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Teaching Thales's theorem: relations between suitable mathematical working spaces and specialised knowledge

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Abstract
In this article, we analyse a lesson on Thales's theorem in a Chilean secondary school classroom through the combination of two theories: Mathematics Teachers' Specialised Knowledge (ThMTSK) and Mathematical Working Spaces (ThMWS). Both theories, first separately and then in relation to one another, are used to analyse two tasks proposed by the teacher in the classroom following a cross-methodology for networking of theories. Through a single case study research design, a content analysis of the transcript of the video recording of the lesson was conducted. The joint analysis of this lesson allows us to better understand the mathematical work taking place in the classroom. In particular, the results show the scope of each model and their complementarity through the detection of meeting points and blind spots, through the role of proof, representations, and the change between geometrical and numerical work in teaching Thales's theorem. This allows for a deeper understanding of a teacher's practice and teaching. Ultimately, relationships between the theoretical elements of both theories are established to show their complementarity. We conclude that networking between theories can contribute to the development of these theories by raising questions that involve examining their foundations and assumptions in greater depth.

Keywords Networking of theories · Mathematics teachers' specialised knowledge · Mathematical working spaces · Thales's theorem

1 Introduction

The teaching of intercept theorem, also known as *Strahlensatz*, or, in South America, as Thales's theorem (TT), enables the interrelation of various mathematical topics (Filloy & Lema, 1996), including similarity, congruence, vectors, homothety, proportionality, or linear functions. This allows for bidirectionality between geometry, numerical, and algebraic domains, which is key in understanding the problems surrounding TT. These relations between distinct domains can generate not only opportunities to learn but also obstacles to learning, according to Montoya-Delgadillo et al. (2014), in changes from geometric to algebraic or numerical. Chile's Ministry of Education (MINEDUC) (where

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this study was developed) considers the study of TT, as derived from homothety, to be an important application of problem-solving (MINEDUC, 2016), which reinvigorates interest in exploring TT and what teachers engage in when teaching it.

Studies on classroom activity based on teachers' knowledge help explain the teacher's actions in terms of tools available to them, specifically their knowledge (Schoenfeld, 2010). This study explores, based on the paradigm of networking theories (Prediger et al., 2008), the case of a secondary-level teacher who teaches TT. Specifically, we use the strategy of *combination* of two constructs: the Theory of Mathematical Working Spaces (ThMWS) (Kuzniak et al., 2022) and the Theory of Mathematics Teachers' Specialised Knowledge (ThMTSK) (Carrillo et al., 2018). We can refer to these constructs as theories because they have characteristics that identify them as such (Radford, 2008; Verdugo-Hernández et al., 2022).

While ThMWS allows exploration of the mathematical activity developed and attempted in the classroom (Kuzniak et al., 2022), ThMTSK provides a detailed analytical framework to examine the teacher's knowledge (Kilpatrick & Spangler, 2016). Both theories have been widely used in mathematics education research in Europe and Latin America (Carrillo et al., 2022; Kuzniak et al., 2022). In this context, we situate the interest in combining these theoretical constructs and their potential to illuminate practical issues (Prediger et al., 2008), particularly regarding teachers' classroom practice.

Other studies addressing the MWS-MTSK relationship have been developed by applying strategies from Prediger et al. (2008), allowing researchers to understand each framework while gaining deeper knowledge of theoretical aspects of the other (e.g., Flores-Medrano et al., 2016). The teacher is recognised as an articulating element of MWS-MTSK connection. Later works (see Espinoza-Vásquez et al., 2022) have established specific relationships between their components, analysing teaching moments in which the mathematical task stands out as another element favouring connection. These studies consider *mathematical tasks* as materials/situations designed to promote complex mathematical activity (or classwork) (Becker & Shimada, 1997), including diverse actions (such as construction of concepts, proofs, and applications [Watson & Thomson, 2015]).

Recent studies (Henríquez-Rivas & Espinoza-Vásquez, 2018; Verdugo-Hernández et al., 2022) explore the *combination* and *coordination* (Prediger et al., 2008) of ThMWS and ThMTSK elements when studying a single dataset. These works advance the study of teachers' knowledge deployed during mathematical work in specific moments of the lesson, confirming teacher and task as key elements in elucidating theoretical connections; they also define study phenomena that can be addressed using both theories (Bikner-Ahsbahs et al., 2014; Rodríguez-Nieto et al., 2023).

While these works address teachers' specific actions, this study addresses a complete lesson about a specific topic. Considering these advances regarding the MWS-MTSK connection, we propose the following research questions: What mathematical work is proposed by a teacher in a lesson on TT? What specialized knowledge does the teacher mobilize while teaching TT? What aspects of theoretical complementarity emerge in the analyses of the teacher's practice based on ThMWS and ThMTSK?

Following Drijvers et al. (2013), we use a lesson on TT as a representative case study of the theories' complementarity. The MWS-MTSK relationship could be useful in describing classroom practices more thoroughly, contributing to formative intervention and enhanced understanding of the teaching of the topic analysed, and the advancement of both theories.

2 Theoretical framework

2.1 Mathematical working spaces (MWS)

ThMWS aims to analyse the mathematical work performed when solving tasks considering the epistemological and cognitive facets of objects studied within mathematical domains (e.g., geometry or analysis) (Kuzniak et al., 2022). The mathematical work developed by an individual can be modelled by distinguishing two planes (Kuzniak et al., 2016). The epistemological plane centres on mathematical objects, while the cognitive plane aims to describe the individual's cognitive activity when acquiring, developing, or utilising the corresponding mathematical contents (Kuzniak et al., 2022). On the epistemological plane, there are three components: *representamen*, borrowed from Peircean semiotics (1978), which refers to the representation of the mathematical object; *artefact*, the materials or symbolic systems that can be used as means of action (Rabardel, 1995); and *theoretical referential*, which specifies mathematical definitions and properties (Fig. 1). Meanwhile, on the cognitive plane, components include the following: *visualisation*, the perception of the mathematical object through different semiotic systems (figures, graphics, algebraic writing, etc.); *construction*, referring to actions triggered by artefacts used (compass, software, formulas, etc.) and associated techniques; and *proof*, referring to discursive reasoning. We differentiate between *iconic visualisation* (figure recognition/comparison with a standard model) and *non-iconic visualisation* (introduction of reorganising lines/marks, construction using instruments, heuristic decomposition of a figure, and dimensional deconstruction into lower-dimension figural units) (Duval, 2005). Likewise, we distinguish between *pragmatic proof*, which draws upon action, and *intellectual proof*, which is supported by properties and their relationships (Balacheff, 1987).

