

Analytic invariant curves and analytic integrability of a planar vector field

A. Algaba, C. García, M. Reyes

*Department of Integrated Science.
Center of Advanced Studies in Physics, Mathematics and Computation.
Huelva University, 21071. Huelva, Spain*

Abstract

We give an expression of the irreducible invariant curves at the singular point. For analytically integrable systems, we give a reduced expression of its first integral. This fact allows us to obtain necessary conditions of analytic integrability at degenerate singular points.

1. Introduction

We deal with a planar vector field $\mathbf{F} = (P, Q)^T$ with P, Q analytic functions at the origin, and the origin is an isolated singular point.

We say that $C(x, y) = 0$ (or simply C) with $C \in \mathbb{C}[[x, y]]$ (ring of formal power series in x, y over \mathbb{C}) and $C(\mathbf{0}) = 0$, is an *invariant curve at the origin* of the vector field \mathbf{F} if there exists $K \in \mathbb{C}[[x, y]]$, named cofactor of C , such that $F(C) = KC$, where $F := P\partial_x + Q\partial_y$ is the first order differential operator associated to the vector field \mathbf{F} .

Moreover, if K is identically zero, we say that C is a *formal first integral* and \mathbf{F} is *formally integrable at the origin*. If $K = \operatorname{div}(\mathbf{F}) := P_x + Q_y$, C is an *inverse integrating factor* (\mathbf{F}/C is an Hamiltonian vector field on the set $\{C \neq 0\}$).

Every invariant curve at the origin $C(x, y) = 0$ of \mathbf{F} yields a formal solution of the differential equation $P(x, y)dy = Q(x, y)dx$, passing through the origin. Conversely, a formal solution of the differential equation is a parameterization of the curve $C(x, y) = 0$ with C a formal invariant curve at the origin of \mathbf{F} . Thus, any formal first integral at the origin vanishes on any formal invariant curve. Moreover, if an analytic vector field is formally integrable, according to the result of Mattei and Moussu [18], it has also an analytic first integral.

Therefore, the study of the existence of its formal invariant curves is key to give an expression of the analytic first integrals of a vector field and, ultimately, to have a better comprehension of the dynamics of the solutions of a vector field.

Email address: colume@uhu.es (M. Reyes)

Sometimes, the existence of invariant curves ensures the existence of a first integral [3, 4]. Also, the existence of a certain number of polynomial invariant curves implies the integrability of the system as well as the existence of an inverse integrating factor (invariant curve at the origin whose cofactor is the divergence of the vector field) [8, 9, 14, 22].

For quasi-homogeneous vector fields, the analytic integrability problem has been solved, see [4, Theorem 3.1] and [4, Theorem 3.2] gives an expression of the first integral.

Usually, the technique for finding the invariant curves of a vector field is through blow-up transformations. These transformations bring the vector field to a linear vector field we know its invariant curves. This technique is sometimes complicated and too tedious. For this, it is interesting to provide a method easier than the blow-up transformations.

As far as we know, for analytic vector fields only the problem of the existence of the analytic invariant curves has been solved for the perturbations of linear vector fields. The problem of the existence of the formal invariant curves of the family $(a_1x + \text{h.o.t.})\partial_x + (b_1y + \text{h.o.t.})\partial_y$ with $a_1b_1 \neq 0$, was solved by Seidenberg [21]. When a_1/b_1 is a real negative number, the invariant curves at the origin are analytic, this follows from Hadamard and Perron Theorem, see [13, Theorem 7.1], theorem of the existence of the analytic invariant manifolds. The existence of analytic invariant curves of the perturbations of $y\partial_x + \sigma x\partial_y$ with $\sigma \neq 0$ was studied by Bruno [6].

For nilpotent systems, i.e. systems $(y + X(x, y))\partial_x + Y(x, y)\partial_y$ with X and Y real analytic functions without constants and linear terms, there are some partial results: Chavarriga *et al.* [7] prove that a first integral of the nilpotent centers are of the form $y^2 + \dots$. When the origin is an isolated nilpotent or degenerate singular point, Algaba *et al.* [1] prove that the integrable vector fields whose first quasi-homogeneous term is Hamiltonian, being H its Hamiltonian function, are formally conjugated to a Hamiltonian vector field whose Hamiltonian function starts by H . Hence, the irreducible first integrals of these analytic perturbations are of the form $H + \dots$. We emphasize that these expressions provide little information in the study of the existence of a first integral of a vector field.

The provided results in this paper are the following ones: Theorem 2 provides the invariant curves at the origin of a quasi-homogeneous vector field.

One of the results more relevant of our study is Theorem 6. We prove that for each irreducible formal invariant curve there exists an irreducible analytic invariant curve at the origin.

Theorem 9 solve the problem of the existence and uniqueness of analytic invariant curves of analytic perturbations of dissipative quasi-homogeneous vector fields. Theorem 10 also solves the problem, through a constants associated to each irreducible factor of the Hamiltonian part of the lowest-degree quasi-homogeneous term of the vector field. In particular, we solve the problem for analytic perturbations of hamiltonian quasi-homogeneous vector fields, Theorem 11. Theorem 12 solves the problem through the Kowalevskaya exponents of the lowest-degree quasi-homogeneous term of the vector field.

We give a new and easier proof of the Camacho and Sad Theorem for the vec-

tor fields whose Hamiltonian associated to the lowest-degree quasi-homogeneous term has only simple factors on $\mathbb{C}[x, y]$, Theorem 13.

We study a link between the existence of analytic invariant curves and the analytic integrability and, in such a case, also give an expression of an analytic first integral, Theorems 14, 15 and 16.

In Section 4, we apply our techniques. We solve the problem of existence and uniqueness of invariant curves at the origin when the origin is a non-degenerate singular point (Theorem 17) and when the origin is a nilpotent singularity (Theorems 18 and 19). We solve the analytic integrability problem for a family whose lowest-degree quasi-homogeneous term is $(y + dx^2)\partial_x + (2x^3 + 2dxy)\partial_y$, i.e. it is non-Hamiltonian, Theorem 20.

Last on, we consider the vector field $y^3\partial_x + (-x^3 + \frac{1}{2}x^2y^2)\partial_y$ whose origin is a degenerate center [19]. In [11], it is proved that the vector field does not have a formal inverse integrating factor [11, Theorem 2]. To carry out its proof, the authors solve *ODE*'s and use Gauss hypergeometric functions. Here, we give an easier proof. We also prove that the vector field does not have a rational inverse integrating factor, Theorem 22.

2. Invariant curves of analytic vector fields

Before of showing our results, we introduce some notation. Given $\mathbf{w} = (w_1, w_2)$ with w_1 and w_2 natural numbers without common factors, a scalar function f of two variables is a *quasi-homogeneous function* of type or weight exponent \mathbf{w} and degree k if $f(\varepsilon^{w_1}x, \varepsilon^{w_2}y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous polynomials of type \mathbf{w} and degree k is denoted by $\mathcal{P}_k^{\mathbf{w}}$. A vector field $\mathbf{F} = (P, Q)^T$ is a *quasi-homogeneous vector field* of type \mathbf{w} and degree k if $P \in \mathcal{P}_{k+w_1}^{\mathbf{w}}$ and $Q \in \mathcal{P}_{k+w_2}^{\mathbf{w}}$. We denote the vector space of the quasi-homogeneous polynomial vector fields of type \mathbf{w} and degree k by $\mathcal{Q}_k^{\mathbf{w}}$. An analytic vector field can be expanded into quasi-homogeneous terms of type \mathbf{w} of successive degrees. Thus, the vector field \mathbf{F} can be written in the form

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \dots,$$

for some integer r , where $\mathbf{F}_j = (P_{j+w_1}, Q_{j+w_2})^T \in \mathcal{Q}_j^{\mathbf{w}}$ and $\mathbf{F}_r \neq \mathbf{0}$. Such expansions are expressed as $\mathbf{F} = \mathbf{F}_r + \text{q-h.h.o.t.}$, where “q-h.h.o.t.” means “quasi-homogeneous higher order terms”. If we select the type $\mathbf{w} = (1, 1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees.

