Manuscript submitted to Discrete and Continuous Dynamical Systems. doi:10.3934/xx.xx.xx.xx
AIMS' Journals
Volume **X**, Number **0X**, XX **200X** pp. **X–XX**

NON-FORMALLY INTEGRABLE CENTERS ADMITTING AN ALGEBRAIC INVERSE INTEGRATING FACTOR

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Abstract. We study the existence of a class of inverse integrating factor for a family of non-formally integrable systems whose lowest-degree quasi-homogeneous term is a Hamiltonian vector field. Once the existence of an inverse integrating factor is established, we study the systems having a center. Among others, we characterize the centers of the perturbations of the system $-y^3\partial_x + x^3\partial_y$ having an algebraic inverse integrating factor.

1. **Introduction.**

One of the classical problems in the qualitative theory of the analytic planar systems is to characterize when a monodromic point (*singular point which is surrounded by orbits of the system*) is a center, usually called elliptic point for integrable dynamical systems in dimension higher than two. For planar vector fields, this problem (so-called center problem) was solved theoretically for a non-degenerate singular point (systems whose linear part evaluated at the singular point have two imaginary nonzero eigenvalues) and for the nilpotent case. The problem is still unsolved for the remaining case, i.e. the systems with linear part identically zero at the singular point, called degenerate singular point.

One of the main tools used to characterize non-degenerate and nilpotent centers is the computation of a normal form [27, 28]. Normal forms for non-degenerate points are also of interest in symplectic geometry [10, 15, 16]. For the degenerate case, it is not surprising that a possible solution might be given by means of the theory of normal forms.

Another question related to the center problem is, once the monodromy is established, to determine the existence of an analytic first integral. Thus, for instance, for a non-degenerate singular point, the analytic integrability and the center problem are equivalent. Namely, the vector field $(-y+\cdots)\partial_x+(x+\cdots)\partial_y$ has a center at the origin if, and only if, it has an analytic first integral of the form $x^2+y^2+\cdots$. Otherwise, for nilpotent or degenerate singular points, in order to determine whether the singular point is a center, the existence of a first integral is a sufficient condition, but not a necessary one.

In this context, the existence of an integrating factor or an inverse integrating factor enables us to provide information about both center and integrability problems.

Key words and phrases. Nonlinear Differential Systems, Integrability, Inverse Integrating Factor, Centers, Normal Form.

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We consider an autonomous system

$$\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T, \ \mathbf{x} \in \mathbb{C}^2, \tag{1}$$

where $\mathbf{F}$ is an analytic planar vector field defined in a neighborhood of the origin $\Omega \subset \mathbb{C}^2$ having a singular point at the origin, i.e. $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $P, Q \in \mathbb{C}[[x, y]]$ (algebra of the power series in x and y with coefficients in $\mathbb{C}$).

A non-null $\mathcal{C}^1$ class function V is *an inverse integrating factor* of the system (1) (or also of $\mathbf{F}$) on Ω if it satisfies the linear partial differential equation $L_{\mathbf{F}}V = \mathrm{div}(\mathbf{F})V$, with $L_{\mathbf{F}}V := P\partial V/\partial x + Q\partial V/\partial y$, the Lie derivative of V with respect to $\mathbf{F}$, and $\mathrm{div}(\mathbf{F}) := \partial P/\partial x + \partial Q/\partial y$, the divergence of $\mathbf{F}$. This name for V comes from the fact that V^{-1} defines on $\Omega \setminus \{V = 0\}$ an integrating factor of the system (1), i.e. $\mathbf{F}/V$ is divergence-free. So, if the system (1) possesses a formal inverse integrating factor V then it is formally integrable on $\Omega \setminus \{V = 0\}$.

The existence of an inverse integrating factor has been used to study the center problem for non-degenerate singular point [26], the center problem and analytic integrability for nilpotent vector fields [7], the Darboux integrability [12, 13, 25], the Hopf bifurcation [19] and the existence of Lie symmetries [24].

The inverse integrating factor also plays an important role in the study of the dynamics of a system because the zero-set $\{V = 0\}$, formed by orbits of the system (1), contains the limit cycles and the homoclinic and heteroclinic connections between hyperbolic saddle equilibria which are in Ω [17, 20, 21, 23]. Moreover, the cyclicity of a limit cycle is related to the vanishing order of V [18].

The problem of the existence of an inverse integrating factor depending on the type of singularity has also been considered. Enciso and Peralta-Salas [17] studied the existence of a smooth inverse integrating factor in a neighborhood of an arbitrary elementary singularity, i.e. for the systems whose linear part at the origin have at least one nonzero eigenvalue. This extends previous results given in [11], [14, Theorem 5.2], where the authors consider elementary singularities that admit analytic orbital normalization. For stationary points with nilpotent or vanishing linearization, we only know the results obtained by Walcher [32]. These are partial results and show that generically there is a non-formal inverse integrating factor. Finally, Algaba *et al.* [7] characterize the nilpotent systems whose lowest-degree quasi-homogeneous term is $(y, \sigma x^n)^T$, $\sigma = \pm 1$, having a formal inverse integrating factor.

In this paper, we study the existence of an algebraic inverse integrating factor (which will be named AIIF) of a nilpotent or degenerate singular point. The class of systems studied here includes, among others, some families of great relevance. For example, the systems $(y+\cdots)\partial_x + (\sigma x^n + \cdots)\partial_y$ with n a natural number and $\sigma = \pm 1$ and also, the wide class of degenerate systems $(-\partial h/\partial y + \cdots)\partial_x + (\partial h/\partial x + \cdots)\partial_y$, with h a homogeneous polynomial of degree 3,4 or 5 having only simple factors on $\mathbb{C}[x, y]$.

This paper is organized as follows. Section 2 is devoted to provide an expansion in quasi-homogeneous terms of an orbital equivalent normal form of systems whose lowest-degree quasi-homogeneous term is a Hamiltonian vector field, Theorem 2.2.

Our results are presented in Section 3. We obtain in Theorem 3.1 a reduced normal form for the considered class of non-integrable systems. Theorem 3.2 characterizes, by means of normal form theory, when a system of this class admits an AIIF. Moreover, if the vector field has an AIIF, we give the expression of the inverse integrating factor (formal or algebraic) and solve the formal integrability problem.

Theorem 3.3 establishes when the origin of such systems is a monodromic point. Theorem 3.4 determines, assuming the existence of an AIIF and the monodromy of the origin, when the origin is either a center or a focus.

In Section 4, we compute the systems with an AIIF for perturbations of quadratic Hamiltonian systems, nilpotent Hamiltonian systems, $y^2\partial_x+x^3\partial_y$ and $-y^3\partial_x+x^3\partial_y$. Furthermore, when the origin is a monodromic singular point, we characterize their centers. So, for example, we prove that the system $(y^2+\frac{3}{4}ax^3)\partial_x+(x^3+ax^2y)\partial_y$, $a\neq 0$, has the AIIF $V=(4y^3-3x^4)^{13/12}$, but it is not formally integrable and does not have any formal inverse integrating factor (see Proposition 4.7).
We also prove that the system $y^3\partial_x+(-x^3+c_3x^2y^2+c_4xy^3)\partial_y$ with $(c_3,c_4)\neq(0,0)$, does not have an AIIF (see Proposition 4.10).

Finally, Section 5 contains the proofs of the results obtained.

2. **Quasi-homogeneous normal forms.**

Given $\mathbf{t}=(t_1,t_2)$ non-null with t_1 and t_2 non-negative integer numbers without common factors, in what follows, we denote the vector spaces of quasi-homogeneous polynomials and vector fields of type $\mathbf{t}$ and degree k by $\mathcal{P}_k^{\mathbf{t}}$ and $\mathcal{Q}_k^{\mathbf{t}}$, respectively, i.e.

$$\mathcal{P}_k^{\mathbf{t}}=\{f\in\mathbb{C}[x,y]:f(\varepsilon^{t_1}x,\varepsilon^{t_2}y)=\varepsilon^k f(x,y)\},\quad k\in\mathbb{N}_0,\ (\mathbb{N}_0=\mathbb{N}\cup\{0\})$$

and

$$\mathcal{Q}_k^{\mathbf{t}}=\{\mathbf{F}=(P,Q)^T:P\in\mathcal{P}_{k+t_1}^{\mathbf{t}},\ Q\in\mathcal{P}_{k+t_2}^{\mathbf{t}}\}\quad k+t_1,k+t_2\in\mathbb{N}_0.$$

Any vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F}$ can be written in the form

$$\mathbf{F}=\mathbf{F}_r+\mathbf{F}_{r+1}+\cdots,$$

for some $r\in\mathbb{Z}$, where $\mathbf{F}_j=(P_{j+t_1},Q_{j+t_2})^T\in\mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r\not\equiv\mathbf{0}$. Such expansions are expressed as $\mathbf{F}=\mathbf{F}_r+$q-h.h.o.t., where "q-h.h.o.t." means "quasi-homogeneous higher order terms".

If we select the type $\mathbf{t}=(1,1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees.

In this paper we will use the following result.

Lemma 2.1. [3] *For fixed* $\mathbf{t}=(t_1,t_2)$,

1. $\mathcal{P}_k^{\mathbf{t}}=\{0\}$, *if* $k\notin\mathcal{I}^{\mathbf{t}}$,
2. *if* $t_1=1$, *for every* $t_2\geq 1$, *the sets* $\mathcal{P}_k^{\mathbf{t}}$ *are nontrivial spaces for all* k,
3. *if* $k>t_1t_2-|\mathbf{t}|$, *then* $k\in\mathcal{I}^{\mathbf{t}}$, *i.e.* $\mathcal{P}_k^{\mathbf{t}}$ *is a nontrivial space,*

where $\mathcal{I}^{\mathbf{t}}=\{k=k_1t_1+k_2t_2+k_3t_1t_2\in\mathbb{N}:k_1,k_2,k_3\in\mathbb{N},k_1<t_2,k_2<t_1\}$.

From Lemma 2.1, there exists $m_0:=\max\{\mathbb{N}_0\setminus\mathcal{I}^{\mathbf{t}}\}$.

Now we illustrate this lemma for the type $\mathbf{t}=(3,4)$.

Example 1. Under the degree, the first quasi-homogeneous polynomials of type $(3,4)$ are:

$$\begin{aligned}
\mathcal{P}_3^{(3,4)}&=\{x\}, & \mathcal{P}_4^{(3,4)}&=\{y\}, & \mathcal{P}_6^{(3,4)}&=\{x^2\},\\
\mathcal{P}_7^{(3,4)}&=\{xy\}, & \mathcal{P}_8^{(3,4)}&=\{y^2\}, & \mathcal{P}_9^{(3,4)}&=\{x^3\},\\
\mathcal{P}_{10}^{(3,4)}&=\{x^2y\}, & \mathcal{P}_{11}^{(3,4)}&=\{xy^2\}, & \mathcal{P}_{12}^{(3,4)}&=\{y^3,\ x^4\}.
\end{aligned}$$

Thus, $\mathbb{N}_0 \setminus \mathcal{I}^{(3,4)} = \{1,2,5\}$ and $m_0 = 5$. The quasi-homogeneous expansion with respect to $\mathbf{t} = (3,4)$ of a vector field $\mathbf{F} = (\sum_{j\geq 1} P_j(x,y), \sum_{j\geq 1} Q_j(x,y))^T$ with $P_j(x,y) = \sum_{j=m+n} a_{mn}x^m y^n$ and $Q_j(x,y) = \sum_{j=m+n} b_{mn}x^m y^n$, is

$$\begin{aligned}\mathbf{F} \;=\; & \overbrace{\begin{pmatrix} 0 \\ b_{10}x \end{pmatrix}}^{\mathbf{F}_{-1}} + \overbrace{\begin{pmatrix} a_{10}x \\ b_{01}y \end{pmatrix}}^{\mathbf{F}_0} + \overbrace{\begin{pmatrix} a_{01}y \\ 0 \end{pmatrix}}^{\mathbf{F}_1} + \overbrace{\begin{pmatrix} 0 \\ b_{20}x^2 \end{pmatrix}}^{\mathbf{F}_2} + \overbrace{\begin{pmatrix} a_{20}x^2 \\ b_{11}xy \end{pmatrix}}^{\mathbf{F}_3} + \overbrace{\begin{pmatrix} a_{11}xy \\ b_{02}y^2 \end{pmatrix}}^{\mathbf{F}_4} \\ & + \overbrace{\begin{pmatrix} a_{02}y^2 \\ b_{30}x^3 \end{pmatrix}}^{\mathbf{F}_5} + \overbrace{\begin{pmatrix} a_{30}x^3 \\ b_{21}x^2y \end{pmatrix}}^{\mathbf{F}_6} + \overbrace{\begin{pmatrix} a_{21}x^2y \\ b_{12}xy^2 \end{pmatrix}}^{\mathbf{F}_7} + \overbrace{\begin{pmatrix} a_{12}xy^2 \\ b_{03}y^3 + b_{40}x^4 \end{pmatrix}}^{\mathbf{F}_8} + \text{q-h.h.o.t.}\end{aligned}$$

The key in the problem of obtaining a normal form of system (1) is to analyze the effect of a near-identity transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$ and a reparametrization of the time by $\frac{\mathrm{dt}}{\mathrm{dT}} = 1 + \tau_k(\mathbf{x})$, where $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ and $\tau_k \in \mathcal{P}_k^{\mathbf{t}}$, with $k \geq 1$.

