*Acta Mathematica Sinica, English Series*
xxx., xxxx, Vol. xx, No. x, pp. 1–20

DOI:
Http://www.ActaMath.com

# Local phase portraits through the Newton diagram of a vector field

**Antonio ALGABA**
*Department of Mathematics. Faculty of Experimental Sciences.*
*Avda. Tres de Marzo s/n, 21071 Huelva, Spain*
*E-mail: algaba@uhu.es*

**Isabel CHECA**
*Department of Mathematics. Faculty of Experimental Sciences.*
*Avda. Tres de Marzo s/n, 21071 Huelva, Spain*
*E-mail: isabel.checa@dmat.uhu.es*

**Cristóbal GARCÍA**
*Department of Mathematics. Faculty of Experimental Sciences.*
*Avda. Tres de Marzo s/n, 21071 Huelva, Spain*
*E-mail: cristoba@uhu.es*

**Manuel REYES**
*Department of Mathematics. Faculty of Experimental Sciences.*
*Avda. Tres de Marzo s/n, 21071 Huelva, Spain*
*E-mail: colume@uhu.es (corresponding author)*

January 23, 2015

**Abstract** The Newton diagram and, in particular, the lowest-degree quasi-homogeneous terms of an analytic planar vector field allow us to determine the existence of characteristic orbits and separatrices of an isolated singular point. We give an easy algorithm for obtaining the local phase portrait near the origin of a bi-dimensional differential system and we provide several examples.

**Keywords** Monodromy, separatrices, phase portraits, quasi-homogeneous vector field

**MR(2010) Subject Classification** 34C05, 34M35

## 1 Introduction

We are interested in the behavior of the trajectories in a neighborhood of a singular point of the analytic planar differential system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}). \tag{1}$$

It is known that any orbit solution of the system (1) that tends to the origin is either a spiral or tends to the origin in a limiting direction, see Bendixon [7] or Andronov *et al.* [5].

---

Supported by *Ministerio de Ciencia y Tecnología, Plan Nacional I+D+I* co-financed with FEDER funds, in the frame of the projects MTM2010-20907-C02-02, and by *Consejería de Educación y Ciencia de la Junta de Andalucía* (FQM-276 and P08-FQM-03770)

The trajectories passing through the origin with a definite tangent at the origin are named characteristic orbits of the singular point.

From Algaba *et al.* [2], the characteristic orbits of an analytic differential system are either paths of the form $y = ax^\lambda + o(x^\lambda)$ with $\lambda, a \in \mathbb{R}, \lambda > 0, a \neq 0$, (or by changing $x$ by $y$), or $x = 0$ or $y = 0$ or a curve solution that asymptotically tends to one of them.

It is known the topological behavior in a neighborhood of a non-monodromic point (a singular point with at least one characteristic orbit). If the jacobian matrix at the singular point is not identically null, the problem was solved by Poincaré [10] (when both eigenvalues of the jacobian matrix are different from zero), by Andronov *et al.* [5] (when one of the eigenvalues equal to zero) and by Andreev [4] (when the matrix is nilpotent). Finally, if the jacobian matrix at the singular point is identically null (in such a case, the singular point is a degenerate singular point), the problem can be solved studying each degenerate singular point case by case, by using the blow-up technique (developed by Dumortier [9]) which consists of performing a series of changes to desingularize the point. However, its application for determining the topological classification of a singular point of a family of vector fields with parameters becomes rather complicated. The desingularization procedure using quasi-homogeneous blow ups has also been applied in the program P4, see Artés *et al.* [6].

An algorithm for determining the monodromy of a singular point can be seen in [1].

In this paper, we give a method, from the first quasi-homogeneous terms of the vector field and its orthogonal field, that allows us to obtain the local phase portrait near the origin .

The paper is organized as follows. In next section, we recall some concepts related with the quasi-homogeneous vector field and the Newton diagram of a planar vector field. Section 3 contains a method for detecting the trajectories with defined quasi-homogeneous direction which leave or enter to the origin. In Section 4 we give a method for ranking the different sectors of the origin. Section 5 contains some examples.

## 2 Newton diagram of a bidimensional vector field.

Let $\mathbf{t} = (t_1, t_2)$ be non-null with $t_1$ and $t_2$ non-negative integer numbers without common factors. A function $f$ of two variables is a quasi-homogeneous function of type $\mathbf{t}$ and degree $k$ if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree $k$ will be denoted by $\mathscr{P}_k^{\mathbf{t}}$.

We note that any non-vanishing quasi-homogeneous polynomial of type $\mathbf{t} = (t_1, t_2)$ with $t_1$ and $t_2$ non-null, can be expressed as $p(x, y) = x^{k_1}y^{k_2}p_0(x^{t_2}, y^{t_1})$ with $0 \le k_1 < t_2,\ 0 \le k_2 < t_1$ and $p_0$ a homogeneous polynomial. So, by abusing the notation, we can write in a compact form $p(x, y) = c\prod_{j=0}^m f_j^{m_j} \prod_{j=0}^n g_j^{n_j}$, where

$$f_j(x, y) = x,\ y \ \text{ or } y^{t_1} - \lambda_j x^{t_2}, \qquad j = 0, ..., m$$

and

$$g_j(x, y) = (y^{t_1} - a_j x^{t_2})^2 + b_j^2 x^{2t_2}, \qquad j = 0, ..., n$$

with $c, \lambda_j, a_j$ and $b_j$ real numbers and $\lambda_j, b_j$ non-zero, for all $j$.

A vector field $\mathbf{F} = (P, Q)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree $k$ if $P \in \mathscr{P}_{k+t_1}^{\mathbf{t}}$ and

$Q \in \mathscr{P}^{\mathbf{t}}_{k+t_2}$. We will denote $\mathcal{Q}^{\mathbf{t}}_k$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree $k$.

The quasi-homogeneous vector monomials can be determined by drawing the lattice $\mathbb{Z}^2_+$, and assigning each point $(m,n)$ to the quasi-homogeneous vector fields $(x^m y^{n-1}, 0)^T$ and $(0, x^{m-1}y^n)^T$. The points with integer coordinates aligned in the straight lines perpendicular to $\mathbf{t}$, $(m-1)t_1 + (n-1)t_2 = k$, determine the quasi-homogeneous vector monomials with the same degree $k$.

We will write the components of the vector field $\mathbf{F}$ in the form $P(x,y) = \sum a_{ij}x^i y^{j-1}$ and $Q(x,y) = \sum b_{ij}x^{i-1}y^j$.

The *support* of (1) and also of $\mathbf{F}$, denoted by *supp*$(\mathbf{F})$, is the set of pairs $(i,j)$ with $(a_{ij}, b_{ij}) \neq (0,0)$. The vector $(a_{ij}, b_{ij})$ is called the *vector coefficient* of $(i,j)$ in the support. Consider the set

$$\mathcal{A}(\mathbf{F}) := \bigcup_{(i,j)\in supp(\mathbf{F})} \left((i,j) + \mathbb{R}^2_+\right),$$

where $\mathbb{R}^2_+$ is the positive quadrant and the union is taken over all points $(i,j)$ in the support. The boundary of the convex hull of $\mathcal{A}(\mathbf{F})$ is made up of two open rays and a polygon, which can be just one point. The polygon together with the rays that do not lie on a coordinate axis, if they existed, is called the *Newton diagram* of the vector field $\mathbf{F}$. The component parts of the Newton diagram (the rays do not lie on a coordinate axis and the segments) are called *edges* and their endpoints are the *vertices* of the Newton diagram.

If a vertex of the Newton diagram does not lie on a coordinate axis, then it is said to be an *inner* vertex; otherwise, it is an *exterior* vertex.

If the Newton diagram of $\mathbf{F}$ has the ray $\{(k,y),\ y \geq 1\}$ ( or $\{(x,k),\ x \geq 1\}$, resp.) then $x = 0$ (or $y = 0$, resp.) is an invariant curve of the system.

For example, we consider the vector field $\mathbf{F}$ given by

$$\begin{aligned}&(a_{14}xy^3 + a_{22}x^2y + a_{33}x^3y^2 + a_{42}x^4y + a_{41}x^4,\\ &b_{14}y^4 + b_{22}xy^2 + b_{33}x^2y^3 + b_{42}x^3y^2 + b_{51}x^3y + b_{70}x^6)^T\end{aligned} \tag{2}$$

with $a_{14}^2 + b_{14}^2 > 0, a_{22}^2 + b_{22}^2 > 0, a_{33}^2 + b_{33}^2 > 0, a_{42}^2 + b_{42}^2 > 0, a_{41}^2 + b_{41}^2 > 0$ and $b_{70} \neq 0$.

Under these conditions, support of $\mathbf{F}$ is *supp*$(\mathbf{F}) = \{(1,4),(2,2),(3,3),(4,2),(4,1),(7,0)\}$. The boundary of the convex hull of $\mathcal{A}(\mathbf{F})$ is made up of two open rays (one of them does not lie on a coordinate axis), and the polygon of vertices $(1,4),(2,2),(4,1)$ (inner vertices) and $(7,0)$ (exterior vertex). So, Newton diagram of (2) has three edges and the ray $\{(1,y),\ y \geq 4\}$. Newton diagram of the vector field (2) is shown in Figure 1

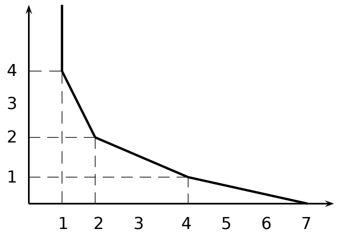

Figure 1 Newton diagram of vector field (2).

Fixed a type $\mathbf{t}=(t_1,t_2)$, every vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F}$ can be written in the form

$$\mathbf{F}=\mathbf{F}_r^{\mathbf{t}}+\mathbf{F}_{r+1}^{\mathbf{t}}+\cdots,$$

for some $r\in\mathbb{Z}$, where $\mathbf{F}_j^{\mathbf{t}}=(P_{j+t_1},Q_{j+t_2})^T\in\mathscr{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r^{\mathbf{t}}\not\equiv\mathbf{0}$. Such expansions will be expressed as $\mathbf{F}=\mathbf{F}_r^{\mathbf{t}}+$ q-h.h.o.t., where "q-h.h.o.t." means quasi-homogeneous higher order term.