ThMWS also considers relations between components of the epistemological and cognitive planes by means of three geneses (Fig. 1). *Semiotic* genesis is based on representation registers, allowing for their identification and processing and conversion between different registers. *Instrumental* genesis enables the operationalisation of

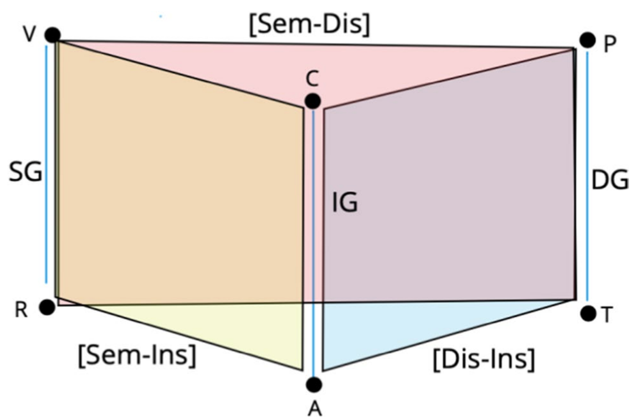


Fig. 1 MWS diagram, components, geneses, and vertical planes. Source: Created by the authors; inspired by (Kuzniak et al., 2016). Note. V: visualisation; C: construction; P: proof; R: representamen; A: artefact; T: theoretical referential. SG: semiotic genesis; IG: instrumental genesis; DG: discursive genesis

artefacts in the construction process. Finally, *Discursive* genesis allows for mathematical reasoning based on definitions and properties (Kuzniak et al., 2022).

In ThMWS, interactions between geneses and their associated components leads to the activation of three *vertical planes* (Kuzniak et al., 2016): *semiotic-instrumental* [Sem-Ins], when artefacts are used to build under certain conditions or to explore semiotic representations; *instrumental-discursive* [Ins-Dis], involving proofs based on experimentation, exploration, or justifying a construction; and *semiotic-discursive* [Sem-Dis], where the proof is coordinated with the visualisation process.

Additionally, the ThMWS addresses *domain changes*, understood as movement or articulation between different mathematical domains (e.g., geometry and algebra), one of origin and another of resolution (Henríquez-Rivas et al., 2021; Montoya & Vivier, 2014). A domain change could imply a non-return to the origin domain.

This study focuses on teachers's actions on the *suitable* or *idoine* MWS (noted by MWS^i), understood as a space linked to the mathematical work undertaken in the task-selection process for teaching, which involves task design, adaptation, and classroom implementation in a given context and institution with the intention of helping students construct their learning (Kuzniak et al., 2016). MWS^i involves mathematical work that the teacher encourages in their teaching, in which we can distinguish between what is planned for teaching, termed potential MWS^i, from what occurs in the classroom, termed actual MWS^i (Henríquez-Rivas et al., 2022).

2.2 Mathematics teacher's specialised knowledge (ThMTSK)

ThMTSK allows for analysis of teachers' knowledge through examination of their teaching practice, referring to specialisation of knowledge because mathematical objects are considered from the perspective of their teaching–learning. The ThMTSK includes knowledge domains inspired by Shulman (1986): Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK) (Fig. 2).

Within the MK domain, *Knowledge of Topics* (KoT) integrates contents that students must learn in greater depth. It includes understanding phenomenology, definitions and properties of a topic and their foundations, along with registers of representation, and procedures associated with a topic. The *Knowledge of the Structure of Mathematics* (KSM) subdomain entails knowledge of inter-conceptual connections between mathematical objects, including increasing or decreasing complexity, transverse, and auxiliary connections. Finally, the *Knowledge of Practices in Mathematics* (KPM) subdomain comprises the means of generating new mathematical knowledge and exploring and communicating it. This includes knowledge about validating and proving, use of formal language to communicate mathematical ideas, and conditions to define mathematical objects, among other topics (Alfaro-Carvajal & Fonseca-Castro, 2024; Delgado-Rebolledo & Zakaryan, 2020).

Meanwhile, PCK includes *Knowledge of Features of Learning Mathematics* (KFLM), covering ways of learning mathematical content, associated strengths and weaknesses, students' interactions with mathematical content, and students' motivations regarding mathematical content. The *Knowledge of Mathematics Teaching* (KMT) subdomain encompasses knowledge of personal or institutional teaching theories, material and virtual resources, and activities, tasks, examples, and support that can be directed toward teaching a topic. Lastly, the *Knowledge of Mathematics Learning Standards* (KMLS) subdomain includes

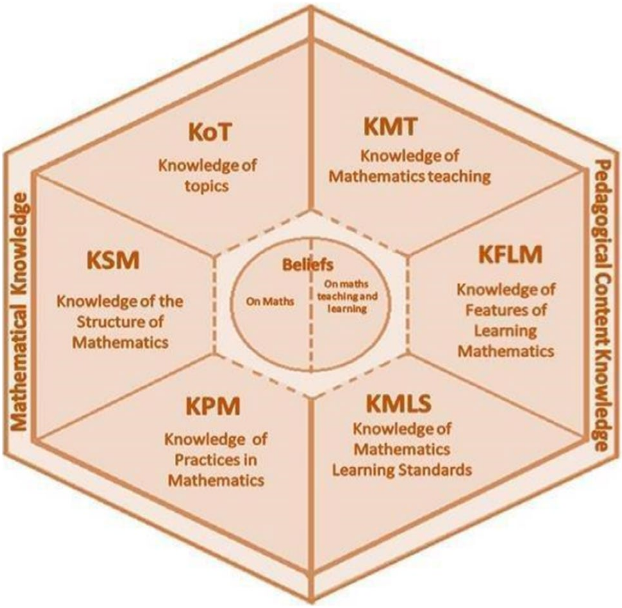


Fig. 2 MTSK diagram, domains, and subdomains. Source: Carrillo et al. (2018)

knowledge of expectations for students' learning at different grade levels, expected levels of conceptual and procedural development, and sequencing of topics.

In the ThMTSK, knowledge is viewed comprehensively as an organic whole, considering both epistemological aspects of mathematical objects and their adaptation to educational contexts (Carrillo et al., 2018). ThMTSK analysis addresses the question of what knowledge is evidenced/needed by the teacher and how this allows their teaching approach to be understood.