The quasi-homogeneous terms of the transformed system $\mathbf{y}' = \frac{\mathrm{d}\mathbf{y}}{\mathrm{dT}} = \mathbf{G}(\mathbf{y})$ agree with the original ones up to degree $r+k-1$ and for the degree $r+k$ it holds

$$\begin{aligned}\mathbf{G}_{r+k} &= \mathbf{F}_{r+k} - (D\mathbf{P}_k\mathbf{F}_r - D\mathbf{F}_r\mathbf{P}_k) + \tau_k\mathbf{F}_r = \mathbf{F}_{r+k} - [\mathbf{P}_k, \mathbf{F}_r] + \tau_k\mathbf{F}_r \\ &= \mathbf{F}_{r+k} - \mathcal{L}_{r+k}(\mathbf{P}_k, \tau_k),\end{aligned}$$

where we have introduced the homological operator under orbital equivalence

$$\begin{aligned}\mathcal{L}_{r+k} \quad : \quad & \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \tau_k) \to \mathcal{L}_{r+k}(\mathbf{P}_k, \tau_k) = [\mathbf{P}_k, \mathbf{F}_r] - \tau_k\,\mathbf{F}_r,\end{aligned}$$

where $[\mathbf{P}_k, \mathbf{F}_r] := D\mathbf{P}_k\mathbf{F}_r - D\mathbf{F}_r\mathbf{P}_k$ is the Lie bracket of $\mathbf{P}_k$ and $\mathbf{F}_r$. Following the ideas of the conventional normal form theory, it is enough to choose $(\mathbf{P}_k, \tau_k) \in \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}}$ adequately in order to simplify the $(r+k)$-degree quasi-homogeneous term in system (1), by annihilating the part belonging to the range of the linear operator $\mathcal{L}_{r+k}$. In other words, we can achieve that $\mathbf{F}_{r+k}$ belongs to a complementary subspace to the range of $\mathcal{L}_{r+k}$. When this has been done, we say that *the corresponding term has been reduced to normal form under orbital equivalence.* So, by means of a sequence of time-reparametrizations and near-identity transformations (by performing the procedure first for $k=1$, then for $k=2$ and so on) the system (1) can be formally reduced to normal form under orbital equivalence, i.e. the system can be transformed into

$$\mathbf{y}' = \frac{\mathrm{d}\mathbf{y}}{\mathrm{dT}} = \mathbf{G}(\mathbf{y}) = \mathbf{G}_r(\mathbf{y}) + \mathbf{G}_{r+1}(\mathbf{y}) + \cdots,$$

with $\mathbf{G}_r \not\equiv \mathbf{0}$ and $\mathbf{G}_{r+k} \in \mathrm{Cor}(\mathcal{L}_{r+k}) \subseteq \mathcal{Q}_{r+k}^{\mathbf{t}}$ where $\mathrm{Cor}(\mathcal{L}_{r+k})$ is any complementary subspace to the range of the homological operator $\mathcal{L}_{r+k}$. We note that such space is not unique, in general.

2.1. **Orbital equivalent normal forms for perturbations of a quasi-homogeneous Hamiltonian system.**

For a type $\mathbf{t}$ fixed, we consider the systems whose quasi-homogeneous expansion is

$$\dot{\mathbf{x}} = \mathbf{X}_h + \text{q-h.h.o.t.}, \tag{2}$$

where $\mathbf{X}_h := (-\partial h/\partial y, \partial h/\partial x)^T \in \mathcal{Q}_r^{\mathbf{t}}$, with r a non-negative integer (thus $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$); i.e. a class of systems which can be considered as perturbations of a Hamiltonian system whose Hamiltonian function h is a quasi-homogeneous polynomial.

For each $j > r$, we define the linear operator

$$\begin{aligned}\ell_j \quad : \quad & \mathcal{P}^{\mathbf{t}}_{j-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_j \\ & \mu_{j-r} \longrightarrow \ell_j(\mu_{j-r}) := \frac{\partial h}{\partial x}\frac{\partial \mu_{j-r}}{\partial y} - \frac{\partial h}{\partial y}\frac{\partial \mu_{j-r}}{\partial x},\end{aligned}$$

(Poisson bracket of h and μ_{j-r}) and denote a complementary subspace to the range of the linear operator ℓ_j (co-range of the operator ℓ_j) by $\mathrm{Cor}(\ell_j)$.

It is always possible to choose the subspaces $\mathrm{Cor}(\ell_j)$ such that $\mathrm{Cor}(\ell_{r+|\mathbf{t}|+j}) = h\mathrm{Cor}(\ell_j)$ for all $j > r$ with $\mathcal{P}^{\mathbf{t}}_{j-r} \neq \{0\}$ [3, Proposition 3.18].

We define the following subsets of $\mathbb{N}_0$:

$$\begin{aligned}\mathcal{J}_1 &= \{j,\ j \le r\},\\ \mathcal{J}_2 &= \{j,\ j \ge r+1 \text{ such that } \mathcal{P}^{\mathbf{t}}_{j-r} = \{0\}\},\\ \mathcal{J} &= \{j \in \mathcal{J}_1 \cup \mathcal{J}_2,\ S_{r+|\mathbf{t}|+j} \neq \{0\}\}.\end{aligned}$$

From Lemma 2.1, there exists $m_0 := \max\{\mathbb{N}_0 \setminus \mathcal{I}^{\mathbf{t}}\}$. Thus, $j \notin \mathcal{J}_2$, for all $j \ge n_0 := 1 + r + m_0$.

An orbital equivalent normal form for perturbations of a quasi-homogeneous Hamiltonian system is provided in the following result.

Theorem 2.2. [2, Theorem 10] *Consider the system (2) with h having only simple factors on $\mathbb{C}[x, y]$. An orbital equivalent normal form of the system (2) becomes*

$$\dot{\mathbf{x}} = \mathbf{X}_{h+g} + \mu \mathbf{D}_0, \tag{3}$$

$\mathbf{D}_0 := (t_1 x, t_2 y)^T \in \mathcal{Q}^{\mathbf{t}}_0$, *where* $g = \sum_{j\in\mathcal{J}} g_{r+|\mathbf{t}|+j}$ *with* $g_{r+|\mathbf{t}|+j} \in S_{r+|\mathbf{t}|+j}$, $S_{r+|\mathbf{t}|+j}$ *a subspace verifying* $Cor(\ell_{r+|\mathbf{t}|+j}) = S_{r+|\mathbf{t}|+j} \oplus h\,Cor(\ell_j)$, *and* $\mu = \sum_{j>r} \mu_j$ *with* $\mu_j \in Cor(\ell_j)$ *and* $\mu_{r+|\mathbf{t}|+j} = h\mu_j$ *for all* $j \notin \mathcal{J}_1 \cup \mathcal{J}_2$ *(i.e.* $j \ge n_0$*).*

Thus, to obtain a normal form, it is enough only to calculate the co-ranges of ℓ_j for $j = r+1, \ldots, n_0 + r + |\mathbf{t}| - 1$.

We emphasize that if the factorization of h on $\mathbb{C}[x, y]$ has only simple factors, then there is, at most, a finite number of monomials g_j in the expression of (3). Here, in this paper, we study the systems satisfying $g_j \equiv 0$, i.e. systems orbitally equivalent to systems of the form $\dot{\mathbf{x}} = \mathbf{X}_h + \mu \mathbf{D}_0$.
Such class of systems is a wide family and contains, among others, the non-degenerate saddle ($h = xy$), linear center ($h = x^2 + y^2$), the nilpotent systems of the form $(\dot{x}, \dot{y}) = (y, \sigma x^n)$ + q-h.h.o.t with $\sigma = \pm 1$ ($h = \frac{\sigma}{n+1}x^{n+1} - \frac{y^2}{2}$), the systems $(-\partial h/\partial y + \cdots)\partial_x + (\partial h/\partial x + \cdots)\partial_y$ with h a homogeneous polynomial of degree 3, 4 or 5 and h having only simple factors on $\mathbb{C}[x, y]$.
Nevertheless, the system (2) for $h = x^6/6 + y^4/4$ is not orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu \mathbf{D}_0$ [4, Theorem 4.18].

3. Main results.

We introduce the systems under consideration in this paper. We define $\mathcal{F}^{\mathbf{t}}_{r+|t|}$ as the set of $h \in \mathcal{P}^{\mathbf{t}}_{r+|t|}$ whose factorization on $\mathbb{C}[x, y]$ has only simple factors and satisfies that

> $\mathcal{J}$ is an empty set, i.e. $h\mathcal{P}^t_j$ is a complementary subspace to the range of $\ell_{r+|\mathbf{t}|+j}$ for $j \le r$ or j such that $\mathcal{P}^t_{j-r} = \{0\}$.

So, under Theorem 2.2, the systems we consider are orbitally equivalent to the system (3) with $g \equiv 0$.

In this section, we characterize the systems (2) with $h \in \mathcal{F}^{\mathbf{t}}_{r+|\mathbf{t}|}$ which have an AIIF and we also determine which ones are centers. To achieve our goal, we need to provide a reduced normal form under orbital equivalence of the normal form given by Theorem 2.2.

Let $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ be a polynomial whose factorization on $\mathbb{C}[x, y]$ only has simple factors. We consider the system

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+N}\mathbf{D}_0 + \sum_{j>r+N} \mu_j \mathbf{D}_0, \tag{4}$$

with $\mu_{r+N} \not\equiv 0$ and $\mu_j \in \mathrm{Cor}(\ell_j),\ j \geq r+N$. The system (4) is a normal form for $h \in \mathcal{F}^{\mathbf{t}}_{r+|t|}$.

We define the linear operator $\ell^{(2)}_{r+j} : \mathcal{P}^{t}_{j} \times \mathrm{Ker}(\ell_{r+j-N}) \longrightarrow \mathcal{P}^{\mathbf{t}}_{r+j}$ as

$$\begin{aligned} \ell^{(2)}_{r+j}(\mu_j, \alpha h^{l_1}) &:= \ell_{r+j}(\mu_j) + \alpha\mu_{r+N} h^{l_1}, \quad &\text{if } l_2 = 0,\\ \ell^{(2)}_{r+j}(\mu_j, 0) &:= \ell_{r+j}(\mu_j), &\text{if } l_2 \neq 0, \end{aligned}$$

where $j = N + l_1(r + |\mathbf{t}|) + l_2$, with $0 \leq l_2 < r + |\mathbf{t}|$.

It is always possible to choose $\mathrm{Cor}(\ell_j)$ and $\mathrm{Cor}(\ell^{(2)}_j)$, co-range of the linear operators ℓ_j and $\ell^{(2)}_j$, respectively, such that $\mathrm{Cor}(\ell^{(2)}_j) \subseteq \mathrm{Cor}(\ell_j)$. Therefore, the normal form provided in the following result is simpler than the normal form (3).

Theorem 3.1 (Reduced normal form of the systems (4))**.** *The following statements are satisfied:*

1. *if $N \neq j(r+|\mathbf{t}|)$ with j any natural number, then a formal normal form under orbital equivalence for the system (4) is*
$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+N}\mathbf{D}_0 + \sum_{j>r+N} \tilde{\mu}_j \mathbf{D}_0,$$
where $\tilde{\mu}_j \in Cor(\ell^{(2)}_j)$,
2. *if $N = s(r + |\mathbf{t}|)$ with s a natural number, then a formal normal form under orbital equivalence for the system (4) is*
$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+N}\mathbf{D}_0 + \sum_{j>r+N} \tilde{\mu}_j \mathbf{D}_0,$$
where $\tilde{\mu}_{r+2N} \in Cor(\ell_{r+2N})$ and $\tilde{\mu}_j \in Cor(\ell^{(2)}_j),\ j \neq r+2N$.