Throughout the paper, we name $\mathbf{D}_0 := (w_1x, w_2y)^T \in \mathcal{Q}_0^{\mathbf{w}}$ (a dissipative quasi-homogeneous vector field) and $\mathbf{X}_h := (-\partial h/\partial y, \partial h/\partial x)^T$ (the Hamiltonian vector field whose Hamiltonian function is the polynomial h). If $h \in \mathcal{P}_{r+|\mathbf{w}|}^{\mathbf{w}}$, $|\mathbf{w}| = w_1 + w_2$, then $\mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{w}}$. Moreover, it is proved that every $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{w}}$ can be expressed as $\mathbf{F}_k = \mathbf{X}_h + \mu\mathbf{D}_0$ with $h = \frac{1}{k+|\mathbf{w}|}(\mathbf{D}_0 \wedge \mathbf{F}_k) \in \mathcal{P}_{k+|\mathbf{w}|}^{\mathbf{w}}$ (the product wedge of both vector fields) and $\mu = \frac{1}{k+|\mathbf{w}|}\text{div}(\mathbf{F}_k) \in \mathcal{P}_k^{\mathbf{w}}$ (the divergence of \mathbf{F}_k), see [1, 2]. This sum is known as *the conservative-dissipative*

splitting of a quasi-homogeneous vector field. The polynomial h is named the Hamiltonian function associated to \mathbf{F}_r .

We say that a function $C \in \mathbb{C}[[x, y]]$ with $C(\mathbf{0}) = 0$ is *reducible* if $C = C_1 C_2$ with $C_1, C_2 \in \mathbb{C}[[x, y]]$ and $C_1(\mathbf{0}) = C_2(\mathbf{0}) = 0$. Otherwise, we say that C is *irreducible*.

The following result gives the expression of the formal invariant curves at the origin of an analytic vector field.

Proposition 1. *Every $C \in \mathbb{C}[[x, y]]$ formal invariant curve at the origin of an analytic vector field \mathbf{F} is given by $C := C_1^{k_1} C_2^{k_2} \dots C_m^{k_m}$ each C_j being an irreducible formal invariant curve at the origin.*

Moreover, its cofactor is $k_1 K_1 + \dots + k_m K_m$ with K_j cofactor of C_j .

Proof. Let C be an invariant curve at the origin whose cofactor is K . The factorization of C on $\mathbb{C}[[x, y]]$ is of the form $C = C_1^{k_1} \tilde{C}$ with C_1 an irreducible function and $\gcd(C_1, \tilde{C}) = 1$. We have that

$$F_r(C_1^{k_1} \tilde{C}) = C_1^{k_1} F_r(\tilde{C}) + k_1 \tilde{C} C_1^{k_1-1} F_r(C_1) = K C_1^{k_1} \tilde{C},$$

that is, $C_1(\tilde{C}K - F_r(\tilde{C})) = k_1 \tilde{C} F_r(C_1)$. As C_1 and \tilde{C} are coprime, we have that C_1 divides to $F_r(C_1)$, i.e. $F_r(C_1) = K_1 C_1$, that is, C_1 is an irreducible invariant curve at the origin.

The second part is obtained easily. \square

The following result proves that every quasi-homogeneous vector field has quasi-homogeneous invariant curves and also gives the expression of them.

Theorem 2. *Consider $\mathbf{F}_r = (P_{r+w_1}, Q_{r+w_2})^T \in \mathcal{Q}_r^{\mathbf{w}}$.*

If $w_1 x Q_{r+w_2} - w_2 y P_{r+w_1}$ is identically zero, then any quasi-homogeneous polynomial is invariant curve at the origin.

Otherwise, every quasi-homogeneous polynomial invariant curve at the origin is given by $f_1^{n_1} f_2^{n_2} \dots f_m^{n_m}$ each f_j being an irreducible polynomial factor of the polynomial $w_1 x Q_{r+w_2} - w_2 y P_{r+w_1}$ and n_j non-negative integer.

Moreover, its cofactor is $n_1 K_1 + \dots + n_m K_m$ with K_j cofactor of f_j .

Proof. By conservative-dissipative splitting of a quasi-homogeneous vector field, we know that $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$ with $h = \frac{1}{r+|\mathbf{w}|}(w_1 x Q_{r+w_2} - w_2 y P_{r+w_1})$ and $\mu = \frac{1}{r+|\mathbf{w}|} \operatorname{div}(\mathbf{F}_r)$.

Assume that h is zero. In this case, for any $f \in \mathcal{P}_s^{\mathbf{w}}$ we have that $F_r(f) = \mu D_0(f) = s\mu f$. Therefore, f is an invariant curve of \mathbf{F}_r .

Otherwise, assume that h is not zero. Let g be a quasi-homogeneous polynomial invariant curve whose cofactor is K . The factorization of g on $\mathbb{C}[x, y]$ is of the form $g := f_1^{k_1} f_2^{k_2} \dots f_m^{k_m}$ each f_j being an irreducible formal invariant curve at the origin. So, $K_1 f_1 = F_r(f_1) = X_h(f_1) + \mu D_0(f_1) = X_h(f_1) + s\mu f_1$. Thus, $X_h(f_1) = (K_r - s\mu) f_1$ and f_1 is an invariant curve of \mathbf{X}_h . So, f_1 divides h and, therefore, f_1 is an irreducible factor of h .

Conversely, we prove that any factor of h is an invariant curve of \mathbf{F}_r . Indeed, if $f \in \mathcal{P}_s^{\mathbf{w}}$ is a factor of h , then $h = fg$ and $F_r(f) = X_{fg}(f) + \mu D_0(f) =$

$fX_g(f) + s\mu f = (X_g(f) + s\mu)f$. Therefore, f is an invariant curve of \mathbf{F}_r . \square

Theorem 2 is geometrically natural because every solution at the origin in the blow-up is associated to an irreducible factor of $w_1xQ_{r+w_2} - w_2yP_{r+w_1}$, i.e. the Hamiltonian function of the lowest-degree quasi-homogeneous term.

The following example shows evidence of the difficulty of the problem of characterizing the invariant curves for non-quasi-homogeneous vector fields.

Example 1. Consider the vector field $F = x\partial_x + (2y+x^2)\partial_y$. The lowest-degree term of its Taylor expansion, i.e. quasi-homogeneous expansion with respect to the type $\mathbf{w} = (1, 1)$, is $F_0 = x\partial_x + 2y\partial_y$, whose Hamiltonian associated is $h = \frac{1}{2}xy$. The vector field \mathbf{F} does not have an invariant curve at the origin of the form $C = y + h.o.t.$. Indeed, let $C = y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \dots$ and $K = k_{00} + k_{10}x + k_{01}y + \dots$. For degree 1, the equation $F(C) = KC$ is $(2 - k_{00})y = 0$. Thus, $k_{00} = 2$. For degree 2, the equation $F(C) = KC$ is $x^2 + (c_{11} - k_{10})xy + (-k_{01} + 2c_{02})y^2 = 0$, i.e. the equation is not satisfied.

Therefore, by Example 1, if f is an irreducible factor of h , it does not imply that there exists an invariant curve at the origin of $\mathbf{F} = \mathbf{F}_r + \text{q-h.h.o.t.}$ of the form $C = f + \text{q-h.h.o.t.}$

So, from Theorem 2 and Proposition 1, the invariant curves of $\mathbf{F} = \mathbf{F}_r + \text{q-h.h.o.t.}$ are of the form $f_1^{n_1} f_2^{n_2} \dots f_m^{n_m} + \text{q-h.h.o.t.}$ each f_j being an irreducible polynomial factor of the polynomial $h = \frac{1}{r+|\mathbf{w}|}(\mathbf{D}_0 \wedge \mathbf{F}_r) \in \mathcal{P}_{r+|\mathbf{w}|}^{\mathbf{w}}$.

