

Characterization of a monodromic singular point of a planar vector field

Abstract

The Newton diagram and the lowest-degree quasi-homogeneous terms of an analytic planar vector field allow us to determine whether an isolated singular point of the vector field is monodromic or has a characteristic trajectory.

Keywords: monodromy, characteristic orbits

1. Introduction.

We are interested in the behavior of the trajectories in a neighborhood of a singular point of the planar analytic differential system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad (1)$$

and, in particular, in determining when a singular point (we can assume the origin to be the singular point) is surrounded by orbits of the system (monodromic singular point).

Each trajectory by lying on a vicinity of a monodromic singular point is either a spiral or an oval. Moreover, from the finiteness theorem for the number of limit cycles, a monodromic point of an analytic planar vector field can be only either a focus or a center, see Il'yashenko [12]. So, the monodromy problem is a prior step to solving the center problem of a vector field which is one of the classical open problems in the qualitative theory of planar differential systems.

If the differential matrix $D\mathbf{F}(\mathbf{0})$ is not identically null, the monodromy problem is completely solved. The problem when the eigenvalues of the matrix are conjugated complex, was solved by Poincaré [16] and when the matrix is nilpotent, by Andreev [6]. Finally, if $D\mathbf{F}(\mathbf{0})$ is identically null (in such a case, O is a degenerate singular point), the monodromy problem can be solved by using the blow-up technique (developed by Dumortier [9]) which consists of performing a series of changes to desingularize the point. However, its application for determining the monodromy of a singular point of a family of vector fields with parameters becomes rather complicated. Some works that use this technique in order to study the monodromy are Medvedeva [15], Gasull *et al.* [11], Mañosa [14], García *et al.* [10]. All of them are only partial results.

In order to show our results, we need to recall the following concepts that we will use throughout the paper: the quasi-homogeneous vector fields (in particular, the conservative-dissipative splitting of a quasi-homogeneous vector field),

the Newton diagram of a vector field and the generalized polar coordinates, introduced by Liapunov [13].

Conservative-dissipative splitting.

Let $\mathbf{t} = (t_1, t_2)$ be non-null with t_1 and t_2 non-negative integer numbers without common factors. A function f of two variables is quasi-homogeneous of type \mathbf{t} and degree k if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous polynomials of type \mathbf{t} and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (F_1, F_2)^T$ is quasi-homogeneous of type \mathbf{t} and degree k if $F_1 \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type \mathbf{t} and degree k .

The quasi-homogeneous vector monomials can be determined by drawing the lattice \mathbb{Z}_+^2 , and assigning each point (m, n) to the quasi-homogeneous vector fields $(x^m y^{n-1}, 0)^T$ and $(0, x^{m-1} y^n)^T$. The points with integer coordinates aligned in the straight lines perpendicular to \mathbf{t} , $(m-1)t_1 + (n-1)t_2 = k$, determine the quasi-homogeneous vector monomials with the same degree k .

Any vector field can be expanded into quasi-homogeneous terms of type \mathbf{t} of successive degrees. Thus, the system (1) can be written in the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots = \sum_{j=0}^{\infty} \mathbf{F}_{r+j}(\mathbf{x}),$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \neq \mathbf{0}$. These expansions are usually considered in the analysis of the topological determination of the singularity by means of the blow-up technique (see Bruno [7], Brunella and Miari [8] and Dumortier [9]). This concept also has been used by Algaba *et al.* [5] as an application of the Normal Form Theory, and for the study of the integrability and of the center problem of systems with a singular point degenerated, i.e. systems whose matrix of the linear part evaluated in the singular point is identically null, see Algaba *et al.* [3, 4].

