathcal{H}_i$ be the linear vector space of the homogeneous polynomial vector field of degree i in x and y. The homological operator which determines the normal form of (1.1) is

$$L_i: \mathcal{H}_i \longrightarrow \mathcal{H}_i, \qquad L_i(F_i) = [F_i, (-y + \lambda x)\partial_x + (x + \lambda y)\partial_y],$$

with $[F, G] := DF \cdot G - DG \cdot F$ being the Lie product of the vector fields F and G, see [3].

If $\lambda \neq 0$, $L_i(\mathcal{H}_i) = \mathcal{H}_i$. Thus, the system (1.1) is smoothly linearizable (see [26]); therefore, O is an isochronous focus.
If $\lambda = 0$, it is easy to prove that $\mathcal{C}_i = Ker\, L_i$ is a subspace in $\mathcal{H}_i$ complementary to $\mathcal{R}_i$, the range of the linear operator L_i. Additionally, $\mathcal{C}_{2i} = \{0\}$ and $(x^2 + y^2)^i(x\partial_x + y\partial_y)$, $(x^2 + y^2)^i(-y\partial_x + x\partial_y)$ is a basis for $\mathcal{C}_{2i+1}$, which we will denote by $(1,0)_{\mathcal{C}_{2i+1}}$, $(0,1)_{\mathcal{C}_{2i+1}}$, respectively. Thus, by the classical normal form Theorem (see [13]), (1.1) can be transformed by means of a smooth near-identity change of

variables into

$$(0,1)_{\mathcal{C}_1} + \sum_{i\geq 1}(\alpha_{2i+1}, \beta_{2i+1})_{\mathcal{C}_{2i+1}} + F(x,y), \tag{1.2}$$

where F is a smooth function in a neighbourhood of O and flat at O. The constants α_{2i+1} and β_{2i+1} are called the i-th radial and azimuthal coefficient of (1.1).

It is known that if $\mathcal{X}$ is analytic and their radial coefficients are zero, then O is a centre (in fact, there is a convergent normalizing transformation, see [7]). If α_{2r+1} is the first one non-zero, that is $\alpha_3 = \cdots = \alpha_{2r-1} = 0, \alpha_{2r+1} \neq 0$, O is a weak focus of order r (but, in this case, the existence of a convergent normalizing transformation is not guaranteed).

Next, we show our results. The first related to the problem of the isochronicity of a weak focus of finite order.

THEOREM 1.2. *The following statements are equivalent:*
(i) origin of (1.1) is a weak isochronous focus of order $r \geq 0$,
(ii) $\lambda = 0, \alpha_3 = ... = \alpha_{2r-1} = 0, \alpha_{2r+1} \neq 0, \beta_3 = \beta_5 = ... = \beta_{2r+1} = 0$, where $\alpha_{2i+1}, \beta_{2i+1}$ are the radial and azimuthal coefficients of order i of a Poincaré-Dulac normal form of (1.1), respectively.

Theorem 1.2 is proved in section 3. We emphasize that origin is an isochronous weak focus of finite order if a finite number of azimuthal coefficients of (1.1) are zero. Nevertheless, if O is a centre with all its azimuthal coefficients equal to zero, does not imply that the origin is an isochronous centre, for example

$$\dot{x} = -y - ye^{-1/(x^2+y^2)}, \qquad \dot{y} = x + xe^{-1/(x^2+y^2)}.$$

Analogously, the origin of

$$\dot{x} = -y + (x-y)e^{-1/(x^2+y^2)}, \qquad \dot{y} = x + (x+y)e^{-1/(x^2+y^2)}$$

is a weak focus of infinity order with all its azimuthal coefficients equal to zero and it is a non-isochronous focus.

Now, we provide an algorithm that we have used to calculate the radial and azimuthal coefficients. Thereby, we define $T_{i_1,i_2,\ldots,i_k} \in \mathcal{H}_{i_1+i_2+\cdots+i_k-k+1}$ by

$$T_{i_1} = \mathcal{X}_{i_1},\ i_1 \geq 2,$$
$$T_{i_1,i_2,\ldots,i_k} = [\mathcal{X}_{i_1}, L^{-1}(T_{i_2,\ldots,i_k})],\ \ k\geq 2,\ i_1,\ldots,\ i_k \geq 2.$$

THEOREM 1.3. *Let vector field (1.1) be with $\lambda = 0,\ \alpha_{2i+1} = \beta_{2i+1} = 0,\ i = 1,\ldots,r-1$. The pair $(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}}$ is given by the projection into $\mathcal{C}_{2r+1}$ of*

$$\sum_{k=0}^{2r-1}\Big(\sum_{j_1=1}^{2r-1}\sum_{\substack{j_1+j_2=1\\ j_2\geq 1}}^{2r-1}\cdots\sum_{\substack{j_1+\cdots+j_k=1\\ j_k\geq 1}}^{2r-1}(2r-j_1-\cdots-j_k)T_{j_1+1,j_2+1,\ldots,j_k+1,2r+1-j_1-\cdots-j_k}\Big), \tag{1.3}$$

where k is the number of j_i greater than or equal to one.

Theorem 1.3 is proved in section 3. Let us note that Theorem 1.3 provides closed conditions so that the origin of (1.1) be isochronous centre.

So, for instance, the expression of (1.3) for $r = 1$ is $2T_3 + T_{2,2}$, and for $r = 2$ is

$$4T_5 + 3T_{2,4} + 2T_{3,3} + T_{4,2} + 2T_{2,2,3} + T_{2,3,2} + T_{3,2,2} + T_{2,2,2,2}.$$

In the particular case of $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_m$, these conditions are very simple, since

$$T_{j_1+1,j_2+1,\dots,j_k+1,n-j_1-j_2-\dots-j_k+1} \equiv 0,$$

except for $j_1 + 1 = j_2 + 1 = \dots = j_k + 1 = n - j_1 - \dots - j_k + 1 = m$, that is, $n = (k+1)(m-1)$. The following corollary provides conditions of isochronicity for these vector fields.

Corollary 1.4. *The origin of (1.1) with $\mathcal{X} = -y\partial_x + x\partial_y + \mathcal{X}_m$ is an isochronous centre if and only if the projection into $\mathcal{C}_{(j+1)(m-1)+1}$ of D_j is zero for every j, where $D_j \in \mathcal{H}_{(j+1)(m-1)+1}$ is defined by*

$$D_1 = \mathcal{X}_m,$$
$$D_j = [\mathcal{X}_m, L^{-1}(D_{j-1})], \quad j \geq 2.$$

Finally, if moreover the vector field is uniformly isochronous (i.e. it has constant angular velocity), $\mathcal{X}_m = H_{m-1}(x\partial_x + y\partial_y)$ with H_{m-1} homogeneous polynomial of degree $m-1$ in the variables x and y, it has that $L^{-1}(D_1) = K_{m-1}(x\partial_x + y\partial_y)$, with K_{m-1} homogeneous polynomial of degree $m-1$. In this case

$$D_2 = [H_{m-1}(x\partial_x + y\partial_y), K_{m-1}(x\partial_x + y\partial_y)] \equiv 0,$$

and so $D_j \equiv 0$, for all $j \geq 2$. Therefore, it has the following result.

Corollary 1.5. *Let (1.1) be with $\mathcal{X} = -y\partial_x + x\partial_y + H_{m-1}(x\partial_x + y\partial_y)$. It holds:*
i) *if m is even, the origin is an isochronous centre,*
ii) *if m is odd, the origin is an isochronous centre if and only if $H_{m-1}(x\partial_x + y\partial_y) \in \mathcal{R}_m$.*

A similar result is given in Conti [8].

Next, we introduce the concept of an isochronous section of a monodromic point, which helps us to analyze several geometric aspects of vector field (1.1) whose origin is an isochronous point. For every $(x, y) \in \mathbf{R}^2$, the flow of vector field (1.1) is denoted by $\Phi_{\mathcal{X}}(t; x, y)$.