The factorization of h on $\mathbb{C}[x, y]$ is of the form

$$h(x, y) = cx^{n_x} y^{n_y} \prod_{i=3}^m (y^{w_1} - \lambda_i x^{w_2})^{n_i}, \quad (1)$$

with $c \in \mathbb{R}$, $\lambda_i \in \mathbb{C} \setminus \{0\}$, $n_x \geq 0$, $n_y \geq 0$ and n_i natural numbers.

Consequently, for h non-zero, the irreducible quasi-homogeneous invariant curves of \mathbf{F}_r can only be $f_1 := x$, $f_2 := y$ or $f_i := y^{w_1} - \lambda_i x^{w_2}$, $i = 3, \dots, m$, whose quasi-homogeneous degrees are w_1 , w_2 or $w_1 w_2$, respectively.

The following result emphasizes that for each solution at the origin, there exist are infinitely many invariant curves associated to the solution.

Proposition 3. If C is an invariant curve at the origin of the analytic vector field \mathbf{F} , then CU with $U(\mathbf{0}) \neq 0$ (a unit function) is also an invariant curve at the origin of \mathbf{F} .

Proof. Let note that any formal function U with $U(\mathbf{0}) \neq 0$, satisfies $F(U) = \hat{K}U$ with $\hat{K} = F(U)/U \in \mathbb{C}[[x, y]]$. Thus, $F(CU) = (K + \hat{K})CU$, where K is the cofactor of C . \square

Let \mathcal{X} be the set of all invariant curves of \mathbf{F} (it is not an empty set, see [21]). We say that two invariant curves C_1 and C_2 are *equivalent*, $C_1 \sim C_2$, if $C_1 = C_2U$ with U a formal unit function. The relation \sim is a equivalence relation on \mathcal{X} . Therefore, from Proposition 3, each solution through the origin

is associated with a unique equivalence class $[C] \in \mathcal{X}/\sim$ with C an invariant curve at the origin of \mathbf{F} .

From now, roughly speaking, we will say that an invariant curve of the form $C = f + \text{q-h.h.o.t.}$ is unique if there is a unique equivalence class such that the representative is of the form $f + \text{q-h.h.o.t.}$.

The following result detects reducible formal invariant curves of a vector field.

Proposition 4. *Assume that $C = f^n g + \text{q-h.h.o.t.}$ is a formal invariant curve at the origin of a vector field \mathbf{F} where f and g are coprime quasi-homogeneous polynomials and f is irreducible. Then, $C = (f^n + \text{q-h.h.o.t.})(g + \text{q-h.h.o.t.})$, i.e. C is reducible.*

Proof. We suppose that $g \in \mathcal{P}_m^{\mathbf{w}}$. We first assume that $f = y^{w_1} - \lambda x^{w_2}$. The quasi-homogeneous blow-up $x = u^{w_1}$, $y = u^{w_2}v$ transforms $y^{w_1} - \lambda x^{w_2}$ into $u^{w_1 w_2}(v^{w_1} - \lambda)$ and $g(x, y)$ into $u^m g(1, v)$, i.e. it transforms $C = f^n g + \text{q-h.h.o.t.}$ into $C = u^{n w_1 w_2 + m}[(v^{w_1} - \lambda)^n g(1, v) + u \tilde{g}_1(u, v)]$.

Making $z = v^{w_1} - \lambda$, the invariant curve at the origin C becomes

$$C = u^{n w_1 w_2 + m}[(z^n g(1, (\lambda + z)^{1/w_1}) + u \tilde{g}_1(u, (\lambda + z)^{1/w_1})],$$

an invariant curve at the origin of the vector field \mathbf{G} resulting from the transformation of \mathbf{F} by means of $x = u^{w_1}$, $y = u^{w_2}(\lambda + z)^{1/w_1}$.

One has that $g(1, \lambda^{1/w_1}) \neq 0$ since f and g are coprime. Thus, $g(1, (\lambda + z)^{1/w_1})$ is a unit element.

Therefore, C can be expressed as

$$C = u^{n w_1 w_2 + m} g(1, (\lambda + z)^{1/w_1}) [(z^n + u \tilde{g}_2(u, (\lambda + z)^{1/w_1})],$$

i.e. C is a product of two formal invariant curves $u^m g(1, (\lambda + z)^{1/w_1})$ and $\tilde{C} = u^{n w_1 w_2} (z^n + u \tilde{g}_2(u, (\lambda + z)^{1/w_1}))$. Undoing the change, we have that $C = (f^n + \text{q-h.h.o.t.})(g + \text{q-h.h.o.t.})$.

For $f = x$, we do the change $x = u^{w_1}v$, $y = v^{w_2}$ and we proceed in similar way. \square

As a direct consequence of Theorem 2 and Proposition 4, we have the following Proposition which gives the first quasi-homogeneous term of an irreducible invariant curve.

Proposition 5. *Assume that $C \in \mathbb{C}[[x, y]]$ is an irreducible invariant curve at the origin of an analytic vector field \mathbf{F} . Then, $C = f^n + \text{q-h.h.o.t.}$ with n a certain natural number and f an irreducible factor of h (Hamiltonian polynomial associated to the lowest-degree quasi-homogeneous term of \mathbf{F}).*

Next, we give one of the results more relevant of our study. We prove that each equivalence class of \mathcal{X}/\sim has a representative that is an analytic invariant curve.

Theorem 6. *Consider the analytic vector field \mathbf{F} . For each irreducible formal invariant curve at the origin, there always exists an irreducible analytic invariant curve that belongs to the same equivalence class.*

To prove this theorem we need the following result given by Mattei and Moussu [18]:

Lemma 7. *[18, Theorem A] Consider $C \in \mathbb{C}[[x, y]]$ with $C(\mathbf{0}) = 0$ and C irreducible function. Then, there exists a function $l \in \mathbb{C}[[x]]$ with $l(0) = 0$ and $l'(0) = 1$ such that $l \circ C$ is analytic at the origin.*

Proof of Theorem 6. We assume that C is a formal invariant curve at the origin. From Lemma 7, there exists $l \in \mathbb{C}[[x]]$ with $l(0) = 0$ and $l'(0) = 1$ such that $l \circ C$ is analytic. One has that $l \circ C = C + a_2C^2 + a_3C^3 + \dots = C(1 + a_2C + a_3C^2 + \dots) = CU$, therefore, from Proposition 3, $l \circ C$ is an analytic irreducible invariant curve that belongs to the same equivalence class. \square

The above result was proved by Mattei and Moussu [18] for first integrals at the origin. Thus, formal integrability is equivalent to analytic integrability

Now, we devote to study the existence of invariant curves of the vector fields whose lowest-degree term is given. We need the following auxiliary result that determines the existence and uniqueness of invariant curves C of the vector field $(ax + \text{h.o.t.})\partial_x + (cx + by + \text{h.o.t.})\partial_y$ satisfying $C(0, y) = y^n + \text{h.o.t.}$ with n a natural number.

Lemma 8. *Consider the vector field $(ax + \text{h.o.t.})\partial_x + (cx + by + \text{h.o.t.})\partial_y$. Assume that $\frac{a}{b} \notin \mathbb{Q} \cap (0, 1)$, and $a \neq 0$ if $b = 0$. Then, for each n a natural number, there is a unique analytic invariant curve C such that $C(0, y) = y^n + \text{h.o.t.}$. Moreover, if $n > 1$, then C is reducible.*

Proof. The vector field \mathbf{F} is $\mathbf{F} = \mathbf{F}_0 + \text{h.o.t.}$ with $F_0 = ax\partial_x + (cx + by)\partial_y$ whose irreducible polynomial invariant curves are x and $f := \frac{c}{b-a}x + y$ ($b - a \neq 0$, by hypotheses).

Fixed n , we consider $\Delta_{j, f^n} = \langle x^{j-i}y^i \rangle$, $i = 0, \dots, n-1$, a complementary subspace to $f^n \mathcal{P}_{j-n}^{(1,1)}$ on $\mathcal{P}_j^{(1,1)}$, $j > n$.