The main result of this paper is stated in the next theorem.

Theorem 3.2. *The system $\dot{\mathbf{x}} = \mathbf{X}_h +$ q-h.h.o.t. with $h \in \mathcal{F}^{\mathbf{t}}_{r+|\mathbf{t}|}$ possesses an AIIF if, and only if, it is formally orbital equivalent to one, and only one, of the following systems:*

1. $$\dot{\mathbf{x}} = \mathbf{X}_h \tag{5}$$
(formally integrable system). Moreover, $f(h)$, with f any algebraic scalar function, is an AIIF.
2. $$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+N}\mathbf{D}_0, \tag{6}$$
with $N \neq j(r+|t|)$ for all natural j and $\mu_{r+N} \in Cor(\ell_{r+N})\backslash\{0\}$ (non-formally integrable system). Moreover, the AIIF is $(h +$ q-h.h.o.t.$)^{1+N/(r+|\mathbf{t}|)}$, up to a multiplicative constant; i.e. it is not formal.

3.
$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+N}\mathbf{D}_0 + \alpha h^s \mu_{r+N}\mathbf{D}_0, \tag{7}$$
where $N = s(r + |\mathbf{t}|)$ *with* s *a natural number,* $\mu_{r+N} \in Cor(\ell_{r+N}) \setminus \{0\}$ *(non-formally integrable system) and* α *a real number. Moreover, the AIIF is* $(h + q\text{-}h.h.o.t.)^{1+s}$*, up to a multiplicative constant; i.e. it is formal.*

We solve the monodromic and the center problems of the system (2). For the monodromy problem, it has the following result.

Theorem 3.3. *The origin of the* $\dot{\mathbf{x}} = \mathbf{X}_h +$ *q-h.h.o.t. with* h *having only simple factors on* $\mathbb{C}[x, y]$*, is monodromic if, and only if,* h *is zero only at the origin.*

Note that if the origin is a monodromic point and the system is formally integrable, then the origin is a center. Last on, we state the result which gives title to this work. It characterizes the centers of the non-formally integrable systems (2) having an AIIF.

Theorem 3.4. *Consider the system* $\dot{\mathbf{x}} = \mathbf{X}_h +$ *q-h.h.o.t. with* $h \in \mathcal{F}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *and* h *vanishes only at the origin. Assume that it is a non-formally integrable and possesses an AIIF. Then, the origin is:*

1. *a center, if* $I = 0$,
2. *an unstable focus, if* $sig(h)I > 0$,
3. *a stable focus, if* $sig(h)I < 0$,

where $I = \int_{h=sig(h)} \mu_{r+N}$ *and* μ_{r+N} *is given by the normal form (4).*

4. **Some examples and applications.** In this section we determine several families of the systems (2) admitting an AIIF. For the monodromic case, we also characterize their centers.

The easiest case appears when the polynomial h is quadratic, that is, the lowest-degree part of the system is linear, i.e. systems (2) with $\mathbf{t} = (1, 1)$, $r = 0$ and $h = (\sigma x^2 - y^2)/2$ (after a change of variable if it would be necessary) with $\sigma = 1$ or -1. The origin of the system (2) is a weak saddle (for $\sigma = 1$) or a non-degenerate center-focus (for $\sigma = -1$). It is easy to check that $h \in \mathcal{F}^{(1,1)}_2$. One has also that $\mathrm{Cor}(\ell_{2k}) = \mathrm{span}\{h^k\}$ and $\mathrm{Cor}(\ell_j) = \{0\}$, otherwise. Thus, from Theorem 2.2, these systems are formally orbital equivalent to
$$(\dot{x}, \dot{y})^T = (y, \sigma x)^T + \sum_{k\geq 1} \alpha_{2k}(\sigma x^2 - y^2)^k (x, y)^T.$$

From Theorem 3.1, a reduced normal form of this normal form is
$$(\dot{x}, \dot{y})^T = (y, \sigma x)^T + \alpha_{2s+1}(\sigma x^2 - y^2)^s (x, y)^T + \alpha_{4s+1}(\sigma x^2 - y^2)^{2s}(x, y)^T,$$
for a certain s, which is the normal form (7) given in Theorem 3.2. Therefore, both have a formal inverse integrating factor.

Moreover, they are integrable systems if, and only if, $\alpha_{2s+1} = \alpha_{4s+1} = 0$, i.e. they are linearizable. Some related partial results can be seen in [8, 11, 31].

A) Perturbations of Hamiltonian quadratic systems. These systems are
$$(\dot{x}, \dot{y})^T = \mathbf{X}_h + \text{h.o.t.}, \quad h = ax^3 + bx^2y + cxy^2 + dy^3. \tag{8}$$
That is, systems (2) with $\mathbf{t} = (1, 1)$ and $r = 1$.

From Theorem 3.3, the origin of these systems is non-monodromic. We are looking for systems (8) with an AIIF. For $d \neq 0$, without loss of generality, we can assume $c = 0$ and $d = 1$, the polynomial h has only simple factors if $27a^2 + 4b^3 \neq 0$, and by Lemma 2.1, the sets $\mathcal{P}_j^{\mathbf{t}}$ are nontrivial spaces for all j. Table 1 shows the range and co-range of the operator $\ell_j,\ j=2,3,4$ for the system (8) with $d\neq 0$. It is straightforward to check that $\mathcal{J}$ is an empty set and therefore the polynomial h belongs to $\mathcal{F}_3^{(1,1)}$ if $27a^2+4b^3\neq 0$.

TABLE 1. Range and co-range of operator ℓ_j for the system (8).

Range(ℓ_2)=span$\{-bx^2-3y^2, 3ax^2+2bxy\}$. If $a\neq 0$, Cor(ℓ_2)=span$\{xy\}$. If $a=0$, Cor(ℓ_2)=span$\{x^2\}$.
Range(ℓ_3)=span$\{-2bx^3-6xy^2, 6ax^3+4bx^2y-3h, 6ax^2y+4bxy^2\}$. Cor($\ell_3$)=span$\{h\}$.
Range(ℓ_4)=span$\{3bx^4+9x^2y^2, -9ax^4-6bx^3y+6xh,$ $-9ax^3y-6bx^2y^2+3yh\}$. Cor(ℓ_4)=span$\{xh, yh\}$.

The normal form (3) of the system (8) becomes

$$(\dot{x},\dot{y})^T=(-bx^2-3y^2, 3ax^2+2bxy)^T+\sum_{j\geq 0} f_j(x,y,h)h^j(x,y)^T,$$

with $f_j\in\text{span}\{h, xh, yh, xyh\}$ if $a\neq 0$, or $f_j\in\text{span}\{h, xh, yh, x^2h\}$ if $a=0$.

Applying Theorem 3.2, we get the following result.

Theorem 4.1. *System (8) with $h=ax^3+bx^2y+y^3$ and $27a^2+4b^3\neq 0$, possesses an AIIF if, and only if, it is formally orbital equivalent to one of the following systems:*

1. $(\dot{x},\dot{y})^T=\mathbf{X}_h$. *In this case, it has an AIIF of the form $g(h+$ q-h.h.o.t.$)$ with g any nonzero function. In particular, there are nonzero inverse integrating factors at the origin.*
2. $(\dot{x},\dot{y})^T=\mathbf{X}_h+\alpha_{3j}h^j(x,y)^T,\ \alpha_{3j}\neq 0, j\geq 1$. *Moreover, the AIIF of the system (8) is of the form $(h+$ q-h.h.o.t.$)^{j+2/3}$, up to a multiplicative constant.*
3. $(\dot{x},\dot{y})^T=\mathbf{X}_h+(\alpha_{3j+1}x+\beta_{3j+1}y)h^j(x,y)^T+\alpha(\alpha_{3j+1}x+\beta_{3j+1}y)h^{2j}(x,y)^T$, *with α a real number and $(\alpha_{3j+1},\beta_{3j+1})\neq(0,0), j\geq 1$. Moreover, the AIIF of the system (8) is of the form $(h+$ q-h.h.o.t.$)^{1+j}$, up to a multiplicative constant; i.e. it is a formal inverse integrating factor.*
4. $(\dot{x},\dot{y})^T=\mathbf{X}_h+\alpha_{3j+2}xyh^j(x,y)^T$ *if $a\neq 0$, or $(\dot{x},\dot{y})^T=\mathbf{X}_h+\alpha_{3j+2}x^2h^j(x,y)^T$ if $a=0$ with $\alpha_{3j+2}\neq 0, j\geq 1$. Moreover, the AIIF of the system (8) is of the form $(h+$ q-h.h.o.t.$)^{j+4/3}$, up to a multiplicative constant.*

B) Centers of perturbations of nilpotent Hamiltonian systems. We consider the nilpotent systems whose quasi-homogeneous expansion is of the form

$$(\dot{x},\dot{y})^T=(y,\sigma x^n)^T+\text{q-h.h.o.t.},\qquad \sigma=\pm 1. \tag{9}$$

From Theorem 3.3, the origin is not monodromic if, and only if, n is even, or n is odd and $\sigma=1$.

We analyze the center problem for the system (9) admitting an AIIF. We assume that the origin is monodromic, i.e. n odd ($n=2m-1$) and $\sigma=-1$. These systems are

$$(\dot{x},\dot{y})^T=(y,-x^{2m-1})^T+\text{q-h.h.o.t.},\quad m\geq 1. \tag{10}$$

They correspond to systems (2) with $\mathbf{t}=(1,m)$, $r=m-1$ and $h=\frac{1}{2m}x^{2m}+\frac{1}{2}my^2\in\mathcal{P}^{\mathbf{t}}_{2m}$.

The center problem for the nilpotent systems was solved theoretically in [9, 27, 30].

We give the following result which characterizes the centers of the nilpotent systems (10) having an AIIF.

Theorem 4.2. *Assume that system (10) possesses an AIIF. Then, the origin is a center if, and only if, it is formally orbital equivalent to*

$$(\dot{x},\dot{y})^T=(y,-x^{2m-1})^T+Ax^{2l-1}h^L(x,my)^T+Bx^{2l-1}h^{2L}(x,my)^T, \tag{11}$$

with $B=0$ if $2l\neq m$.

Proof. From [5, Theorem 2], if the system (10) possesses an AIIF then it is formally orbital equivalent either to $(\dot{x},\dot{y})^T=(y,-x^{2m-1})^T$ which is a center, or to

$$(\dot{x},\dot{y})^T=(y,-x^{2m-1})^T+Ax^Mh^Lf(h)(x,my)^T, \tag{12}$$

with A a nonzero real number, f a function with $f(0)=1$, L a non-negative integer, and $M\in\{0,1,\dots,2m-2\}$ if $L>0$ or $M\in\{m,m+1,\dots,2m-2\}$ if $L=0$.

By applying Theorem 3.2, we obtain a further reduction of the normal form (12) of the system (10):

$$(\dot{x},\dot{y})^T=(y,-x^{2m-1})^T+Ax^Mh^L(x,my)^T+Bx^Mh^{2L}(x,my)^T, \tag{13}$$

with $B=0$ if $M\neq m-1$.

In order to get the centers, it is enough to compute the integral I given by Theorem 3.4. In this case $I=A\int_0^T \mathrm{Cs}^M(\theta)d\theta$, where $(\mathrm{Cs}(\theta),Sn(\theta))^T$ is the solution of the initial value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\theta}=\mathbf{X}_h(\mathbf{x}),\quad \mathbf{x}(0)=(1,0)^T,$$

and T is a minimal period of both functions.

It is known that the integral I is different from zero if, and only if, M is even. So, $M=2l-1$; that is, the system (13) becomes the system (11).

The sufficient condition follows since the system (11) is time-reversible and its origin is a monodromic singular point. ♢

We note that if $2l\neq m$ and $A\neq 0$, the inverse integrating factor is not formal. Consequently, the centers of the systems (10) having an AIIF are formally orbital-equivalent to time-reversible systems but not all of them have a formal inverse integrating factor.