Definition 1.6. *An isochronous section of (1.1) at O is a smooth curve η, transversal to $\mathcal{X}$, defined in $[0, 1)$, verifying $\eta(0) = O$, $\eta'(0) \in \mathbf{R}^2 \setminus \{(0, 0)\}$ and such that:*
(i) given $s \in (0, 1)$, there exists $\bar{s} \in (0, 1)$ with $\Phi_{\mathcal{X}}(2\pi; \eta(s)) = \eta(\bar{s})$,
(ii) for every $t \in (0, 2\pi)$, $s \in (0, 1)$, it has that $\Phi_{\mathcal{X}}(t; \eta(s)) \notin \{\eta(s), \ s \in (0, 1)\}$.

This definition is more demanding than the one given by Sabatini [24] and used in Giné and Grau [15], since we also impose that the curve arrives at origin with defined direction and the return time is 2π.

The following result relates the concepts of isochronous point, normalized vector field and isochronous sections at origin.

THEOREM 1.7. *The origin is an isochronous point of (1.1) if and only if there exists a smooth vector field* $\mathcal{Y} = x\partial_x + y\partial_y + \mathcal{O}(2)$ *such that* $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{Y}$, *where* μ *is a smooth scalar function with* $\mu(O) = 0$, *i.e.* $\mathcal{Y}$ *is a normalized vector field by* $\mathcal{X}$. *Moreover, every orbit of* $\mathcal{Y}$ *contained in a neighbourhood of* O *is an isochronous section of* (1.1) *at the origin.*

Theorem 1.7 is proved in section 3 (the first part of this theorem is proved in [5]; for completeness, we have also included its demonstration). We emphasize that the vector field $\mathcal{Y}$ is not unique. We give the following relation between two normalized vector fields by $\mathcal{X}$.

COROLLARY 1.8. *The following statements hold:*
(i) If $\mathcal{Y}_1 = x\partial_x + y\partial_y + \mathcal{O}(2)$ *and* $\mathcal{Y}_2 = x\partial_x + y\partial_y + \mathcal{O}(2)$, *are normalized vector fields by* $\mathcal{X}$, *then* $\mathcal{Y}_1 = \alpha\mathcal{Y}_2$, *where* α *is a smooth scalar function with* $\alpha(O) = 1$.
(ii) If $\mathcal{Y}_1 = \alpha\mathcal{Y}_2$ *where* α *is a smooth scalar function with* $\alpha(O) = 1$ *and* $\mathcal{Y}_1$ *is a normalized vector field by* $\mathcal{X}$, *then* $\mathcal{Y}_2$ *is also a normalized vector field by* $\mathcal{X}$.

Corollary 1.8 is proved in section 3. The following result shows that if O is a focus, the study of its isochronicity is reduced to prove the existence of a normalized vector field up to certain order. So, in such a case, the problem is easier than if O were a centre.

THEOREM 1.9. *Let* O *be a weak focus of order* r *of (1.1).* O *is an isochronous focus if and only if there exists a polynomial vector field* $\mathcal{Y}_p$ *of degree* $2r + 1$ *of the form* $\mathcal{Y}_p = x\partial_x + y\partial_y + \mathcal{O}(2)$ *such that* $[\mathcal{X}, \mathcal{Y}_p] = \mu_p\mathcal{Y}_p + \mathcal{O}(2r + 2)$, *where* μ_p *is a polynomial of degree* $2r$ *with* $\mu_p(O) = 0$, *i.e.* $\mathcal{Y}_p$ *is a normalized vector field by* $\mathcal{X}$ *up to order* $2r + 1$.

Theorem 1.9 is proved in section 3. Now, we offer a similar result to the one obtained in [15].

THEOREM 1.10. *The origin is an isochronous point of (1.1) if and only if (1.1) has an infinite number of isochronous sections.*
Moreover, if the origin is a focus, it has that:
(a) the isochronous sections are disjoint two to two and they fill a neighbourhood of the origin,
(b) given a non-zero vector, there is an unique isochronous section which arrives at the origin with that direction.

Theorem 1.10 is proved in section 3.

The remainder of the paper is organized as follows. In the second section, we show several applications. We determine the weak isochronous foci for quadratic systems and systems with cubic non-linearities. We calculate the azimuthal constants of a three-parameter family of Rayleigh equations. That allows us to prove that there are systems of the family whose return time has four local critical points. We show a class of reduced Kukles systems with a isochronous point (finding a normalized vector field) and obtain the first terms of the isochronous section of derivative one at origin. Finally we prove the theorems in the last section.

2. APPLICATIONS.

2.1. Quadratic isochronous points.

The quadratic systems can be transformed by means of a rotation of axis into the form provided by Bautin [6],

$$\begin{aligned} \dot{x} &= -y + \lambda x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= \ \ x + \lambda y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{aligned} \tag{2.4}$$

He found the weak foci and their orders, and their centres.
Later, Loud [19] obtained their isochronous centres. The following result completes the work of Loud, characterizing the isochronous foci of (2.4) and their orders.

THEOREM 2.1. *The origin of system (2.4) is an isochronous focus if and only if one of the following five series of conditions holds:*
i) $\lambda \neq 0$ *(Strong focus).*
ii) $\lambda = 0,\ \lambda_5 \neq 0,\ \lambda_3 \neq \lambda_6,\ (\lambda_5 + 4\lambda_2)^2 = -\lambda_4^2 - 18\lambda_3^2 - 10\lambda_6^2 - 9\lambda_3\lambda_4 + 12\lambda_3\lambda_6 + \lambda_4\lambda_6 \geq 0.$ *(Weak focus of order 1)*
iii) $\lambda = 0,\ \lambda_5 = 0,\ \lambda_3 \neq \lambda_6,\ \lambda_6 \neq 0,\ \lambda_4 = 6(\lambda_6 - \lambda_3),\ \lambda_2 \neq 0,\ \lambda_6(3\lambda_3 - 5\lambda_6) = 2\lambda_2^2$ *(Weak focus of order 2)*
iv) $\lambda = 0,\ \lambda_5 = 0,\ \lambda_3 \neq \lambda_6,\ \lambda_6 \neq 0,\ \lambda_4 \neq 0,\ \lambda_2 \neq 0,\ 14\lambda_4 = 55\lambda_6 - 71\lambda_3 + sgn(\lambda_6)\sqrt{1065\lambda_6^2 - 1650\lambda_3\lambda_6 + 841\lambda_3^2}$, *with* $\lambda_3 > 4\lambda_6$ *if* $\lambda_6 > 0$ *or* $\lambda_3 < 4\lambda_6$ *if* $\lambda_6 < 0$ *(Weak focus of order 2)*
v) $\lambda = 0,\ \lambda_5 = 0,\ \lambda_3 \neq \lambda_6,\ \lambda_6 \neq 0,\ \lambda_4 \neq 0,\ \lambda_2 \neq 0,\ 14\lambda_4 = 55\lambda_6 - 71\lambda_3 - sgn(\lambda_6)\sqrt{1065\lambda_6^2 - 1650\lambda_3\lambda_6 + 841\lambda_3^2}$, *with* $(15 - \sqrt{105})\lambda_6 < 4\lambda_3 < (15 + \sqrt{105})\lambda_6$ *if* $\lambda_6 > 0$, *or* $(15 + \sqrt{105})\lambda_6 < 4\lambda_3 < (15 - \sqrt{105})\lambda_6$ *if* $\lambda_6 < 0$, *(Weak focus of order 2)*
Moreover, there isn't an isochronous focus of order greater than or equal to three.

Theorem 2.1 is proved in section 3. As a consequence, it has that an isochronous focus of system (2.4) with order greater than two is an isochronous centre.

2.2. Isochronous points of systems with non linearities cubic.

The cubic systems centre-focus type without quadratic terms can be written by means of a rotation of axes in the following form given by Sibirskii [25],

$$\begin{aligned} \dot{x} &= -y + \lambda x + (\mu_3 - \mu_1 - \mu_2)x^3 + (3\mu_5 - \mu_4)x^2 y \\ &\qquad + (3\mu_2 - 3\mu_1 - 2\mu_3 + \mu_6)xy^2 + (\mu_7 - \mu_5)y^3, \\ \dot{y} &= \ \ x + \lambda y + (\mu_5 + \mu_7)x^3 + (3\mu_1 + 3\mu_2 + 2\mu_3)x^2 y \\ &\qquad + (\mu_4 - 3\mu_5)xy^2 + (\mu_1 - \mu_2 - \mu_3)y^3. \end{aligned} \tag{2.5}$$

He gave the weak foci and their orders, and their centres.
Later, Pleshkan [21] found the systems (2.5) with an isochronous centre at the origin. We now give the systems whose origin is a weak isochronous focus and their orders.