First on, we analyze the existence and uniqueness of invariant curves of the form $\tilde{C} = f^n + \sum_{j=n+1} \tilde{C}_j$ with $\tilde{C}_j \in \Delta_{j, f^n}$ whose cofactor is $K = K_0 + \text{h.o.t.}$ with $K_0 = nb$.

We check by induction method that such equation is satisfied degree by degree, for which we choose appropriately \tilde{C}_j and K_{j-n} in each case.

For degree n , the equation is satisfied since $F_0(f^n) - nbf^n = 0$. We assume that the equation is satisfied for $j-1$ and we want to prove that it is also satisfied for degree j with $j > n$. The quasi-homogeneous term of degree j of $F(\tilde{C}) - \tilde{C}K$ with $j > n$, by applying Euler Theorem for quasi-homogeneous functions, is

$$(F(\tilde{C}) - \tilde{C}K)_j = (F_0 - \frac{nb}{j}D_0)(\tilde{C}_j) - \tilde{C}_n K_{j-n} + P_j,$$

where $P_j = \sum_{i=1}^{j-n} F_i(\tilde{C}_{j-i}) - \sum_{i=1}^{j-n-1} \tilde{C}_{j-i} K_i$, i.e. P_j is a quasi-homogeneous polynomial of degree j determined by $\tilde{C}_n, \dots, \tilde{C}_{j-1}$ and K_1, \dots, K_{j-n-1} . By writing $P_j = q_j + f^n p_{j-n}$, with $q_j \in \Delta_{j, f^n}$, we have that

$$(F(\tilde{C}) - \tilde{C}K)_j = (F_0 - \frac{nb}{j}D_0)(\tilde{C}_j) - f^n(K_{j-n} - p_{j-n}) + q_j,$$

with $p_{j-n} \in \mathcal{P}_{j-n}^{(1,1)}$ and $q_j = \sum_{i=0}^{n-1} \beta_i x^{j-i} y^i \in \Delta_{j, f^n}$.
If $\tilde{C}_j = \sum_{i=0}^{n-1} \alpha_i x^{j-i} y^i$, one has that

$$(F_0 - \frac{nb}{j}D_0)(\tilde{C}_j) = \sum_{i=0}^{n-1} ci\alpha_i x^{j-i+1} y^i + [(j-i)a - (n-i)b] \alpha_i x^{j-i} y^i \in \Delta_{j, f^n}.$$

Notice that, on the hypotheses given, one has that $(j-i)a - (n-i)b \neq 0$ for $i < n < j$. Therefore, the above equation has a unique solution \tilde{C}_j . So, in order to satisfy the equation $F(\tilde{C}) - \tilde{C}K = 0$ for degree j , we choose $K_{j-n} = p_{j-n}$ and \tilde{C}_j .

We now prove that if C is an invariant curve at the origin of the form $C = f^n + \text{h.o.t.}$ then there exists a unique element unit U such that $C = \hat{C}U$. We check that the equation $C = \hat{C}U$ is satisfied degree by degree, for which we choose appropriately $U_{j-n}, j \geq n$, in each case.

For degree n , the equation $C = \hat{C}U$ leads to $U_0 = 1$, and for degree $n+1$ is $C_{n+1} = \hat{C}_{n+1} + f^n U_1$, it which is satisfied by taking U_1 such that $f^n U_1 = \text{Proj}_{f^n \mathcal{P}_1^{\mathbf{w}}} (C_{n+1})$.

For degree $i > n$, the equation is $C_i = \hat{C}_i + f^n U_{i-n} + \sum_{j=n+1}^{i-1} \hat{C}_j U_{i-j}$, and we choose U_{i-n} such that $f^n U_{i-n} = \text{Proj}_{f^n \mathcal{P}_{i-n}^{\mathbf{w}}} (C_i - \sum_{j=n+1}^{i-1} \hat{C}_j U_{i-j})$.

Summarizing, for each natural number n , there exists a unique invariant curve at the origin of the form $C = (\frac{c}{b-a}x + y)^n + \text{h.o.t.}$, that is, $C(0, y) = y^n$. In particular, for $n = 1$, there is a unique invariant curve at the origin $\tilde{C} = \frac{c}{b-a}x + y + \text{h.o.t.}$. So, \tilde{C}^n and C are the same invariant curve., i.e. C is reducible.

The analyticity of the invariant curve follows by Theorem 6. \square

We have the following result when the lowest-degree term is a quasi-homogeneous dissipative vector field ($h \equiv 0$). In this case, there is an infinitely of analytic invariant curves starting by a quasi-homogeneous polynomial given.

Theorem 9. *Consider $\mathbf{F} = \mu \mathbf{D}_0 + q\text{-h.o.t.}$ with μ a quasi-homogeneous polynomial. If $f \in \mathbb{C}[x, y]$ is an irreducible quasi-homogeneous polynomial and f is not a factor of μ , then there exists a unique C an irreducible analytic invariant curve at the origin starting by f .*

Moreover, if \tilde{C} is an analytic invariant curve starting by f^n , with n a natural number, then $\tilde{C} = C^n U$, with U a formal unit element.

Proof. Let $f = y^{w_1} - \lambda x^{w_2}$, $\lambda \in \mathbb{C}$ and $\mu \in \mathcal{P}_r^{\mathbf{w}}$. (For $f = x$, we reason in a similar way). The blow-up $x = u^{w_1}$, $y = u^{w_2}v$ and the rescaling $dt = \frac{w_1}{u^r} d\tau$,

transforms the system into

$$\begin{aligned} u' &= \frac{du}{d\tau} = uw_1\mu(1, v) + u^2\Phi(u, v), \\ v' &= \frac{dv}{d\tau} = u\Psi(u, v) \end{aligned} \quad (2)$$

where Φ, Ψ are analytic functions. Applying Lemma 8 for $n = 1$, as $b := 0$ and $a := w_1\mu(1, \lambda^{1/w_1}) \neq 0$ since f is not a factor of μ , we have that there exists a unique analytic invariant curve of the form $C = f + q$ -h.h.o.t. Moreover, C is irreducible. Otherwise, $C = f\bar{C}$, this contradicts that f is not a factor of μ . The second part follows from Lemma 8 for $n > 1$. \square

Any quasi-homogeneous polynomial p can be univocally written as $p(x, y) = cx^n y^m p_0(x^{w_2}, y^{w_1})$ where p_0 is a homogeneous polynomial without the factors x and y . The polynomial p_0 is named *homogeneous polynomial associated to p* .

From now, we consider $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ with h non-zero and assume that the origin of \mathbf{F}_r is an isolated singular point. We define the rational function $\eta(x, y) := \frac{\mu_0(x, y)}{x^{\delta_x} y^{\delta_y} h_0(x, y)}$, with h_0 and μ_0 the homogeneous polynomials associated to h and μ , being $\delta_x = 1$ ($\delta_y = 1$, resp.) if x (y , resp.) is a factor of h ; otherwise $\delta_x = 0$ ($\delta_y = 0$, resp.).

The poles of $\eta(1, y)$ are:

- ∞ , if x is a simple factor of h since $\deg(h_0(1, y)) = \deg_{\mathbf{w}}(\mu_0(1, y)) + 1$. Moreover, $\text{Res}[\eta(1, y), \infty] = -\lim_{y \rightarrow \infty} y\eta(1, y)$,
- 0 , if y is a simple factor of h ,
- $a \in \mathbb{C} \setminus \{0\}$, if $y^{w_1} - ax^{w_2}$ is a simple factor of h .

For each f an irreducible simple factor of h , we define the number $r(f)$ as

$$r(f) := \frac{1}{r+|\mathbf{w}|} \left(1 - \frac{w_1 w_2}{\deg_{\mathbf{w}}(f)} \text{Res}[\eta(1, y), \lambda] \right), \quad (3)$$

being λ the pole of $\eta(1, y)$ associated to f and $\deg_{\mathbf{w}}(f)$ denotes the degree of the quasi-homogeneous polynomial with respect to the type \mathbf{w} .