C) Perturbations of the system $y^2\partial_x+x^3\partial_y$. We consider the degenerate systems of the form

$$(\dot{x},\dot{y})^T=(y^2+\sum_{j\geq 3}P_j(x,y),\sum_{j\geq 3}Q_j(x,y))^T, \tag{14}$$

with P_j and Q_j homogeneous polynomials of degree j and $Q_3(1,0)\neq 0$ (without loss of generality, we can assume $Q_3(1,0)=1$).

The quasi-homogeneous expansion with respect to $\mathbf{t} = (3,4)$ of the system (14) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \overbrace{\begin{pmatrix} y^2 \\ x^3 \end{pmatrix}}^{\mathbf{F}_5} + \overbrace{\begin{pmatrix} a_{30}x^3 \\ b_{21}x^2y \end{pmatrix}}^{\mathbf{F}_6} + \overbrace{\begin{pmatrix} a_{21}x^2y \\ b_{12}xy^2 \end{pmatrix}}^{\mathbf{F}_7} + \overbrace{\begin{pmatrix} a_{12}xy^2 \\ b_{03}y^3 + b_{40}x^4 \end{pmatrix}}^{\mathbf{F}_8} + \text{q-h.h.o.t.}, \tag{15}$$

see Example 1; i.e. the system (2) with $r = 5$ and $h = x^4/4 - y^3/3$ (h has only simple factors on $\mathbb{C}[x,y]$). Also, $m_0 = 5$, $n_0 = 11$, $\mathcal{J}_1 = \{1,2,3,4,5\}$ and $\mathcal{J}_2 = \{6,7,10\}$.

TABLE 2. Range and co-range of operator ℓ_j for the system (15)

Range(ℓ_6)=span$\{0\}$,	Cor(ℓ_6)=span$\{x^2\}$
Range(ℓ_7)=span$\{0\}$,	Cor(ℓ_7)=span$\{xy\}$
Range(ℓ_8)=span$\{y^2\}$,	Cor(ℓ_8)=span$\{0\}$
Range(ℓ_9)=span$\{x^3\}$,	Cor(ℓ_9)=span$\{0\}$
Range(ℓ_{10})=span$\{0\}$,	Cor(ℓ_{10})=span$\{x^2y\}$
Range(ℓ_{11})=span$\{xy^2\}$,	Cor(ℓ_{11})=span$\{0\}$
Range(ℓ_{12})=span$\{7x^4 - 12h\}$,	Cor(ℓ_{12})=span$\{h\}$
Range(ℓ_{13})=span$\{x^3y\}$,	Cor(ℓ_{13})=$\{0\}$
Range(ℓ_{14})=span$\{x^2y^2\}$,	Cor(ℓ_{14})=$\{0\}$
Range(ℓ_{15})=span$\{x^3 - 6xh\}$,	Cor(ℓ_{15})=span$\{xh\}$
Range(ℓ_{16})=span$\{11x^4y - 12yh\}$,	Cor(ℓ_{16})=span$\{yh\}$
Range(ℓ_{17})=span$\{x^3y^2\}$,	Cor(ℓ_{17})=$\{0\}$
Range(ℓ_{18})=span$\{13x^6 - 36x^2h\}$,	Cor(ℓ_{18})=span$\{x^2h\}$
Range(ℓ_{19})=span$\{7x^5 - 12xyh\}$,	Cor(ℓ_{19})=span$\{xyh\}$
Range(ℓ_{22})=span$\{17x^6y - 9x^2yh\}$,	Cor(ℓ_{22})=span$\{x^2yh\}$

As above, we observe that h belongs to $\mathcal{F}_{12}^{(3,4)}$. So, from Theorem 2.2, a normal form of the system (15) becomes

$$(\dot{x}, \dot{y})^T = (y^2, x^3)^T + \sum_{j \geq 0} f_j(x,y,h) h^j (3x, 4y)^T, \tag{16}$$

with $f_j \in \text{span}\{x^2, xy, x^2y, h, xh, yh\}$.

As a consequence of Theorem 3.2, we give the following result which characterizes the systems (14) with an AIIF.

Theorem 4.3. *System (14) possesses an AIIF if, and only if, it is formally orbital equivalent to one of the following systems:*

1. $(\dot{x}, \dot{y})^T = \mathbf{X}_h$. *The AIIF of the system (14) is $g(h + \text{q-h.h.o.t.})$ with g any nonzero function (in particular, there are nonzero inverse integrating factors at the origin).*
2. $(\dot{x}, \dot{y})^T = \mathbf{X}_h + \alpha_{12j+6} x^2 h^j (3x, 4y)^T$. *Moreover, the AIIF of the system (14) is of the form $(h + \text{q-h.h.o.t.})^{1+j+1/12}$, up to a multiplicative constant.*

3. $(\dot{x},\dot{y})^T = \mathbf{X}_h + \alpha_{12j+7}xyh^j(3x,4y)^T$. *Moreover, the AIIF of the system (14) is of the form* $(h + q\text{-}h.h.o.t.)^{1+j+1/6}$.
4. $(\dot{x},\dot{y})^T = \mathbf{X}_h + \alpha_{12j+10}x^2yh^j(3x,4y)^T$. *Moreover, the AIIF of the system (14) is of the form* $(h + q\text{-}h.h.o.t.)^{1+j+5/12}$*, up to a multiplicative constant.*
5. $(\dot{x},\dot{y})^T = \mathbf{X}_h + \alpha_{12j+12}h^{j+1}(3x,4y)^T$. *Moreover, the AIIF of the system (14) is of the form* $(h + q\text{-}h.h.o.t.)^{1+j+7/12}$*, up to a multiplicative constant.*
6. $(\dot{x},\dot{y})^T = \mathbf{X}_h + \alpha_{12j+15}xh^{j+1}(3x,4y)^T$. *Moreover, the AIIF of the system (14) is of the form* $(h + q\text{-}h.h.o.t.)^{1+j+10/12}$*, up to a multiplicative constant.*
7. $(\dot{x},\dot{y})^T = \mathbf{X}_h + \alpha_{12j+16}yh^{j+1}(3x,4y)^T$. *Moreover, the AIIF of the system (14) is of the form* $(h + q\text{-}h.h.o.t.)^{1+j+11/12}$*, up to a multiplicative constant*

with $\alpha_k \neq 0$ *and* $j \geq 0$.

We claim that the AIIF's of the nonformally integrable systems (14) are algebraic but not formal. Consequently, we get the following result.

Theorem 4.4. *System (14) is formally integrable if, and only if, it admits a formal inverse integrating factor.*

Next, we give necessary conditions for the existence of an AIIF for the system (14). The first two coefficients of the right-hand side of (16) are

$$\alpha_6 = 3a_{30} + b_{21}, \tag{17}$$

$$\alpha_7 = 13(a_{21} + b_{12}) + (3a_{30} + b_{21})(4a_{30} - 3b_{21}). \tag{18}$$

We have obtained these coefficients of the quasi-homogeneous normal form by using the procedure given in [1].
From Theorem 4.3, we have the following result.

Theorem 4.5. *System (14) with* $3a_{30} + b_{21} \neq 0$ *is not formally integrable. Moreover, if it possesses an AIIF, then* $13(a_{21}+b_{12})+(3a_{30}+b_{21})(4a_{30}-3b_{21}) = 0$*. The AIIF is* $(4y^3 - 3x^4 + q - h.h.o.t.)^{13/12}exp(u)$*, for some series* u *which is unique up to an additive constant.*

Example 2. We study a particular case of systems (14). We consider the family of systems (14) with $P_j = Q_j \equiv 0$ for $j > 3$, and $P_3(1,0) = 0$ ($a_{30} = 0$), that is,

$$(y^2 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3)\partial_x + (x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3)\partial_y. \tag{19}$$

We give the following result.

Proposition 4.6. *Assume that the system (19) possesses an AIIF. The following statements are satisfied:*

1. *if* $b_{21} \neq 0$*, then* $13(a_{21} + b_{12}) = 3b_{21}^2$ *(non-formally integrable case),*
2. *if* $b_{21} = 0$*, then the system (19) has a formal inverse integrating factor (integrable case).*
   Moreover, in such a case, the system (19) is one of the following systems
   (a) $b_{21} = a_{21} + b_{12} = a_{12} + 3b_{03} = 0$ *(Hamiltonian case).*
   (b) $a_{21} = a_{03} = b_{21} = b_{12} = 0, a_{12} + 3b_{03} \neq 0$ *(non-Hamiltonian, not axis-reversible case).*

Proof. The first part follows from the above theorem. We assume that $b_{21} = 0$ ($\alpha_6 = 0$). If $a_{21} + b_{12} \neq 0$ ($\alpha_7 \neq 0$). It is easy to check that $\alpha_{10}, \alpha_{12}, \alpha_{15}$ and α_{16} are not zero simultaneously. Therefore, from Theorem 3.2, the system (19) does not have an AIIF. Otherwise, $a_{21} + b_{12} = 0$ ($\alpha_6 = \alpha_7 = 0$). The coefficient α_{10} is

$\alpha_{10} = (3b_{12} - 4a_{21})(a_{12} + 3b_{03})$. If it is not zero, the following coefficients under the cancellation of the above ones are

$$\begin{aligned}
\alpha_{12} &= (a_{12} + 3b_{03})(98a_{03} + (3b_{12} - 4a_{21})^2),\\
\alpha_{15} &= (3b_{12} - 4a_{21})^2(a_{12} + 3b_{03})(5b_{03} - 4a_{12}),\\
\alpha_{16} &= (3b_{12} - 4a_{21})(a_{12} + 3b_{03})((289/1372)(3b_{12} - 4a_{21})^3\\
&\qquad +(11/25)(a_{12} + 3b_{03})^2),\\
\alpha_{18} &= (3b_{12} - 4a_{21})^2(a_{12} + 3b_{03})^3.
\end{aligned}$$

Thus, α_{18} is nonzero and therefore the system (19) does not have an AIIF. Otherwise, $(3b_{12} - 4a_{21})(a_{12} + 3b_{03}) = 0$ ($\alpha_6 = \alpha_7 = \alpha_{10} = 0$). If $a_{12} + 3b_{03} = 0$, the system (19) is a Hamiltonian system whose Hamiltonian is a polynomial inverse integrating factor and a first integral. So, the system is formally integrable and has a formal inverse integrating factor (family 2.(a)). If $a_{12}+3b_{03} \neq 0$ and $3b_{12}-4a_{21} = 0$, then

$$\begin{aligned}
\alpha_{12} &= (a_{12} + 3b_{03})a_{03},\\
\alpha_{15} &= a_{03}(a_{12} + 3b_{03})(11b_{03} - 8a_{12}),\\
\alpha_{16} &= a_{03}^2(a_{12} + 3b_{03}).
\end{aligned}$$

If $a_{03} \neq 0$, then α_{12} and α_{16} are nonzero. The existence of an AIIF leads to $a_{03} = 0$, i.e. family 2.(b). It is straightforward to check that

$$\begin{aligned}
V = \;& 1 + (a_{12} + 3b_{03})x + (3/2)b_{03}(a_{12} + 3b_{03})x^2\\
&-(1/2)b_{03}(a_{12} - 3b_{03})(a_{12} + 3b_{03})x^3\\
&+(1/2)a_{12}b_{03}(-3b_{03} + 2a_{12})(a_{12} - 3b_{03})x^4\\
&-(1/2)b_{03}(-b_{03} + a_{12})(a_{12} - 3b_{03})(-3b_{03} + 2a_{12})y^3\\
&-(1/2)b_{03}(-b_{03} + a_{12})(a_{12} - 3b_{03})(-3b_{03} + 2a_{12})a_{12}xy^3,
\end{aligned}$$

is a polynomial inverse integrating factor for family 2.(b). As $V(0,0) = 1 \neq 0$, the system is formally integrable. $\diamondsuit$

Remark 1. If the system (2) has an AIIF which does not have the form $(h + \text{q-h.h.o.t.})^{1+j/(r+|\mathbf{t}|)}$ then the system is formally integrable. For instance, $V = (y^3/3 - x^4/4 - 3\lambda x^5)^{6/5}$ is an inverse integrating factor of $(y^2 + 60\lambda xy^2)\partial_x + (x^3 + 100\lambda y^3)\partial_y$ and, from Proposition 4.6, it is a formally integrable system.

Example 3. Last on, we study the problem for the systems (14) given by

$$(y^2 + a_{30}x^3)\partial_x + (x^3 + b_{21}x^2y + b_{03}y^3)\partial_y, \tag{20}$$

with $a_{30} \neq 0$, (case $a_{30} = 0$, studied before).