Abusing of notation, we will write $\mathbf{F}_j$ instead $\mathbf{F}_j^{\mathbf{t}}$ while this does not cause confusion.

If $\mathbf{F}_j$ is the quasi-homogeneous term of degree $j$ of $\mathbf{F}$, i. e. $\mathbf{F}_j\in\mathscr{Q}_j^{\mathbf{t}}$, we consider the polynomials $\mu_j:=\operatorname{div}(\mathbf{F}_j)\in\mathscr{P}_j^{\mathbf{t}}$ (the divergence of $\mathbf{F}_j$) and $h_{j+|\mathbf{t}|}:=\mathbf{D}_0\wedge\mathbf{F}_j=t_1xQ_{j+t_2}-t_2yP_{j+t_1}\in\mathscr{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$ (the wedge product of $\mathbf{D}_0$ and $\mathbf{F}_j$) where $\mathbf{D}_0(x,y):=(t_1x,t_2y)^T$ (a dissipative quasi-homogeneous vector field of type $\mathbf{t}$ and degree 0) and $|\mathbf{t}|=t_1+t_2$.

For example, Newton diagram of the system

$$\dot{x}=ay^3+cx^2y,\quad \dot{y}=dxy^2+bx^7, \tag{3}$$

with $ab(c^2+d^2)\neq0$, consists of two exterior vertices $(0,4)$ associated to $(ay^3,0)^T$ and $(8,0)$ associated to vector field $(0,bx^7)^T$ and an inner vertex $(2,2)$ associated to $(cx^2y,dxy^2)^T$. The Newton diagram has two edges, one associated to the type $\mathbf{t}=(1,1)$ and another of type $\mathbf{s}=(1,3)$ and the hamiltonians of the conservative parts of least degree are $h_4^{(1,1)}=y^2[(d-c)x^2-ay^2]$ and $h_8^{(1,3)}=x^2[(d-3c)y^2+bx^6]$, respectively.

Algaba *et al.* [3] proved the conservative-dissipative decomposition of a quasi-homogeneous vector field which establishes that

$$(j+|\mathbf{t}|)\mathbf{F}_j=\mathbf{X}_{h_{j+|\mathbf{t}|}}+\mu_j\mathbf{D}_0 \tag{4}$$

where $\mathbf{X}_h:=(-\partial h/\partial y,\partial h/\partial x)^T$ (the Hamiltonian vector field associated to the polynomial $h$).

The polynomial $h_{r+|\mathbf{t}|}\in\mathscr{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, hamiltonian of the lowest-degree quasi-homogeneous term of type $\mathbf{t}$ of $\mathbf{F}$, will be useful in what follows and will play a main role in our analysis.

**Definition 2.1** *Fixed a type* $\mathbf{t}$*, we say that the origin is a* $\mathbf{t}$*-dicritical singular point of system (1) if it is an isolated singular point such that* $h_{r+|\mathbf{t}|}$ *is identically zero (i.e.* $(r+|\mathbf{t}|)\mathbf{F}_r=\mu_r\mathbf{D}_0$*).*

We emphasize that if the origin is a $\mathbf{t}$-dicritical singular point then the system (1) is not locally integrable, see [3].

## 3 Determining the characteristic orbits of a singular point

In this section, we give a method for finding the characteristic orbits of system (1) from the edges and vertices of its Newton diagram.

First, we give a sufficient condition of monodromy of a singular point, i.e. condition of non-existence of characteristic orbits. In case of analyticity, monodromy equivalents to a center or a focus.

**Lemma 3.1** *Fixed a type* $\mathbf{t}$*, if* $h_{r+|\mathbf{t}|}(x,y)\neq0$*, for all* $(x,y)\neq(0,0)$*, then the origin of system (1) is monodromic.*

The following result gives necessary conditions for the existence of characteristic orbits of system (1) from the edges and vertices of its Newton diagram.

**Theorem 3.2** *We assume that system (1) has an orbit of the form* $y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1})$ *where* $\tilde{a}$ *is a non-zero real number. Then, either the origin is a* $(t_1, t_2)$*-dicritical singular point, or* $y^{t_1} - \tilde{a}x^{t_2}$ *is a factor of* $h_{r+|\mathbf{t}|}$ *with* $\mathbf{t} = (t_1, t_2)$ *is a type associated to an edge of its Newton diagram.*

We note that the origin can be a $(t_1, t_2)$-dicritical singular point and $(t_1, t_2)$ a type no associated to an edge of its Newton diagram. See example 5.2.

Let $\gamma$ a non-spiral orbit at origin. From Bendixson [7] and Algaba *et al.* [2], $\gamma$ is either $x = 0,\ y = 0,\ y = \lambda x^{a/b} + o(x^{a/b})$ formal, $x = \lambda y^{a/b} + o(y^{a/b})$ formal ($\lambda \neq 0, ab > 0$), or a flat curve solution that tends asymptotically to one of them, i.e. it is of the form either $y = \tau(x),\ x = \sigma(y),\ y = x^{a/b}\tau(x) + o(x^{a/b}\tau(x))$ or $x = y^{a/b}\sigma(y) + o(y^{a/b}\sigma(y))$ with $\lim_{x\to 0}\tau(x) = \lim_{y\to 0}\sigma(y) = 0$, and $\lim_{x\to 0}\frac{\tau(x)}{x^{1/n}} = \lim_{y\to 0}\frac{\sigma(y)}{y^{1/n}} = +\infty$, for all $n \in \mathbb{N}$.

**Lemma 3.3** *If system (1) has a flat orbit at origin of the form* $y = \tau(x)$ *with* $\lim_{x\to 0}\tau(x) = 0$ *and* $\lim_{x\to 0}\frac{\tau(x)}{x^{1/n}} = +\infty$*, for all* $n \in \mathbb{N}$*, then* $y = 0$ *is an orbit of system (1).*

*In general, if system (1) has a flat orbit at origin of the form* $y = x^{a/b}\tau(x) + o(x^{a/b}\tau(x))$ *with* $a, b \in \mathbb{N},\ \lim_{x\to 0}\tau(x) = 0$ *and* $\lim_{x\to 0}\frac{\tau(x)}{x^{1/n}} = +\infty$*, for all* $n \in \mathbb{N}$*, then it has a formal orbit of the form* $y = \lambda x^{a/b} + o(a/b)$ *with* $\lambda \neq 0$*.*

Therefore, from Lemma 3.3, it is enough to study the existence and the dynamic of the formal orbits to determine the local phase portrait at origin.

**Remark 3.4** Following the proof of Theorem 2 of [2], if system (1) has an orbit of the form $y = ax^{\lambda} + o(\lambda)$ with $\lambda$ a non-rational real number, from the density of $\mathbb{Q}$ in $\mathbb{R}$, system (1) has a flat orbit given in Lemma 3.3 and as a consequence the system has a formal characteristic orbit.

We note that the orbits associated to a factor $y^{t_1} - \bar{a}x^{t_2}$ (if they existed) are two half-paths $\gamma^+$ and $\gamma^-$. Concretely, if $t_1$ is odd, a parameterization of both characteristic orbits is $\gamma^+ = \{y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1}),\ x \geq 0\}$ and $\gamma^- = \{y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1}),\ x \leq 0\}$. If $t_2$ is odd, it has that $\gamma^+ = \{x = ((1/\bar{a})y^{t_1})^{1/t_2} + o(y^{t_1/t_2}),\ y \geq 0\}$ and $\gamma^- = \{x = ((1/\bar{a})y^{t_1})^{1/t_2} + o(y^{t_1/t_2}),\ y \leq 0\}$. Analogously, if $x = 0$ is a factor, gives the half-line paths $\{x \leq 0\}$ and $\{x \geq 0\}$.

**Definition 3.5** *We say that a polynomial of the form* $x,\ y$ *or* $y^{t_1} - \lambda x^{t_2},\ \lambda \neq 0$*, is a strong factor of* $\mathbf{F}$ *associated to the type* $\mathbf{t}$*, or simply a strong factor of* $h_{r+|\mathbf{t}|}$*, if it satisfies one of the following properties:*

- **(i)** *it is a factor of* $h_{r+|\mathbf{t}|}$ *of odd multiplicity order,*
- **(ii)** *it is a factor of* $h_{r+|\mathbf{t}|}$ *of even multiplicity order (*$2m$*) and, either it is not a factor of* $\mu_r$ *with* $\mu_r \not\equiv 0$ *or is a factor of* $\mu_r$ *with even multiplicity order (*$2n$*) with* $0 < n < m$*.*

*Otherwise, we say that it is an exceptional factor.*

From Lemma 3.3, if system (1) has a flat orbit at origin of the form $y = \tau(x)$, then $y = 0$ is an invariant curve of system (1), that is $Q(x, 0) \equiv 0$. And if it has a flat orbit at origin of the form $x = \tau(y)$, then $x = 0$ is an invariant curve of system (1), that is $P(0, y) \equiv 0$. Therefore, we center our study in the existence of characteristic orbit of the form $y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1})$ with $\bar{a} \neq 0$.

**Proposition 3.6** *We assume that the origin is an isolated singular point of system (1).*

1. *If $y^{t_1} - \bar{a}x^{t_2}$, with $\bar{a} \neq 0$, is a strong factor of $h_{r+|\mathbf{t}|} \not\equiv 0$, there is exactly one path (two semipaths) of the form $y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1})$.*
2. *If the origin is a $(t_1, t_2)$-dicritical point, then for each $\bar{a} \neq 0$ such that $y^{t_1} - \bar{a}x^{t_2}$ is not a factor of $\mu_r$, the divergence of $\mathbf{F}_r^{\mathbf{t}}$, there is exactly one path (two semipaths) of the form $y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1})$.*

Proposition 3.6 provides a method for determining the characteristic orbits of the origin:

1. we check if $x = 0$ or $y = 0$ are invariant curves of system (1),
2. if the origin is a $\mathbf{t}$-dicritical singular point, for each $\bar{a} \neq 0$ such that $y^{t_1} - \tilde{a}x^{t_2}$ is not a factor of $\mu_r$, there two semipaths of the form $y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1})$,
3. each strong factor $y^{t_1} - \tilde{a}x^{t_2}$ gives two semipaths of the form $y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1})$ which are solution curves of (1).