The following example clarifies the above concept.

Example 2. Consider the vector field

$$(2x^3 - 7xy^3 + q\text{-h.h.o.t.})\partial_x + (5x^2y - y^4 + q\text{-h.h.o.t.})\partial_y.$$

The Hamiltonian and dissipative parts of the lowest-degree quasi-homogeneous term with respect to the type $\mathbf{w} = (3, 2)$ are $h = (x^2 + y^3)$ and $\mu = x^2 - y^3$. Thus, the homogeneous polynomials associated to h and μ are $h_0 = x + y$ and $\mu_0 = x - y$ and consequently, $\eta(x, y) = \frac{x-y}{xy(x+y)}$. One has that,

$$\begin{aligned} \text{for } f = x, \quad r(f) &:= \frac{1}{11} \left(1 + 2 \lim_{y \rightarrow \infty} y\eta(1, y) \right) = -\frac{1}{11}, \\ \text{for } f = y, \quad r(f) &:= \frac{1}{11} \left(1 - 3 \lim_{y \rightarrow 0} y\eta(1, y) \right) = -\frac{2}{11}, \\ \text{for } f = y^3 + x^2, \quad r(f) &:= \frac{1}{11} \left(1 - \lim_{y \rightarrow -1} (y+1)\eta(1, y) \right) = \frac{3}{11}. \end{aligned}$$

The following result determines the existence of an analytic invariant curve at the origin whose lowest-degree term quasi-homogeneous is irreducible throughout the expression of the lowest-degree quasi-homogeneous term of the vector field.

Theorem 10. *Consider $\mathbf{F} = \mathbf{F}_r + q$ -h.h.o.t. with $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ and $h \neq 0$. If $f \in \mathbb{C}[x, y]$ is an irreducible simple factor of h such that $f(x, y) = 0$ is not a curve of singular points and $r(f) \notin \mathbb{Q} \cap (-1, 0)$, then there is a unique irreducible analytic invariant curve at the origin C starting by f .*

Moreover, if \tilde{C} is an invariant curve starting by f^n , with n a natural number, then $\tilde{C} = C^n U$, with U a formal unit element.

Proof. Let $f = y^{w_1} - \lambda x^{w_2}$, $\lambda \in \mathbb{C}$. (For $f = x$, we reason in a similar way). By applying the blow-up $x = u^{w_1}$, $y = u^{w_2}v$ and by rescaling the time $dt = \frac{w_1}{u^r} d\tau$, the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is transformed into

$$\begin{aligned} u' &= \frac{du}{d\tau} = u(-h_y(1, v) + w_1\mu(1, v)) + u^2\Phi(u, v), \\ v' &= \frac{dv}{d\tau} = (r + |\mathbf{w}|)h(1, v) + u\Psi(u, v) \end{aligned} \quad (4)$$

where Φ, Ψ are analytic functions. Moreover, $(u, v) = (0, \lambda^{1/w_1})$ is a singular point of the system (4) since f is a factor of h , $b := (r + |\mathbf{w}|)h_y(1, \lambda^{1/w_1}) \neq 0$ since f is a simple factor of h and $a := -h_y(1, \lambda^{1/w_1}) + w_1\mu(1, \lambda^{1/w_1}) \neq 0$ since, otherwise, f is a factor of both components of \mathbf{F}_r , that is, it is a curve of singular points of \mathbf{F}_r .

Applying the change $z = v - \lambda^{1/w_1}$, the system (4) is transformed into $(au + \dots)\partial_u + (bz + \dots)\partial_z$. We now see that $a/b = -r(f)$. Indeed,

$$\begin{aligned} \frac{a}{b} &= \frac{-h_y(1, \lambda^{1/w_1}) + w_1\mu(1, \lambda^{1/w_1})}{(r + |\mathbf{w}|)h_y(1, \lambda^{1/w_1})} = -\frac{1}{r + |\mathbf{w}|} \left(1 - w_1 \frac{\mu(1, \lambda^{1/w_1})}{h_y(1, \lambda^{1/w_1})} \right) \\ &= -\frac{1}{r + |\mathbf{w}|} \left(1 - \lim_{y \rightarrow \lambda^{1/w_1}} w_1 \frac{(y - \lambda^{1/w_1})\mu(1, y)}{h(1, y)} \right). \end{aligned}$$

To prove the result, it is enough to check that the above limit is $\text{Res}[\eta(1, y), \lambda]$. This is followed from $\lim_{y \rightarrow \lambda^{1/w_1}} \frac{y^{w_1} - \lambda}{(y - \lambda^{1/w_1})y^{w_1 - 1}} = w_1$ and $\frac{\mu(1, y)}{h(1, y)} = y^{w_1 - 1}\eta(1, y^{w_1})$. From Lemma 8, if $r(f) \notin \mathbb{Q} \cap (-1, 0)$, there is a unique analytic invariant curve at the origin C starting by z . Also, C is irreducible. Otherwise, $C = f\tilde{C}$, this contradicts that $f(x, y) = 0$ is not a curve of singular points. Moreover, if \tilde{C} is an invariant curve at the origin starting by z^n , with n a natural number, then $\tilde{C} = C^n$. Undoing the change, the result follows. \square

The following example shows that the above theorem is not true when $f(x, y) = 0$ is a curve of singular points of \mathbf{F}_r .

Example 3. *Consider the vector field $F = (2x^2 - 3xy + y^2 + x^3)\partial_x + (x^2 - xy)\partial_y$. Respect to the type $\mathbf{w} = (1, 1)$, $F = (x - y)F_0 + F_2$ with $F_0 = (2x - y)\partial_x + x\partial_y$ and $F_2 = x^3\partial_x$. In this case, $(x - y)$ is an invariant curve at the origin of \mathbf{F}_0 and $(x - y)\mathbf{F}_0$. It is easy to check that \mathbf{F} does not have any invariant curve at the origin whose lowest degree quasi-homogeneous term is $x - y$.*

As a direct consequence of Theorem 10, we solve the problem of the existence and uniqueness of analytic invariant curves for the vector fields whose lowest-degree quasi-homogeneous term is Hamiltonian.

Theorem 11. *Consider $\mathbf{F} = \mathbf{X}_h + q$ -h.h.o.t. with h a quasi-homogeneous polynomial. If $f \in \mathbb{C}[x, y]$ is an irreducible simple factor of h , then there exists a unique irreducible analytic invariant curve at the origin C starting by f . Moreover, if \tilde{C} is an analytic invariant curve starting by f^n , with n a natural number, then $\tilde{C} = C^n U$, with U a formal unit element.*

Proof. For each f a simple factor of $h \in \mathcal{P}_{r+|\mathbf{w}|}^{\mathbf{w}}$, we have that $r(f) = \frac{1}{r+|\mathbf{w}|} \in (0, 1)$. Thus, applying Theorem 10, the result follows. \square

The following example shows that Theorem 11 is not true when f is a multiple factor of h .

Example 4. *Consider the vector field $(-x^2+y^3)\partial_x + 2xy\partial_y$, i.e. a non-homogeneous vector field whose lowest-degree term is $X_h = -x^2\partial_x + 2xy\partial_y$, a Hamiltonian vector field whose Hamiltonian function is $h = x^2y$.*

On the one hand, y is a simple factor of h . So, from Theorem 11, the vector field has an analytic invariant curve at the origin starting by y .

On the other hand, x is a multiple factor of h . It is easy to check that the condition $F(C) - KC = 0$ is not satisfied for degree 3. Therefore, the vector field does not have any invariant curve at the origin starting by x .

The following result gives a link between the Kowalevskaya exponents of a planar vector field and the existence of analytic invariant curves. More information about the Kowalevskaya exponents can be found in [10, 12, 15, 16].