Next, we cite the splitting of a quasi-homogeneous vector field as a sum of two quasi-homogeneous vector fields, a conservative one (having zero-divergence) and a dissipative one (in the sense of the non-conservative part that fully captures the divergence of the vector field) that will be useful in what follows and will play a main role in our analysis. Throughout this paper, the Hamiltonian system associated to the \mathcal{C}^1 function f is denoted by \mathbf{X}_f , i.e. $\mathbf{X}_f = (-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})^T$. Algaba *et al.* [4] proved that any quasi-homogeneous vector field $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_j = \mathbf{X}_{h_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0, \quad (2)$$

where $\mathbf{D}_0(x, y) := (t_1 x, t_2 y)^T$ (a dissipative quasi-homogeneous vector field of type \mathbf{t} and degree 0), $\mu_j := \frac{1}{j+|\mathbf{t}|} \operatorname{div}(\mathbf{F}_j) \in \mathcal{P}_j^{\mathbf{t}}$ (the divergence of \mathbf{F}_j), $h_{j+|\mathbf{t}|} := \frac{1}{j+|\mathbf{t}|} (t_1 x Q_{j+t_2} - t_2 y P_{j+t_1}) \in \mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$ (the wedge product of \mathbf{D}_0 and \mathbf{F}_j) and $|\mathbf{t}| = t_1 + t_2$.

We note that any non-vanishing quasi-homogeneous polynomial of type $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 non-null, in particular $h_{r+|\mathbf{t}|}$, can be expressed as $p(x, y) = x^{k_1} y^{k_2} p_0(x^{t_2}, y^{t_1})$ with $0 \leq k_1 < t_2$, $0 \leq k_2 < t_1$ being p_0 a homogeneous polynomial. So, by abusing the notation, we can write any quasi-homogeneous polynomial of type \mathbf{t} in a compact form $p(x, y) = c \prod_{j=0}^m f_j^{m_j} \prod_{j=0}^n g_j^{n_j}$, where

$$f_j(x, y) = x, y \text{ or } y^{t_1} - \lambda_j x^{t_2}, \quad j = 0, \dots, m$$

and

$$g_j(x, y) = (y^{t_1} - a_j x^{t_2})^2 + b_j^2 x^{2t_2}, \quad j = 0, \dots, n$$

with c, λ_j, a_j and b_j real numbers and λ_j, b_j non-zero, for all j .

If $h_{r+|\mathbf{t}|} \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ and $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$ are the polynomials associated to the lowest-degree quasi-homogeneous term of type \mathbf{t} of \mathbf{F} , we will say that a polynomial of the form x, y or $y^{t_1} - \lambda x^{t_2}$, $\lambda \neq 0$, is a *strong factor of \mathbf{F} associated to the type \mathbf{t}* , or simply a *strong factor of $h_{r+|\mathbf{t}|}$* , if it satisfies one of the following properties:

- (i) it is a factor of $h_{r+|\mathbf{t}|}$ of odd multiplicity order,
- (ii) it is a factor of $h_{r+|\mathbf{t}|}$ of even multiplicity order ($2m$) and, either it is not a factor of μ_r with $\mu_r \neq 0$ or is a factor of μ_r with even multiplicity order ($2n$) with $0 < n < m$.

Newton diagram.

We will write the components of the vector field \mathbf{F} in the form $P(x, y) = \sum a_{ij} x^i y^{j-1}$ and $Q(x, y) = \sum b_{ij} x^{i-1} y^j$. The *support* of (1) and also of \mathbf{F} , denoted by $\text{supp}(\mathbf{F})$, is the set of pairs (i, j) with $(a_{ij}, b_{ij}) \neq (0, 0)$. The vector (a_{ij}, b_{ij}) is called the *vector coefficient* of (i, j) in the support. Consider the set

$$\bigcup_{(i,j) \in \text{supp}(\mathbf{F})} ((i, j) + \mathbb{R}_+^2),$$

where \mathbb{R}_+^2 is the positive quadrant and the union is taken over all points (i, j) in the support. The boundary of the convex hull of this set is made up of two open rays and a polygon, which can be just one point. The polygon together with the rays that do not lie on a coordinate axis, if they existed, is called the *Newton diagram* of the vector field \mathbf{F} . The component parts of the Newton diagram are called *edges* and their endpoints are the *vertices* of the Newton diagram.