2.3 MWS-MTSK relationship

Both theories give a central role to mathematics, constituting a first meeting point between them (Bosch et al., 2017). In previous studies, as established above, theoretical connections are made between components of ThMWS and ThMTSK based on the paradigm of networking theories (Prediger et al., 2008). For example, KoT has been related to the *representamen* and *theoretical referential* components of the ThMWS, while KPM has been related to the discursive aspects of the *proof* component. The teacher's knowledge of auxiliary connections has been linked to the use of objects from two mathematical domains as domain change (Verdugo-Hernández et al., 2022). Such connections encourage reinterpretation of what each theory detects in a phenomenon, sharing observation of the teacher's practice as methodology and raising questions about a phenomenon that can be addressed through joint analysis.

Here, we refer to connections between theories in the study of the MWS-MTSK relationship because each has principles (P), methodologies (M), and pragmatic research questions (Q), characteristics that would typically be seen in a theory (Radford, 2008). The connections between theories can likewise be established through these characteristics.

Following Verdugo-Hernández et al. (2022), ThMWS is aimed at understanding of mathematical activity, regarding these as human activities. The questions (Q) this model addresses include characteristics of tasks, the mathematical work they promote, and the work encouraged in the classroom or developed by a particular individual (teachers, students, or others). Such questions involve the organisation of a cognitive dimension linked to epistemological aspects of the discipline (P). ThMWS studies employ observation of individuals during the development or resolution of tasks (M).

Meanwhile, ThMTSK seeks to provide answers regarding the mathematics teacher's knowledge. Specifically, its objective is to understand the knowledge possessed or demonstrated by the teacher, which is mobilised to undertake their teaching work. The ThMTSK examines what and how the teacher knows about the discipline, their teaching, and their learning (Q). For example, it questions relations between different knowledge areas (Carrillo et al., 2022) or which knowledge the teacher needs to guide a teaching-leaning process. Therefore, ThMTSK considers knowledge domains (disciplinary and didactic-disciplinary) and the domain of beliefs regarding mathematics and its teaching–learning (P). The study of the teacher's specialised knowledge is proposed in the context of teaching practice in the classroom, planning, or interacting with other teachers (M).

3 Methodology

To address the research questions, an interpretive qualitative research approach was developed (Bassey, 1999). Specifically, a single case study (Yin, 2009) of a secondary-school teacher (P1) participating in a continuing education programme at a Chilean university. This design is appropriate because teaching took place in a real school context, and a deep analysis of the teacher in the classroom was the intended outcome.

The continuing education P1 participated in was organised into voluntary teachers' workshops during a single semester; their purpose was to have teachers reflect on their practice. The teachers presented tasks habitually utilised when teaching a geometric topic, followed by the reformulation of their tasks considering elements of ThMWS and ThMTSK. This study examines the implementation of TT-related tasks reformulated by P1.

3.1 Case selection

P1 is a mathematics teacher with an early background in accounting. P1 identified himself as a traditional teacher (expository teaching), and initially his teaching of TT consisted of presenting his demonstration to the students and then applying it to exercises. During task reformulation, without intervention by researchers, P1 proposed the incorporation of GeoGebra and greater student participation. Here, P1's selections correspond with the criteria of Yin (2009): *representativeness* regarding teaching of TT; *accessibility* for data collection; participant's *availability* and *commitment* (voluntary); and a *revealing* case. P1 was the only teacher who proposed the incorporation of GeoGebra, but had no previous training with geometrical software.

3.2 Data collection and analysis

The data source was a video recording (non-participant observation) of a lesson in a Chilean public school with 40 students aged 14–15. The resulting audio transcript was transferred to a spreadsheet, where the interventions of P1 and the students were identified using sequential numbering.

The analysis process incorporated the steps of the cross-methodology (Bikner-Ahsbahs & Kildron, 2015) for networking theories:

a) Researchers cooperatively identified the tasks (*t*) that P1 proposed to students and the episodes (*e*) developed for each task, following Kuzniak and Nechache (2021). Researchers were experts on ThMTSK or ThMWS and had experience using both to analyse teachers' practices.
b) Selection of tasks refers to the different teacher's goals observed in class: establishing proportional relations of TT (empirical proof), proof of TT, and applications of TT. This study only considers the first two. Episodes in each task were defined by the sequence of mathematical actions used to solve the task or by subgoals in its resolution.
c) Separate processing of data using each theory was achieved through content analysis (Bardin, 1996). Units of analysis were defined as the teacher's oral or written interventions. Tables 1 and 2 describe the protocols utilised. The data was analysed by two subgroups of researchers based on expertise.
d) Results were exchanged between subgroups, with each working with them in turn to identify common elements and blind spots regarding the phenomenon for each model. The aim was to identify emerging relationships, contributing to complementarity through the combination of said elements.
e) Finals results were then reworked considering the research questions.
f) Collaborative meetings were held aiming to build consensus regarding the work completed.

Ultimately, findings were validated using data triangulation consistent with data collection from recordings, transcripts, and photographs of the class, allowing us to confirm or discard evidence of P1's knowledge or mathematical work elements, considering criteria of pertinence (Denzin, 1978). Additionally, researcher triangulation was applied via participation of the full research team in data analysis and achieving consensus in results.

Table 1 Protocol for MWS analysis

Coordination between components and geneses	Component	Description
Semiotic genesis (SG)	Representamen	Links mathematical objects and signifying elements
	Visualisation	Interprets and links mathematical objects with semiotic representation registers (identification, treatments, conversions)
Instrumental genesis (IG)	Artefact	Utilises material/technological artefacts or a symbolic system
	Construction	Based on processes resulting from actions triggered by the artefacts utilised and associated usage techniques
Discursive genesis (DG)	Referential	Utilises definitions, properties, or theorems
	Proof	Discursive reasoning, based on distinct forms of justification, argumentation, or demonstration
Vertical plane	[Sem-Ins]	Artefacts are used in the construction of results under certain conditions or for the exploration of semiotic representations
	[Ins-Dis]	The proof process is based on experimentation with the use of an artefact, or alternatively on the validation of a construction
	[Sem-Dis]	The process of visualisation of the objects represented is coordinated with discursive reasoning in order to prove