We have the following result.

Proposition 4.7. *System (20), with $a_{30} \neq 0$, admits an AIIF if, and only if, one of the following series of conditions is satisfied:*

1. $3a_{30} + b_{21} = b_{03} = 0$ *(Hamiltonian system),*
2. $3b_{21} - 4a_{30} = 0$ *and* $b_{03} = 0$ *(non-formally integrable system).*
   Moreover, in this case, the AIIF is $V = (4y^3 - 3x^4)^{13/12}$.

Proof. We assume that the system (20) with $a_{30} \neq 0$, possesses an AIIF. The first two coefficients of the quasi-homogeneous normal form of the system (20) are given by (17) and (18) for $a_{21} = b_{12} = 0$. Therefore, if $3a_{30} + b_{21} \neq 0$, we obtain

$4a_{30} - 3b_{21} = 0$. In such a case, the following coefficient of the normal form is $\alpha_{10} = a_{30}^2 b_{03}$. Hence, $b_{03} = 0$. It is easy to check that $V = (4y^3 - 3x^4)^{13/12}$ is an AIIF of the system.
Otherwise, $3a_{30} + b_{21} = 0$. The coefficients α_6 and α_7 are zero and $\alpha_{10} = a_{30}^2 b_{03}$. This leads to $b_{03} = 0$, the system (20) is a Hamiltonian system. $\diamondsuit$

D) Perturbations of the system $-y^3\partial_x + x^3\partial_y$. The systems are

$$(\dot{x}, \dot{y})^T = \mathbf{X}_h + \text{h.o.t.}, \quad h = x^4/4 + y^4/4, \tag{21}$$

i.e. the system (2) with $\mathbf{t} = (1,1)$ and $r = 2$. From Lemma 2.1, the sets $\mathcal{P}_j^{\mathbf{t}}$ are nontrivial spaces for all j, hence $n_0 = 1 + r = 3$. So, to provide a normal form, it is enough to compute the subspaces $\text{Cor}(\ell_j)$, $j = 3, 4, 5, 6$, which are given in Table 3.

TABLE 3. Range and co-range of operator ℓ_j for the system (21).

$\text{Range}(\ell_3)=\text{span}\{x^3, y^3\}$. $\text{Cor}(\ell_3)=\text{span}\{x^2y, xy^2\}$.
$\text{Range}(\ell_4)=\text{span}\{xy^3, x^4 + 2h, x^3y\}$. $\text{Cor}(\ell_4)=\text{span}\{x^2y^2, h\}$.
$\text{Range}(\ell_5)=\text{span}\{x^2y^3, 3x^5 + 8xh, 3x^4y + 4yh, x^3y^2\}$. $\text{Cor}(\ell_5)=\text{span}\{xh, yh\}$.
$\text{Range}(\ell_6)=\text{span}\{x^3y^3, x^6 + 3x^2h, x^5y + 2xyh, x^4y^2 + y^2h\}$. $\text{Cor}(\ell_6)=\text{span}\{x^2h, xyh, y^2h\}$.

We note that $h \in \mathcal{F}_2^{(1,1)}$. A normal form of the system (21) is

$$(\dot{x}, \dot{y})^T = (-y^3, x^3)^T + \sum_{j\geq 0} f_j(x, y, h)h^j(x, y)^T, \tag{22}$$

with $f_j \in \text{span}\{x^2y, xy^2, h, x^2y^2, xh, yh, x^2h, xyh, y^2h\}$.

From Theorem 3.3, the origin is a monodromic singular point. To characterize the centers of the system (22), it is necessary to compute the integrals $I_{n,k} = \int_0^T \text{Cs}^n(\theta)Sn^k(\theta)d\theta$, $n, k \in \{0, 1, 2\}$, with $g(\theta) = (\text{Cs}(\theta), \text{Sn}(\theta))$, $\theta \in [0, T)$is a parametrization of the closed curve $h = 1$, $(\text{Cs}(\theta), \text{Sn}(\theta))$ is the solution of the initial value problem

$$\begin{cases} \frac{\text{dCs}\theta}{\text{d}\theta} = -\text{Sn}^3\theta, \\ \frac{\text{dSn}\theta}{\text{d}\theta} = \text{Cs}^3\theta, \end{cases}$$

with $(\text{Cs}(0), \text{Sn}(0)) = (1, 0)$ and T is a minimal period of both functions.

We cite some properties of these integrals. For the shake of shortness, we omit their proofs in this paper.

Lemma 4.8. *For every $n, k \geq 0$, it holds:*

1. $I_{2n+1,k} = I_{n,2k+1} = 0$,
2. $I_{2n+2,2k+2} = \frac{(2n+1)(2k+1)}{4(n+k+2)(n+k+1)} I_{2n,2k}$.

So,

$$I_{1,0} = I_{0,1} = I_{1,1} = I_{2,1} = I_{1,2} = 0, \quad I_{2,0} = I_{0,2}, \quad I_{0,0} = 8I_{2,2}.$$

Other properties of these integrals can be found in [22].

Applying Theorems 3.2 and 3.4, we have the following result.

Theorem 4.9. *System (21) possesses an AIIF if, and only if, it is formally orbital equivalent to one of the following systems:*

1. $(\dot{x},\dot{y})^T = \mathbf{X}_h$. *The AIIF is $g(h)$ with g any nonzero function (in particular, there are nonzero inverse integrating factors at the origin).*
   In this case, the origin is a center.
2. $(\dot{x},\dot{y})^T = \mathbf{X}_h + (\alpha_{4j+3}x^2y + \beta_{4j+3}xy^2)h^j(x,y)^T$, $(\alpha_{4j+3},\beta_{4j+3}) \neq (0,0), j \geq 0$.
   The AIIF is $(h + q\text{-h.h.o.t.})^{2+j+1/4}$, up to a multiplicative constant.
   In this case, the origin is a center.
3. $(\dot{x},\dot{y})^T = \mathbf{X}_h + (\alpha_{4j+4}h + \beta_{4j+4}x^2y^2)h^j(x,y)^T$, $(\alpha_{4j+4},\beta_{4j+4}) \neq (0,0), j \geq 0$.
   The AIIF is $(h + q\text{-h.h.o.t.})^{2+j+1/2}$, up to a multiplicative constant.
   In this case, the origin is a center if, and only, if $8\alpha_{4j+4} + \beta_{4j+4} = 0$.
4. $(\dot{x},\dot{y})^T = \mathbf{X}_h + (\alpha_{4j+5}xh + \beta_{4j+5}yh)h^j(x,y)^T$, $(\alpha_{4j+5},\beta_{4j+5}) \neq (0,0), j \geq 0$.
   The AIIF is $(h + q\text{-h.h.o.t.})^{2+j+3/4}$, up to a multiplicative constant.
   In this case, the origin is a center.
5. $(\dot{x},\dot{y})^T = \mathbf{X}_h + (\alpha_{4j+6}x^2h + \beta_{4j+6}xyh + \gamma_{4j+6}y^2h)h^j(x,y)^T + A(\alpha_{4j+6}x^2h + \beta_{4j+6}xyh + \gamma_{4j+6}y^2h)h^{2j+1}(x,y)^T$, *with* $(\alpha_{4j+6},\beta_{4j+6},\gamma_{4j+6}) \neq (0,0,0)$, $j \geq 0$.
   The AIIF is $(h + q\text{-h.h.o.t.})^{3+j}$, up to a multiplicative constant; i.e. it is formal.
   In this case, the origin is a center if, and only if, $\alpha_{4j+6} + \gamma_{4j+6} = 0$.

Example 4. We analyze the system

$$y^3\partial_x + (-x^3 + c_3x^2y^2 + c_4xy^3)\partial_y. \tag{23}$$

This system for $c_3 = 1/2$ and $c_4 = 0$ was studied in Moussu [27] by showing that it is a degenerate analytic center without formal first integral. Moreover, Giné and Peralta-Salas [24] have proved that it does not have a formal inverse integrating factor.

The first coefficients of the normal form (22) are $\alpha_3 = 2c_3$ and $\beta_3 = 3c_4$. If both c_3 and c_4 are zero, the system is Hamiltonian whose first integral is $h = x^4 + y^4$. Otherwise, the system is not formally integrable. Moreover,

- if $c_3.c_4 \neq 0$, the fourth-order coefficients of the normal form are $\alpha_4 = 0$ and $\beta_4 = 2c_3c_4$. Thus, from Theorem 4.9, the system does not have an AIIF,
- if $c_3 = 0$ and $c_4 \neq 0$, the fourth-order coefficients of the normal form are zero but $\alpha_5 = 6/5c_4^3$. And if $c_3 \neq 0$ and $c_4 = 0$, then $\beta_5 = 16/45c_3^3$. Therefore, from Theorem 4.9, the system does not have an AIIF.

These results are summarized below.

Proposition 4.10. *System (23) with $(c_3, c_4) \neq (0,0)$ does not have an AIIF.*

5. **Proofs of the main results.** We consider the non-integrable system (4).

It is possible to compute a reduced normal form by proceeding as follows. Firstly one carries out a re-parametrization of the time $dt/dT = 1 - \nu_k(\mathbf{x})$ and a coordinate transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$, with $\nu_k \in \mathrm{Cor}(\ell_k)$, $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$, that puts the k-degree quasi-homogeneous terms in normal form as it was explained above. Then, one carries out a new re-parametrization of the time $dT/d\tau = 1 - \tilde{\nu}_{k-N}(\mathbf{y})$ and a near-identity transformation $\mathbf{y} = \mathbf{z} + \tilde{\mathbf{P}}_{k-N}(\mathbf{z})$ with $(\tilde{\mathbf{P}}_{k-N}, \tilde{\nu}_{k-N}) \in \mathrm{Ker}(\mathcal{L}_{r+k-N}) \subset \mathcal{Q}_{k-N}^{\mathbf{t}} \times \mathrm{Cor}(\ell_{k-N})$. This process defines the two-step homological operator:

Definition 5.1. For every $k \geq 1$, we define the linear operator

$$\begin{aligned}
\mathcal{L}^{(2)}_{r+k} \quad : \quad & \mathcal{Q}^{\mathbf{t}}_k \times \mathrm{Cor}(\ell_k) \times \mathrm{Ker}(\mathcal{L}_{r+k-N}) \longrightarrow \mathcal{Q}^{\mathbf{t}}_{r+k} \\
& (\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{k-N}, \tilde{\nu}_{k-N})) \to \mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{k-N}, \tilde{\nu}_{k-N})) \\
& = [\mathbf{P}_k, \mathbf{F}_r] - \nu_k \mathbf{F}_r + [\tilde{\mathbf{P}}_{k-N}, \mathbf{F}_{r+N}] - \tilde{\nu}_{k-N}\mathbf{F}_N \\
& = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + [\tilde{\mathbf{P}}_{k-N}, \mathbf{F}_{r+N}] - \tilde{\nu}_{k-N}\mathbf{F}_{r+N}. \qquad (24)
\end{aligned}$$

It is clear that the quasi-homogeneous terms of the transformed system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$ remain unchanged up to degree $r+k-1$ and for the degree $r+k$ get

$$\mathbf{G}_{r+k} = \mathbf{F}_{r+k} - \mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{k-N}, \tilde{\nu}_{k-N})).$$

To obtain the expression of a reduced normal form of the system (4) with μ_{r+N} nonzero, it is necessary to know the bases of $\mathrm{Ker}(\ell_{r+k})$ and $\mathrm{Ker}(\mathcal{L}_{r+k})$ which will allow us to get the expression of a complementary subspace to the range of the operator $\mathcal{L}^{(2)}_{r+k}$.

Lemma 5.2. *Given $k = l_1(r+|\mathbf{t}|) + l_2$, with $0 \leq l_2 < r+|\mathbf{t}|$, it holds:*

1. *$Ker(\ell_{r+k}) = span\{h^{l_1}\}$, if $l_2 = 0$.*
   Otherwise, $Ker(\ell_{r+k})$ is a trivial space.
2. *$Ker(\mathcal{L}_{r+k}) = span\{(\mathbf{0}, h^{l_1}\mathbf{D}_0, -rh^{l_1})\}$, if $l_2 = 0$ and $r \neq 0$,*
   $Ker(\mathcal{L}_{r+k}) = span\{(\mathbf{X}_{h^{l_1+1}}, \mathbf{0}, 0)\}$, if $l_2 = r$ and $r \neq 0$,
   $Ker(\mathcal{L}_{r+k}) = span\{(\mathbf{0}, h^{l_1}\mathbf{D}_0, 0),\ (\mathbf{X}_{h^{l_1+1}}, \mathbf{0}, 0)\}$, if $l_2 = 0$ and $r = 0$.
   Otherwise, $Ker(\mathcal{L}_{r+k})$ is a trivial space.