THEOREM 2.2. *The origin of system (2.5) is an isochronous focus if and only if one of the following five series of conditions holds:*
i) $\lambda \neq 0$ *(Strong focus).*
ii) $\lambda = \mu_4 = 0,\ \mu_6 \neq 0$ *(Weak focus of order 1)*
iii) $\lambda = \mu_4 = \mu_6 = 0,\ (4\mu_1 + \mu_3)(6\mu_1 - \mu_3) = -6(\mu_2^2 + \mu_5^2 + \mu_7^2),\ \mu_3 \neq 0,\ \mu_7 \neq 0,$

(Weak focus of order 2).
iv) $\lambda = \mu_4 = \mu_6 = 0,\ \mu_7 = 0,\ \mu_5 = 0,\ (4\mu_1 + \mu_3)(6\mu_1 - \mu_3) = -6\mu_2^2,\ \mu_3 \neq 0,\ \mu_1 \neq 0,\ \mu_2 \neq 0,$ *(Weak focus of order 3).*
v) $\lambda = \mu_4 = \mu_6 = 0,\ \mu_7 = 0,\ \mu_3 = -24\mu_1,\ \mu_2^2 + \mu_5^2 = 100\mu_1^2,\ \mu_3 \neq 0,\ \mu_1 \neq 0,\ \mu_2 \neq 0,$ *(Weak focus of order 3).*
Moreover, there isn't an isochronous focus of order greater than or equal to four.

Theorem 2.2 is proved in section 3. As a consequence, an isochronous focus of system (2.5) with order greater than three is an isochronous centre.

2.3. A three-parameters family of Rayleigh equations with four local critical periods.

Aside from its interest in physical applications, the study of the period function is essential for approaching some problems of differential equations. So, for instance, the monotonicity of the period function is strictly related to the existence and uniqueness of solutions of some boundary values, bifurcation or perturbation problems.
Our following application shows how the calculation of the azimuthal coefficients of a normal form of the system allows us to solve the problem of determining the number of local critical points of the period function (*local critical periods*) which can appear by perturbation of a system in the neighbourhood of a centre.

Let us consider the family of second order differential equations, so-called Rayleigh equations, $\ddot{x} + h(\dot{x}) + x = 0$, where $h(x) = a_2x^2 + a_4x^4 + a_6x^6$. Each differential equation of the family will be denoted by $\mathrm{R}(a_2, a_4, a_6)$. It is easy to prove that O is a centre of $\mathrm{R}(a_2, a_4, a_6)$, for all a_2, a_4, a_6 real numbers. We have the following result.

THEOREM 2.3. *The following properties hold:*
i) *There are, at the most, four critical periods of the family* $\mathrm{R}(a_2, a_4, a_6)$ *in a neighbourhood of the origin, for* $(a_2, a_4, a_6) \neq (0, 0, 0)$.
ii) *Moreover, given a neighbourhood of the origin, there are values* $a_2, a_4, a_6 \in \mathbf{R}$ *such that* $\mathrm{R}(a_2, a_4, a_6)$ *has exactly four critical periods in a neighbourhood of the origin.*

Theorem 2.3 is proved in section 3. The azimuthal constants have been obtained by using Theorem 1.3.

2.4. An example of an isochronous focus of a cubic Kukles system.

We consider the following family of cubic Kukles systems depending on the parameters a_1 and b_2,

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= \ x + a_1(x^2 + y^2) + a_1^2(x^2 + 5y^2)x + b_2(x^2 + y^2)y, \end{aligned} \tag{2.6}$$

with $a_1 b_2 \neq 0$. The first radial coefficient for this family is $\alpha_3 = b_2$. Hence, when $b_2 > 0$ the origin is an unstable weak focus and when $b_2 < 0$ the origin is a stable weak focus. We have the following result.

THEOREM 2.4. *The family (2.6) has a weak isochronous focus at the origin. The isochronous section of derivative 1 is*

$$\eta(x) = x - (\frac{3}{8}a_1^2 + \frac{1}{4}b_2)x^3 + \mathcal{O}(4).$$

Theorem 2.4 is proved in section 3.

3. PROOFS.

This first result shows two properties of the Lie bracket of two vector fields, which are easily obtained.

LEMMA 3.1. *Let* $\mathcal{H}_i^R = \{P_{i-1}(x\partial_x + y\partial_y) \in \mathcal{H}_i,\ deg(P_{i-1}) = i-1,\ i \geq 1\}$. *The following properties hold:*
(a) if $[-y\partial_x + x\partial_y, U_{i+2}] \in \mathcal{H}_{i+1}^R$ *then* $U_{i+2} \in \mathcal{H}_{i+2}^R$,
(b) if $U_i \in \mathcal{H}_i^R,\ V_j \in \mathcal{H}_j^R$ *then* $[U_i, V_j] \in \mathcal{H}_{i+j-1}^R$.

We now provide some properties of the Poincaré-Dulac normal form of (1.1). We will use lemmas 3.2 and 3.3 in order to prove Theorem 1.2.
In what follows, we denote by ϕ_* and ϕ^* the push-forward and pull-back defined by the smooth diffeomorphism ϕ, respectively (see [20]).

LEMMA 3.2. *Let* $\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + (\alpha_{2r+1}, 0)_{\mathcal{C}_{2r+1}} + F$ *with* $\alpha_{2r+1} \neq 0$ *and* F *a smooth function of order greater than* $2r+1$. *Then,* $\hat{\mathcal{X}}$ *is smoothly conjugated to a smooth vector field of the form* $(0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}}$, *with* $\alpha_{2i+1} \in \mathbf{R}$.

Proof. From (3.7), the Lie transform whose generator is $U = U_{2k+1} = (0, B)_{\mathcal{C}_{2k+1}}$, with $k \geq 1$, transforms $\hat{\mathcal{X}}$ into $\phi_{U^*}\hat{\mathcal{X}}$, which is also in normal form, since $[\hat{\mathcal{X}}_{2i+1}, U_{2k+1}] \in \mathcal{C}_{2k+2i+1}$.
As $U_{2k+1} \in \mathcal{C}_{2k+1} = Ker\, L_{2k+1}$, we have that the degree of $[\hat{\mathcal{X}}, U]$ is greater than or equal to $2k+2r+1$. Therefore the transformed vector field remains unaltered up to order $2k+2r-1$, and the term of order $(2k+2r+1)$ is $\hat{\mathcal{X}}_{2k+2r+1} + [\hat{\mathcal{X}}_{2r+1}, U_{2k+1}]$, that is $(\alpha_{2k+2r+1}, \beta_{2k+2r+1} - 2kB\alpha_{2r+1})_{\mathcal{C}_{2k+2r+1}}$. Thus, taking $B = \frac{\beta_{2k+2r+1}}{2k\alpha_{2r+1}}$, the $(k+r)$-th azimuthal constant of $\phi_{U^*}\hat{\mathcal{X}}$ is annihilated.
Making successive changes of variables over $\hat{\mathcal{X}}$ and by Borel's Theorem (see [18]), we can assert that there exists a smooth diffeomorphism ϕ such that

$$\phi_*\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}} + \hat{F},$$

where $\hat{F}$ is a smooth function in a neighbourhood of the origin and flat in O.
By Tokarev [28], there exists a smooth diffeomorphism ψ such that

$$\psi_*\phi_*\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}} + \bar{f}(x^2+y^2)(1,0)_{\mathcal{C}_1} + \bar{g}(x^2+y^2)(0,1)_{\mathcal{C}_1}$$

where $\bar{f}, \bar{g}$ are smooth functions in a neighbourhood of 0 and flat in 0.
Using polar coordinates, $\psi_*\phi_*\hat{\mathcal{X}}$ has the expression

$$\psi_*\phi_*\hat{\mathcal{X}} = (rS(r^2) + r\bar{f}(r^2))\partial_r + (1 + \bar{g}(r^2))\partial_\theta$$

where $S(r^2) = \sum_{i\geq r}\alpha_{2i+1}r^{2i}$. By Takens [27], there is a smooth change of variables φ of the form $(r,\theta) \to \varphi(r,\theta) = (r + \bar{\varphi}(r^2), \theta)$ such that

$$\varphi_*\psi_*\phi_*\hat{\mathcal{X}} = rS(r^2)\partial_r + (1 + h(r^2))\partial_\theta$$

where h is a smooth function in a neighbourhood of 0 and flat in 0.
Finally, we complete the proof, by performing the smooth change $(r,\theta) \to (r,\theta + G(r^2))$ where G is the smooth function and flat in 0 given by $G(z) = -\int_0^z \frac{h(\bar{z})}{S(\bar{z}^2)} d\bar{z}$.
■

The following lemma provides a normal form of the vector fields with constant angular speed.