If a vertex of the Newton diagram does not lie on a coordinate axis, then it is said to be an *inner* vertex; otherwise, it is an *exterior* vertex. The *exponent* of a vertex $V = (i, j)$ will be denoted by α_V and it is given by

$$\alpha_V := \begin{cases} b_{ij}/a_{ij}, & \text{if } a_{ij} \neq 0; \\ \infty, & \text{if } a_{ij} = 0. \end{cases} \quad (3)$$

The *exponent* of a bounded edge ℓ of the Newton diagram is a positive rational number t_2/t_1 , equal to the tangent of the angle between the edge and the ordinate axis. The exponent of ℓ will be denoted by α_ℓ , and the pair $\mathbf{t} = (t_1, t_2)$

plane $(x, y) \in \mathbb{R}^2$, as

$$x = u^{t_1} \text{Cs}(\theta), \quad y = u^{t_2} \text{Sn}(\theta), \quad (6)$$

with $u \geq 0$ and $\theta \in [0, T_{\mathbf{t}})$.

The quoted transformation carries the region of the plane (u, θ) given by the rectangle $R = [0, \epsilon) \times [0, T_{\mathbf{t}})$, $\epsilon > 0$, to the neighborhood of the origin $W = \{(x, y) \in \mathbb{R}^2, x^{2t_2} + y^{2t_1} \leq \epsilon^{2t_1 t_2}\}$. Also, it transforms the curve $y^{t_1} = ax^{t_2}$ onto $\theta = \theta^*$ with $a = \frac{\text{Sn}^{t_1}(\theta^*)}{\text{Cs}^{t_2}(\theta^*)}$.

A characteristic orbit of system (1) is a trajectory that enters or leaves of the singular point (the singular point is an ω or α limit point of any point of the edge of the type \mathbf{t} with the fixed angle of the Newton diagram) *characteristic orbit of system (1) is a trajectory that enters or leaves of the singular point (the singular point is an ω or α limit point of any point of the edge of the type \mathbf{t} with the fixed angle of the Newton diagram) in a neighborhood of the origin $(0, 0)$ and is transformed by the change of coordinates (6) into (u, θ) and $(\mathbf{t}, \mathcal{R})$. Figure 1 shows two distinct Newton diagrams with two edges.*

- $\lim_{\tau \rightarrow +\infty} u(\tau) = 0$ (or $\lim_{\tau \rightarrow -\infty} u(\tau) = 0$), and
- there exists $\lim_{\tau \rightarrow +\infty} \theta(\tau) = \theta^* \in [0, T_{\mathbf{t}})$ (or $\lim_{\tau \rightarrow -\infty} \theta(\tau) = \theta^* \in [0, T_{\mathbf{t}})$).

Next, we cite the results obtained. The first result ranks the different orbits of (1) that tend to the origin and give the relationship among them. It is an easy extension of the well-known result given by Zhang *et al.* [[17], Theorem 3.10].

Theorem 1. *Let $\mathbf{x}(t)$ be an orbit of analytic system (1) verifying $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$. Then it is either a spiral orbit of (1) (in such a case, the origin is a focus) or it is a \mathbf{t} -characteristic orbit of (1) for any type \mathbf{t} . Otherwise, if there is no orbit of (1) such that $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$, then the origin is a center.*

As a consequence, we have the following well-known characterization of monodromic singular point.

Corollary 1. *The following properties are equivalent:*

- 1) *the origin is a monodromic singular point of system (1),*
- 2) *system (1) does not have any characteristic orbits,*
- 3) *system (1) does not have any \mathbf{t} -characteristic orbits for any type \mathbf{t} .*

Next, we give our main result, Theorem 2, which describes system (1) with a characteristic orbit from the edges and vertices of its Newton diagram, and provides the shape of the orbit.

Theorem 2. *If system (1) has a characteristic orbit in the first quadrant different from $x = 0$ and $y = 0$, then it has one, and only one, of the following situations:*

(e1) *There exists an edge of the Newton diagram of (1) with type $\mathbf{t} = (t_1, t_2)$ such that $h_{r_{\mathbf{t}+|\mathbf{t}|}}(x, y) \equiv 0$. In this case, the characteristic orbit is of the form $y = \tilde{a}x^{t_2/t_1} + o(x^{t_2/t_1})$, for a certain \tilde{a} non-zero real.*

(e2) *There exists an edge of the Newton diagram of (1) with type $\mathbf{t} = (t_1, t_2)$ and a certain \tilde{a} non-zero real such that $y^{t_1} - \tilde{a}x^{t_2}$ is a factor of $h_{r_{\mathbf{t}+|\mathbf{t}|}}(x, y)$. In this case, the characteristic orbit is of the form $y = \tilde{a}x^{t_2/t_1} + o(x^{t_2/t_1})$.*