For each exceptional factor of type $\mathbf{t} = (t_1, t_2)$,

- if $t_1$ is odd, applying the directional blow-up $x = u^{t_1}$, $y = u^{t_2}(\bar{y} + \tilde{a}^{1/t_1})$ and the reparameterization $dt = (t_1/u^r)d\tau$, we obtain the system

$$\begin{aligned} u' &= u\sum_{j=0}^{\infty} P_{r+j+t_1}(1, \bar{y} + \tilde{a}^{1/t_1})u^j, \\ \bar{y}' &= \sum_{j=0}^{\infty} h_{r+j+|\mathbf{t}|}(1, \bar{y} + \tilde{a}^{1/t_1})u^j. \end{aligned} \tag{5}$$

- if $t_1$ is even, applying the directional blow-up $x = u^{t_1}(\bar{x} + \tilde{a}^{-1/t_2})$, $y = u^{t_2}$ and the reparameterization $dt = (t_2/u^r)d\tau$, we obtain the system

$$\begin{aligned} u' &= -u\sum_{j=0}^{\infty} Q_{r+j+t_2}(\bar{x} + \tilde{a}^{-1/t_2}, 1)u^j, \\ \bar{x}' &= \sum_{j=0}^{\infty} h_{r+j+|\mathbf{t}|}(\bar{x} + \tilde{a}^{-1/t_2}, 1)u^j. \end{aligned} \tag{6}$$

We apply Proposition 3.6 for the new system (5) or (6) (which axis $u = 0$ is invariant for).

We remark that the orbits $\bar{y} = (\bar{b}u^{s_2})^{1/s_1} + o(u^{s_2/s_1})$ of the systems (5) and (6) are orbits of system (1) of the form

$$y = (\bar{a}x^{t_2})^{1/t_1} + ((\bar{b}x^{s_2})^{1/s_1})^{1/t_1}x^{t_2/t_1} + o(x^{t_2/t_1 + s_2/(s_1 t_1)}).$$

Fixed a type $\mathbf{t}$, we define the set

$$W_{\mathbf{t}} := \bigcup_{\sigma_1, \sigma_2 \in \{0,1\}} W_{\mathbf{t}}^{(\sigma_1, \sigma_2)},$$

with

$$W_{\mathbf{t}}^{(\sigma_1,\sigma_2)} = \{(x,y)\in\mathbb{R}^2 \mid \epsilon x^{t_2/t_1} \le y \le \frac{1}{\epsilon}x^{t_2/t_1},\ (-1)^{\sigma_1}x \ge 0,\ (-1)^{\sigma_2}\epsilon > 0\},$$

with $\sigma_1, \sigma_2 \in \{0,1\}$.

These sets play a main role since if the origin is a $\mathbf{t}$-dicritical singular point, $W_{\mathbf{t}}$ is filled of orbits of system (1) of the form $y = (\bar{a}x^{t_2})^{1/t_1} + o(x^{t_2/t_1})$. So, for instance, the shaded zones of Figures 5 and 7 are the *dicritical zones* of the origin of the systems (10) and (14).

## 4 Behavior of the orbits near the origin

We analyze the stability and dynamic of the orbits near the origin.

The below result allows to analyze when the orbits leave or enter origin.

**Theorem 4.1** *We assume that system (1) has an orbit* $\gamma$ *associated to the type* $(t_1, t_2)$*. It has that* $\nabla H^{\mathbf{t}} \cdot \mathbf{F} \not\equiv 0$ *and*

1. *if* $\nabla H^{\mathbf{t}} \cdot \mathbf{F} < 0$ *on* $\gamma$ *for* $x$ *enough small, then the path enters origin as* $t \to +\infty$ *(we say that the dynamic of* $\gamma$ *is attractive),*

2. *if* $\nabla H^{\mathbf{t}} \cdot \mathbf{F} > 0$ *on* $\gamma$ *for* $x$ *enough small, then the path enters origin as* $t \to -\infty$ *(we say that the dynamic of* $\gamma$ *is repulsive),*

*where* $H^{\mathbf{t}}$ *is the Hamiltonian* $H^{\mathbf{t}}(x,y) = x^{2t_2} + y^{2t_1} \in \mathscr{P}^{\mathbf{t}}_{2t_1t_2}$.

A neighborhood of an equilibrium may be partitioned into regions of three distinct types: elliptic, hyperbolic and parabolic regions. We recall that an elliptic sector is a filled region with orbits starting and ending at the equilibrium itself (and these orbits forming ever smaller loops converging on the origin). A hyperbolic sector, recognizable by orbits roughly resembling hyperbolas. These sectors are separated in one case, by separatrices (orbits with one end at the singular point separating the sectors). Finally, a parabolic sector is a region filled with orbits having just one end at the equilibrium, and surrounded by other orbits of the same kind.

We denote by $\gamma_1, \gamma_2, \ldots, \gamma_s$ the characteristic orbits of system (1). Two half-paths $\gamma_i$ and $\gamma_{i+1}$ are consecutive if the region $\Omega_i$ between both paths does not contain any $\gamma_j$ in a neighborhood of the origin, for all $j$.

Each region $\Omega_i$ contains, at least, either an elliptic or hyperbolic or parabolic sector.

We note that if $\Omega_i$ contains an elliptic or hyperbolic sector, then contains a curve orthogonal to the solutions of system (1) in a neighborhood of the origin. Otherwise, if there is not any orthogonal curve then $\Omega_i$ is a parabolic sector. Figure 2 shows the three possible situations which may arise. These cases correspond to parabolic, elliptic and hyperbolic region, respectively.

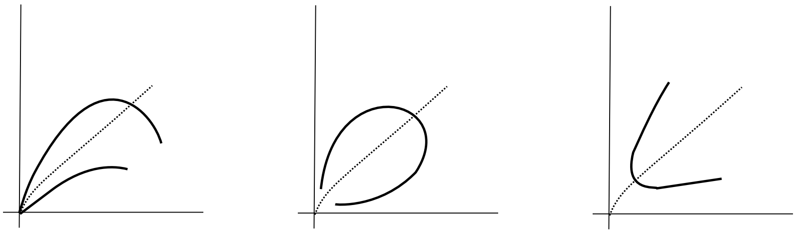

Figure 2 Behaviour of orthogonal curves.

We consider the system

$$\dot{\mathbf{x}} = \mathbf{F}^{\perp}(\mathbf{x}) = (-Q(x,y), P(x,y))^T. \tag{7}$$

The solution curves of (1) are orthogonal to the solution curves of (7) and vice versa. We denote $h^{\perp}_{\bar r+\bar{\mathbf{t}}}$ to the hamiltonian of the lowest-degree quasi-homogeneous term of type $\bar{\mathbf{t}} = (\bar t_1, \bar t_2)$ of $\mathbf{F}^{\perp}$.

The following result gives sufficient conditions for the existence of curve orthogonal to the solutions curves of system (1) from the edges and vertices of the Newton diagram of system (7). We also give the expression of $h^{\perp}_{\bar r+|\bar{\mathbf{t}}|}$ when $\bar{\mathbf{t}}$ is a type of both Newton diagrams.

**Theorem 4.2** *We assume that the origin is an isolated singular point of system (7). If* $y^{\bar t_1} - \bar a x^{\bar t_2}$, *with* $\bar a \neq 0$, *is a strong factor of* $h^{\perp}_{\bar r+|\bar{\mathbf{t}}|} \not\equiv 0$, *there is exactly one path (two semipaths),* $\Gamma$, *of the form* $y = (\bar a x^{\bar t_2})^{1/\bar t_1} + o(x^{\bar t_2/\bar t_1})$ *orthogonal to the solutions curves of (1).*
*Moreover if* $\mathbf{t}$ *is a type of both Newton diagrams, then* $2x^{k_1}y^{k_2}h^{\perp}_{\bar r+|\mathbf{t}|} = \nabla H^{\mathbf{t}} \cdot \mathbf{F}^{\mathbf{t}}_r$, *i.e. both functions have the same factors, by exception* $x$ *and* $y$.

Theorems 4.1 and 4.2 allow to characterize the different sectors in the vicinity of the origin, see Figure 3.

| $\gamma_i$ | $\gamma_{i+1}$ | sig($h^{\perp}_{\bar r+|\bar{\mathbf{t}}|}$) on $\Gamma_-$ | sig($h^{\perp}_{\bar r+|\bar{\mathbf{t}}|}$) on $\Gamma_+$ | $\Omega_i$ |
|---|---|---|---|---|
| atr | atr | | | parabolic |
| rep | rep | | | parabolic |
| rep | atr | neg | pos | elliptic |
| rep | atr | pos | neg | hyperbolic |
| atr | rep | neg | pos | hyperbolic |
| atr | rep | pos | neg | elliptic |

Figure 3 Study of the sectors of the origin.

The following results are useful for categorizing the different sectors of the origin.

**Theorem 4.3** *Let* $\gamma_i$ *and* $\gamma_{i+1}$ *be two consecutive orbits of system (1) with* $\gamma_i$ *associated to the type* $\mathbf{t} = (t_1, t_2)$ *and* $\gamma_{i+1}$ *to the type* $\mathbf{s} = (s_1, s_2)$. *It has that:*

1. *The region* $\Omega_i$ *is a parabolic sector of the origin if and only if both orbits have the same dynamic.*

2. *We assume that both orbits have different dynamic. It has that either* $\nabla H^{\mathbf{t}} \cdot \mathbf{F}$ *or* $\nabla H^{\mathbf{s}} \cdot \mathbf{F}$ *change the sign on* $\Omega_i$, *or* $(\nabla H^{\mathbf{t}} \cdot \mathbf{F})(\nabla H^{\mathbf{s}} \cdot \mathbf{F}) < 0$, *for* $x$ *enough small.*

**Theorem 4.4** *Let* $\gamma_i$ *and* $\gamma_{i+1}$ *be two consecutive orbits of system (1) with* $\gamma_i$ *associated to the type* $\mathbf{t} = (t_1, t_2)$ *and* $\gamma_{i+1}$ *to the type* $\mathbf{s} = (s_1, s_2)$. *It has that if* $h^{\mathbf{t}}$ *or* $h^{\mathbf{s}}$ *change of sign on* $\Omega_i$, *then* $\Omega_i$ *contains an elliptic sector of the origin.*

From Proposition 3.6, if the origin is a $\mathbf{t}$-dicritical singular point, $W_{\mathbf{t}}$ is filled by curves solution of system (1) which enter or leave the origin, except for a finite number of curves at most.