Theorem 12. *Consider $\mathbf{F} = \mathbf{F}_r + q$ -h.h.o.t. with $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$, $r > 0$, $h \neq 0$, $\mu \neq 0$ and $f \in \mathbb{C}[x, y]$ simple factor of h . If the Kowalevskaya exponent ρ different from -1 associated to f satisfies $\rho^{-1} \notin \mathbb{Q} \cap (-r, 0)$, then there exists C a unique irreducible analytic invariant curve at the origin C starting by f . Moreover, if \tilde{C} is an invariant curve starting by f^n , with n a natural number, then $\tilde{C} = C^n U$, with U a formal unit element.*

Proof. The Kowalevskaya exponent $\rho \neq -1$ associated to f , a simple factor of h , is given by $r\rho = \frac{1}{r(f)}$ [3, Proposition 5]. Applying Theorem 10, the result follows. \square

Seidenberg [21] proved the existence of a formal invariant curve at the origin and Camacho and Sad [17] proved the existence of an analytic invariant curve. Here we give a new and easier proof of the Camacho and Sad Theorem for the vector fields whose Hamiltonian h associated to the lowest-degree quasi-homogeneous term has only simple factors on $\mathbb{C}[x, y]$.

Theorem 13. *Consider the analytic vector field $\mathbf{F} = \mathbf{X}_h + \mu\mathbf{D}_0 + q$ -h.h.o.t. with h having only simple factors on $\mathbb{C}[x, y]$ and whose origin is an isolated singular point. Then, there exists an irreducible analytic invariant curve at the origin.*

Proof. The irreducible quasi-homogeneous factors of h can only be $f_1 := x$, $f_2 := y$ or $f_i := y^{w_1} - \lambda_i x^{w_2}$, $i = 3, \dots, m$, whose quasi-homogeneous degrees are w_1 , w_2 or $w_1 w_2$, respectively. Thus,

$$w_1 \delta_x r(f_1) + w_2 \delta_y r(f_2) + w_1 w_2 \sum_{j=3}^m r(f_j) = \frac{1}{r + |\mathbf{w}|} (w_1 \delta_x + w_2 \delta_y + (m-2)w_1 w_2 - w_1 w_2 \sum_{j=1}^m \text{Res}[\eta(1, y), \lambda_j])$$

where $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_j = 2, \dots, m$ are the poles of $\eta(1, y)$. From Residues Theorem and as $r + |\mathbf{t}| = \delta_x w_1 + \delta_y w_2 + (m-2)w_1 w_2$ (the degree of h), we have that $w_1 \delta_x r(f_1) + w_2 \delta_y r(f_2) + w_1 w_2 \sum_{j=3}^m r(f_j) = 1$. Therefore, there exists at least a j_0 such that $r(f_{j_0}) \notin \mathbb{Q} \cap (-1, 0)$, i.e. from Theorem 11, there exists an irreducible analytic invariant curve at the origin starting by f_{j_0} . \square

3. First integrals of analytic vector fields

In this section, we study a link between the existence of analytic invariant curves and the analytic integrability and, in such a case, we give an expression of an analytic first integral.

The following result proves that the polynomial integrability of the lowest-degree quasi-homogeneous vector field implies the existence and uniqueness of invariant curves starting by an irreducible factor of h .

Theorem 14. *Consider $\mathbf{F} = \mathbf{F}_r + q \cdot h$. h.o.t., assume that the origin is an isolated singular point of \mathbf{F}_r and \mathbf{F}_r is polynomially integrable. Then, for each f , an irreducible factor of h , there exists a unique invariant curve at the origin of \mathbf{F} starting by f . Moreover, if \tilde{C} is an invariant curve at the origin starting by f^n , with n a natural number, then $\tilde{C} = C^n$.*

Proof. If \mathbf{F}_r is formally integrable, then h is non-zero and \mathbf{F}_r has a polynomial first integral. On the one hand, if μ is identically zero, then $\mathbf{F}_r = \mathbf{X}_h$ with h having only simple factors since the origin is an isolated singular point of \mathbf{F}_r . So, from Theorem 11, for each factor f_j there is a unique g_j irreducible analytic invariant curve at the origin of \mathbf{F} starting by f_j .

On the other hand, if μ is non-zero, from Algaba *et al.* [3, Theorem 2], all the factors of h are simple. Also, the constants $r(f)$, for all f , are non-negative rational numbers. So, from Theorem 10, the result follows. \square

Next result gives the expression of an analytic first integral of an analytically integrable vector field whose lowest-degree quasi-homogeneous term is known. From Algaba *et al.* [3, Theorem 2], the Hamiltonian part h of the lowest-degree quasi-homogeneous term of an formally integrable vector field has, at least, two irreducible factors and all are simple.

Theorem 15. Consider $\mathbf{F} = \mathbf{F}_r + q\text{-h.h.o.t.}$, $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ with h having only simple factors on $\mathbb{C}[x, y]$ f_j , $j = 1, \dots, m$. Assume that the origin of \mathbf{F}_r is an isolated singular point and \mathbf{F} is analytically integrable. Then $r(f_j) \in \mathbb{Q}^+$, for any $j = 1, \dots, m$ (where $r(f)$ is defined in (3)), and a primitive analytic first integral of \mathbf{F} is of the form $I = U \prod_{j=1}^m g_j^{m_j}$, where g_j is an irreducible analytic invariant curve at the origin of \mathbf{F} starting by f_j , U a unit element and $m_j = r(f_j)M$ with M a natural number.

Proof. If \mathbf{F} is formally integrable, then \mathbf{F}_r is also formally integrable, μ is non-zero and, a first integral of \mathbf{F} is of the form $I = I_p + q\text{-h.h.o.t.}$ with I_p a first integral primitive of \mathbf{F}_r . Moreover, $r(f_j) \in \mathbb{Q}^+$, for any $j = 1, \dots, m$, and the first integral of \mathbf{F}_r is $I_p = \prod_{j=1}^m f_j^{r(f_j)M}$, being $M = \text{lcm}(\text{denominator}(r(f_j)))$ [3, Theorem 2]. Also from Theorem 10, $I = U \prod_{j=1}^m g_j^{n_j}$, where g_j is an irreducible analytic invariant curve at the origin of \mathbf{F} starting by f_j and U a unit element. The result follows. \square

As a consequence of above result, we study the analytic integrability of the vector fields whose lowest-degree quasi-homogeneous term is Hamiltonian, $\mathbf{F} = \mathbf{X}_h + q\text{-h.h.o.t.}$ and whose origin is an isolated singular point of \mathbf{X}_h , i.e. h has only simple factors in its decomposition on $\mathbb{C}[x, y]$.

Theorem 16. Consider $\mathbf{F} = \mathbf{X}_h + q\text{-h.h.o.t.}$ with h a quasi-homogeneous polynomial having simple factors f_j , $j = 1, \dots, m$. If \mathbf{F} is analytically integrable, a primitive analytic first integral is of the form $I = U \prod_{j=1}^m g_j$, where g_j is an invariant curve at the origin of \mathbf{F} starting by f_j and U a unit element.

Proof. The numbers $r(f_j)$ are $1/(r + |\mathbf{w}|)$. Applying Theorem 15, we obtain the result. \square

4. Applications

The first application is the study of the existence and uniqueness of analytic invariant curves for the well-known class of vector fields whose origin is an isolated singular point and whose differential evaluated at origin has some eigenvalue non-zero. In this case, the origin is called a non-degenerate singular point.

For this class, if the Hamiltonian associated to the lowest-degree term is zero, these vector fields can be transformed, by means of a linear change of variables, into $(ax + \dots)\partial_x + (ay + \dots)\partial_y$ with $a \neq 0$. In this case the origin is a star node and, from Theorem 9 for $\mathbf{w} = (1, 1)$, any analytic perturbation of a star node has an infinitely of invariant curves. Otherwise, these vector fields can be transformed, by means of a linear change of variables and by rescaling the time variable, into $(dx + y + \dots)\partial_x + (cx + dy + \dots)\partial_y$, with $c \in \{-1, 0, 1\}$ and $(c, d) \neq (0, 0)$. If $c = 0$, the Hamiltonian part of the lowest-degree term has only a factor, y , and it is a multiple factor. Next, we study the existence of analytic invariant curves.