LEMMA 3.3. *Let $\mathcal{X} = (0,1)_{\mathcal{C}_1} + H(1,0)_{\mathcal{C}_1}$ with H a smooth scalar function in a neighbourhood of the origin. $\mathcal{X}$ is smoothly conjugated either to $(0,1)_{\mathcal{C}_1}$ or to a smooth vector field of the form $(0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1},0)_{\mathcal{C}_{2i+1}}$ with $\alpha_{2i+1} \in \mathbf{R}$ and $\alpha_{2r+1} \neq 0$.*

Proof. From Lemma 3.1, the generator U which transforms $\mathcal{X}$ into $\hat{\mathcal{X}} = \phi_{U^*}\mathcal{X}$ can be chosen such that $U_i \in \mathcal{H}_i^R$ and $\hat{\mathcal{X}}_i \in \mathcal{H}_i^R$ too.
So, if all $\hat{\mathcal{X}}_i$ are zero, $\mathcal{X}$ is smoothly conjugated to $(0,1)_{\mathcal{C}_1}$. Otherwise, $\mathcal{X}$ is smoothly conjugated to $(0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1},0)_{\mathcal{C}_{2i+1}}$ with $\alpha_{2r+1} \neq 0$. ■

Proof of Theorem 1.2. (i)⇒(ii) If O is a weak isochronous focus of order r, there exists a change of variables ϕ such that $\phi_*\mathcal{X}$ takes the form $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a scalar smooth function. From Lemma 3.3, we have $\lambda = 0, \alpha_3 = ... = \alpha_{2r-1} = 0, \alpha_{2r+1} \neq 0$ and $\beta_{2i+1} = 0$, for all i. In particular, the first r azimuthal coefficients are zero.
(ii)⇒(i) From Lemma 3.2, we have that there exists a normal form of $\mathcal{X}$ whose azimuthal coefficients are zero. That is, O is an isochronous focus of $\mathcal{X}$. ■

To prove Theorem 1.3 we will use the transformation theory based on Lie transforms, which provides an efficient procedure to obtain the expression of the transformed vector field by means of a near-identity change of variables. Basically, this theory consists of performing a change of variables $\phi_U(x,y) = u(x,y,1)$ where u is the unique solution of the initial-value problem

$$\frac{\partial}{\partial\epsilon}u(x,y,\varepsilon) = U(u(x,y,\varepsilon)), \quad u(x,y,0) = (x,y),$$

with U a smooth vector field and $U(O) = O$. The vector field $\mathcal{X}$ is transformed into

$$\mathcal{Y} = \mathcal{X} + [\mathcal{X},U] + \frac{1}{2!}[[\mathcal{X},U],U] + \frac{1}{3!}[[[\mathcal{X},U],U],U] + \cdots, \tag{3.7}$$

see Algaba et al. [2]. The key in this approach is that, if $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \cdots$, $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2 + \cdots$ and $U = U_2 + \cdots$, are their decompositions into homogeneous polynomial in the variables x, y, respectively, for each $k \geq 1$, we can determine $\mathcal{Y}_k$ from $\mathcal{X}_1, ..., \mathcal{X}_k; U_2, ..., U_{k-1}$, by defining the sequence $\{V_{k,l}\}$ by

$$\begin{aligned} V_{k,0} &= k!\mathcal{X}_{k+1}, \; k \geq 0, \\ V_{k,l} &= V_{k,l-1} + \sum_{j=0}^{k-l} \binom{k-l}{j} [V_{k-j-1,l-1}, U_{j+2}], \; 1 \leq l \leq k. \end{aligned}$$

The above recurrent succession of vector fields satisfies $k!\mathcal{Y}_{k+1} = V_{k,k}$, for all $k \geq 0$. The computation of the above succession can be accomplished by means of the Lie triangle:

$$\begin{array}{ccccc} V_{0,0} & & & & \\ V_{1,0} & V_{1,1} & & & \\ V_{2,0} & V_{2,1} & V_{2,2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ V_{k,0} & V_{k,1} & V_{k,2} & \dots & V_{k,k} \end{array}$$

where each row can be computed from the previous rows. It is easy to see that the elements of the $(k+1)$-th row can be written as $V_{k,l} = W_{k,l} - L_{k+1}(U_{k+1})$, where $W_{k,l}$ only depends on U_2, ..., U_k. In particular, $k!\mathcal{Y}_{k+1} = V_{k,k} = W_{k,k} - L_{k+1}(U_{k+1})$. So, we can choose U_{k+1} to obtain a normal form up to order $k+1$, see Algaba et al. [1].

Proof of Theorem 1.3. We are interested in computing α_{2r+1} and β_{2r+1} of vector field (1.1), by assuming that $\lambda = 0$; $\alpha_{2i+1} = \beta_{2i+1} = 0,\ i = 1, ..., r-1$. Thereby, we make the following considerations:

The Lie transform which brings $\mathcal{X}$ to $(0,1)_{\mathcal{C}_1} + \sum_{i \geq r} (\alpha_{2i+1}, \beta_{2i+1})_{\mathcal{C}_{2i+1}}$ verifies:

- the first column of Lie triangle, up to order $2r$, is

$$V_{0,0} = 0!\mathcal{X}_1,\ ...,\ V_{2r-1,0} = (2r-1)!\mathcal{X}_{2r},$$

- from $V_{i,i} = 0,\ 1 \leq i \leq 2r-1$, it has that the i-column, $2 \leq i \leq 2r-1$, up to order $2r$ are zero. Therefore, $V_{i,1} = V_{i,2} = ... = V_{i,i} = 0,\ 1 \leq i \leq 2r-1$.
  In particular, the second column, up to order $2r+1$, is given by

$$\begin{aligned} V_{1,1} &= \mathcal{X}_2 + [\mathcal{X}_1, U_2], \\ V_{2,1} &= 2!\mathcal{X}_3 + [\mathcal{X}_2, U_2] + [\mathcal{X}_1, U_3], \\ V_{3,1} &= 3!\mathcal{X}_4 + 2[2!\mathcal{X}_3, U_2] + 2[\mathcal{X}_2, U_3] + [\mathcal{X}_1, U_4]. \end{aligned}$$

  So, $V_{n,1},\ 1 \leq n \leq 2r$ is

$$V_{n,1} = (n-1)! \left(n\mathcal{X}_{n+1} + \sum_{i=1}^{n} \frac{1}{(i-1)!} [\mathcal{X}_{n+1-i}, U_{i+1}] \right).$$

  From Lie triangle, it follows easily that $V_{2r,2r} = V_{2r,1}$. So, if we split $V_{2r,1} = V_{2r,1}^r + V_{2r,1}^c$, where $V_{2r,1}^r \in \mathcal{R}_{2r+1}$ and $V_{2r,1}^c \in \mathcal{C}_{2r+1}$, we have that

$$(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}} = V_{2r,1}^c.$$

Basically, to obtain the expression (1.3), it is enough to take into account, on the one hand, that for each j, with $2 \leq j \leq 2r$, by means of a recursive procedure we obtain U_j verifying

$$L_j(U_j) = (j-2)! \left((j-1)\mathcal{X}_j + \sum_{i=1}^{j-2} \frac{1}{(i-1)!} [\mathcal{X}_{j-i}, U_{i+1}] \right).$$

on the other hand, that

$$(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}} = Proy_{\mathcal{C}_{2r+1}} \left((2r)\mathcal{X}_{2r+1} + \sum_{i=1}^{2r-1} \frac{1}{(i-1)!} [\mathcal{X}_{2r+1-i}, U_{i+1}] \right). \ \blacksquare$$