Source: Henríquez-Rivas et al. 2021

Table 2 Protocol for MTSK analysis

Subdomain	Description
Mathematical knowledge	
Knowledge of topics (KoT)	Corresponds to teachers' knowledge of the topic, its conceptual network and applications
Knowledge of the structure of mathematics (KSM)	Corresponds to knowledge of the connections between current topic and others, or its epistemological progression
Knowledge of practices in mathematics (KPM)	Corresponds to knowledge of the manners in which mathematics is produced, explored, and communicated
Pedagogical content knowledge	
Knowledge of mathematics teaching (KMT)	Corresponds to knowledge of topics as objects of teaching
Knowledge of features of learning mathematics (KFLM)	Corresponds to knowledge of topics as objects of learning
Knowledge of mathematics learning standards (KMLS)	Corresponds to knowledge of what the student should or can achieve at a given level

Source: Espinoza-Vásquez and Verdugo-Hernández (2022)

4 Results

Beginning the lesson, P1 indicates its topic and objective, commenting that TT will be used to solve problems after its proof has been demonstrated. He presents tasks that show the numerical connection of the theorem using graphic manipulation software. Afterward, he conducts a demonstration of TT and finalises the lesson with a series of applied exercises. Analysis is organised by tasks (*t*) and episodes (*e*), starting with a brief description of its development, including extracts (E) of the teacher's interventions.

4.1 Task 1: incorporating GeoGebra

Task 1 (*t1*) involves calculation and comparison of quotients between segment measurements to establish the proportionality posed by TT, supported by technological resources. This task includes four episodes (*e1*, *e2*, *e3*, *e4*), which are presented in Table 3.

P1 begins *e1* by projecting a GeoGebra image (Fig. 3 in Table 3), signalling the class objective:

E1 P1: Understanding and solving TT exercises […], first we're going to look at it empirically, and then, we're going to look at it through a proof, which is Thales's theorem itself

Table 3 Task 1 and its episodes

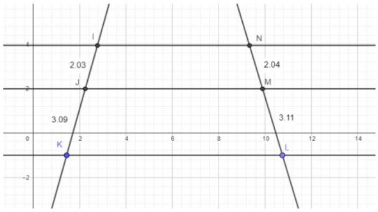
<table>
<tr><td>Task (t)</td><td>t1: To compare the value of the ratio of segment measurements between parallel and non-parallel lines with a calculator.</td></tr>
<tr><td>e1: Linking TT with the similarity of the projected figure.


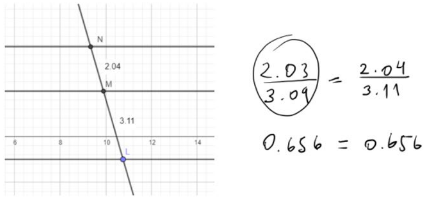
Figure 3: Initial work of P1.</td><td>e2: Comparing the value of two ratios of segment measurements between parallel lines using a calculator.


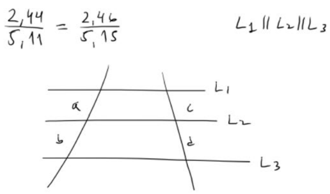
Figure 4: P1 compares the value of segment measurement ratios using a calculator.</td></tr>
<tr><td>e3: Illustrating the condition of parallelism of TT.


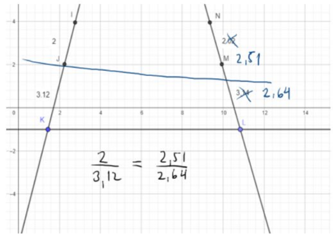
Figure 5: P1 compares new segment measurements and illustrates this freehand.</td><td>e4: Comparing the value of two ratios of segment measurements between non-parallel lines using a calculator.


Figure 6: P1 compares the values of ratios of segment measurements resulting from non-parallel lines.</td></tr>
</table>

Source: Created by the authors. Reproduction of original images from lesson

Here, he indicates the coming procedures and demonstration of TT. In *e1*, P1 links TT with similarity, specifically similarity of triangles:

E2 P1: In truth, TT is derived, basically, from the similarity of figures. Now, I want us to do this together. Do you remember how you did it with similarities?

In *e2*, P1 recalls how they proceeded when working with similarity of triangles and asks students to compare ratios between segments ($\frac{IJ}{JK} = \frac{NM}{ML}$), checking equality (Fig. 4 in Table 3) using calculators:

E3 P1: But here we're going to make a proportionality. And we have two ratios, right? Now we're going to do the same, but when this portion of [this] segment… is divided by this segment, and it will give us two values. […] calculators, phones, we're dividing this by this. Ah, this happens if and only if this line is parallel to this line and is parallel to this

E4 P1: …here we're going to do something similar, we're going to compare this with this, make the fraction and write it out […] Okay. Now, this segment, divided by this segment […], could you give me the value of this fraction?

In *e3*, P1 presents a situation through a figure using GeoGebra and uses freehand drawing on the whiteboard. The intermediate line moves in a parallel manner, generating new measurements for the segments. P1 seeks to infer the condition of parallelism. The ratios obtained using the calculator are different decimal expressions, which was not anticipated. P1 attributes this to a GeoGebra error, since it provides the measurements with two decimals.

E5 P1: That is, empirically, if I have a situation like this, with this segment, first it must be true that the 3 lines are parallel. If this line is worth *a,* and this is *b*, *c*, *d*, it is true that *a* divided by *b* equals *c* divided by *d*, is that clear?

At this point, P1 achieves what he calls *empirical proof*. In *e4*, P1 questions the condition of parallelism and asks what will happen if it changes. Then, he tries to move the middle line and produce a non-parallel line, but fails because of his difficulties with GeoGebra, so he draws it freehand (Fig. 6 in Table 3). The students check whether $\frac{IJ}{JK} = \frac{NM}{ML}$ holds with the new measurements, concluding that it does not and that the three lines must be parallel.

E6 P1: Now, what happens if this line stops being parallel, it wouldn't work?

E7 P1: … what happens if I draw a line and now, I slide it, and I see… it doesn't want to move… […] I'm going to cheat […] What if we increase this number, let's say it's *2.51*

4.1.1 ThMWS-based analysis of t1

P1 starts class (E1) by offering a glimpse of his actual MWS^{i}: the transition from empirical to formal proof of TT, placing importance on the intellectual proof.

In *e1*, P1 uses the typical TT configuration and relates TT with the similarity of figures (E2). This relationship is developed through representations using GeoGebra, which activates the artefact component and semiotic genesis. P1 has planned the task with this use of the artefact; however, the utilisation of software is scarce in his MWS^{i} and hampered by difficulties, highlighting shortcomings in his planning of his potential MWS^{i} related to use of technology. Thus, iconic visualisation of the figures projected with GeoGebra is salient in this episode (Fig. 3 in Table 3).