Theorem 17. *The vector field $(dx + y + \dots)\partial_x + (cx + dy + \dots)\partial_y$, with $c = \pm 1$, has one or two irreducible analytic invariant curves at the origin. Moreover, one has that:*

- (a) *If $c = -1$, there exists a unique analytic invariant curve at the origin C starting by $y - ix$ and other curve \bar{C} , conjugated of C , starting by $y + ix$; i.e. the vector field has an analytic invariant real curve at the origin starting by $y^2 + x^2$.*
- (b) *If $c = 1$, there exists a unique irreducible analytic invariant curve at the origin starting by $y - x$ ($y + x$, resp.) if $d \notin \mathbb{Q} \cap (-3, -1)$ ($d \notin \mathbb{Q} \cap (1, 3)$, resp.).*

Proof. For this case, $r = 0$, $\mathbf{w} = (1, 1)$, $h = \frac{1}{2}(cx^2 - y^2)$ and $\mu = d$.

If $d = 0$, the lowest degree term is an Hamiltonian vector field whose Hamiltonian function has only simple factors. So, from Theorem 11, the result follows.

Assume that $d \neq 0$. The factorization of h on $\mathbb{C}[x, y]$ has only simple factors and $\eta(1, y) = \frac{2d}{c-y^2}$ whose poles are $\pm\sqrt{c}$.

For $f = y - \sqrt{c}x$, one has that $r(f) = \frac{1}{2}(1 + \frac{d}{\sqrt{c}})$. And for $f = y + \sqrt{c}x$, one has that $r(f) = \frac{1}{2}(1 - \frac{d}{\sqrt{c}})$. From Theorem 10, one has the conditions on d . Moreover, for any value of d , at least one of the conditions on d is satisfied. \square

Now, we solve the above problem when the origin is a nilpotent singularity, i.e. we consider the nilpotent vector field $(y + X(x, y))\partial_x + Y(x, y)\partial_y$ with X and Y real analytic functions without constants and linear terms defined in a neighborhood of the origin and assume that the origin is an isolated singularity.

The vector field can be transformed, by means of a polynomial change of variables, into $\mathbf{F} = \mathbf{F}_r + q$ -h.h.o.t. where

$$F_r = y\partial_x + x^{2n}\partial_y, \text{ i.e. } \mathbf{w} = (2, 2n + 1) \text{ and } r = 2n - 1, \quad (5)$$

or

$$F_r = (y + dx^{n+1})\partial_x + (n+1)(cx^{2n+1} + dx^n y)\partial_y, \text{ i.e. } \mathbf{w} = (1, n+1) \text{ and } r = n \quad (6)$$

with $c = \pm 1$.

Theorem 18. *The vector field $\mathbf{F} = \mathbf{F}_r + q$ -h.h.o.t. with \mathbf{F}_r given by (5) has a unique irreducible analytic invariant curve at the origin and it starts by $y^2 - x^{2n+1}$.*

Proof. The result follows from Theorem 11. \square

Theorem 19. *The vector field $\mathbf{F} = \mathbf{F}_r + q$ -h.h.o.t. with \mathbf{F}_r given by (6) has one or two irreducible analytic invariant curves. Moreover, one has that:*

- (a) *If $c = -1$, there exists a unique analytic invariant curve at the origin C starting by $y - ix^{n+1}$ and other curve \bar{C} , conjugated of C , starting by $y + ix^{n+1}$; i.e. the vector field has a unique real analytic invariant curve at the origin starting by $y^2 + x^{2n+2}$.*
- (b) *If $c = 1$, there exists a unique irreducible analytic invariant curve at the origin starting by $y - x^{n+1}$ ($y + x^{n+1}$, resp.) if $d \notin \mathbb{Q} \cap (-2n - 3, -1)$ ($d \notin \mathbb{Q} \cap (1, 2n + 3)$, resp.).*

Proof. The proof is similar to the proof of Theorem 17. Now, for this case, $r = n$, $\mathbf{w} = (1, n + 1)$, $h = \frac{1}{2}(cx^{2n+2} - y^2)$ and $\mu = dx^n$. For $f = y - \sqrt{c}x^{n+1}$, one has that $r(f) = \frac{1}{2n+2}(1 + \frac{d}{\sqrt{c}})$. And for $f = y + \sqrt{c}x^{n+1}$, one has that $r(f) = \frac{1}{2n+2}(1 - \frac{d}{\sqrt{c}})$. Moreover, for any value of d , at least one of the conditions on d is satisfied. \square

As an application, we solve the analytic integrability problem for a family whose origin is a nilpotent singular point. Consider the system

$$(y + dx^2 + a_{11}xy)\partial_x + (2x^3 + 2dxy + b_{40}x^4 + b_{02}y^2)\partial_y \quad (7)$$

i.e. it is a perturbation of the nilpotent vector field (6) for $n = 1$ and $c = 1$.

Theorem 20. *System (7) is analytically integrable if, and only if, one of the following series of conditions is satisfied:*

- (a) $d = 0$,
- (b) $d \neq 0, d \in \mathbb{Q} \cap (-1, 1), a_{11} = b_{40} + b_{02} = 0$.

Proof. We denote the vector field associated to the system (7) by \mathbf{F} . So, $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ with $\mathbf{F}_i \in \mathcal{Q}_i^{\sharp}$, $i = 1, 2$ and $\mathbf{w} = (1, 2)$. The lowest-degree term of \mathbf{F} is $\mathbf{F}_1 = \mathbf{X}_h + \mu\mathbf{D}_0$ with $h = \frac{1}{2}(x^4 - y^2)$ and $\mu = 2dx$.

We first assume that $d = 0$. In this case, the lowest-degree quasi-homogeneous term of \mathbf{F} is a Hamiltonian vector field. If $a_{11} = -2b_{02}$, then \mathbf{F} is Hamiltonian and thus the Hamiltonian function is a polynomial first integral. If $a_{11} = 0$, The vector field \mathbf{F} has the inverse integrating factor $\exp(-2b_{02}x)$ which is non-zero at the origin; therefore, the system is analytically integrable. Otherwise, $a_{11}(a_{11} + 2b_{02}) \neq 0$, the analytic function V^a , with $V = 1 + a_{11}x$ and $a = \frac{a_{11} + 2b_{02}}{a_{11}}$ is an inverse integrating factor non-zero at the origin, therefore, the system is also analytically integrable.

We now assume that $d \neq 0$. If \mathbf{F} is analytically integrable, then \mathbf{F}_1 also is integrable. From Theorem 14, there are two analytic invariant curves of the form $C_1 = y + x^2 + \text{q-h.h.o.t.}$ and $C_2 = y - x^2 + \text{q-h.h.o.t.}$ of the vector field \mathbf{F} . Also, for $f = y - x^2$, one has that $r(f) = \frac{1}{4}(1 + d)$. And for $f = y + x^2$, $r(f) = \frac{1}{4}(1 - d)$. So, from Theorem 15, $1 + d$ and $1 - d$ are positive rational numbers, that is, $|d| < 1$. A primitive analytic first integral is of the form $I = UC_1^{n_1}C_2^{n_2}$, with $n_1 = \frac{M(1-d)}{4}$, $n_2 = \frac{M(1+d)}{4}$ where n_1 and n_2 coprime natural numbers and U a unit element.

As $F(I) = UC_1^{n_1}C_2^{n_2} \left(\frac{F(U)}{U} + n_1K_1 + n_2K_2 \right)$, by denoting $\log(U)$ by V , the integrability condition becomes

$$F(V) + n_1K_1 + n_2K_2 = 0, \quad (8)$$

with K_1 and K_2 cofactors of C_1 and C_2 , respectively. Expanding the expression (8) degree to degree in quasi-homogeneous terms with respect to the type $\mathbf{w} = (1, 2)$, we are going to obtain necessary conditions of integrability.