Proof of Theorem 1.7. We prove the first part. We assume that O is an isochronous point of (1.1). Then there exists a smooth change of variables ϕ with $D\phi(O) = I$ such that $\phi_*\mathcal{X} = -y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a smooth scalar function. The vector field $x\partial_x + y\partial_y$ verifies $[\phi_*\mathcal{X}, x\partial_x + y\partial_y] = \mu(x\partial_x + y\partial_y)$ with $\mu(x, y) = (xH_x + yH_y)H(x, y)$, where H_x, H_y stand the partial derivatives of H. Therefore, the smooth vector field $\phi^*(x\partial_x + y\partial_y)$ verifies

$$[\mathcal{X}, \phi^*(x\partial_x + y\partial_y)] = \nu\phi^*(x\partial_x + y\partial_y)$$

with $\phi^*(x\partial_x + y\partial_y) = x\partial_x + y\partial_y + \mathcal{O}(2)$ and $\nu(O) = 0$.

Conversely, from the Sternberg's Theorem [26], the vector field $\mathcal{Y}$ is linearizable, then there exists a change of variables $(x, y) \to \psi(x, y) = (x, y) + \mathcal{O}(2)$ such that $\psi_*\mathcal{Y} = x\partial_x + y\partial_y$. Thus, $[\psi_*\mathcal{X}, x\partial_x + y\partial_y] = \sigma(x\partial_x + y\partial_y)$. Therefore, $\psi_*\mathcal{X}$ transforms every ray of the origin $R_\xi = \{(r, \xi),\ \theta = \xi\}$ in other ray of the origin $R_{\bar{\xi}}$, that is $\psi_*\mathcal{X}$ has constant angular speed.
Now we undertake the second part. The change of variables ϕ which brings $\mathcal{X}$ into $\phi_*\mathcal{X} = -y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$, brings $\mathcal{Y}$ into $\phi_*\mathcal{Y} = x\partial_x + y\partial_y + \mathcal{O}(2)$. From Sternberg [26], $\phi_*\mathcal{Y}$ is linearizable. Also it is easy to prove that the change of variables φ which transforms $\phi_*\mathcal{Y}$ into $\varphi_*\phi_*\mathcal{Y} = (1 + \alpha)(x\partial_x + y\partial_y)$, where α is a smooth scalar function with $\alpha(0, 0) = 0$, can be chosen radial. Thus, $\varphi_*\phi_*\mathcal{X} = -y\partial_x + x\partial_y + K(x\partial_x + y\partial_y)$. The trajectories of $\varphi_*\phi_*\mathcal{Y}$ (straight-lines passing by O) are isochronous sections of $\varphi_*\phi_*\mathcal{X}$. So, the orbits of $\mathcal{Y}$ are isochronous sections of vector field (1.1). ■

Proof of Corollary 1.8. (i) From Theorem 1.7, the orbits of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are isochronous sections of $\mathcal{X}$ which fill a neighbourhood of the origin. Therefore $\mathcal{Y}_1$ and $\mathcal{Y}_2$ have the same orbital structure at O, thus $\mathcal{Y}_1 = \alpha\mathcal{Y}_2$, where α is a smooth scalar function with $\alpha(0, 0) = 1$,
(ii) From Theorem 1.7, the statement easily follows since under such conditions the orbits of $\mathcal{Y}_2$ also are isochronous sections of $\mathcal{X}$. ■

Proof of Theorem 1.9. The sufficient condition is deduced from Theorem 1.7.
To prove the necessary condition, we impose the existence of $\mathcal{Y}_p$ a smooth normalized vector field by $\mathcal{X}$ up to order $2r + 1$, i.e. $\mathcal{J}^{2r+1}[\mathcal{X}, \mathcal{Y}_p] = \mu_p\mathcal{Y}_p$, where $\mathcal{J}^k f$ denotes the k-jet of f at origin.

As the origin of vector field (1.1) is a weak focus of order r, there exists a smooth near-identity change ϕ such that $\phi_*\mathcal{X}$ takes the form $\phi_*\mathcal{X} = \hat{\mathcal{X}} = (0, 1)_{\mathcal{C}_1} + \sum_{i\geq 1}(\alpha_{2i+1}, \beta_{2i+1})_{\mathcal{C}_{2i+1}}$ with $\alpha_3 = \alpha_5 = ... = \alpha_{2r-1} = 0,\ \alpha_{2r+1} \neq 0$, and $\phi_*\mathcal{Y}_p = \hat{\mathcal{W}} = (1, 0)_{\mathcal{C}_1} + \cdots$. Moreover, it has that $\mathcal{J}^{2r+1}[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = \hat{\mu}\hat{\mathcal{W}}$, with $\hat{\mu}(O) = 0$. Writing $\hat{\mathcal{X}} = \hat{\mathcal{X}}_1 + \hat{\mathcal{X}}_2 + ...$ and $\hat{\mathcal{W}} = \hat{\mathcal{W}}_1 + \hat{\mathcal{W}}_2 + ...$ with $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{W}}_i$ homogeneous polynomial vector fields of order $i \geq 1$ and $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2 + ...$, with $\hat{\mu}_i$ homogeneous polynomial of order $i \geq 1$, we have
1. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_1] \equiv 0$,
2. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_2] = \hat{\mu}_1\hat{\mathcal{W}}_1$. From Lemma 3.1, this leads to $\hat{\mathcal{W}}_2 \in \mathcal{H}_2^R$.
In a similar way, for order $2i$ we have $\hat{\mathcal{W}}_{2i} \in \mathcal{H}_{2i}^R$, for $i \leq r$.
3. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_3] + [\hat{\mathcal{X}}_3, \hat{\mathcal{W}}_1] = \hat{\mu}_1\hat{\mathcal{W}}_2 + \hat{\mu}_2\hat{\mathcal{W}}_1$. Then

$$L_3(\hat{\mathcal{W}}_3) = 2\beta_3(x^2 + y^2)(0, 1)_{\mathcal{C}_1} - \hat{\mu}_1\hat{\mathcal{W}}_2 - \hat{\mu}_2\hat{\mathcal{W}}_1.$$

Projecting the above equality onto the range of L_3 and onto $\mathcal{C}_3$, we deduce that $\beta_3 = 0$, and hence $\hat{\mathcal{W}}_3 \in \mathcal{H}_3^R$.

4. Analogously, taking into account the $(2i+1)$-th order terms of $[\hat{\mathcal{X}}, \hat{\mathcal{W}}]$, we have that $\beta_{2i+1} = 0,\ \hat{\mathcal{W}}_{2i+1} \in \mathcal{H}^R_{2i+1}$, for $i \leq r-1$.
5. For order $2r+1$, we get

$$[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2r+1}] + [\hat{\mathcal{X}}_{2r+1}, \hat{\mathcal{W}}_1] = \sum_{j=1}^{2r} \hat{\mu}_j \hat{\mathcal{W}}_{2r+1-j},$$

that is, $L_{2r+1}(\hat{\mathcal{W}}_{2r+1}) = 2r(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}} + \sum_{j=1}^{2r} \hat{\mu}_j \hat{\mathcal{W}}_{2r+1-j}$. Then $\beta_{2r+1} = 0$. Thus, from Theorem 1.2, the origin is a weak isochronous focus. ■