Proof. The basis of $\mathrm{Ker}(\ell_{r+k})$ is obtained in [3].
Now we are going to obtain the expression of $\mathrm{Ker}(\mathcal{L}_{r+k})$. First, we assume that $r \neq 0$. Let $(\mathbf{X}_{g_{k+|\mathbf{t}|}}, \mu_k\mathbf{D}_0, \nu_k) \in \mathrm{Ker}(\mathcal{L}_{r+k})$ be. It holds that

$$\ell_{r+k+|\mathbf{t}|}(g_{k+|\mathbf{t}|}) = \tfrac{r(r+|\mathbf{t}|)}{r+k+|\mathbf{t}|}\mu_k h + \tfrac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|}\nu_k h \qquad (25)$$

$$(k+|\mathbf{t}|)\mu_k - \nu_k \in \mathrm{Ker}(\ell_{r+k}). \qquad (26)$$

We suppose that $l_2 \neq 0$. From (26), as $\mathrm{Ker}(\ell_{r+k}) = \{0\}$, it turns out that $\nu_k = (k+|\mathbf{t}|)\mu_k$; i.e. $\mu_k \in \mathrm{Cor}(\ell_k)$. So, (25) becomes $\ell_{r+k+|\mathbf{t}|}(g_{k+|\mathbf{t}|}) = (r+|\mathbf{t}|)\mu_k h \in \mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}) \cap \mathrm{Range}(\ell_{r+k+|\mathbf{t}|})$, that is, $\mu_k = \nu_k = 0$. Moreover, $g_{k+|\mathbf{t}|} \in \mathrm{Ker}(\ell_{r+k+|\mathbf{t}|})$. Consequently, $g_{k+|\mathbf{t}|} = \alpha h^{l_1+1}$ if $l_2 = r$, and $g_{k+|\mathbf{t}|} = 0$ if $l_2 \neq r$.
Now, we suppose that $l_2 = 0$. So, $\mathrm{Ker}(\ell_{r+k}) = \mathrm{span}\{h^{l_1}\}$ and $\mathrm{Ker}(\ell_{r+k+|\mathbf{t}|}) = \{0\}$. Hence, by (26), $\nu_k = (k+|\mathbf{t}|)\mu_k + \alpha h^{l_1}$ (ensures that $\mu_k \in \mathrm{Cor}(\ell_k)$ since $\nu_k, h^{l_1} \in \mathrm{Cor}(\ell_k)$) and by (25), $\ell_{r+k+|\mathbf{t}|}(g_{k+|\mathbf{t}|}) = (r+|\mathbf{t}|)\mu_k h + \frac{k+|\mathbf{t}|}{r+k+|\mathbf{t}|}h^{l_1+1} \in \mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}) \cap \mathrm{Range}(\ell_{r+k+|\mathbf{t}|})$. So, by (25) and (26), $g_{r+k+|\mathbf{t}|} = 0$ and $\mu_k = -\frac{\alpha}{r+k+|\mathbf{t}|}h^{l_1}$, $\nu_k = \frac{\alpha r}{r+k+|\mathbf{t}|}h^{l_1}$.
For $r = 0$, this result can be similarly proved. $\diamondsuit$

The operator $\ell^{(2)}_{r+k}$ is well-defined and provides an expression of $\mathrm{Cor}(\mathcal{L}^{(2)}_{r+k})$.

Proposition 5.3. *Consider the system (4).*

1. *$Cor(\mathcal{L}^{(2)}_{r+k}) = Cor(\ell^{(2)}_{r+k})\mathbf{D}_0$, if $N \neq l_1(r+|\mathbf{t}|)$, for any l_1 natural number.*
2. *For $N = s(r+|\mathbf{t}|)$, it holds:*
   (a) *$Cor(\mathcal{L}^{(2)}_{r+k}) = Cor(\ell^{(2)}_{r+k})\mathbf{D}_0$, if $k \neq 2s(r+|\mathbf{t}|)$.*
   (b) *Otherwise, $Cor(\mathcal{L}^{(2)}_{r+2s(r+|\mathbf{t}|)}) = Cor(\ell_{r+2s(r+|\mathbf{t}|)})\mathbf{D}_0$.*

Proof. Any natural number $k \geq N$ is written as $k = N + l_1(r + |\mathbf{t}|) + l_2$ with $0 \leq l_2 < r + |\mathbf{t}|$.
We first assume that $r \neq 0$ and we distinguish two cases:
On the one hand, we suppose that $l_2 \neq 0$. As $\text{Ker}(\ell_{r+k-N})$ is a trivial space, then $\ell^{(2)}_{r+k} = \ell_{r+k}$.
Moreover, if $l_2 \neq r$, then $\text{Ker}(\mathcal{L}_{r+k-N})$ is actually a trivial space and thus $\mathcal{L}^{(2)}_{r+k} = \mathcal{L}_{r+k}$. Otherwise, $l_2 = r$ and $\text{Ker}(\mathcal{L}_{r+k-N}) = \text{span}\{(\mathbf{X}_{h^{l_1+1}}, \mathbf{0}, 0)\}$.

As $[p\mathbf{F}, \mathbf{G}] = (L_{\mathbf{G}}p)\mathbf{F} + p[\mathbf{F}, \mathbf{G}]$ with p a smooth scalar function and $\mathbf{F}$ and $\mathbf{G}$ smooth vector fields,

$$\begin{aligned}
&\mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, \alpha(\mathbf{X}_{h^{l_1+1}}, \mathbf{0}, 0)) \\
&\qquad = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha[\mathbf{X}_{h^{l_1+1}}, \mu_{r+N}\mathbf{D}_0] \\
&\qquad = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha(l_1+1)\mathcal{L}_{r+k}(h^{l_1}\mu_{r+N}\mathbf{D}_0, -(k-r-N)h^{l_1}\mu_{r+N}).
\end{aligned}$$

Therefore, $\text{Range}(\mathcal{L}^{(2)}_{r+k}) = \text{Range}(\mathcal{L}_{r+k})$. For both cases, $l_2 \neq r$ or $l_2 = r$,

$$\text{Cor}(\mathcal{L}^{(2)}_{r+k}) = \text{Cor}(\mathcal{L}_{r+k}) = \text{Cor}(\ell_{r+k})\mathbf{D}_0 = \text{Cor}(\ell^{(2)}_{r+k})\mathbf{D}_0.$$

On the other hand, we suppose that $l_2 = 0$. We have that $\text{Ker}(\mathcal{L}_{r+k-N}) = \{(\mathbf{0}, h^{l_1}\mathbf{D}_0, -rh^{l_1})\}$. Hence,

$$\begin{aligned}
&\mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, (\mathbf{0}, \alpha h^{l_1}\mathbf{D}_0, -\alpha r h^{l_1})) \\
&\quad = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + [\alpha h^{l_1}\mathbf{D}_0, \mu_{r+N}\mathbf{D}_0] + \alpha r \mu_{r+N} h^{l_1}\mathbf{D}_0 \\
&\quad = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha\left((\nabla h^{l_1} \cdot (\mu_{r+N}\mathbf{D}_0)\mathbf{D}_0 + h^{l_1}[\mathbf{D}_0, \mu_{r+N}\mathbf{D}_0]\right) + \alpha r \mu_{r+N} h^{l_1}\mathbf{D}_0 \\
&\quad = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha\left(l_1(r+|\mathbf{t}|)\mu_{r+N}h^{l_1}\mathbf{D}_0 - h^{l_1}(\nabla\mu_{r+N}\mathbf{D}_0)\mathbf{D}_0\right) + \alpha r \mu_{r+N} h^{l_1}\mathbf{D}_0 \\
&\quad = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha(l_1(r+|\mathbf{t}|) - N)\mu_{r+N}h^{l_1}\mathbf{D}_0,
\end{aligned}$$

with α a real number.
Consequently, if there exists a natural number $l_1 := s$ such that $N = s(r + |\mathbf{t}|)$, i.e. $r + k = r + 2s(r + |\mathbf{t}|)$, then $\text{Range}(\mathcal{L}^{(2)}_{r+2s(r+|\mathbf{t}|)}) = \text{Range}(\mathcal{L}_{r+2s(r+|\mathbf{t}|)})$ and

$$\text{Cor}(\mathcal{L}^{(2)}_{r+2s(r+|\mathbf{t}|)}) = \text{Cor}(\mathcal{L}_{r+2s(r+|\mathbf{t}|)}) = \text{Cor}(\ell_{r+2s(r+|\mathbf{t}|)})\mathbf{D}_0.$$

Otherwise,

$$\text{Range}(\mathcal{L}^{(2)}_{r+k}) = \text{Range}(\mathcal{L}_{r+k}) + \text{span}\{h^{l_1}\mu_{r+N}\mathbf{D}_0\}.$$

Moreover, as the system (4) is a normal form, then

$$\text{Range}(\mathcal{L}_{r+k}) = \mathbf{X}_{\mathcal{P}^{\mathbf{t}}_{r+k+|\mathbf{t}|}} + \text{Range}(\ell_{r+k})\mathbf{D}_0.$$

Thus, $\text{Cor}(\mathcal{L}^{(2)}_{r+k}) = \text{Cor}(\ell^{(2)}_{r+k})\mathbf{D}_0$.
For $r = 0$, this result can be similarly proved. $\diamondsuit$

Proof of Theorem 3.1. Theorem 3.1 is a straightforward consequence of Proposition 5.3. $\diamondsuit$

From Prelle & Singer [29, Propositions 1 and 2] and particularizing in our context, if a system has an AIIF then it also admits an inverse integrating factor V over $\mathbb{C}((x, y))$ where $\mathbb{C}((x, y))$ denotes the quotient field of the algebra of the power series $\mathbb{C}[[x, y]]$. Moreover, if V exists, from Algaba *et al.* [5], then the system also has an inverse integrating factor of the form W^q with $W \in \mathbb{C}[[x, y]]$ and q a rational number.

The following results are used for the proof of Theorem 3.2.

Proposition 5.4. [5, Propositions 10 and 13] *Consider the system (4). If V is an AIIF then V is given by $V=(\sum_{j\geq 1} b_j h^j)^{1+N/(r+|\mathbf{t}|)}$, with $b_1=1$. Furthermore, it is unique, up to a multiplicative constant.*
Moreover, the real numbers b_j satisfy the recursive relation

$$0=\sum_{i=0}^{j-1}\left[\tfrac{N+(1+i)(r+|\mathbf{t}|)}{r+|\mathbf{t}|+N}-(j-i)\right]b_{j-i}h^{j-i}\mu_{r+N+i(r+|\mathbf{t}|)}. \tag{27}$$

Furthermore, if $\mu=\lambda f(h)+\nu$ with $\lambda\in Cor(\ell_{r+N})\setminus\{0\}$, f a scalar function, $f(0)=1$, and $\nu=\sum_{j>N}\nu_{r+j}$, $\nu_j\in Cor(\ell_j)$, $\nu_{r+N+l(r+|\mathbf{t}|)}\equiv 0$ for all non-negative integer l, then, under these conditions, the system possesses an AIIF if, and only if, $\nu\equiv 0$.

Proposition 5.5. [7, Proposition 19] *Consider the system $\dot{\mathbf{x}}=\mathbf{X}_h+\mu\mathbf{D}_0$ with $\mu=\sum_{j>r}\mu_j$, $\mu_j\in Cor(\ell_j)$ and $h\in\mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ having only simple factors in its factorization on $\mathbb{C}[x,y]$ and $\mu\not\equiv 0$. We assume that V is an inverse integrating factor V of the system. Then, $V=h^{m+1}+\sum_{j>m+1}b_jh^j$, for a certain m natural number.*
Moreover, it holds

$$0=\sum_{l=1}^{k-(m+1)(r+|\mathbf{t}|)}(k-(r+|\mathbf{t}|)-2l)V_{k-l}\mu_{r+l}, \tag{28}$$

with V_j the quasi-homogeneous term of degree j of V.