By Proposition 6.1 and Proposition 6.2, the analysis of the signs of $\nabla H^{\mathbf{t}} \cdot \mathbf{F}$ and $h^{\mathbf{t}}$ together, will allow to characterize the different sectors of the origin. So, if the origin is a $\mathbf{t}$-dicritical singular point,

1. if $y^{t_1} - \lambda x^{t_2}$ is not a factor of $\mu_r$, then $\gamma$ is contained into an open set $\Omega$, being $\Omega$ a region filled of orbits with the same dynamic that $\gamma$.
   Moreover, if the sign of $h^{\mathbf{t}}$ changes on $\Omega$, then $\Omega$ is formed by two parabolic sectors, each one of them may be subregions of two elliptic sectors (a global analysis would determine it).

2. We suppose that $y^{t_1} - \lambda x^{t_2}$ is a factor of $\nabla H^{\mathbf{t}} \cdot \mathbf{F}_r = \frac{1}{r+|\mathbf{t}|} H^{\mathbf{t}} \mu_r$, that is it is a factor of $\mu_r$. If $\nabla H^{\mathbf{t}} \cdot \mathbf{F}$ preserves the sign on $\Omega$, open set containing to $\gamma$, then $\Omega$ is a region filled of orbits with the same dynamic that $\gamma$.
   Otherwise, $\Omega$ is an elliptic sector if $h^{\mathbf{t}}$ changes of sign on $\Omega$. And $\Omega$ is a hyperbolic sector, if $h^{\mathbf{t}}$ preserves it.
   We emphasize that for one analysis more detailed and precise, we must perform the blow-up transformations given in section 3.

## 5 Applications and examples

**Example 5.1** We study the local behavior near origin of the orbits of the quadratic system

$$\dot{x} = y^2 - 7xy + 11x^2, \qquad \dot{y} = 2y^2 - 9xy. \tag{8}$$

Newton diagram of (8) has one exterior vertex $(0,3)$, one interior vertex $(2,1)$, i.e. it has one bounded edge of type $\mathbf{t} = (1,1)$ and one unbounded edge of type $\mathbf{s} = (0,1)$, i.e. $y = 0$ is an invariant curve of the system, see Figure 4. The polynomial $h_2^{(1,1)}$ is

$$h_2^{(1,1)} = (x,y)^T \wedge (y^2 - 7xy + 11x^2, 2y^2 - 9xy)^T = -y(y-4x)(y-5x).$$

The polynomials $y$, $y-4x$ and $y-5x$ are simple factors of $h_2^{(1,1)}$, therefore they are strong factors. The curve $y = 0$ is an invariant curve of system (8) and by Proposition 3.6, there are two paths of the form $y = 4x + o(x)$ and $y = 5x + o(x)$, respectively. We conclude that there are six formal semipaths which enter or leave to origin. We denote them by $\gamma_1 = \{y = 0, x > 0\}$, $\gamma_2 = \{y = 4x + \cdots, x > 0\}$, $\gamma_3 = \{y = 5x + \cdots, x > 0\}$, $\gamma_4 = \{y = 0, x < 0\}$, $\gamma_5 = \{y = 4x + \cdots, x < 0\}$ and $\gamma_6 = \{y = 5x + \cdots, x < 0\}$.

The polynomial $\nabla H^{(1,1)} \cdot \mathbf{F}$, where $H^{(1,1)} = x^2 + y^2$ and $\mathbf{F} = \mathbf{F}_2^{(1,1)}$ is

$$\nabla H^{(1,1)} \cdot \mathbf{F} = 2x(y^2 - 7xy + 11x^2) + 2y(2y^2 - 9xy) = 4y^3 - 16xy^2 - 14x^2y + 22x^3.$$

Thus,

$$\nabla H^{(1,1)} \cdot \mathbf{F}(\gamma_2) = \nabla H^{(1,1)} \cdot \mathbf{F}(\gamma_5) = -34x^3 + \cdots,$$

$$\nabla H^{(1,1)} \cdot \mathbf{F}(\gamma_3) = \nabla H^{(1,1)} \cdot \mathbf{F}(\gamma_6) = 52x^3 + \cdots.$$

And $\nabla H^{(0,1)} \cdot \mathbf{F} = 2x(y^2 - 7xy + 11x^2)$, that is

$$\nabla H^{(0,1)} \cdot \mathbf{F}(\gamma_1) = \nabla H^{(0,1)} \cdot \mathbf{F}(\gamma_4) = 22x^3.$$

From Theorem 4.1, $\gamma_1, \gamma_3$ and $\gamma_5$ have repulsive dynamic and $\gamma_2, \gamma_4$ and $\gamma_6$ attractive.

We analyze the sectors surrounding the origin. We consider the system

$$\dot{x} = -2y^2 + 9xy, \qquad \dot{y} = y^2 - 7xy + 11x^2, \tag{9}$$

system whose solutions are orthogonal trajectories to the trajectories of system (8).

Newton diagram of system (9) only has two exterior vertices, $(0,3)$ and $(3,0)$, i.e. it has an edge of type $\mathbf{t} = (1,1)$, which also is a type associated to an edge of Newton diagram of system (8), see Figure 4. By applying Theorem 4.2 as $2h^{\perp} = \nabla H^{(1,1)} \cdot \mathbf{F}_2^{(1,1)} = (y - r_1x)(y - r_2x)(y - r_3x)$ with $r_1 \approx 0.88,\ r_2 \approx 4.5$ and $r_3 \approx -1.38$, we have that $y = r_ix + \cdots$, are curve orthogonal to the orbits of the system (8).

We denote by $\Gamma_1 = \{y = r_1x + \cdots, x > 0\},\ \Gamma_2 = \{y = r_2x + \cdots, x > 0\},\ \Gamma_3 = \{y = r_3x + \cdots, x < 0\},\ \Gamma_4 = \{y = r_1x + \cdots, x < 0\},\ \Gamma_5 = \{y = r_2x + \cdots, x < 0\}$ and $\Gamma_6 = \{y = r_3x + \cdots, x > 0\}$. By following the table of Figure 3, it has that all $\Omega_i$, region between both paths $\gamma_i$ and $\gamma_{i+1}$ are hyperbolic sectors of the origin, see Figure 4.

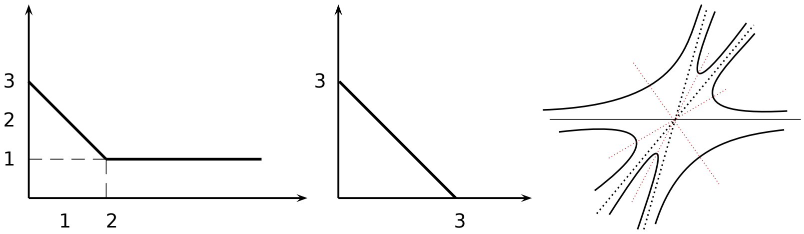

Figure 4 Newton diagrams of systems (8) and (9) and scheme of the origin of system (8).

**Example 5.2** We consider the system

$$\dot{x} = xy + \sigma(x+y)^4, \qquad \dot{y} = y^2 - \sigma(x+y)^4,\ \sigma = \pm 1. \tag{10}$$

Newton diagram of (10) has two exterior vertices ($(0,5)$ and $(5,0)$) and one interior vertex $(1,2)$, i.e. it consists of two bounded edges of type $\mathbf{t} = (3,1)$ and of type $\mathbf{s} = (1,2)$, see Figure 5.

For the type $\mathbf{t} = (3,1)$, the lowest-degree quasi-homogeneous term of the vector field is $\mathbf{F}_1^{(3,1)} = (xy + \sigma y^4, y^2)^T$, that is,

$$h_5^{(3,1)} = (3x, y)^T \wedge \mathbf{F}_1^{(3,1)} = -y^2(y^3 - 2\sigma x), \qquad \mu_1^{(3,1)} = 3y.$$

So, $y^3 - 2\sigma x$ is a strong factor of $h_5^{(3,1)}$.

For the type $\mathbf{s} = (1,2)$, the lowest-degree quasi-homogeneous term of the vector field is $\mathbf{F}_2^{(1,2)} = (xy, y^2 - \sigma x^4)^T$, thus,

$$h_5^{(1,2)} = (x, 2y)^T \wedge \mathbf{F}_2^{(1,2)} = -x(y^2 + \sigma x^4), \qquad \mu_2^{(1,2)} = 3y.$$

So, for $\sigma = 1$, $x$ is a strong factor associated to the type $(1,2)$. And for $\sigma = -1$, there are three strong factors $x,\ y - x^2$ and $y + x^2$.

On the other hand, system (10) is $(1,1)$-dicritical. As the divergence of $\mathbf{F}_1^{(1,1)} = (xy, y^2)^T$, lowest-degree quasi-homogeneous term of $\mathbf{F}$ with respect to the type $(1,1)$, is $\mu_1^{(1,1)} = 3y$, from Proposition 3.6, we have that for each $\lambda \neq 0$ there is exactly one path (two semipaths) of the form $y = \lambda x + o(x)$.

For $\sigma = 1$, the formal orbits of system (10) at origin are of the form $x = 1/2y^3 + o(y^3)$ and $y = \lambda x + o(x)$, for any $\lambda \neq 0$.

Moreover, $y = 0$ is not a orbit of system (10) and $\nabla H^{(1,1)} \cdot \mathbf{F}_1^{(1,1)} = 2y(x^2 + y^2)$. Therefore, origin is surrounded by orbits of the form $y = \lambda x + o(x)$, for any $\lambda \neq 0$.
By analyzing the sign of $\mu_2 = y$ and $h^{(1,1)} = h_5 = -(x+y)^5$, we have that a neighborhood of the origin is formed by two elliptic sectors which they are separated by the orbit $y = -x + o(x)$. The semipath $y = -x + \cdots,\ x > 0$, has attractive dynamic and the semipath $y = -x + \cdots,\ x < 0$, has repulsive dynamic since $\nabla H^{(1,1)} \cdot \mathbf{F}(x, -x + \cdots) = -4x^3 + \cdots$

We analyze the formal trajectories of the orthogonal system associated. Newton diagram of the system

$$\dot{x} = -y^2 + \sigma(x+y)^4, \qquad \dot{y} = xy + \sigma(x+y)^4 \tag{11}$$

has two edge of types $(1,1)$ and $(1,3)$ whose hamiltonians of the lowest-degree quasi-homogeneous term are $y(x^2+y^2)$ and $x^2(y+\sigma x^3)$, respectively. Thus, $y + \sigma x^3$ is a strong factor and therefore there is an orthogonal trajectory of the form $y = -\sigma x^3 + \cdots$

So, by following the table of Figure 3, we have that the regions between the semipaths $y = -x + \cdots,\ x > 0$ and $y = -x + \cdots,\ x < 0$ are elliptic sectors of the origin.