In *e2*, P1 utilises GeoGebra to measure the segments and present a proportion based on two ratios. P1 appears to privilege numerical proportionality, emphasising the ratio between numerical values. The notion of the ratio is observed as a symbolic artefact operating on the segments determined by the parallel lines, focusing on their numerical representations (Fig. 4 in Table 3). In *e2*, P1 incorporates a second artefact (calculator) for the students to carry out calculations associated with the ratios, which appears essential for the development of this task. Thus, the actual MWS^{i} manifests through instrumental genesis using the calculator (technological artefact) and the ratio (symbolic artefact) for the numerical calculations between segment measurements, provided by the representation and visualisation of TT. This activates the [Sem-Ins] plane in P1's work.

Moreover, when P1 indicates the numerical work and the use of the calculator in the checking of the theorem through equality between ratios, a type of pragmatic proof is observed ("empirical" for P1), which activates the [Ins-Dis] plane. Likewise, this proof is based on the iconic visualisation of figures (Fig. 3 in Table 3) and the condition of parallelism related to the referential of the epistemological plane (although he does not explicitly mention that this is the hypothesis of the TT), activating the [Sem-Dis] plane.

Similarly, in *e3* the work involves numerical calculations, calculator use, and the iconic visualisation of a freehand figure, activating the [Sem-Ins] plane. However, P1 had not incorporated, in his potential MWS[i], the results of the new ratios based on software measurements, which evinces a barrier in the [Ins-Dis] plane (Henríquez-Rivas & Montoya-Delgadillo, 2016) between the use of the technological artefact (software) and the theoretical referential of TT.

In *e4*, P1's limitations in the use and potentiality of the artefact are again evident in relation to the software's tools and dynamic aspects that could support the validation of TT (Lagrange & Richard, 2022; Richard et al., 2019). This difficulty limits the precision of the example in which TT fails (E7). Thus, P1 emphasises iconic visualisation, pragmatic proof ("empirical"), and calculator use as an artefact for numerical calculations. Furthermore, in this episode (E6), part of the referential on the parallelism condition of TT is observed, wherein the software mainly plays an illustrative role; P1 does not aim to enhance learning through GeoGebra since the students do not have the opportunity to work with this tool. Therefore, in *e4*, the work favours the [Sem-Ins] vertical plane and the intention to prove the theorem, which leads to activation of the [Sem-Dis] plane, emphasising that the ratios involved be treated as fractions, producing a domain change in P1's work from the origin domain (geometry) to the resolution domain (numerical).

4.1.2 ThMTSK-based analysis of t1

The lesson's objective provides an indication of what P1 expects his students to achieve: solving exercises using TT (KMLS). In *e1*, P1 relates TT using the similarity of figures (E2); he uses the concept of similarity to build to TT. Referencing the connection between similarity and TT demonstrates his knowledge of the foundations of TT (KoT) and of the curricular progression between both topics (KMLS). The figures projected with GeoGebra (Fig. 3 in Table 3) show his knowledge of a prototypical representation of TT (KoT) and of the use of a technological resource (KMT).

Likewise, in *e2*, P1 exhibits knowledge of the software (KMT) through the possibility of measuring the segments and knowledge of numerical proportionality as a support for TT (numerical proportionality serves as a tool to proceed through TT) as we see in E3, which can be interpreted as an auxiliary connection within his KSM. P1 privileges numerical over geometric treatment and exhibits his organisational approach to TT, highlighting procedural aspects that students should carry out (KMLS, expected learning results). The verification of equality between ratios is aligned with the expected procedural handling of TT. Likewise, this selection of tasks shows P1's knowledge of a transversal connection between the numerical and geometric domains (KSM), since P1 gives the same treatment to ratios and segments through proportionality (E3 and E4).

Although P1 indicates the parallelism hypothesis, he does not explicitly state it as a hypothesis, providing insights into his knowledge of this topic (the formulation of TT [KoT]—definitions, properties, and their foundations) and the logical structure of the theorem (KPM).

In E4, P1 refers to the quotients forming the proportion as *fractions*, illustrating his manner of understanding the proportion (KoT). P1 explicitly refers to the ratio as a fraction multiple times (E4, E5). The ratios treated as fractions and their numerical operations help establish the relationships of TT, showing a connection between rational numbers and TT (auxiliary, KSM) that supports emphasis on the numerical over the geometric.

In *e3*, P1 uses the projected configuration (Fig. 3 in Table 3) to test the parallelism condition. He relies on similarity of figures as a previously studied topic, demonstrating his knowledge of said topics (parallelism, proportionality, similarity of figures) (KMLS, topic sequencing) and his TT teaching strategy based on similarity as a grounding and support for TT (KoT). Here, P1 recognises the limitations of the software (KMT, virtual resources), which produces measurements with two decimals that prevent the equality of TT ratios from being established.

In *e4*, P1 knows the conditions of TT when applying the theorem if the parallelism condition is satisfied (KoT), but difficulty with the software (KMT) limits its use in attempting to show an example where the conclusion of TT fails (E7).

Table 4 synthesises the mathematical work and specialised knowledge identified in *t1*.

Table 4 ThMWS and ThMTSK-based analysis of *t1*

MWS[i]	MTSK
Episode 1	
P1's actual MWS[i] begins with iconic visualisation of figures and artefact usage (software). Semiotic genesis predominates	Similarity as foundation of TT (KoT) Similarity prior to TT (KMLS) Prototypical graphic representation of TT (KoT) GeoGebra as resource to represent geometry (KMT)
Episode 2	
Use of artefacts to measure (software) and operate on the segments (symbolic), conducting numerical calculations (technological) to prove TT Instrumental genesis predominates [Sem-Ins] plane is activated	Auxiliary connection between numerical proportionality and TT (KSM) Proportionality as a transversal connection in the numerical and geometric domains (KSM) Knowledge of use of GeoGebra to measure (KMT) Learning expectations linked to TT with procedural emphasis (KMLS)
Pragmatic proof of TT based on calculator use, a technological artefact [Ins-Dis] plane is activated	Formulation of TT and hypothesis of parallelism (KoT) Logical structure of theorem (hypothesis, thesis) (KPM)
Iconic visualisation of figures, importance of the parallelism condition (referential) [Sem-Dis] plane is activated	Proportion as fraction (KoT) Auxiliary connection between fractions and TT (KSM)
Episode 3	
Iconic visualisation and use of calculator as artefact for numerical calculations [Sem-Ins] plane is activated Blockage of [Ins-Dis] plane by limitations in use of software	Similarity as foundation of TT (KoT) Similarity previous to TT (KMLS) Software limitations (KMT)
Episode 4	
Iconic visualisation and use of calculator as artefact for numerical calculation [Sem-Ins] plane is activated	P1's difficulties with the software (KMT)
Pragmatic proof of TT through iconic visualisation and use of calculator [Sem-Dis] plane is activated	Parallelism condition of TT (KoT)
In general, a domain change is evident (geometric to numerical)	––-

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4.2 Task 2: proof of Thales's theorem

Task 2 (*t2*) corresponds to the proof of TT, with three episodes (*e1*, *e2*, *e3*) detailed in Table 5.