(v1) *There exists an inner vertex V of the Newton diagram of (1) with $j_0 > 0$ and $\beta_V < 0$. In this case, the characteristic orbit has the form $y = x^{\alpha_\ell} \tau(x) + o(x^{\alpha_\ell} \tau(x))$, where ℓ is the adjacent upper edge with $\lim_{x \rightarrow 0} \tau(x) = 0$ and $\lim_{x \rightarrow 0} \frac{\tau(x)}{x^{1/n}} = +\infty$*

for all $n \in \mathbb{N}$.

(v2) *There exists an inner vertex V of the Newton diagram of (1) with $i_0 > 0$ and $\beta_V < 0$.*

In this case, the characteristic orbit has the form $x = y^{1/\alpha_{\tilde{\ell}}}\tau(y) + o(y^{1/\alpha_{\tilde{\ell}}}\tau(y))$, where $\tilde{\ell}$ is the adjacent lower edge, with $\lim_{y \rightarrow 0} \tau(y) = 0$ and $\lim_{y \rightarrow 0} \frac{\tau(y)}{y^{1/n}} = +\infty$ for all $n \in \mathbb{N}$.

(v3) *There exists an inner vertex V of the Newton diagram of (1) with $i_0 = j_0 = 0$ and $\beta_V < 0$.*

In this case, the characteristic orbit has the form either $y = \lambda x^{b/a} + o(x^{b/a})$, or $y = x^{b/a}\tau(x) + o(x^{b/a}\tau(x))$ or $x = y^{a/b}\tau(y) + o(y^{a/b}\tau(y))$ where (a, b) is the vector coefficient of V with $ba > 0$, $\lambda \neq 0$, $\lim_{x \rightarrow 0} \tau(x) = 0$ and $\lim_{x \rightarrow 0} \frac{\tau(x)}{x^{1/n}} = +\infty$ for all $n \in \mathbb{N}$.

Theorem 2 can be used to analyze the topological equivalence near a singular point, and, in particular, to solve the monodromy problem. The two following results, Theorems 3 and 4, provide necessary and sufficient conditions of monodromy for a singular point.

Theorem 3. *If the origin of system (1) is monodromic then the Newton diagram of (1) verifies:*

- 1) *all its vertices have even coordinates,*
- 2) *it has two exterior vertices. Moreover, if $(a, 0)$ and $(0, b)$ are the vector coefficients of the exterior vertices, then $ab < 0$,*
- 3) *all its inner vertices V verify $\beta_V > 0$,*
- 4) *For each bounded edge, its associated Hamiltonian is non-null and does not have any strong factor.*

Theorem 4. *If the Newton diagram of (1) verifies:*

- 1) *all its vertices have even coordinates,*
- 2) *it has two exterior vertices. Moreover, if $(a, 0)$ and $(0, b)$ are the vector coefficients of the exterior vertices, then $ab < 0$,*
- 3) *all its inner vertices V verify $\beta_V > 0$,*
- 4) *For each bounded edge, its associated Hamiltonian is non-null and does not have any factor of the form $y^{t_1} - \tilde{a}x^{t_2}$ with \tilde{a} non-zero real,*

then the origin of system (1) is monodromic.

Summarizing, from Theorems 3 and 4, it follows that Newton diagram of (1) determines the monodromy of the origin except the case of a non-strong factor of \mathbf{F} different from x and y existing. For these cases, there is an algorithm which deals with blow-up techniques, see Algaba *et al.* [1].

The rest of the paper is organized as follows: Sections 2 and 3 are devoted to detect characteristic orbits from the lowest-degree quasi-homogeneous vector fields and from its inner vertices, respectively. In Section 4, we show some

examples which have already been considered by García *et al.* [10], by revealing the straightforwardness of our method for characterizing the monodromy of a singular point. Section 5 contains the proofs of the results.