For $\sigma = -1$, by applying Theorem 4.1, as

$$\nabla H^{(1,2)} \cdot \mathbf{F} = 4x^4 y + 2y^3 + (x+y)^4(2y - 4x^3),$$

we have that there are two semipaths of the form $y = x^2 + o(x^2), x < 0$ and $y = x^2 + o(x^2), x > 0$ with repulsive dynamic, since $\nabla H^{(1,2)} \cdot \mathbf{F}(x, x^2 + \cdots) = 8x^6 + \cdots > 0$. And there are two semipaths of the form $y = -x^2 + o(x^2), x < 0$ and $y = -x^2 + o(x^2), x > 0$ attractive, since $\nabla H^{(1,2)} \cdot \mathbf{F}(x, -x^2 + \cdots) = -8x^6 + \cdots < 0$.

Moreover, for $x > 0,\ y = x^3 + \cdots$ is between $y = -x^2 + \cdots$ and $y = x^2 + \cdots$. From table of Figure 3, the region is a hyperbolic sector of the origin. Analogously, the region between $y = x^2 + \cdots, y = -x^2 + \cdots$ and $x < 0$ is a hyperbolic sector.

Qualitative portraits of the paths in a neighborhood of the origin for the system (10) are shown in Figure 5.

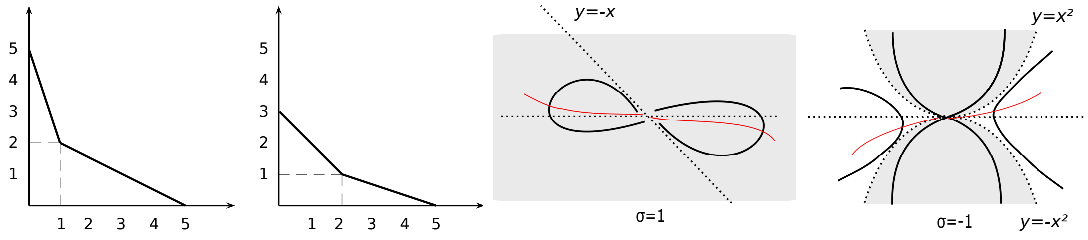

Figure 5 Newton diagrams of systems (10) and (11) and scheme of the origin of system (10).

**Example 5.3** We analyze the system

$$\dot{x} = y^3 + xy^2 + x^2y + x^4, \qquad \dot{y} = y^3 + 2xy^2 + 2x^3y - x^5. \tag{12}$$

Its Newton diagram consists of the two exterior vertices $(0,4)$ and $(6,0)$, the interior vertex $(2,2)$, and two bounded edges of type $\mathbf{t} = (1,1)$ and $\mathbf{s} = (1,2)$. Newton diagram of the system

$$\dot{x} = -y^3 - 2xy^2 - 2x^3y + x^5, \qquad \dot{y} = y^3 + xy^2 + x^2y + x^4, \tag{13}$$

consists of two edges whose types are $\mathbf{t} = (1,1)$ and $\mathbf{s} = (1,2)$, i.e. both diagrams has the same types associated to the edges, see Figure 6.

For the type $\mathbf{t} = (1,1)$, the lowest-degree quasi-homogeneous Hamiltonian of the vector field is

$$h_3^{(1,1)} = (x,y)^T \wedge (y^3 + xy^2 + x^2y, y^3 + 2xy^2)^T = -y^2(y-x)(y+x).$$

Moreover,

$$2h_3^{\perp} = \nabla H^{(1,1)} \cdot \mathbf{F}_2^{(1,1)} = 2y(y^3 + 3xy^2 + x^2y + x^3),$$

that is, $y - x$ and $y + x$ are strong factors and $\nabla H^{(1,1)} \cdot \mathbf{F}_2^{(1,1)}$ changes of sign on $y = 0$ and $y = -2.769x$.

For the type $\mathbf{s} = (1,2)$, the lowest-degree quasi-homogeneous Hamiltonian of the vector field is

$$h_6^{(1,2)} = (x,2y)^T \wedge (x^2y + x^4, 2xy^2 + 2x^3y - x^5)^T = -x^6,$$

and

$$2h_6^{\perp} = \nabla H^{(1,2)} \cdot \mathbf{F}_3^{(1,2)} = 2x(2x^6 + x^4y + 2x^2y^2 + 2y^3).$$

So, origin does not have any strong factors associated to the type $(1,2)$ and $\nabla H^{(1,2)} \cdot \mathbf{F}_3^{(1,2)}$ changes of sign on $x = 0$ and $y = -1.24x^2$.

The formal orbits of system (12) at origin are of the form $y = x + o(x)$ and $y = -x + o(x)$. The study of the sign of $\nabla H^{(1,1)} \cdot \mathbf{F}$ on both curves, it determines the dynamic of them.

By analyzing the sign of $\nabla H^{(1,1)} \cdot \mathbf{F}$ in a neighborhood of the path $y = -2.769x$ and the sign of $\nabla H^{(1,2)} \cdot \mathbf{F}$ in a neighborhood of the path $y = -1.24x^2$, we have that a neighborhood of the origin is formed by two elliptic sectors and two hyperbolic sectors, see Figure 6.

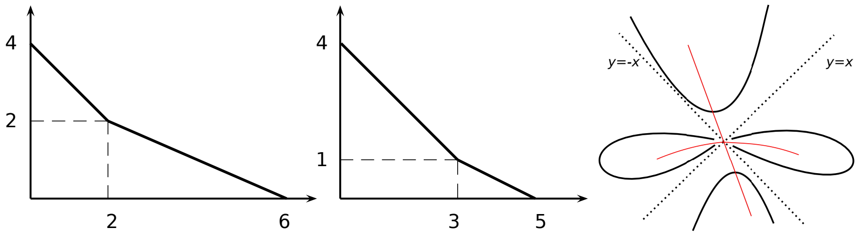

Figure 6 Newton diagrams of systems (12) and (13) and scheme of the origin of system (12).

**Example 5.4** We consider the system

$$\begin{aligned} \dot{x} &= -xy^3 + 3x^2y^2 + xy^2 - 2x^3y - 3x^2y + 2x^3, \\ \dot{y} &= y^3 + x^2y^2 - 3xy^2 - 2x^4y - 3x^3y + 2x^2y + 2x^4 + 3x^5. \end{aligned} \tag{14}$$

The origin of system (14) is an isolated singular point and $x = 0$ is an invariant curve. Newton diagram of system (14) has one exterior vertex, $(5, 0)$, and two interior vertices, $(1, 3)$ and $(3, 1)$. So, its Newton diagram has two bounded edges of type $\mathbf{t} = (1, 1)$ and $\mathbf{s} = (1, 2)$ and one unbounded edge of type $(1, 0)$, see Figure 6.

The origin of system (14) is $(1, 1)$-dicritical. As the divergence of $\mathbf{F}_2^{(1,1)} = (xy^2 - 3x^2y + 2x^3, y^3 - 3xy^2 + 2x^2y)^T$, lowest-degree quasi-homogeneous term of $\mathbf{F}$ with respect to the type $(1, 1)$, is $\mu_3^{(1,1)} = 4(y-x)(y-2x)$, from Proposition 3.6, we have that for each $\lambda \neq 0,\ \lambda \neq 1,\ \lambda \neq 2$ there is exactly one path (two semipaths) of the form $y = \lambda x + o(x)$. Moreover, by applying Theorem 4.1, the dynamic of the semipaths is attractive if $1 < \lambda < 2$ and repulsive otherwise. So, $y = x$ with $x < 0$, $y = x$ with $x > 0$, $y = 2x$ with $x < 0$ and $y = 2x$ with $x > 0$ are orthogonal orbits of system (14) and they are contained in four elliptic sectors.

For the type $(1, 2)$, the lowest-degree quasi-homogeneous term of the vector field is $\mathbf{F}_2^{(1,2)} = (2x^3, 2x^2y + 2x^4)^T$, that is,

$$h_5^{(1,2)} = (x, 2y)^T \wedge \mathbf{F}_2^{(1,2)} = -2x^3(y - x^2).$$

So, $y - x^2$ is a strong factor of $h_5^{(1,2)}$. And $x = 0$ is an invariant curve of system (14).

The polynomial $\nabla H^{(1,2)} \cdot \mathbf{F}_2^{(1,2)} = 4x^2(2x^4 + y^2 + x^2y)$ is positive for all $(x, y)$ different from $(0, 0)$, therefore, from Theorem 4.1, the dynamics of $y = x^2 + o(2),\ x < 0$ and $y = x^2 + o(2),\ x > 0$ are repulsive.

On the other hand, it is easy to check that $x = 0$ has repulsive dynamic, see Figure 7.

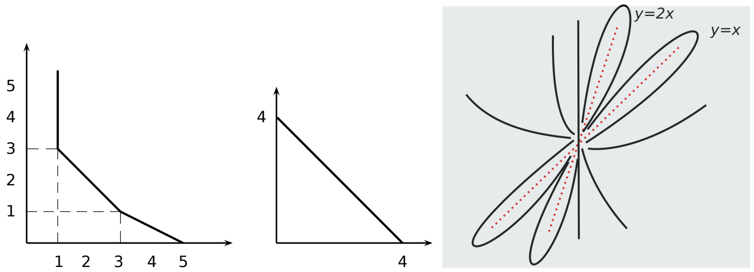

Figure 7 Newton diagram and scheme of the origin of system (14) and of the orthogonal system associated.

**Example 5.5** We study the system

$$\dot{x} = xy^{d-1} + \sum_{i=2}^{d} a_i x^{2i-1} y^{d-i}, \qquad \dot{y} = 2y^d + 2\sum_{i=2}^{d} a_i x^{2i-2} y^{d+1-i} + \sum_{i=2}^{d} b_i x^{2i+2} y^{d-i} \tag{15}$$

where $a_i, b_i$ satisfy

$$\lambda^{d-1} + \sum_{i=2}^{d} a_i \lambda^{d-i} = \prod_{j=1}^{d-1} (\lambda - j), \qquad \sum_{i=2}^{d} b_i \lambda^{d-i} = \frac{d}{d\lambda} \prod_{j=1}^{d-1} (\lambda - j),\ d \geq 3.$$

This system has been studied by S. Schecter and M.F. Singer in [11].