Commencing *t2*, P1 announces that the class will conduct a demonstration.

E8 P1: So, guys, now let's go to the proof […] I know that many don't like proofs, but I love knowing where things come from, and the reason why. So, using this empirical method it can be practically demonstrated…

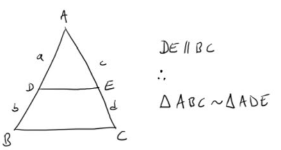
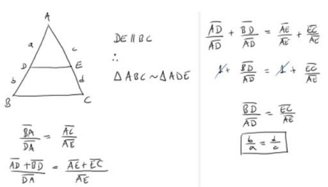
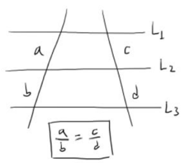
Using a freehand representation of a prototypical figure (Hershkowitz, 1989) to illustrate the similarity of triangles (Fig. 7 in Table 5), P1 establishes that triangles *ABC* and *ADE* are similar (*e1*). He then identifies proportional relationships (*e2*), concluding that $\frac{BD}{AD} = \frac{EC}{AE}$ through the sum of ratios as fractions. After, he substitutes the previous segments for *a*, *b*, *c*, and *d*, concluding that $\frac{b}{a} = \frac{d}{c}$.

E9 P1: Now, I can separate that fraction into two fractions, right? [referring to $\frac{BA}{DA}, \frac{AC}{AE}$]. Remember that when I add fractions with the same denominator […] the denominator doesn't change, and the numerators are added

In *e3*, P1 indicates that he has demonstrated a special case of TT, explaining the transition from the general case (Fig. 9 in Table 5). P1 considers the theorem to have been formalised, underscoring that they have gone "from the empirical to the mathematical":

E10 P1: Although this is basically not TT [alluding to the work from *e2*], it is a special case, because TT is basically this: L1 parallel to L2, parallel to L3 [Fig. 9 in Table 5] if I have *a, b, c, d,* okay? It will be the same […] Because to prove it I can draw a line here and find the same thing there, and from there I can get to this, but I won't spend the whole class proving, so *a* divided by *b* results in *c* divided by *d*

Table 5 Task 2 and its episodes

Task (*t*)	*t2:* To establish proportional relationships from the similarity of triangles to prove TT.	
e1: To draw a figure of the similarity of triangles on the board.  **Figure 7.** Prototypical example of the similarity of triangles.	*e2:* To establish proportional relationships from the figure drawn.  **Figure 8.** Proportional relationships.	*e3:* To prove TT through a hand-drawn figure.  **Figure 9.** Proving TT.

Source: created by the authors. Reproduction of original images from lesson

4.2.1 ThMWS-based analysis of t2

In *e1*, P1 presents the similarity between triangles ADE and ABC to visualise the TT through his freehand figural representation (Fig. 7 in Table 5). This episode involves the visualisation of the figure and instrumental genesis through the construction of the prototypical figure of similarity of triangles. That is, the [Sem-Ins] vertical plane is activated, and the theoretical referential is present as the definition of similarity of triangles.

In *e2*, P1 establishes proportional geometric relationships between the segments, which arise from the similarity of triangles (Fig. 8 in Table 5). This work, based on the theoretical referential (definition of similarity) and supported by algebraic representation of the ratios, seeks to present a proof of TT to the students (E8), activating discursive genesis. From the semiotic perspective, P1's work entails the conversion from the figural representation register to the algebraic by establishing such proportional relationships, which he writes using letters to represent magnitudes in a generalised manner. Visualisation, meanwhile, is of a non-iconic type, as it involves deconstructing the initial figure into its figural units to establish the proportional relationships written algebraically.

P1 continues his work in the algebraic register on the fractions associated with the proportionality of the segments (Fig. 8 in Table 5). The latter evidences a domain change from the geometric (origin domain) to the numerical (resolution domain [E9]) (Montoya-Delgadillo & Vivier, 2014). This change occurs based on the treatment of the proportional relationships of the similarity of triangles as numerical relationships developed in the algebraic register.

Additionally, in the algebraic approaches of *e2*, P1 considers the similarity of triangles as part of his theoretical referential, which is justified with the theorem on angle measurements between parallel lines. This indicates that the actual MWS^i privileges the [Sem-Dis] vertical plane, manifested in the visualisation of similarity and the completion of a proof of TT.

In *e3*, P1 draws a typical TT figure freehand and explains through gestures (moving his arm diagonally) how to draw a secant line in the figural representation (Fig. 9 in Table 5). This is related to the proof of TT (E10) planned. In particular, P1 acknowledges TT in the initial configuration (*e2*) but identifies this as a corollary of TT (Fig. 9 in Table 5), calling it "special case". Thus, the activation of the [Sem-Dis] plane in this episode reflects the components of visualisation and the proof, which characterises the work of P1.

Finally, P1 underscores the importance of the formal demonstration, and although he does not present it as such, he resorts (discursively and through gestures) to the construction of an auxiliary geometric element (secant line) to show that, in the *e3* configuration, the theorem is equally valid.

4.2.2 ThMTSK-based analysis of t2

P1 shows how the similarity of triangles underpins, for him, TT (KoT). Likewise, by presenting different forms of TT (Fig. 7 and Fig. 9 in Table 5), he exhibits knowledge of representations of TT (KoT). P1 knows students do not like proofs (KFLM, student attitudes about mathematics).

P1 refers to the "empirical proof" (developed in *t1*) (E8) with the purpose of convincing the students of its veracity, presenting it as a way of validating TT, although he seems to prioritise the formal proof (showing his knowledge of a formal proof of TT [KoT], the differentiation between proof and checking, and the role he assigns to the proof [KPM]).