2. Detecting characteristic orbits from the lowest-degree quasi-homogeneous vector field

The system (1) by means of the change (6) and rescaling the time by $dt = (2t_1 t_2 / u^r) d\tau$, becomes

$$\begin{aligned} \dot{u} &= du/d\tau = u \sum_{j=0}^{\infty} \left[-h'_{r+j+|\mathbf{t}|}(\theta) + 2t_1 t_2 \mu_{r+j}(\theta) \right] u^j, \\ \dot{\theta} &= d\theta/d\tau = \sum_{j=0}^{\infty} (r+j+|\mathbf{t}|) h_{r+j+|\mathbf{t}|}(\theta) u^j, \end{aligned} \quad (7)$$

where we have denoted

$$h_{r+j+|\mathbf{t}|}(\theta) = h_{r+j+|\mathbf{t}|}(\text{Cs}(\theta), \text{Sn}(\theta)), \quad \mu_{r+j}(\theta) = \mu_{r+j}(\text{Cs}(\theta), \text{Sn}(\theta)),$$

see Algaba *et al.* [2]. We note that the quoted transformation does not change the direction of the orbits, since $dt/d\tau > 0$.

The next results give sufficient conditions of non-monodromy or monodromy of a singular point.

Proposition 5. *Let us assume that the lowest-degree quasi-homogeneous term of type \mathbf{t} of \mathbf{F} is $\mathbf{F}_r = \mu_r \mathbf{D}_0$, (i.e. $h_{r+|\mathbf{t}|} \equiv 0$ and $\mu_r \neq 0$). Then, the origin of the system (1) is a node (the origin is surrounded by parabolic sectors).*

As a consequence, it has that, in such a case, the system (1) is not locally integrable.

Proposition 6. *Let $h_{r+|\mathbf{t}|} \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ be the Hamiltonian associated to the lowest-degree quasi-homogeneous term of type \mathbf{t} of \mathbf{F} . If $h_{r+|\mathbf{t}|}(\theta) \neq 0$, for all θ , the origin of system (1) is monodromic.*

From the proposition given above, if the system has characteristic orbits then the polynomial $h_{r+|\mathbf{t}|}(\theta)$ must have some real roots. Thus, if there were \mathbf{t} -characteristic orbits, the polynomial $h_{r+|\mathbf{t}|}(x, y)$ must have factors x , y or $y^{t_1} - \lambda x^{t_2}$, with $\lambda \neq 0$.

Specifically, it has the following result.

Proposition 7. *Let $\mathbf{x}(t)$ be a \mathbf{t} -characteristic orbit of the analytic system (1), $\gamma(\tau) = (u(\tau), \theta(\tau)) \in [0, \epsilon) \times [0, T_{\mathbf{t}})$ the orbit $\mathbf{x}(t)$ of (1) transformed by (6) and $\theta^* = \lim_{t \rightarrow \infty} \theta(t)$.*

If $h_{r+|\mathbf{t}|}(x, y) \not\equiv 0$, it has one and only one of the following situations:

- a) *if $\theta^* = T_{\mathbf{t}}/4, 3T_{\mathbf{t}}/4$, then x is a factor of $h_{r+|\mathbf{t}|}(x, y)$ and $\lim_{t \rightarrow +\infty} \frac{y(t)^{t_1}}{x(t)^{t_2}} = \pm\infty$,*

- b) if $\theta^* = 0, T_{\mathbf{t}}/2$, then y is a factor of $h_{r+|\mathbf{t}|}(x, y)$ and $\lim_{t \rightarrow +\infty} \frac{y(t)^{t_1}}{x(t)^{t_2}} = 0$,
c) if $\theta^* \neq 0, T_{\mathbf{t}}/4, T_{\mathbf{t}}/2, 3T_{\mathbf{t}}/4$, then $y^{t_1} - \tilde{a}x^{t_2}$, with $\tilde{a} = \frac{S n^{t_1}(\theta^*)}{C s^{t_2}(\theta^*)}$, is a factor of $h_{r+|\mathbf{t}|}(x, y)$.

Next, we show that the existence of a strong factor of \mathbf{F} implies the existence of characteristic orbits.

Proposition 8. *If there exists a strong factor of \mathbf{F} associated to a type \mathbf{t} , then there exists a \mathbf{t} -characteristic orbit of system (1).*

With some abuse of the language, we will say that the characteristic orbit is associated to the strong factor. It has the following necessary condition of monodromy.