Its Newton diagram consists of two interior vertices $(1, d)$ and $(2d-1, 1)$, the exterior vertex $(2d + 3, 0)$, thus, it has two bounded edges of type $(1, 2)$ and $(1, 4)$ and the unbounded edge $x = 1$.

On the other hand, Newton diagram of the orthogonal system associated consists of the two exterior vertices $(0, d+1)$ and $(2d, 0)$ and the interior vertex $(2, d-1)$, thus, it has two bounded edges of type $(1,1)$ and $(1,2)$, see Figure 8.

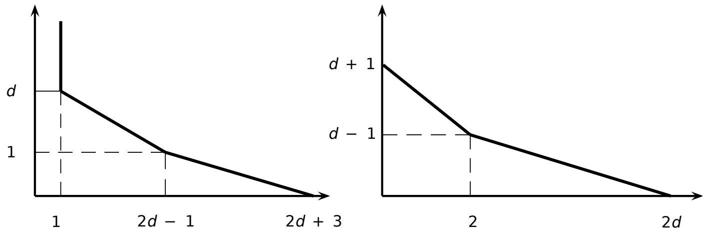

Figure 8 Newton diagrams of systems (15) and its orthogonal system.

The lowest-degree quasi-homogeneous vector field of type $(1,2)$ is

$$\mathbf{F}_{2d-2}^{(1,2)} = (xy^{d-1} + \sum_{i=2}^{d} a_i x^{2i-1} y^{d-i}, 2y^d + 2\sum_{i=2}^{d} a_i x^{2i-2} y^{d+1-i})^T.$$

Its quasi-homogeneous Hamiltonian of type $(1,2)$ is identically zero, i.e. the origin of the system (15) is a $(1,2)-$dicritical.

And its divergence is

$$(1+2d)(y^{d-1} + \sum_{i=2}^{d} a_i x^{2i-2} y^{d-i}) = (1+2d)\prod_{j=1}^{d-1}(y - jx^2).$$

The lowest-degree quasi-homogeneous Hamiltonian of type $(1,2)$ non-null is

$$h_{2d+3}^{(1,2)} = (x, 2y)^T \wedge (0, \sum_{i=2}^{d} b_i x^{2i+2} y^{d-i})^T = \sum_{i=2}^{d} b_i x^{2i+3} y^{d-i}.$$

It has that

$$h_{2d+3}^{(1,2)}(x, \lambda x^2) = x^{2d+3}\sum_{i=2}^{d} b_i \lambda^{d-i} = x^{2d+3}\frac{d}{d\lambda}\prod_{j=1}^{d-1}(\lambda - j).$$

Therefore, $h_{2d+3}^{(1,2)}$ changes of sign between the paths $y = jx^2$ and $y = (j+1)x^2,\ j = 1, ..., d-1$. Analyzing the sign and by applying Theorem 4.2, it has that $y = \lambda x^2 + o(x^2)$ is a separatrix of the origin, where $\lambda$ is a root of the polynomial $\frac{d}{d\lambda}\prod_{j=1}^{d-1}(\lambda - j)$.

For the type $\mathbf{t} = (1,4)$, the lowest-degree quasi-homogeneous Hamiltonian of the vector field is

$$h_{2d+3}^{(1,4)} = (x, 4y)^T \wedge (a_d x^{2d-1}, 2a_d x^{2d-2} y + b_d x^{2d+2})^T = x^{2d-1}(-2a_d y + b_d x^4),$$

that is, $-2a_d y + b_d x^4$ is a strong factor.

We end by analyzing the orthogonal trajectories of type $(1,1)$. As

$$h_{d+1}^{\perp} = (x, y)^T \wedge (-2y^d, xy^{d-1})^T = y^{d-1}(x^2 + 2y^2),$$

there is not any orthogonal curve of type $(1,1)$. Thus, we conclude that the origin is surrounded by orbits of the form $y = ax^2 + o(x^2)$ with $a$ different from $1, 2, \ldots, d-1$. Also, $x = 0$ is invariant curve of the system ant it is repulsive. There are orthogonal trajectories of the form $y = ax^2$

with $a = 1, 2, \dots, d-1$. From Theorem 4.2, they are contained into a hyperbolic sectors. So, there are $2(d-1)$ hyperbolic sectors and $4(d-1)$ separatrices. The region $y < 0$ is a parabolic sector of the origin.

## 6 Proofs

**Proof of Lemma 3.3.**

We suppose that $y = 0$ is not solution. Then, system (1) can be written as

$$\dot{x} = P(x, y), \qquad \dot{y} = b_k x^k + b_{k+1} x^{k+1} g_1(x) + y g_2(x, y)$$

where $P(0,0) = g_1(0) = g_2(0,0) = 0$ and $b_k \neq 0$.
We see that $y = \tau(x)$ with $\lim_{x\to 0} \tau(x) = 0$ and $\lim_{x\to 0} \frac{\tau(x)}{x^{1/n}} = +\infty$, for all $n \in \mathbb{N}$, is not solution of system (1). Indeed, if it is a solution, we have that

$$0 = \dot{y} - \tau'(x)\dot{x} = b_k x^k + b_{k+1} x^{k+1} g_1(x) + \tau(x) g_2(x, \tau(x)) - \tau'(x) P(x, \tau(x)).$$

Therefore, the least-degree coefficient in $x$ of the right-hand side is zero, i.e. $b_k = 0$. This fact arrives to contradiction.

In general, we suppose that system (1) is a formal orbit of the form $y = \lambda x^{a/b} + o(a/b)$ with $\lambda > 0$. The directional blow-up $x = u^b$, $y = u^a(\bar{y} + \lambda)$ and the reparameterization $dt = (t_1/u^r)d\tau$, transform system (1) into system (5) for $t_1 = b$, $t_2 = a$, which is invariant to $\bar{y} = 0$. Next, reasoning as before, the result follows. ∎

Before proving our results we need to recall the generalized polar coordinates.
Fixing a type $\mathbf{t}$, define the *generalized trigonometric functions,* $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$, as the unique solution of the Cauchy problem

$$\frac{d\mathbf{x}}{d\theta} = \mathbf{X}_{H^{\mathbf{t}}}(\mathbf{x}), \qquad \mathbf{x}(\mathbf{0}) = (1, 0)^T,$$

where $H^{\mathbf{t}}$ is the Hamiltonian $H^{\mathbf{t}}(x, y) = x^{2t_2} + y^{2t_1} \in \mathscr{P}^{\mathbf{t}}_{2t_1t_2}$. These functions are periodic and $T_{\mathbf{t}}$ will denote their minimal period. Moreover, they satisfy the equality $\mathrm{Cs}^{2t_2}(\theta) + \mathrm{Sn}^{2t_1}(\theta) = 1$. We introduce the *generalized polar coordinates*, $u$ and $\theta$ of the real plane $(x, y) \in \mathbb{R}^2$, as

$$x = u^{t_1}\mathrm{Cs}(\theta), \qquad y = u^{t_2}\mathrm{Sn}(\theta), \tag{16}$$

with $u \geq 0$ and $\theta \in [0, T_{\mathbf{t}})$.

The quoted transformation carries the region of the plane $(u, \theta)$ given by the rectangle $R = [0, \epsilon) \times [0, T_{\mathbf{t}})$, $\epsilon > 0$, to the neighborhood of the origin $W = \{(x, y) \in \mathbb{R}^2,\ x^{2t_2} + y^{2t_1} \leq \epsilon^{2t_1t_2}\}$. Also, it transforms the curve $y^{t_1} = a x^{t_2}$ onto $\theta = \theta^*$ with $a = \frac{\mathrm{Sn}^{t_1}(\theta^*)}{\mathrm{Cs}^{t_2}(\theta^*)}$.

The stability of the orbits and the dynamic near the origin is given by the analysis of the sign of the derivatives of the radial and azimuthal variables, $u$ and $\theta$, on the solution curves in a neighborhood of the origin.

**Proposition 6.1** *Let $\mathbf{t} = (t_1, t_2)$ be a type and $(u, \theta)$ the generalized polar coordinates given by (16). The derivative of $u$ with respect to the time and $\nabla H^{\mathbf{t}} \cdot \mathbf{F}$ have the same sign on any solution curve of system (1).*

**Proposition 6.2** *Let* $\mathbf{t}$ *be a type and* $(u,\theta)$ *the generalized polar coordinates given by (16). The derivative of* $\theta$ *with respect to the time and* $h^{\mathbf{t}} = h_{r+|\mathbf{t}|} + h_{r+1+|\mathbf{t}|} + \cdots$, *where* $h_j \in \mathscr{P}^{\mathbf{t}}_{j+|\mathbf{t}|}$. *is the hamiltonian part of* $\mathbf{F}_j$, *have the same sign on any solution curve of system (1).*

**Proof of Proposition 6.1.**

By differentiating of (16) respect to the time, we have that

$$\begin{aligned} \dot{x} &= t_1 u^{t_1-1} Cs(\theta)\dot{u} - 2t_1 u^{t_1} Sn^{2t_1-1}(\theta)\dot{\theta}, \\ \dot{y} &= t_2 u^{t_2-1} Sn(\theta)\dot{u} + 2t_2 u^{t_2} Cs^{2t_2-1}(\theta)\dot{\theta}. \end{aligned}$$

By denoting $\mathbf{x}=(x,y)$ and $\mathbf{u}=(u,\theta)$ we have the matrix equation

$$\dot{\mathbf{x}} = (\frac{1}{u}\mathbf{D}_0, \frac{1}{u^{2t_1t_2-|\mathbf{t}|}}\mathbf{X}_{H^{\mathbf{t}}}) \cdot \dot{\mathbf{u}}. \tag{17}$$

where the right-hand side is evaluated for $(u^{t_1}Cs(\theta),\ u^{t_2}Sn(\theta))$.