In *e2*, P1 continues with the treatment and algebraic operations (Fig. 8 in Table 5, E9), mobilising knowledge of operations with fractions to address the proportionality of segments (auxiliary connection between fractions and TT [KSM], adding ratios as fractions). P1 works with the proportional relationships of TT as generalisations of the numerical relationships established in *t1*. We again observe P1's knowledge of proportionality as a notion transversal to numerical (fractions), geometric (proportional segments), and now algebraic (proportions) topics as part of his KSM, allowing for the treatment of one topic as another. There is emphasis on the algebraic-procedural process as a way of illustrating deductive procedures, indicating his knowledge of the demonstration in relation to its role of convincing (KPM). For this, P1 draws upon a theorem on angle measures in similar triangles to establish the proportional relationships of TT (Fig. 8 in Table 5), exhibiting knowledge of similarity as a foundation of TT (KoT), as well as demonstration of the theorem (KoT) and the sequence between both topics (KMLS).

In *e3*, P1 relies upon gestures to show how to draw a secant line and links this construction to the proof of TT. P1 emphasises this as a special case of TT (E10). In this manner, these two representations of TT (KoT) are differentiated, which affects the organisation of the teaching presented (KMT) in terms of the sequence of these representations.

Table 6 synthesises the mathematical work and specialised knowledge identified in *t2*.

Table 6 ThMWS and ThMTSK-based analysis of *t2*

MWS[i]	MTSK
Episode 1	
Visualisation of TT through similar triangles and the construction of a prototypical figure [Sem-Ins] vertical plane is activated	Similarity of triangles as foundation of TT (KoT) Different representations of TT (KoT)
Similarity as part of the theoretical referential	Students do not like proofs (KFLM) Proofs of TT (pragmatic and formal proof) (KoT) The proof's role of convincing (KPM) Difference between proof and checking (KPM)
Episode 2	
Non-iconic visualisation of the figure for proof The definition of similarity of triangles as part of the theoretical referential [Sem-Dis] plane is activated	Similarity of triangles as foundation of TT (KoT) Proofs of the theorem (KoT) Similarity previous to TT (KMLS)
Conversion from the register of figural representation to the register of algebraic presentation, and then operations on the fractions associated with the proportionality of segments Domain change is evident (geometric to numerical)	Auxiliary connection between fractions and TT (KSM) Proportionality as a notion transversal to numerical, geometric, and algebraic topics (KSM) The proof's role of convincing (KPM)
Episode 3	
Non-iconic visualisation (auxiliary line) to prove TT [Sem-Dis] plane is activated	General and particular cases of TT (KoT) Teaching sequence of representations of TT (KMT)

Source: Created by the authors

4.3 Complementarity between MWS^i and MTSK

Tables 4 and 6 allow us to observe points of comparison, including the visualisation and representations of TT, as well as meeting points, like the role of the GeoGebra for teaching, the demonstration of TT, and the domain change. Below, we focus on three of these points to illustrate the complementarity of MWS^i and MTSK.

4.4 The role of representations

The two configurations of TT that P1 presents (Figs. 7 and 9 in Table 5), key to the lesson, reflect his knowledge of prototypical representations, part of his teaching strategy and MWS^i, highlighting semiotic genesis in his work. P1 relates both representations through auxiliary geometric construction as an artefact, activating the [Ins-Dis] plane. Here, he places importance on iconic visualisation—based on these representations, and the use of GeoGebra—activating the [Sem-Ins] plane. However, the software is restricted by limitations in his KMT, which helps explain the blockage of the [Ins-Dis] plane. This iconic visualisation is based on similarity as a foundation of TT, linked to his knowledge of the proof of TT, which permits the demonstration of the hypothesis of parallelism and activation of the [Sem-Dis] plane.

The treatment of ratios and proportionality show semiotic aspects of P1's work. The geometric arguments (similarity as a referential to activate discursive genesis, leading to proportionality among segments) as well as algebraic (sum of fractions) show his knowledge about representations and ways of understanding ratios (KoT). This treatment is supported by P1's KMLS (sequencing of topics) and MK (foundations of TT; auxiliary connections with fractions; proportionality as transversal connection; and the role of proof) regarding TT and reflects the kind of work he encourages from his students. However, there is a domain change (geometric to numerical) from the ThMWS perspective when transitioning from segment measurements to the purely algebraic treatment of their values.

4.5 The phenomenon of domain change

P1's knowledge of GeoGebra (KMT) allows him to use it as a teaching resource and as a non-material artefact to measure segments, which supports the activation of instrumental genesis. The software illustrates the domain change from geometric to numerical, with numerical proportionality as a tool and auxiliary connection coming from another referential, which affords him the connection and transition.

The notion of the ratio, viewed as a fraction, also reflects the domain change. It is used by P1 as an auxiliary tool to operate algebraically on the geometric configuration. The change is reflected by a change of referential and the treatment of ratios through different representations and their comprehension.

Here, domain change is based on mathematical connections considered in KSM, involving broad concepts, like proportionality, that are transversal to both domains. The emphasis on numerical procedures seems consistent with P1's learning expectations (KMLS), which could encourage a non-return to the geometric domain.

4.6 The proof of TT

The proof of TT has a role of validation and convincing (KPM) to enable its application (KMLS). The calculator, as a resource/artefact that aids in the generalisation of relationships, allows developing a type of pragmatic proof. This proof is based on the iconic visualisation of figures (representations [KoT]) and activating discursive genesis for these validations through checking in the [Sem-Dis] and [Ins-Dis] planes. Although P1 alludes to the parallelism condition (referential component), demonstrating his KoT and KPM, he does not explicitly mention it as a hypothesis.

The pragmatic proof in *t1* can be understood as having the aim of exploring and establishing conjectures (KPM), while the transition between the pragmatic and the intellectual in *t2* is viewed as part of P1's KoT, intended as a demonstration. The proof of TT displays the discursive genesis linked with the KPM. However, knowledge of this demonstration is limited to P1's KoT. P1 mobilises knowledge on the representation of TT, similarity, and fractions (KSM). Thus, discursive genesis could be related to both KoT and KPM, in this case.

While actual MWS^{i} includes discursive genesis to construct the demonstration of TT, ThMTSK differentiates within the knowledge mobilised in the demonstration and the two types of proof (empirical and formal), linking KoT and KPM.