Proof of Theorem 1.10. By definition, there is a diffeomorphism ϕ such that $\phi_*\mathcal{X}$ has constant angular velocity. The ray $R_\xi = \{(r,\xi),\ \theta = \xi\}$ is an isochronous section of $\phi_*\mathcal{X}$. Therefore, the curve counter-images of the ray, $\phi^{-1}(R_\xi)$ is an isochronous section of (1.1), for every $\xi \in [0, 2\pi]$.
These curves are transversal to the vector field and their derivatives are equal to the derivatives of the rays. Moreover, in the case of a focus, these curves are disjoint, since, by Theorem 1.7, they are different trajectories of a smooth vector field.
Now, we prove the sufficient condition. Let η be a isochronous section, verifying $\eta'(0) = (1,0)$.
The curve $\eta_\xi :\ [0,1) \to \mathbf{R}^2$ defined by $\eta_\xi(s) = \Phi_{\mathcal{X}}(\xi; \eta(s))$ is a transversal isochronous section of (1.1) at O, for every $\xi \in [0, 2\pi)$.
We define the function $\Psi(\xi, s) = \Phi_{\mathcal{X}}(\xi; \eta(s)),\ \xi \in [0, 2\pi),\ s \in (0,1)$. We have

$$\begin{aligned} &\tfrac{\partial}{\partial \xi}\Psi(\xi,s) = \tfrac{\partial}{\partial \xi}\Phi_{\mathcal{X}}(\xi;\eta(s)) = \mathcal{X}(\Phi_{\mathcal{X}}(\xi;\eta(s)) = \mathcal{X}(\eta_\xi(s)), \\ &\tfrac{\partial}{\partial s}\Psi(\xi,s) = \tfrac{\partial}{\partial s}\Phi_{\mathcal{X}}(\xi;\eta(s)) = \eta'_\xi(s).\eta'(s). \end{aligned}$$

Thus, $|D\Psi(t,s)| = \mathcal{X}(\eta_\xi(s)) \wedge \eta'_\xi(s) \neq 0$, by transversality. Therefore Ψ is a diffeomorphism. Also, $\Psi_*\mathcal{X}$ verifies

$$\Phi_{\Psi_*\mathcal{X}}(\xi; R_t) = R_{\xi+t}.$$

So, $\Psi_*\mathcal{X}$ is an uniformly isochronous focus, therefore $\mathcal{X}$ has an isochronous focus at O. ■

Proof of Theorem 2.1. If $\lambda \neq 0$, then (2.4) is a strong focus which is smoothly linearizable, see Sternberg [26]; therefore, O is an isochronous focus, (case **i**)).
The first radial coefficient is $\alpha_3 = -\lambda_5(\lambda_3 - \lambda_6)$. So, the systems (2.4) whose origin is an isochronous focus of order 1, must hold $\lambda = 0,\ \lambda_3 \neq \lambda_6, \lambda_5 \neq 0$. Under these conditions, it has $\beta_3 = \frac{1}{24}(r - (\lambda_5 + 4\lambda_2)^2)$, with

$$r := r(\lambda_3, \lambda_4, \lambda_6) = -\lambda_4^2 - 18\lambda_3^2 - 10\lambda_6^2 - 9\lambda_3\lambda_4 + 12\lambda_3\lambda_6 + \lambda_4\lambda_6.$$

Therefore $\beta_3 = 0$ if and only if $r = (\lambda_5 + 4\lambda_2)^2$ (case **ii**)).
We now obtain systems (2.4) whose origin is a weak isochronous focus of order 2. By imposing $\alpha_3 = 0$, it has that

$$\alpha_5 = \lambda_2\lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5\lambda_3 - 5\lambda_6).$$

That is, from Theorem 1.2, such systems must verify

$$\lambda = \lambda_5 = 0,\ \lambda_2 \neq 0, 0 \neq \lambda_4 \neq 5(\lambda_6 - \lambda_3) \neq 0,$$

and $\beta_3 = \beta_5 = 0$. In this case $r = 16\lambda_2^2 > 0$ and

$$\beta_5 = \frac{1}{768}(\lambda_3 - \lambda_6)(\lambda_4 + 6\lambda_3 - 6\lambda_6)\Sigma(\lambda_3, \lambda_4, \lambda_6)$$

where $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 70\lambda_6^2 - 220\lambda_6\lambda_3 - 55\lambda_6\lambda_4 + 150\lambda_3^2 + 71\lambda_3\lambda_4 + 7\lambda_4^2$.
If λ_6 were zero, we would have

$$r = -(3\lambda_3 + \lambda_4)(6\lambda_3 + \lambda_4), \quad \beta_5 = -\frac{1}{768}\lambda_3(50\lambda_3 + 7\lambda_4)r.$$

As $\lambda_3 \neq 0$ and $r > 0$, $\beta_5 = 0$ only if $\lambda_4 = -50/7\lambda_3$; in this case, r is negative. Therefore, without loss of generality, we can assume that $\lambda_6 \neq 0$.
The azimuthal coefficient of order 2, β_5, is zero if and only if either $\lambda_4 = 6(\lambda_6 - \lambda_3)$ or $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 0$.
In the first case, that is $\lambda_4 = 6(\lambda_6 - \lambda_3)$, it has $r = 8\lambda_6(3\lambda_3 - 5\lambda_6)$, therefore if also $\lambda_6(3\lambda_3 - 5\lambda_6) = 2\lambda_2^2$, the origin is a weak isochronous focus of order 2 (case **iii**)).
In the second case, $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 0$, O is a weak isochronous focus of order 2 if and only if λ_3, λ_4 and λ_6 also verify $r(\lambda_3, \lambda_4, \lambda_6) > 0$. Making $\bar{\lambda}_3 = \frac{\lambda_3}{\lambda_6}$, $\bar{\lambda}_4 = \frac{\lambda_4}{\lambda_6}$ we have

$$\Sigma(\lambda_3, \lambda_4, \lambda_6) = \lambda_6^2\bar{\Sigma}(\bar{\lambda}_3, \bar{\lambda}_4), \qquad r(\lambda_3, \lambda_4, \lambda_6) = \lambda_6^2\bar{r}(\bar{\lambda}_3, \bar{\lambda}_4),$$

therefore O will be isochronous focus on the region where $\bar{\lambda}_3$ and $\bar{\lambda}_4$ verify $\bar{\Sigma}(\bar{\lambda}_3, \bar{\lambda}_4) = 0$ and $\bar{r}(\bar{\lambda}_3, \bar{\lambda}_4) > 0$.
It is easy to prove that the hyperbolas $\bar{\Sigma}(\bar{\lambda}_3, \bar{\lambda}_4) = 0$ and $\bar{r}(\bar{\lambda}_3, \bar{\lambda}_4) = 0$ meet at the points $P_0(4, -10)$, $P_1(\frac{1}{4}(15+\sqrt{105}), -\frac{1}{4}(85+7\sqrt{105}))$ and $P_2(\frac{1}{4}(15-\sqrt{105}), -\frac{1}{4}(85-7\sqrt{105}))$. The point P_0 is on the branch of $\bar{\Sigma}$

$$14\bar{\lambda}_4 = 55 - 71\bar{\lambda}_3 + sgn(\lambda_6)\sqrt{1065 - 1650\bar{\lambda}_3 + 841\bar{\lambda}_3^2}$$

and P_1, P_2 are on the branch

$$14\bar{\lambda}_4 = 55 - 71\bar{\lambda}_3 - sgn(\lambda_6)\sqrt{1065 - 1650\bar{\lambda}_3 + 841\bar{\lambda}_3^2}.$$

Studying the relative position of both conics, it leads to the cases **iv**) and **v**).
Finally, we compute the systems (2.4) whose origin is a weak isochronous focus of order 3. It has that

$$v_7 = -\lambda_2\lambda_4(\lambda_3 - \lambda_6)^2(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2).$$

So, it arrives at

$$\lambda = \lambda_5 = 0, \lambda_4 = 5(\lambda_6 - \lambda_3) \neq 0, \lambda_2^2 \neq \lambda_6(\lambda_3 - 2\lambda_6).$$

Under these conditions, also it must hold $\beta_3 = \beta_5 = \beta_7 = 0$. In this case, β_3 is zero if and only if $8\lambda_2^2 = \lambda_3^2 + 6\lambda_6\lambda_3 - 15\lambda_6^2$. Substituting λ_2^2 in β_5 it has $\beta_5 = -\frac{5}{128}(\lambda_3 - \lambda_6)^4$, which is nonzero. Therefore, O cannot be a weak isochronous focus of order 3. ∎

Proof of Theorem 2.2. If $\lambda \neq 0$ the system (2.5) is a strong focus which is smoothly linearizable (see [26]); therefore, O is an isochronous focus, (case **i**)).