By (17), we have that

$$\nabla H^{\mathbf{t}} \cdot \mathbf{F} := \dot{\mathbf{x}} \wedge \mathbf{X}_{H^{\mathbf{t}}} = \frac{1}{u}(\mathbf{D}_0 \wedge \mathbf{X}_{H^{\mathbf{t}}})\dot{u},$$

with $\mathbf{D}_0 \wedge \mathbf{X}_{H^{\mathbf{t}}} := t_1 x(2t_2 x^{2t_2-1}) - t_2 y(-2t_1 y^{2t_1-1}) = 2t_1t_2 H^{\mathbf{t}}(x,y)$. This polynomial is positive for any $(x,y) \neq (0,0)$. Thus, $\dot{u}$ and $\nabla H^{\mathbf{t}} \cdot \mathbf{F}$ have same sign. ■

**Proof of Theorem 4.1.**

The function $\nabla H^{\mathbf{t}} \cdot \mathbf{F}$ is not identically zero, since otherwise, $H^{\mathbf{t}}$ would be a first integral and in consequence the origin would be a center. This fact contradicts that there exists a characteristic orbit.

The sign of the derivative of the radial component $u$ with respect to the time on the orbit $\gamma$, allows us to determine the stability of the orbit, that is, if $\dot{u} > 0$ on $\gamma$ for $x$ enough small, the orbit verifies $lim_{t\to-\infty}\gamma(t) = 0$, the path arrives at origin and if $\dot{u} < 0$ on $\gamma$ for $x$ enough small, then $lim_{t\to+\infty}\gamma(t) = 0$, the path enters origin. ■

**Proof of Theorem 4.2.** The first part is a consequence of Proposition 3.6.
We see the second part. We assume that $\mathbf{t} = (t_1, t_2)$ is a type associated to an edge bounded of the Newton diagram of (1) and of (7).
The change $u = x^{t_2}, v = y^{t_1}$ transforms $\dot{\mathbf{x}} = \mathbf{F}^{\mathbf{t}}_r(\mathbf{x}) = (P_{r+t_1}, Q_{r+t_2})^T$ into a homogeneous system.

It is known that if p is a quasi-homogeneous polynomial of degree $r$ then $r$ can be expressed as $r = k_1t_1 + k_2t_2 + k_3t_1t_2$ with $0 \le k_1 \le t_2$ and $0 \le k_2 \le t_1$ and $p(x,y) = x^{k_1}y^{k_2}p^{hom}_r(u,v)$, where $p^{hom}_r$ is a homogeneous polynomial of degree $r$.

We distinguish the following cases:
Case $k_1 < t_2 - 1$, $k_2 < t_1 - 1$. It has that $r + t_1 = (k_1+1)t_1 + k_2t_2 + k_3t_1t_2$, $r + t_2 = k_1t_1 + (k_2+1)t_2 + k_3t_1t_2$. Thus,

$$\dot{u} = t_2 x^{t_2-1} x^{k_1+1} y^{k_2} P^{hom}, \qquad \dot{v} = t_1 y^{t_1-1} x^{k_1} y^{k_2+1} Q^{hom}.$$

So, the system whose solutions are orthogonal to the trajectories of $\dot{\mathbf{x}} = \mathbf{F}_r^{\mathbf{t}}(\mathbf{x})$ is

$$\dot{u} = -t_1 v Q^{hom}, \qquad \dot{v} = t_2 u P^{hom}.$$

We have a homogeneous system of degree $r+1$, i.e. a quasi-homogeneous system of type $(1,1)$ and degree $r$. So,

$$\begin{aligned} h_r^{\perp} &= t_2 u^2 P^{hom}(u,v) + t_1 v^2 Q^{hom}(u,v) \\ &= \tfrac{1}{2x^{k_1}y^{k_2}}(2t_2 x^{2t_2-1} P_{r+t_1} + 2t_1 y^{2t_1-1} Q_{r+t_2}) = \tfrac{1}{2x^{k_1}y^{k_2}}(\nabla H^{\mathbf{t}} \cdot \mathbf{F}_r). \end{aligned}$$

It is easy to prove the result for the remaining cases:
Case $k_1 = t_2 - 1,\ k_2 < t_1 - 1$. It has that $r + t_1 = k_2 t_2 + (k_3+1)t_1 t_2,\ r + t_2 = k_1 t_1 + (k_2+1)t_2 + k_3 t_1 t_2$.
Case $k_1 < t_2 - 1,\ k_2 = t_1 - 1$. In this case, $r + t_1 = (k_1+1)t_1 + k_2 t_2 + k_3 t_1 t_2,\ r + t_2 = k_1 t_1 + (k_3+1)t_1 t_2$.
Case $k_1 = t_2 - 1,\ k_2 = t_1 - 1$. It has that $r + t_1 = k_2 t_2 + (k_3+1)t_1 t_2,\ r + t_2 = k_1 t_1 + (k_3+1)t_1 t_2$. ∎

**Proof of Proposition 6.2.** On the one hand, by (17), we have that

$$\mathbf{D}_0 \wedge \dot{\mathbf{x}} = \frac{1}{u^{2t_1t_2-|\mathbf{t}|}}(\mathbf{D}_0 \wedge \mathbf{X}_{H^{\mathbf{t}}})\dot{\theta} = \frac{2t_1t_2u^{2t_1t_2}}{u^{2t_1t_2-|\mathbf{t}|}}\dot{\theta} = 2t_1t_2u^{|\mathbf{t}|}\dot{\theta} \tag{18}$$

On the other hand, by applying (4) and Euler theorem for quasi-homogeneous function, we have that

$$\begin{aligned} \mathbf{D}_0 \wedge \dot{\mathbf{x}} &= \mathbf{D}_0 \wedge \mathbf{F} = \textstyle\sum_{j\geq r} \frac{1}{j+|\mathbf{t}|}\mathbf{D}_0 \wedge \mathbf{X}_{h_{j+|\mathbf{t}|}} \\ &= \textstyle\sum_{j\geq r} \frac{1}{j+|\mathbf{t}|}\nabla h_{j+|\mathbf{t}|} \cdot \mathbf{D}_0 = \sum_{j\geq r} h_{j+|\mathbf{t}|} = h^{\mathbf{t}}. \end{aligned} \tag{19}$$

From (18) and (19), $\dot{\theta}$ and $h^{\mathbf{t}}$ have same sign. ∎

**Proof of Theorem 4.3.** If the orbits $\gamma_i$ and $\gamma_{i+1}$ have the same dynamic, the region $\Omega_i$ is a parabolic sector of the origin. Reciprocally, if $\Omega_i$ is a parabolic sector, then all the solution curves has the same dynamic, in particular $\Gamma_i$ and $\gamma_{i+1}$.

We see the second part. If $\nabla H^{\mathbf{t}} \cdot \mathbf{F}$ and $\nabla H^{\mathbf{s}} \cdot \mathbf{F}$ have same sign on the region, it has that $\dot{r}$ does not change of sign on $\Omega_i$ and therefore the dynamic on the region is either attractive or repulsive, according the sign of $\dot{r}$. So, the region would a parabolic sector and this one arrives to contradiction with that $\gamma_i$ and $\gamma_{i+1}$ have different dynamic. ∎

From Theorem 3.2, the types which determine the expression of the orbits of a system whose origin is a non-$(t_1,t_2)$-dicritical are the types of the edges of the Newton diagram of vector field $\mathbf{F}$ since, otherwise, if it is not a type of an edge of its Newton diagram, the lowest-degree term of type $\mathbf{t}$ is $\mathbf{F}_r = (x^{p+1}y^{q-1}, x^p y^q)^T$ with $t_1(p+1) + t_2 q = r + |\mathbf{t}|$; so, the polynomial $h_{r+|\mathbf{t}|}(x,y)$ gives a monomial in $x$ and $y$, i.e. $h_{r+|\mathbf{t}|}$ does not have any factor of the form $y^{t_1} - \tilde{a}x^{t_2}$ where $\tilde{a}$ is a non-zero real number.

**Proof of Proposition 3.6.** By (17), (18) and (19), system (1) by means of the change (16) becomes

$$\begin{aligned} \dot{u} &= \tfrac{u}{2t_1t_2u^{|\mathbf{t}|}}\nabla H^{\mathbf{t}}\cdot\mathbf{F}(u^{t_1}Cs(\theta),u^{t_2}Sn(\theta)) \\ \dot{\theta} &= \tfrac{u}{2t_1t_2u^{|\mathbf{t}|}}h^{\mathbf{t}}(u^{t_1}Cs(\theta),u^{t_2}Sn(\theta)). \end{aligned} \tag{20}$$

If the quasi-homogeneous expansions with respect to the type $\mathbf{t}$ of $\nabla H^{\mathbf{t}}\cdot\mathbf{F}$ and $h^{\mathbf{t}}$ are $\nabla H^{\mathbf{t}}\cdot\mathbf{F}(x,y)=\sum_{j\geq 0}g_{2t_1t_2+r+j}(x,y)$ and $h^{\mathbf{t}}(x,y)=\sum_{j\geq 0}h_{r+|\mathbf{t}|+j}(x,y)$, respectively, system (20) becomes

$$\begin{aligned} \dot{u} &= \tfrac{u}{2t_1t_2u^{|\mathbf{t}|}}u^r\textstyle\sum_{j\geq 0}u^jg_{2t_1t_2+r+j}(\theta) \\ \dot{\theta} &= \tfrac{u^r}{2t_1t_2}\textstyle\sum_{j\geq 0}u^jh_{r+|\mathbf{t}|+j}(\theta), \end{aligned} \tag{21}$$

where we have denoted

$$g_{2t_1t_2+r+j}(\theta)=g_{2t_1t_2+r+j}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)),\quad h_{r+j+|\mathbf{t}|}(\theta)=h_{r+j+|\mathbf{t}|}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)).$$

Moreover,

$$\begin{aligned} g_{2t_1t_2+r+j}(\theta) &= \nabla H^{\mathbf{t}}\cdot\mathbf{F}_{r+j}(Cs(\theta),Sn(\theta)) \\ &= \tfrac{1}{r+j+|\mathbf{t}|}(\nabla H^{\mathbf{t}}\cdot\mathbf{X}_{h_{r+j+|\mathbf{t}|}}(Cs(\theta),Sn(\theta))+\mu_j\nabla H^{\mathbf{t}}\cdot\mathbf{D}_0(Cs(\theta),Sn(\theta))) \\ &= \tfrac{1}{r+j+|\mathbf{t}|}(-\tfrac{d}{d\theta}h(Cs(\theta),Sn(\theta))+2t_1t_2\mu_j(Cs(\theta),Sn(\theta))) \\ &= \tfrac{1}{r+j+|\mathbf{t}|}(-h'(\theta)+2t_1t_2\mu_j(\theta)). \end{aligned} \tag{22}$$

Let us prove the first part.