The role attributed to proof reflects part of the proposed work and exhibits mathematical and teaching knowledge regarding TT, allowing for relating elements of both theories.

5 Discussion

P1's teaching of TT displays links between different mathematical topics (Filloy & Lema, 1996) as domain change or connections within TT, allowing us to characterise his specialised knowledge as well as the mathematical work he promotes. According to Rodríguez-Nieto et al. (2023), proof, arguments, and mathematical connections represent an opportunity for advancement in the understanding of the teacher's practice.

P1 relates TT with similarity and utilises certain symbolic artefacts (ratio, proportionality, and operations with fractions), giving learning a procedural purpose, which seems to explain the domain changes from geometric to numerical and algebraic. In this sense, the learning goal outlined by P1 differs from the Chilean curriculum, which proposes TT linked to homothecy and problem-solving (MINEDUC, 2016).

Regarding the use of software, we find that GeoGebra is presented statically (instrumental genesis), without considering its use by the students. P1 shows limitations in the application of its dynamic potential and the cognitive processes that could be facilitated (in the sense of Arzarello et al., 2002; Richard et al., 2019). This reflects the potential and actual MWS^{i} and P1's knowledge of the resource (KMT), indicating a deficient instrumentalisation in the sense of Rabardel (1995). The calculator use also indicates a gap between both MWS^{i}, which is difficult to demonstrate solely through the ThMTSK.

P1 prefers/promotes semiotic work, while discursive reasoning is not fully utilised. Knowledge of the formulation of TT (KoT) and its logical structure (KPM), together with the communicative and convincing function attributed to the demonstration of TT (De Villiers, 2012), seem to explain the emphasis given to the proof. Following Alfaro-Carvajal and Fonseca-Castro (2024), P1's actions regarding geometric elements and the composition and role of the demonstration reflect a restricted conception of the proof as an act to validate the functionality of TT. In some instances, the proof is supported by technological and symbolic artefacts, activating the [Ins-Dis] plane. However, this plane is scarcely used to coordinate visualisation and proof in the origin domains (geometric).

Visualisation varies between *t1* (iconic) and *t*2 (non-iconic). While in *t1*, it is linked to the observation of figures, in *t2*, it is related to the decomposition of the units of the figure. In *t2*, the [Sem-Dis] plane is activated to demonstrate TT. The use of gestures remains an open topic that can contribute to the study of semiotic aspects of ThMWS and ThMTSK in initial or primary education.

Additionally, the proof serves as a support for working from the operational perspective that P1 encourages. From both theories, proof is recognised as a mathematical practice (Carrillo et al., 2018; Kuzniak, 2022). These practices constitute a *meeting point* in MWS-MTSK relationship, which allows for distinction between type of proof, its roles, and knowledge mobilisation.

The joint use of both theories allows emerging *blind spots* to be identified. From the ThMWS perspective, the phenomenon of domain change, from an origin domain (geometric) to a resolution domain (numerical), is observable without return to the origin. Meanwhile, the ThMTSK analysis contributes through the identification of knowledge of connections (KSM) and P1's learning expectations (KMLS), including the procedural emphasis. In this sense, the complementarity extends this interpretation of domain change from the ThMTSK perspective, which has been focused until now on auxiliary connection (Verdugo-Hernández et al., 2022).

Another blind spot exists regarding the pedagogical aspects addressed only by ThMTSK. In our analysis, we posit that PCK affects the structuring of the MWS^i, which still needs to be examined in greater depth (Espinoza-Vásquez et al., 2018). In particular, the design of the MWS^i could be understood regarding the KFLM subdomain (Flores-Medrano et al., 2016) or his teaching strategy. A deeper examination might be formulated distinguishing between *potential* and *actual* MWS^i (Henríquez-Rivas et al., 2022), which is difficult to research solely from the ThMTSK perspective.

Meanwhile, in terms of MWS-MTSK relationships, each theory considers tasks differently. While tasks are the *raison d'être* of the ThMWS which they contribute to activate and form (Kuzniak, 2022), for the ThMTSK they are part of the teaching (KMT). This differentiation allows their analysis to be complemented by both approaches. The study of the mathematical task is another meeting point (Espinoza-Vásquez et al., 2022), which contributes to understanding of teacher's practices.

Finally, the combination of both theories allows various functions of complementarity: *deepening* or *refinement* through the combination of its elements from similar perspectives (e.g., semiotic aspects); *broadening* through the combination of elements from differing perspectives (tools and connections); and *expansion* through the recognition of blind spots (PCK in MWS^i or domain changes).

6 Conclusions

The activation of geneses and planes of the MWS^i would evidence a *complete* MWS^i (Kuzniak et al., 2016), which seems to be accompanied by greater knowledge mobilisation in distinct subdomains (Tables 4 and 6). This study intends to contribute new explanations for the teacher's practices, the mathematical work proposed, and specialised knowledge, presenting the MWS-MTSK relationship as a tool for doing so through its complementarity. The joint use of the theories allows us to understand the type of work promoted based on semiotic, instrumental, and discursive aspects, and the MTSK mobilised. Moreover, it enables us to understand the emphasis and purposes of this work based on how P1 comprehends the mathematics involved.

The combination of theoretical elements has also allowed for the identification of meeting points and blind spots. The study of tasks, demonstration, or domain change through both theories opens the possibility of refining analyses and improving understanding of the teacher's practice and knowledge.

The exercise of connecting two theories may not result in a linear process, as outlined by Prediger et al. (2008). The comparison of the results (Tables 4 and 6) leads to the combination, but it requires observing how the principles (P) and research questions (Q) of each theory point to new questions. The comparison entails returning to *mutual knowledge* and examining their scopes to then identify complementarities. Thus, an open line of research on MWS-MTSK relationship consists of establishing connections at the level of their epistemological principles and assumptions (P), which requires another type of study regarding their cores.

Finally, as in Thanheiser et al. (2021), the results of the combination allow for an image of the classroom that would not be possible using only one theory, revealing its complexity more effectively. The MWS-MTSK relationship raises questions regarding both theories, for example, whether transition between certain geneses mobilises knowledge in relation to certain subdomains. This could stimulate the development of both or new theories through subsequent networking strategies (as local integration [Prediguer et al., 2008]), and make us more conscious of their reach and limitations (Bosch et al., 2017; Drijvers et al., 2013). Examples could include further development of semiotics as aspect of both considering PCK in the study of MWS^{i} or including the student's personal MWS to delve into the KFLM.

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Declarations

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