For degree 2, we have the integrability condition

$$(6q^2 + 14q + 6)a_{11} + 3(q + 1)^2 (b_{40} + b_{02}) = 0$$

where we have named $q = \frac{n_2}{n_1} \in \mathbb{Q}^+$. Thus, $b_{40} = -\frac{2(3q^2+7q+3)}{3(q+1)^2}a_{11} - b_{02}$.

For degree 3 and 4, the integrability condition (8) is $a_{11}p_2(b, q) = 0$ with

$$p_2 := (330q^2 + 737q + 330) + (690q^2 + 1880q + 690)A + (540q^2 + 972q + 540)A^2$$

and $b = \frac{b_{02}}{a_{11}}$.

Assume that $a_{11} = 0$, thus $b_{40} = -b_{02}$. In this case, the system has the first integral $I = (y + x^2)^{n_1}(y - x^2)^{n_2} \exp(-b_{02}(n_1 + n_2)x)$.

Assume now that $a_{11} \neq 0$. Then, $p_2 = 0$. The equation (8) for the degrees 6 and 8 become $p_4(b, q)$ and $p_6(b, q)$, respectively, where p_4 and p_6 are polynomials of degrees 4 and 6, respectively, with coefficients in q . Let $r_{24}(q)$, $r_{26}(q)$ the resultants with respect to b of the polynomials p_2 , p_4 and p_2 , p_6 , respectively. The unique common solutions of both resultants are $q(3q + 2)(2q + 3)(3q + 4)(4q + 3)(2q + 5)(5q + 2)(2q + 1)(q + 2) = 0$. It is not possible since q is a positive rational number. \square

The analytic integrability of the vector fields $\mathbf{F} = \mathbf{X}_h + q$ -h.h.o.t. was studied by Alga *et al.* in [1]. It was solved when the factorization of h on $\mathbb{C}[x, y]$ has only simple factors. The problem remains open when h has multiple factors and for the vector fields whose lowest-degree quasi-homogeneous term is not Hamiltonian vector field.

Here, we show a vector field whose quasi-homogeneous expansion with respect to the type $\mathbf{w} = (2, 3)$ is of the form

$$\mathbf{F} = \mathbf{F}_7 + \mathbf{F}_9, \tag{9}$$

where $\mathbf{F}_7 = (2y^3 + 2dx^3y, 3x^5 + 3dx^2y^2)^T$ is a non-Hamiltonian vector field and $\mathbf{F}_9 = (-2byx^4 + 2ax^4y, 4bx^3y^2 - 7bx^6 + 3ax^3y^2)^T$.

Theorem 21. *The vector field (9) is analytically integrable if, and only if, there exist two coprime natural numbers n_1 and n_2 such that $(n_1 + n_2)d = n_1 - n_2$ (i.e. $|d| \leq 1$) and $4(n_1 - n_2)b = -(3n_1 + 4n_2)a$.*

Moreover, in such a case, a primitive analytic first integral is

$$I = (y^2 - x^3)^{n_1} \left(y^2 + x^3 + \frac{7(n_1 + n_2)}{4(n_1 - n_2)} ax^4 \right)^{n_2}.$$

Proof. We study the integrability of the vector field (9). We prove the necessary condition. The hamiltonian and dissipative part of \mathbf{F} are $h = \frac{1}{2}(x^3 - y^2)(x^3 + y^2)$ and $\mu = dx^2y$, thus $\eta(1, y) = \frac{2dy}{(1-y)(1+y)}$.

For the factor $f = x^3 - y^2$, $r(f) = \frac{1}{12}(1+d)$ and for $f = x^2 - y^3$, $r(f) = \frac{1}{12}(1-d)$. So, from Theorem 15, if \mathbf{F} is analytically integrable then $1 + d$ and $1 - d$ are positive rational numbers, and a primitive analytic first integral, if it exists, is of

the form $I = C_1^{n_1} C_2^{n_2} U$ with $C_1 := y^2 - x^3 + \text{q-h.h.o.t.}$, $C_2 := y^2 - x^3 + \text{q-h.h.o.t.}$ analytic irreducible invariant curves, n_1, n_2 coprime such that $(1+d)n_2 = (1-d)n_1$ and U of the form $U = 1 + u_{10}x + u_{01}y + \text{h.o.t.}$. It is easy to check that $y^2 - x^3$ is a polynomial invariant curve and we write C_2 of the form $C_2 = y^2 + x^3 + c_{21}x^2y + c_{40}x^4 + c_{12}xy^2 + \text{q-h.h.o.t.}$. By imposing $F(\mathbf{I})=0$, we have that \mathbf{F} is integrable if

$$n_1 C_2 U F(y^2 - x^3) + n_2 (y^2 - x^3) U F(C_2) + (y^2 - x^3) C_2 F(U) = 0.$$

The first quasi-homogeneous term of the above equation is of degree 27 and it leads to $c_{21} = 0$. The term of degree 28 implies that $c_{12} = 0$, $c_{40} = \frac{7(n_1+n_2)}{4(n_1-n_2)}a$, $b = -\frac{3n_1+4n_2}{4(n_1-n_2)}a$ and $u_{10} = 0$. So, the necessary condition is proved. The sufficient condition follows easily. \square

Our techniques can be used to study the existence of inverse integrating factors. For instance, we apply our study to the vector field

$$y^3 \partial_x + (-x^3 + \frac{1}{2}x^2y^2) \partial_y.$$

It is a degenerate center without formal first integral [19]. Moreover, it does not have a formal inverse integrating factor [11, Theorem 2].

From Theorem 11, we know that the formal inverse integrating factors are of the form $(x^4 + y^4 + \dots)^p$ with $p \in \mathbb{N}$. This fact allows us to give a new and easier proof of this result and extend the result by proving that it does not have an algebraic inverse integrating factor.

Theorem 22. *The vector field $F = y^3 \partial_x + (-x^3 + \frac{1}{2}x^2y^2) \partial_y$ does not have any algebraic inverse integrating factor.*

Proof. The lowest-degree term of the vector field with respect to the type $\mathbf{w} = (1, 1)$ is $F_2 = y^3 \partial_x - x^3 \partial_y$ whose Hamiltonian is $h = x^4 + y^4$ whose factorization on $\mathbb{C}[x, y]$ has four simple complex factors, $f_1, \bar{f}_1, f_2, \bar{f}_2$. From Theorem 11, the inverse integrating factor (an invariant curve at the origin whose cofactor is the divergence of the vector field) are of the form $V = (f_1 + \dots)^{n_1} (\bar{f}_1 + \dots)^{m_1} (f_2 + \dots)^{n_2} (\bar{f}_2 + \dots)^{m_2} U$ with U a unit element. By the lowest-degree term of the equation $F(V) = \text{div}(\mathbf{F})V$, we have that $n_1 = m_1 = n_2 = m_2 = p$. So, $V = (x^4 + y^4 + \dots)^p := (x^4 + y^4 + C_5 + C_6 + C_7 + \dots)^p$. with $C_i \in \mathcal{P}_i^{(1,1)}$. The term of degree 7 of the equation $F(V) = \text{div}(\mathbf{F})V$ leads to $C_5 = -\frac{2}{5}x^3y^2$ and $p = \frac{5}{4}$. Thus, V is, at most, an irrational function, i.e. it is not formal.

From Prelle & Singer [20, Propositions 1 and 2] and particularizing in our context, if a system has an algebraic inverse integrating factor, then it also admits an inverse integrating factor V over $\mathbb{C}((x, y))$ where $\mathbb{C}((x, y))$ denotes the quotient field of the algebra of the power series $\mathbb{C}[[x, y]]$. Moreover, if V exists, from Algaba *et al.* [5], then the system also has an inverse integrating factor of the form W^q with $W \in \mathbb{C}[[x, y]]$ and q a rational number.

To degree 8 the equation leads to $C_6 = \frac{1}{75}x^6$ and to degree 9 one has that the equation is not satisfied, for any C_7 . Therefore, the system does not have any

algebraic inverse integrating factor. \square

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