If $h_{r+|\mathbf{t}|}\not\equiv 0$, we have (after dividing by $\frac{u^r}{2t_1t_2}$) the system

$$\begin{aligned} \dot{u} &= u\textstyle\sum_{j\geq 0}\tfrac{u^j}{r+j+|\mathbf{t}|}(-h'(\theta)+2t_1t_2\mu_j(\theta)) \\ \dot{\theta} &= h_{r+|\mathbf{t}|}(\theta)+\textstyle\sum_{j\geq 1}u^jh_{r+|\mathbf{t}|+j}(\theta). \end{aligned} \tag{23}$$

We observe that each real root of $h_{r+|\mathbf{t}|}(\theta)$ corresponds an irreducible factor $x$, $y$ or $y^{t_1}-\lambda_jx^{t_2}$, $\lambda_j\in\mathbb{R}$, $\lambda_j\neq 0$, with the same multiplicity order as the order of the root.

If $\theta_0$ is a root of $h_{r+|\mathbf{t}|}(\theta)$, the change $\alpha=\theta-\theta_0$ takes the singular point $(u,\theta)=(0,\theta_0)$ of the system (20) to the origin.

For all $j\geq 0$, we can write

$$\begin{aligned} h_{r+|\mathbf{t}|+j}(\alpha) &= \sum_{i\geq 0}c_i^{(j)}\alpha^i,\quad c_0^{(0)}=0, \\ 2t_1t_2\mu_{r+j}(\alpha) &= \sum_{i\geq 0}d_i^{(j)}\alpha^i. \end{aligned}$$

System (23), in the new coordinates $(u,\alpha)$, becomes

$$\begin{aligned} \dot{u} &= u\sum_{j\geq 0}\left[\sum_{i\geq 0}\tfrac{1}{r+|\mathbf{t}|+j}(d_i^{(j)}-(i+1)c_{i+1}^{(j)})\alpha^i\right]u^j, \\ \dot{\alpha} &= \sum_{i\geq 0}c_{i+1}^{(0)}\alpha^{i+1}+\sum_{j\geq 1}\left[\sum_{i\geq 0}c_i^{(j)}\alpha^i\right]u^j. \end{aligned} \tag{24}$$

We analyze the following cases separately:

If there is a factor of $h_{r+|\mathbf{t}|}$ of odd multiplicity order, then it has that $c^{(0)}_{2m+1} \neq 0$ and $c^{(0)}_i = 0$, for $0 \leq i \leq 2m$. The system (24) has the form

$$\begin{array}{rcl} \dot{u} & = & u\Psi(u,\alpha), \\ \dot{\alpha} & = & c^{(0)}_{2m+1}\alpha^{2m+1} + \alpha^{2m+2}\Phi_1(\alpha) + u\Phi_2(u,\alpha), \end{array}$$

with $\Psi, \Phi_1, \Phi_2$ analytic functions.
Next, we prove that this system has orbits which tend to the origin when $t \to \pm\infty$ and therefore the system (1) has characteristic orbits. Actually, the axis $u = 0$ is invariant and its dynamic is determined in a neighborhood of the origin, repulsive if $c^{(0)}_{2m+1} > 0$ or attractive if $c^{(0)}_{2m+1} < 0$. Also, there are at least two vertical isoclines (the axis $u = 0$ for $u > 0$ and for $u < 0$).
From Malgrange theorem, as the lowest power in $\alpha$ of $\dot{\alpha}$ has exponent odd then there will be at least one curve of horizontal isoclines with odd multiplicity order. Therefore, the vertical components of the field on both sides of the horizontal isocline have opposite directions.

We can assume, by means a change, that the horizontal isocline is $\alpha = u$ and, by clarity and simplicity we suppose that the vertical isoclines closer to $\alpha = u$ on both sides are $\alpha = 1/2u$ and $\alpha = 2u$. We can also assume (by changing the sign of the time, if necessary) that the direction of the field is $(0, 1)$ on the vertical isocline $\alpha = 1/2u$ and $(0, -1)$ on $\alpha = 2u$.

If the direction of the vector field on $\alpha = u$ is $(1, 0)$, then there is a characteristic orbit in the sector $\frac{1}{2} < \alpha < u$ which tends to the origin when $t \to -\infty$, see Zhang *et al.* [12], pages 68–69. Otherwise, if the direction of the vector field on $\alpha = u$ is $(-1, 0)$ then all the orbits lying in the sector $u < \alpha < 2u$ tend to the origin when $t \to +\infty$.

If there is a factor of $h_{r+|\mathbf{t}|}$ of even multiplicity order $(2m)$ and it is a factor of $\mu_r$ with even multiplicity order $(2n)$ with $0 \leq n < m$, it has that the polynomial $h_{r+|\mathbf{t}|}$ verifies $c^{(0)}_j = 0$ for $0 \leq j \leq 2m-1$, $c^{(0)}_{2m} \neq 0$ and $\mu_r$ verifies $d^{(0)}_j = 0$ for $0 \leq j \leq 2n-1$, $d^{(0)}_{2n} \neq 0$. Now, the system (24) has the form

$$\begin{array}{rcl} \dot{u} & = & \dfrac{d^{(0)}_{2n}}{r+|\mathbf{t}|}\alpha^{2n}u + \alpha^{2n+1}u\Psi_1(\alpha) + u^2\Psi_2(u,\alpha), \\ \dot{\alpha} & = & \alpha^{2n+2}\Phi_1(\alpha) + u\Phi_2(u,\alpha). \end{array}$$

Taking $\mathbf{t} = (2n+1, 1)$, the lowest-degree quasi-homogeneous term of $\mathbf{F}$ is

$$\mathbf{F}_{2n}(u,\alpha) = (\frac{d^{(0)}_{2n}}{r+|\mathbf{t}|}\alpha^{2n}u, 0)^T,$$

with $h_{4n+2}(u,\alpha) = -\frac{1}{4n+2}\frac{d^{(0)}_{2n}}{r+|\mathbf{t}|}\alpha^{2n+1}u$ which has a factor with odd multiplicity order and thus the first part is proved.

When $h_{r+|\mathbf{t}|} \equiv 0$, we have (after dividing by $\frac{u^{r+1}}{2t_1t_2}$) the system

$$\begin{array}{ll} \dot{u} & = \nabla H^{\mathbf{t}} \cdot \mathbf{F}_r(\theta) + \sum_{j\geq 1} u^j g_{2t_1t_2+r+j}(\theta) \\ \dot{\theta} & = \sum_{j\geq 1} u^{j-1} h_{r+|\mathbf{t}|+j}(\theta), \end{array} \tag{25}$$

with $\nabla H^{\mathbf{t}} \cdot \mathbf{F}_r(\theta) = H^{\mathbf{t}}\mu_r(\theta)$.

In this case, By extending the results of Bendixson for generalized polar coordinates, for all $\theta^*$ not satisfying $\mu_r(\theta) = 0$, we have exactly one semipath tending to the origin in that limiting direction. So, the result follows. ■

## Proof of Lemma 3.1.

As $h_{r+|\mathbf{t}|}(x, y) \neq 0$, for all $(x, y) \neq (0, 0)$, we have that $h_{r+|\mathbf{t}|}(\theta) \neq 0$ for all $\theta$. Thus, by rescaling the time by $dt = (u^r/2t_1t_2)h_{r+|\mathbf{t}|}(\theta)d\tau$, system (21) is transformed into

$$\begin{array}{llll} u' & = \mathrm{d}u/\mathrm{d}\tau & = & \frac{u}{h_{r+|\mathbf{t}|}(\theta)} \sum_{j\geq 0} u^j g_{2t_1t_2+r+j}(\theta) \\ \theta' & = \mathrm{d}\theta/\mathrm{d}\tau & = & 1 + \sum_{j\geq 1} u^j \frac{h_{r+|\mathbf{t}|+j}(\theta))}{h_{r+|\mathbf{t}|}(\theta)}. \end{array} \tag{26}$$

It is easy to check that te system does not have a equilibrium in the axis $u = 0$, hence the origin of (1) is monodromic. ■

## References

[1] A. Algaba; C. García; M. Reyes, *A new algorithm for determining the monodromy of a planar differential system.* Appl. Math. Comput., **237**, 419-429, (2014).

[2] A. Algaba; C. García; M. Reyes, *Characterization of a monodromic singular point of a planar vector field.* Nonlinear Anal., **74**, 5402-5414, (2011).

[3] A. Algaba; C. García; E. Gamero, *The integrability problem for a class of planar systems,* Nonlinearity, **22**, (2009), 395-420.

[4] A. F. Andreev, *Investigation of the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point,* Trans. Amer. Math. Soc. **8**, (1958), 187-207.

[5] A.A. Andronov; E.A. Leontovich; I.I. Gordon; A.G. Maier, *Qualitative theory of second order dynamic systems,* Ed. Wiley and Sons, (1973).

[6] J.C. Artés; F. Dumortier; C. Herssens; J. Llibre; P. De Maesschalck, *Computer program P4 to study phase portraits planar ploynomial differential equations,* (2005). `http://mat.uab.es/~artes/p4/p4.htm`

[7] I. Bendixson, *Sur les courbes définies par des equations différentielles,* Acta Math., **24**, (1901), 1-88.

[8] M. Brunella; M. Miari, *Topological equivalence of a plane vector field with its principal part defined through Newton polyhedra,* J. Differential Equations, **85**, 2, (1990), 338-366.

[9] F. Dumortier, *Singularities of vector fields on the plane,* J. Differential Equations, **23**, (1977), 53-106.

[10] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles,* J. Math. **37**, (1881), 375-422.

[11] S. Schecter; M.F. Singer, *Separatrices at singular points of planar vector fields,* Acta Math., **145**, (1980), 47-78.

[12] Z.F. Zhang; T.R. Ding; W.Z. Huang; Z.X. Dong, *Qualitative theory of differential equations,* Trans. Math. Monographs, **101**, (1992), Am. Math. Soc., Providence, RI.