Proposition 5.6. *Consider the system $\dot{\mathbf{x}}=\mathbf{X}_h+\lambda g(h)\mathbf{D}_0$ where $h\in\mathcal{P}_{r+|\mathbf{t}|}$, $\lambda\in\mathcal{P}_{r+N}$ and g a $\mathcal{C}^1$ class function with $g(0)=1$. Then, the function $h^{1+N/(r+|\mathbf{t}|)}g(h)$ is an inverse integrating factor of the system.*

Proof. Applying the Euler theorem for quasi-homogeneous functions, one has that

$$L_{\mathbf{F}}V=V'(h)L_{\mathbf{F}}h=(r+|\mathbf{t}|)\lambda h g(h)V'(h),$$

and

$$\begin{aligned}\operatorname{div}(\mathbf{F}) &= \operatorname{div}(\lambda g(h)\mathbf{D}_0)\\ &= g(h)L_{\mathbf{D}_0}\lambda+\lambda g'(h)L_{\mathbf{D}_0}h+|\mathbf{t}|\lambda g(h)\\ &= (r+N)\lambda g(h)+(r+|\mathbf{t}|)\lambda g'(h)h+|\mathbf{t}|\lambda g(h)\\ &= (r+|\mathbf{t}|+N)\lambda g(h)+(r+|\mathbf{t}|)\lambda g'(h)h.\end{aligned}$$

Substituting $V'(h)=\left(1+\frac{N}{r+|\mathbf{t}|}\right)h^{\frac{N}{r+|\mathbf{t}|}}g(h)+h^{1+\frac{N}{r+|\mathbf{t}|}}g'(h)$ in the expression of $L_{\mathbf{F}}V$ we arrive to $L_{\mathbf{F}}V=V\operatorname{div}(\mathbf{F})$. This completes the proof. $\diamondsuit$

Proof of Theorem 3.2. Necessary condition. From Theorem 2.2, a normal form of the system (2) is of the form $\dot{\mathbf{x}}=\mathbf{X}_h+\mu\mathbf{D}_0$, with $\mu=\sum_{j>r}\mu_j$, $\mu_j\in\mathrm{Cor}(\ell_j)$.

If $\mu_j\equiv 0$, for all j then the system (2) is formally orbital-equivalent to a Hamiltonian system, i.e. there exist a diffeomorphism Φ and a function η on $U\subset\mathbb{C}^2$ with $\det D\Phi\neq 0$ on U and $\eta(\mathbf{0})\neq 0$, such that $\Phi_*(\eta\mathbf{F})=\mathbf{X}_h$, where we have denoted as Φ_* the push-forward defined by Φ. As $\mathbf{X}_h$ is a Hamiltonian vector field, $f(h)$ is a first integral for any f non-constant. In particular, $f(h)$ is an inverse integrating factor. So, the pull-back Φ^* brings $f(h)$ on the inverse integrating factor of the system (2), V=$f(h+\cdots)+\cdots$, i.e. it is not unique. Also, if $f(0)\neq 0$, V would be a formal inverse integrating factor with $V(\mathbf{0})\neq 0$.

Otherwise, let $N=\min\{j,\ \mu_{r+j}\neq 0\}$. We distinguish two cases:

- N is not multiple of $r+|\mathbf{t}|$.

By Theorem 3.1, the system (3) is formally orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+N}\mathbf{D}_0 + \sum_{j>N} \tilde{\mu}_{r+j}\mathbf{D}_0$, with $\tilde{\mu}_j \in \mathrm{Cor}(\ell_j^{(2)})$.

From Proposition 5.4, $V = (h + \sum_{j>1} b_j h^j)^{1+N/(r+|\mathbf{t}|)}$ with b_j verifying (27); that is, V is not formal. We see that $b_j = 0$ and $\tilde{\mu}_{r+N+(j-1)(r+|\mathbf{t}|)} = 0$, for any $j > 1$. Indeed, we assume the contrary and let $j_0 = min\{j > 1 : \ b_j \neq 0\}$. By (27), for $j = j_0$, it satisfies that

$$b_{j_0} h^{j_0} \mu_{r+N} - \tfrac{r+|\mathbf{t}|}{r+|\mathbf{t}|+N} h \tilde{\mu}_{r+N+(j_0-1)(r+|\mathbf{t}|)} = 0.$$

Consequently, $b_{j_0} h^{j_0-1} \mu_{r+N} \in \mathrm{Cor}(\ell^{(2)}_{r+N+(j_0-1)(r+|\mathbf{t}|)}) \setminus \{0\}$, but also

$$b_{j_0} h^{j_0-1} \mu_{r+N} = \ell^{(2)}_{r+N+(j_0-1)(r+|\mathbf{t}|)}(0, b_{j_0} h^{j_0-1}),$$

which is a contradiction. So, $b_{j_0} = 0$ and $\tilde{\mu}_{r+N+(j-1)(r+|\mathbf{t}|)} = 0$, for all $j > 1$.

Applying Proposition 5.4, for $\lambda = \mu_{r+N}, \ f(h) = 1$ and $\nu_{r+j} = \mu_{r+j}$, it arrives to $\mu_{r+j} = 0$, for any $j > N$.

• $N = s(r + |\mathbf{t}|)$ with $s > 0$.
By Theorem 3.1, the system (3) is formally orbital-equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu_{r+s(r+|\mathbf{t}|)}\mathbf{D}_0 + \sum_{j>s(r+|\mathbf{t}|)} \tilde{\mu}_{r+j}\mathbf{D}_0$, with $\tilde{\mu}_{r+2s(r+|\mathbf{t}|)} \in \mathrm{Cor}(\ell_{r+2s(r+|\mathbf{t}|)})$ and $\tilde{\mu}_j \in \mathrm{Cor}(\ell_j^{(2)})$, for $j \neq r + 2s(r + |\mathbf{t}|)$.

From Proposition 5.5, $V = h^{m+1} + \sum_{j>m+1} b_j h^j$, for a certain natural number m.

We shall prove that $V = h^{s+1} + b_{2s+1} h^{2s+1}$ and $\mu = \mu_{r+s(r+|\mathbf{t}|)} + b_{2s+1} h^s \mu_{r+2s(r+|\mathbf{t}|)}$. We do the proof in several steps:

Step 1. We see that $m = s$.

Equation (28), for $k = (s + m + 1)(r + |\mathbf{t}|)$, is

$$\begin{aligned} 0 &= ((s+m+1)(r+|\mathbf{t}|) - r - |\mathbf{t}| - 2s(r+|\mathbf{t}|)) V_{(m+1)(r+|\mathbf{t}|)}\, \mu_{s(r+|\mathbf{t}|)} \\ &= (m-s)(r+|\mathbf{t}|) h^{m+1}\, \mu_{r+s(r+|\mathbf{t}|)}. \end{aligned}$$

Therefore, $m = s$. Thus, up to a multiplicative constant the leading term in the quasihomogeneous expansion of any formal inverse integrating factor is h^{s+1}.

Step 2. We prove that $b_j = 0$ and $\tilde{\mu}_{r+(j-1)(r+|\mathbf{t}|)} = 0$, for $j = s+2, ..., 2s$. Indeed, we assume the contrary, i.e. there exists $j_0 = min\{j, \ b_j \neq 0, \ s + 2 \leq j \leq 2s\}$.

The equality (28), for $k = (j_0 + s)(r + |\mathbf{t}|)$, has only two components $V_{k-l} \neq 0$ with $1 \leq l \leq k - (s + 1)(r + |\mathbf{t}|)$. In particular, for $l = (j_0 - 1)(r + |\mathbf{t}|)$ we have $V_{k-l} = V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$ and for $l = s(r + |\mathbf{t}|)$, $V_{k-l} = V_{j_0(r+|\mathbf{t}|)} = b_{j_0} h^{j_0}$. Then, the equality (28), for $k = (j_0 + s)(r + |\mathbf{t}|)$, becomes

$$\begin{aligned} 0 &= (j_0-s-1)(r+|\mathbf{t}|) b_{j_0} h^{j_0} \mu_{r+s(r+|\mathbf{t}|)} - (j_0-s-1)(r+|\mathbf{t}|) h^{s+1} \tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} \\ &= (j_0-s-1)(r+|\mathbf{t}|) h^{s+1} \left[b_{j_0} h^{j_0-s-1} \mu_{r+s(r+|\mathbf{t}|)} - \tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} \right]. \end{aligned}$$

Consequently, by Theorem 3.1,

$$\tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} = b_{j_0} h^{j_0-s-1} \mu_{r+s(r+|\mathbf{t}|)} \in \mathrm{Cor}(\ell^{(2)}_{r+(j_0-1)(r+|\mathbf{t}|)}) \setminus \{0\}, \qquad (29)$$

but also $b_{j_0} h^{j_0-s-1} \mu_{r+s(r+|\mathbf{t}|)} = \ell^{(2)}_{r+(j_0-1)(r+|\mathbf{t}|)}(0, b_{j_0} h^{j_0-s-1})$, which is a contradiction.

Step 3. We see that $\tilde{\mu}_{r+2s(r+|\mathbf{t}|)} = b_{2s+1}h^s\mu_{r+s(r+|\mathbf{t}|)}$.
From expression (29), for $j_0 = 2s+1$ it holds that $\tilde{\mu}_{r+2s(r+|\mathbf{t}|)} = b_{2s+1}h^s\mu_{r+s(r+|\mathbf{t}|)}$.

Step 4. We prove that $b_j = 0$, and $\tilde{\mu}_{r+(j-1)(r+|\mathbf{t}|)} = 0$, for all $j \geq 2s+2$.
Indeed, we assume the contrary, i.e. there exists $j_0 = min\{j,\ b_j \neq 0,\ j \geq 2s+2\}$. Thus, there exists $m_0 \geq 2$ such that $j_0 \in \{m_0 s+2, ..., (m_0+1)s+1\}$.

The equality (28), for $k = (j_0+s)(r+|\mathbf{t}|)$, has only three components $V_{k-l} \neq 0$ with $1 \leq l \leq k-(s+1)(r+|\mathbf{t}|)$. In particular, for $l = (j_0-1)(r+|\mathbf{t}|)$ we have $V_{k-l} = V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$, for $l = (j_0-s-1)(r+|\mathbf{t}|)$, $V_{k-l} = V_{(2s+1)(r+|\mathbf{t}|)} = b_{2s+1}h^{2s+1}$ and for $l = s(r+|\mathbf{t}|)$, $V_{k-l} = V_{j_0(r+|\mathbf{t}|)} = b_{j_0}h^{j_0}$.
The term $V_{k-l} = V_{(2s+1)(r+|\mathbf{t}|)}$ is multiplying by $\mu_{r+(j_0-s-1)(r+|\mathbf{t}|)}$ which is zero. So, the equality (28) gets (29) for $j_0 \in \{m_0 s+2, ..., (m_0+1)s+1\}$. Thus, by Theorem 3.1, this brings us to the contradiction.
Step 5. We prove that $\tilde{\mu}_j = 0$, for all $j > 2s(r+|\mathbf{t}|)$ and $j \neq r+m(r+|\mathbf{t}|)$, for any m. We suppose, contrary to our claim, that there exists $j_0 = min\{j > 2s(r+|\mathbf{t}|),\ \tilde{\mu}_j \neq 0\}$. Thus, let $m_0 \geq 2$ be such that $j_0 \in \{r+m_0 s(r+|\mathbf{t}|)+1, ..., (m_0+1)s(r+|\mathbf{t}|)-1\}$.

For $k = j_0 - r + (s+1)(r+|\mathbf{t}|)$, the equality (28) has two factors $V_{(s+1)(r+|\mathbf{t}|)}\tilde{\mu}_{j_0}$ and $V_{(2s+1)(r+|\mathbf{t}|)}\tilde{\mu}_{j_0-s(r+|\mathbf{t}|)}$. This second term is zero. Thus, Equation (28) yields

$$
\begin{aligned}
0 &= \sum_{l=1}^{j_0-r}(j_0 - r + s(r+|\mathbf{t}|) - 2l)V_{j_0-r+(s+1)(r+|\mathbf{t}|)-l}\,\mu_{r+l} \\
&= [s(r+|\mathbf{t}|) - (j_0-r)]V_{(s+1)(r+|\mathbf{t}|)}\,\mu_{j_0},
\end{aligned}
$$

and, as $V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$ and $j_0 - r \neq s(r+|\mathbf{t}|)$, we obtain $\mu_{j_0} = 0$, a contradiction.