Now, we assume that $\lambda = 0$. The first radial constants, up to a positive factor, are

$$\begin{array}{l} \alpha_3 = \mu_6, \\ \alpha_5 = -\mu_3\mu_7, \text{ if } \alpha_3 = 0, \\ \alpha_7 = \mu_3\mu_2\mu_1, \text{ if } \alpha_3 = \alpha_5 = 0, \\ \alpha_9 = \mu_3^2\mu_2\mu_4, \text{ if } \alpha_3 = \alpha_5 = \alpha_7 = 0, \\ \alpha_{11} = -\mu_3^2\mu_2[4(\mu_2^2 + \mu_5^2) - \mu_3^2], \text{ if } \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = 0, \\ \alpha_{2k+1} = 0, \ k \geq 6, \text{ if } \alpha_{2i+1} = 0, \ i = 1, 2, 3, 4, 5. \end{array}$$

The first azimuthal constant is, up to a positive factor, $\beta_3 = \mu_4$. Therefore, from Theorem 1.2, O is a weak isochronous focus of order 1 if and only if $\mu_4 = 0, \ \mu_6 \neq 0$ (case **ii**)).
The system (2.5) has a weak focus of order 2 at the origin if $\alpha_3 = 0$ and $\alpha_5 \neq 0$, i.e. $\mu_6 = 0, \mu_7 \neq 0, \ \mu_3 \neq 0$, and it will be isochronous if also $\beta_3 = \beta_5 = 0$, that is $\mu_4 = 0$ and in this case,

$$\beta_5 = -\frac{1}{8}[6(\mu_2^2 + \mu_5^2 + \mu_7^2) + (4\mu_1 + \mu_3)(6\mu_1 - \mu_3)]$$

is zero (case **iii**)).
From the expression for α_5 and α_7 we have that O is focus of order 3 if and only if $\mu_6 = \mu_7 = 0, \ \mu_1 \neq 0, \ \mu_2 \neq 0$ and $\mu_3 \neq 0$. In this case, O is an isochronous focus if $\beta_3 = 0$ (i.e. $\mu_4 = 0$) and

$$\begin{array}{l} \beta_5 = -\frac{1}{8}[6(\mu_2^2 + \mu_5^2) + (4\mu_1 + \mu_3)(6\mu_1 - \mu_3)], \\ \beta_7 = -\frac{5}{16}\mu_1\mu_5(\mu_3 + 24\mu_1) \end{array}$$

are zero.
If $\mu_5 = 0$, then $(4\mu_1 + \mu_3)(6\mu_1 - \mu_3) = -6\mu_2^2$ holds (case **iv**)), and if $\mu_3 = -24\mu_1$ we have that $\mu_2^2 + \mu_5^2 = 100\mu_1^2$ (case **v**)).
The origin is a weak focus of order 4 if it verifies $\alpha_3 = \alpha_5 = \alpha_7 = 0$ and $\alpha_9 \neq 0$, that is $\mu_6 = \mu_7 = \mu_1 = 0, \ \mu_4 \neq 0, \ \mu_2 \neq 0$ and $\mu_3 \neq 0$. So, O cannot be isochronous since $\beta_3 = 0$ only if $\mu_4 = 0$.
Finally, we compute the systems whose origin is a weak isochronous focus of order five. They must verify

$$\mu_6 = \mu_7 = \mu_1 = \mu_4 = 0, \ \mu_2 \neq 0, \ \mu_3 \neq 0, \ \mu_3^2 \neq 4(\mu_2^2 + \mu_5^2),$$

and $\mu_4 = 0$ ($\beta_3 = 0$); $\mu_3^2 = 6(\mu_2^2 + \mu_5^2)$ ($\beta_5 = 0$). This contradicts that $\mu_3 \neq 0$. Therefore, there is not a weak isochronous focus of order 5. ■

Proof of Theorem 2.3. A plane differential system associated to the Rayleigh equations $\mathrm{R}(a_2, a_4, a_6)$ is the Liénard system

$$\dot{x} = -y, \qquad \dot{y} = x + a_2y^2 + a_4y^4 + a_6y^6. \tag{3.8}$$

An analysis on the monotonicity of the period function of a Liénard system can be seen in [23].
This system is time-reversible, therefore O is a centre.
If we denote by $r(\theta, \rho)$ the periodic solution, expressed in polar coordinates, such that $r(0, \rho) = \rho$, where ρ is small enough, the period $T(\rho)$ of this periodic orbit is an analytic function of the form $T(\rho) = 2\pi + \sum_{n=1}^{\infty} T_n\rho^{2n}$, see [11, 3]. The constants T_n are called period constants of system (3.8).

Algaba et al. [3], provide a relation between the period constants and the azimuthal coefficients of a normal form of the system. Concretely,

$$T_n = 2\pi \sum_{\substack{n_1+\cdots+n_l=2n \\ n_i \text{ even, } l\geq 1}} (-1)^l \beta_{n_1+1}\cdots\beta_{n_l+1}.$$

Thus, $T_n = -2\pi\beta_{2n+1}$ if the above period constants, $T_i,\ i = 1...n-1$, are zero. The first azimuthal coefficients of (3.8) are

$$\begin{aligned}
\beta_3 = &\ -\tfrac{1}{6}a_2^2 \\
\beta_5 = &\ -\tfrac{13}{144}a_2^4 - \tfrac{1}{3}a_2a_4 \\
\beta_7 = &\ -\tfrac{6937}{77760}a_2^6 - \tfrac{2711}{4320}a_4a_2^3 - \tfrac{5}{16}a_6a_2 - \tfrac{7}{40}a_4^2 \\
\beta_9 = &\ -\tfrac{71053}{746496}a_2^8 - \tfrac{154129}{155520}a_4a_2^5 - \tfrac{239}{270}a_6a_2^3 - \tfrac{1399}{960}a_4^2a_2^2 - \tfrac{27}{80}a_4a_6 \\
\beta_{11} = &\ -\tfrac{106311847}{895795200}a_2^{10} - \tfrac{60925411}{38707200}a_4a_2^7 - \tfrac{212609}{161280}a_2^5a_6 - \tfrac{25518229}{5806080}a_4^2a_2^4 \\
&\ -\tfrac{3825733}{967680}a_6a_4a_2^2 - \tfrac{297}{1792}a_6^2 - \tfrac{111769}{86400}a_4^3a_2.
\end{aligned}$$

If $(a_2, a_4, a_6) = (0, 0, 0)$, is an isochronous centre. Otherwise, by applying the Malgrange Theorem, to the equation $T'(\rho) = 0$, for $0 < \rho << 1$, we arrive at that 4 local critical periods, at the most, can bifurcate from the origin in the family $\mathrm{R}(a_2, a_4, a_6)$.
We prove the second part. The system $\mathrm{R}(0, 0, 1)$ verifies $T_1 = T_2 = T_3 = T_4 = 0$ and $T_5 \neq 0$. If we take the following perturbation of $\mathrm{R}(0, 0, 1)$,

$$a_2 = \epsilon^{2n},\ a_4 = -\epsilon^n,\ a_6 = 1,\ n \text{ sufficiently big},$$

it has that

$$\begin{aligned}
&\beta_3 = -\tfrac{1}{6}\epsilon^{4n}, \\
&\beta_5 = \tfrac{48}{144}\epsilon^{3n} + o(\epsilon^{8n}), \\
&\beta_7 = -(\tfrac{7}{40} + \tfrac{5}{16})\epsilon^{2n} + o(\epsilon^{7n}), \\
&\beta_9 = \tfrac{27}{80}\epsilon^n + o(\epsilon^{6n}), \\
&\beta_{11} = -\tfrac{297}{1792} + o(\epsilon^{5n}),
\end{aligned}$$

Taking n sufficiently big, it has that $0 < |T_1| << |T_2| << |T_3| << |T_4| << |T_5|$, and the period constants alternate sign, then they form a Sturm sequence. Therefore, we can assert that 4 critical points of the period function bifurcate from the origin. ∎

Proof of Theorem 2.4. The origin of (2.6) is a weak focus of order 1. The field $\mathcal{Y}_p = (x + R)\partial_x + (y + S)\partial_y$ where

$$\begin{aligned}
R = &\ -a_1(x^2 + y^2) - \tfrac{1}{4}a_1^2x(5x^2 + y^2) - \tfrac{1}{2}b_2y(y^2 + x^2) \\
S = &\ -\tfrac{1}{4}a_1^2y(5x^2 - 7y^2) - \tfrac{1}{2}b_2x(x^2 + y^2),
\end{aligned}$$

is a normalized vector field up to order 3, i.e. $[\mathcal{X}, \mathcal{Y}_p] = \mu\mathcal{Y}_p + \mathcal{O}(4)$ with $\mu = b_2(x^2 + y^2)$. From Theorem 1.9, the origin of (2.6) is a weak isochronous focus of order 1.
The expression of the isochronous section is obtained from (3.13), see appendix. ∎

APPENDIX: ISOCHRONOUS SECTIONS AT THE ORIGIN FROM A NORMALIZED VECTOR FIELD.