Therefore, the existence of a formal inverse integrating factor implies that a normal form under orbital-equivalence of the system (2) is the system (7).

Sufficient condition. We suppose that the system (2) is orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h$. Any function $f(h)$ is a first integral and, in particular, it is an inverse integrating factor. Thus, if we perform the transformation which brings $\dot{\mathbf{x}} = \mathbf{X}_h$ to the system (2), we have that the system (2) admits an AIIF but it is not unique.

Otherwise, let $N = \min\{j,\ \mu_{r+j} \neq 0\}$. The normal forms (6) and (7) are of the form $\dot{\mathbf{x}} = \mathbf{X}_h + \lambda g(h)\mathbf{D}_0$, with $\lambda = \mu_{r+N} \in \mathcal{P}_{r+N}$ and $g(0) = 1$. From Proposition 5.6, the function $h^{1+\frac{N}{r+|\mathbf{t}|}}g(h)$ is an AIIF of the system (6) and (7) (formal if N is a multiple of $r+|\mathbf{t}|$). Thus, the system (2) has the AIIF, $(h + \text{q-h.h.o.t.})^{1+\frac{N}{r+|\mathbf{t}|}}$, up to a multiplicative constant. $\diamondsuit$

Proof of Theorem 3.3. Assume that the factorization of $h \in \mathcal{P}^t_{r+|\mathbf{t}|}$ only has simple factors. Therefore, we can write in a compact form $h = c\prod_{j=1}^n f_j \prod_{j=1}^m g_j$, where $f_j = x,\ y$ or $y^{t_1} - \lambda_j x^{t_2}$, $j = 1, \ldots, n$, $g_j(x,y) = (y^{t_1} - a_j x^{t_2})^2 + b_j^2 x^{2t_2}$, $j = 1, \ldots, m$ with c, λ_j, a_j and b_j real numbers and λ_j, b_j nonzero, for all j.

We shall prove the necessary condition. We assume the contrary one. Thus, there exists a f_j and it is a simple real factor of h. Therefore, there exists a real orbit of the system which leaves or enters at the origin [6, Proposition 8]. Consequently, the origin is not monodromic.

We see the sufficient condition. If h is different from zero for any $(x, y) \neq (0, 0)$, one has that the origin is monodromic [6, Proposition 6]. $\diamondsuit$

Proof of Theorem 3.4. As the system (2) is non-integrable and possesses an AIIF, its reduced normal form is either (6) or (7). First, we assume that the system (2) is formally orbital-equivalent to the system (6).

We also can assume that $h(x, y)$ is positive for all $(x, y) \neq (0, 0)$ since, from Theorem 3.3, if the origin is monodromic, h preserves its sign and, if h is negative by changing the time t by $-t$, h becomes $-h$.

A parametrization of the closed curve $h = 1$ is given by $g(\theta) = (\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$, $\theta \in [0, T)$ where $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))^T$ is the solution of the initial value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\theta} = \mathbf{X}_h(\mathbf{x}), \quad \mathbf{x}(0) = (1, 0)^T,$$

and T is a minimal period of both functions.

We consider the transformation

$$x = u^{t_1}\mathrm{Cs}(\theta), \qquad y = u^{t_2}\mathrm{Sn}(\theta), \tag{30}$$

where $u > 0, \theta \in [0, T)$.

Differentiating (30) with respect to the time, we get $\dot{\mathbf{x}} = \frac{1}{u}\mathbf{D}_0\dot{\mathbf{u}} + \frac{1}{u^r}\mathbf{X}_h\dot{\theta}$. From this, we obtain

$$\dot{\mathbf{x}} \wedge \mathbf{X}_h = \frac{1}{u}\mathbf{D}_0 \wedge \mathbf{X}_h \dot{u}, \qquad \mathbf{D}_0 \wedge \dot{\mathbf{x}} = \frac{1}{u^r}\mathbf{D}_0 \wedge \mathbf{X}_h \dot{\theta}. \tag{31}$$

We note that $\mathbf{D}_0 \wedge \mathbf{X}_h(x, y) = \nabla h(x, y) \cdot \mathbf{D}_0 = (r + |\mathbf{t}|)h(x, y) \neq 0$, for all $(x, y) \neq (0, 0)$.

For the system (6) it holds $\dot{\mathbf{x}} \wedge \mathbf{X}_h = u^{r+N}\mu_{r+N}(\theta)\mathbf{D}_0 \wedge \mathbf{X}_h$ and $\mathbf{D}_0 \wedge \dot{\mathbf{x}} = \frac{1}{u^r}\mathbf{D}_0 \wedge \mathbf{X}_h$ (we have denoted $\mu_{r+N}(\theta) := \mu_{r+N}(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$). So, the system (6) is

$$\left\{ \begin{array}{l} \dot{u} = u^{r+N+1}\mu_{r+N}(\theta), \\ \dot{\theta} = u^r. \end{array} \right. \tag{32}$$

It can be further simplified by rescaling the time by $\mathrm{dt} = \frac{1}{\mathrm{u^r}}\mathrm{d}\tau$, which yields to

$$\left\{ \begin{array}{l} u' = \frac{\mathrm{du}}{\mathrm{d}\tau} = u^{N+1}\mu_{r+N}(\theta), \\ \theta' = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 1. \end{array} \right. \tag{33}$$

Finally, we make the change $z = -\frac{1}{N}u^{-N}$, that converts system (33) into

$$\left\{ \begin{array}{l} z' = \mu_{r+N}(\theta), \\ \theta' = 1. \end{array} \right. \tag{34}$$

A Poincaré map for the system (34) is $\Pi(z_0) = z(T, z_0) = z_0 + I$. Consequently, the result follows for N not multiple of $r + |\mathbf{t}|$.

On the other hand, for the system (7), by rescaling the time by $\mathrm{dt} = \frac{1}{\mathrm{u^r}}\mathrm{d}\tau$, we obtain

$$\left\{ \begin{array}{l} u' = \frac{\mathrm{du}}{\mathrm{d}\tau} = u^{N+1}\mu_{r+N}(\theta) + \alpha u^{2N+1}\mu_{r+N}(\theta), \\ \theta' = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 1. \end{array} \right.$$

The change $z = -\frac{1}{N}u^{-N}$ transforms the system into

$$\left\{ \begin{array}{l} z' = \left(1 - \frac{\alpha}{Nz}\right)\mu_{r+N}(\theta), \\ \theta' = 1, \end{array} \right. \tag{35}$$

and, finally, the change $v = z + \frac{\alpha}{N}\log(Nz - \alpha)$ leads to

$$\begin{cases} v' = \mu_{r+N}(\theta), \\ \theta' = 1. \end{cases}$$

Following the same reasoning as before, we complete the proof. $\diamondsuit$

Acknowledgments. This work has been partially supported by *Ministerio de Ciencia y Tecnología, Plan Nacional I+D+I* co-financed with FEDER funds, in the frame of the project MTM2014-56272-C2, and by *Consejería de Educación y Ciencia de la Junta de Andalucía* (FQM-276 and P12-FQM-1658).

REFERENCES

[1] A. Algaba, E. Freire, E. Gamero and C. García. *Quasihomogeneous normal forms.* J. Comput. Appl. Math., **150**, (2003), 193-216.

[2] A. Algaba, N. Fuentes, C. García and M. Reyes. *A class of non-integrable systems admitting an inverse integrating factor.* J. Math. Anal. Appl. **420**, 2, (2014), 1439-1454.

[3] A. Algaba, E. Gamero and C. García. *The integrability problem for a class of planar systems.* Nonlinearity **22**, 2, (2009), 395-420.

[4] A. Algaba, C. García and J. Giné. *Analytic integrability for some degenerate planar vector fields.* J. Differential Equations, **257**, (2014), 549-565.

[5] A. Algaba, C. García and M. Reyes. *Nilpotent systems admitting an algebraic inverse integrating factor over* $\mathbb{C}((x,y))$. Qualitative Theory of Dynamical Systems, **10**, 2, (2011), 303-316.

[6] A. Algaba, C. García and M. Reyes. *Characterization of a monodromic singular point of a planar vector field.* Nonlinear Analysis, **74**, (2011), 5402-5414.

[7] A. Algaba, C. García and M. Reyes. *Existence of an inverse integrating factor, center problem and integrability of a class of nilpotent systems.* Chaos Solitons & Fractals, **45**, (2012), 869-878.

[8] A. Algaba. C. García and M. Reyes. *Like-linearizations of vector fields.* Bulletin des Sciences Mathématiques, **133**, (2009), 806-816.

[9] M. Berthier and R. Moussu. *Réversibilité et classification des centres nilpotents.* Ann. Inst. Fourier **44**, 2, (1994), 465-494.

[10] A.V. Bolsinov and A.T. Fomenko. *Integrable Hamiltonian Systems; Geometry, Topology, Classification.* Chapman and Hall, 2004.

[11] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre. *On the integrability of two-dimensional flows.* J. Differential Equations **157**, (1999), 163-182.

[12] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre. *Darboux integrability and the inverse integrating factor.* J. Differential Equations **194**, (2003), 116-139.

[13] C.J. Christopher and J. Llibre. *Integrability via invariant algebraic curves for planar polynomial differential systems.* Ann. Differential Equations **16**, (2000), 5-19.

[14] C. Christopher, P. Mardesic and C. Rousseau. *Normalizable, integrable, and linealizable saddle points for complex quadratic systems in* $\mathbb{C}^2$. J. Dyn. Control Syst. **9**, (2003), 311-363.

[15] H.R. Dullin and A. Pelayo. *Generating hyperbolic singularities in semitoric systems via Hopf bifurcations.* J. Nonlinear Science **26**, 3, (2016), 787-811.

[16] H. Eliasson. *Normal forms for Hamiltonian systems with Poisson commuting, integrals - elliptic case.* Comm. Math. Helv. **65**, (1990), 4-35.

[17] A. Enciso and D. Peralta-Salas. *Existence and vanishing set of inverse integrating factors for analytic vector fields.* Bull. London Math. Soc. **41**, (2009), 1112-1124.

[18] I. García, H. Giacomini and M. Grau. *The inverse integrating factor and the Poincaré map.* Trans. Amer. Math. Soc. **362**, 7, (2010), 3591-3612.

[19] I. García, H. Giacomini and M. Grau. *Generalized Hopf Bifurcation for planar vector fields via the inverse integrating factor.* J. Dyn. Differ. Equat. **23**, 2, (2011), 251-281.

[20] I. García and M. Grau. *A survey on the inverse integrating factor.* Qual. Theory Dyn. Sist. **9**, 1-2, (2010), 115-166.

[21] I. García and D. Shafer. *Integral invariants and limit sets of planar vector fields.* J. Differential Equations **217**, 2, (2005), 363-376.

[22] A. Gasull and J. Torregrosa. *Center problem for several differential equations via Cherkas' method.* J. Math. Anal. Appl. **228**, (1998), 322-343.

[23] H. Giacomini, J. Llibre and M. Viano. *On the nonexistence, existence and uniqueness of limit cycles.* Nonlinearity **9**, (1996), 501-516.

[24] J. Giné and D. Peralta-Salas. *Existence of inverse integrating factors and Lie symmetries for degenerate planar centers.* J. Differential Equations **252**, (2012), 344-357.

[25] R.E. Kooij and C.J. Christopher. *Algebraic invariant curves and the integrability of polynomial systems.* Appl. Math. Lett. **6**, (1993), 51-53.

[26] L. Mazzi and M. Sabatini. *A characterization of centers via first integrals.* J. Differential Equations, **76**, (1988), 222-237.

[27] R. Moussu. *Symétrie et forme normaledes centres et foyers dégénérés.* Ergodic Theory Dynam. Sys. **2**, (1982), 241-251.

[28] H. Poincaré. *Mémoire sur les courbes définies par les équations différentielles.* J. Math. **37**, (1881), 375-422.

[29] M.J. Prelle and M.F. Singer. *Elementary first integrals of of differential equations.* Trans. Amer. Math. Soc. **279**, (1983), 215-229.

[30] Sadovskii, A.P. *Problem of distinguishing a center and a focus for a system with a nonvanishing linear part.* Diff. Urav. **12**, 7, (1976), 1238-1246 (in Russian).

[31] S. Walcher. *On the Poincaré problem.* J. Differential Equations **166**, (2000), 51-78.

[32] S. Walcher. *Local integrating factors.* J. Lie Theory **13**, (2003), 279-289.

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