We assume that O is an isochronous focus of (1.1) and we know the expression of a normalized vector field up to finite order, $(x+R)\partial_x + (y+S)\partial_y$ with S of order greater than or equal to two.

Our objective is to provide a recursive algorithm to allow us to compute the coefficients of Taylor expansion at the origin of the isochronous section η whose derivative at the origin is known. We will denote $(t, \eta(t))$, $t \in [0,1)$, a parametrization of this isochronous section, i.e. $y = \eta(x)$.

LEMMA 3.4. *Let η be isochronous section at the origin with $\eta'(0) = A_1$. If $R(x,\eta(x)) = \sum_{j\geq 2} r_j x^j$ and $S(x,\eta(x)) = \sum_{j\geq 2} s_j x^j$, it has that $\eta(x) = A_1 x + A_2 x^2 + A_3 x^3 + \cdots$ where*

$$\begin{aligned} &A_2 = s_2, \\ &A_j = \tfrac{1}{j-1}\big(s_j - \textstyle\sum_{k=1}^{j-2}(k+1)A_{k+1}r_{j-k}\big),\ j \geq 3. \end{aligned} \tag{3.9}$$

Proof of Lemma 3.4. The curve $y = \eta(x) = A_1 x + A_2 x^2 + A_3 x^3 + \cdots$ is the trajectory of $(x+R)\partial_x + (y+S)\partial_y$ with $\eta'(0) = A_1$. So, $\dot{y} = \eta'(x)\dot{x}$, that is

$$\eta(x) + S(x,\eta(x)) = \eta'(x)(x + R(x,\eta(x))).$$

Therefore, it has that

$$\begin{aligned} \textstyle\sum_{j\geq 1} A_j x^j + \sum_{j\geq 2} s_j x^j &= \textstyle\sum_{j\geq 1} jA_j x^j + \big(\sum_{j\geq 0}(j+1)A_{j+1}x^j\big)\big(\sum_{j\geq 2} r_j x^j\big) \\ &= \textstyle\sum_{j\geq 1} jA_j x^j + \sum_{j\geq 2}\big(\sum_{k=1}^{j-2}(k+1)A_{k+1}r_{j-k}\big)x^j. \ \blacksquare \end{aligned}$$

We now offer a formula to obtain the coefficients r_m and s_m from the expressions of R and S and $A_1, A_2, ..., A_{m-1}$.

LEMMA 3.5. *Let η isochronous section at the origin be with $\eta'(0) = A_1 \neq 0$. If $R(x,y) = c_{20}x^2 + c_{11}xy + c_{02}y^2 + \cdots$, $S(x,y) = d_{20}x^2 + d_{11}xy + d_{02}y^2 + \cdots$ and $\eta(x) = A_1 x + A_2 x^2 + A_3 x^3 + \cdots$, it has that $R(x,\eta(x)) = \sum_{j\geq 2} r_j x^j$ and $S(x,\eta(x)) = \sum_{j\geq 2} s_j x^j$ with*

$$r_m = c_{m0} + \sum_{k=1}^{m-1} c_{k1}A_{m-k} + \sum_{j=2}^{m}\sum_{k=0}^{m-j} c_{kj}C^{(j)}_{m-k-j+1}, \quad m \geq 2. \tag{3.10}$$

$$s_m = d_{m0} + \sum_{k=1}^{m-1} d_{k1}A_{m-k} + \sum_{j=2}^{m}\sum_{k=0}^{m-j} d_{kj}C^{(j)}_{m-k-j+1}, \quad m \geq 2. \tag{3.11}$$

where the constants $C^{(j)}_m$ are given by

$$\begin{aligned} &C^{(j)}_1 = A_1^j, \\ &C^{(j)}_m = \tfrac{1}{(m-1)A_1}\textstyle\sum_{k=1}^{m-1}(kj - m + k + 1)A_{k+1}C^{(j)}_{m-k}, \quad m \geq 2. \end{aligned} \tag{3.12}$$

To prove Lemma 3.5, we use the following result, which can be seen in [16].

LEMMA 3.6. *If $(\sum_{m=1} A_m x^m)^j = x^j(\sum_{m=1} C^{(j)}_m x^{m-1})$, $A_1 \neq 0$, $j \geq 2$, it has that $C^{(j)}_m$ satisfy (3.12).*

Proof of Lemma 3.5. Rewriting $R(x,\eta(x))$ of the form

$$\begin{aligned}R(x,\eta(x)) &= c_{20}x^2 + c_{11}x\eta(x) + c_{02}\eta(x)^2 + c_{30}x^3 + c_{21}x^2\eta(x) + \cdots\\ &= c_{20}x^2 + c_{30}x^3 + c_{40}x^4 + \cdots\\ &\quad + c_{11}x\eta(x) + c_{21}x^2\eta(x) + c_{31}x^3\eta(x) + \cdots\\ &\quad + c_{02}\eta(x)^2 + c_{12}x\eta(x)^2 + c_{22}x^2\eta(x)^2 + \cdots\\ &= c_{20}x^2 + c_{30}x^3 + c_{40}x^4 + \cdots\\ &\quad + (c_{11}x + c_{21}x^2 + c_{31}x^3 + \cdots)\eta(x)\\ &\quad + (c_{02} + c_{12}x + c_{22}x^2 + \cdots)\eta(x)^2 + \cdots\\ &\quad + (c_{0n} + c_{1n}x + c_{2n}x^2 + \cdots)\eta(x)^n + \cdots,\end{aligned}$$

and applying the above lemma, we arrive at (3.10). Analogously it has (3.11). ∎

To compute the isochronous section with derivative A_1 at the origin, we use the following recursive procedure:
Step 1. Given A_1, from (3.12), we calculate $C_1^{(2)}$ and from (3.10) and (3.11), we compute r_2, s_2; and from (3.9) we obtain A_2.
Step 2. In general, given $A_1, A_2, ..., A_k, C_m^{(n)}$ with $m \leq k-1,\ 2 \leq n \leq k$, and $(r_2, s_2), ..., (r_k, s_k)$, from (3.12) we compute $C_m^{(k+1)},\ m = 1, \cdots, k$ and $C_k^{(n)},\ n = 2, \cdots, k$. From (3.10) and (3.11), we calculate r_{k+1}, s_{k+1}; and from (3.9) we obtain A_{k+1}.
The coefficients A_2 and A_3 are:

$$\begin{aligned}A_2 = &\ d_{20} + d_{11}A_1 + d_{02}A_1^2,\\ A_3 = &\ \tfrac{1}{2}c_{30} + \tfrac{1}{2}d_{11}d_{20} - d_{20}c_{20}\\ &+ (\tfrac{1}{2}d_{21} + \tfrac{1}{2}d_{11}^2 + d_{02}d_{20} - d_{20}c_{11} - d_{11}c_{20})A_1\\ &+ (\tfrac{3}{2}d_{11}d_{02} + \tfrac{1}{2}d_{12} - d_{20}c_{02} - d_{11}c_{11} - d_{02}c_{20})A_1^2\\ &+ (d_{02}^2 + \tfrac{1}{2}d_{03} - d_{11}c_{02} - d_{02}c_{11})A_1^3\\ &- d_{02}c_{02}A_1^4.\end{aligned} \tag{3.13}$$

Recently, Guillamón and Huguet [17] have computed the isochronous sections near a limit cycle.

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