Corollary 2. *If the origin of system (1) is monodromic then all the factors, of the lowest-degree Hamiltonian $h_{r+|\mathbf{t}|}$, of the form x, y or $y^{t_1} - \lambda x^{t_2}$, $\lambda \neq 0$, are not strong factors of \mathbf{F} , for any type \mathbf{t} ,*

The reciprocal is not true. The system $\dot{x} = -x^2y, \dot{y} = 3xy^2 + 4x^3 + y^4$, is non-monodromic because the axis $x = 0$ is a characteristic orbit. Its Newton diagram contains two edges whose types are $(2, 1)$ and $(1, 1)$. The lowest-degree Hamiltonian and divergence associated to $(1, 1)$ are $(x^2 + y^2)x^2$ and xy , respectively, i.e. x is a non-strong factor.

The following property is also a consequence of Proposition 8. It is a condition which must hold the support point of a vertex of system (1) so that there exist characteristic orbits different from the axes $x = 0$ and $y = 0$.

Proposition 9. *Let (m, n) a vertex of the Newton diagram of system (1). If the axis $x = 0$ (or $y = 0$) is not invariant and m (or n) is odd, then there is a characteristic orbit of (1) different from the coordinate axes.*

3. Characteristic orbits and blow-up of vertices of the Newton diagram.

The purpose of this section is to find the characteristic orbits of system (1) that are related to the vertices of the Newton diagram.

Let V be an inner vertex of the Newton diagram of system (1) with α_ℓ and $\alpha_{\bar{\ell}}$ the exponents of the adjacent edges, upper and lower, respectively. Let $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$, with $\alpha_\ell = t_2/t_1 < s_2/s_1 = \alpha_{\bar{\ell}}$ and $h_{r_{\mathbf{t}}+|\mathbf{t}|}, h_{r_{\mathbf{s}}+|\mathbf{s}|}$ be the Hamiltonians associated to the lowest-degree quasi-homogeneous terms of \mathbf{F} of type \mathbf{t} and \mathbf{s} . We define the sets

$$W_{\mathbf{t}, \mathbf{s}}^{(\sigma_1, \sigma_2)} = \{(x, y) \in \mathbb{R}^2 \mid \epsilon x^{s_2/s_1} \leq y \leq \frac{1}{\epsilon} x^{t_2/t_1}, (-1)^{\sigma_1} x \geq 0, (-1)^{\sigma_2} \epsilon > 0\},$$

with $\sigma_1, \sigma_2 \in \{0, 1\}$. The following result provides a necessary condition both of monodromy and non-monodromy.

Proposition 10. *Let V be an inner vertex of the Newton diagram of system (1) with $h_{r_{\mathbf{t}}+|\mathbf{t}|}h_{r_{\mathbf{s}}+|\mathbf{s}|} \neq 0$. It has that:*

- a) $W_{\mathbf{t},\mathbf{s}}^{(0,0)}$ is a parabolic sector of (1), if $\alpha_V \in (t_2/t_1, s_2/s_1)$,
- b) $W_{\mathbf{t},\mathbf{s}}^{(0,0)}$ is a hyperbolic sector of (1), if $\alpha_V \notin [t_2/t_1, s_2/s_1]$.

If we perform the reflections $\bar{x} = (-1)^{\sigma_1}x$, $\bar{y} = (-1)^{\sigma_2}y$, $\sigma_1, \sigma_2 \in \{0, 1\}$, it is easy to check that α_V does not change and, therefore, the above proposition can be used for the study of a characteristic orbit lying on another quadrant different from the positive. However, Proposition 10 has the disadvantage that it can not be always applied for $\alpha_V = \frac{t_2}{t_1}$ or $\alpha_V = \frac{s_2}{s_1}$ and thus the exponent of an inner vertex can not guarantee whether there is, or not, a characteristic orbit.

We now study the existence of characteristic orbits contained in $W_{\mathbf{t},\mathbf{s}}^{(0,0)}$ and later we extend the results for the remaining quadrants. Such orbits lying on $W_{\mathbf{t},\mathbf{s}}^{(\sigma_1,\sigma_2)}$ will be named characteristic orbits associated to the vertex V .

Proposition 11. *Let V be an inner vertex of the Newton diagram of system (1) with $h_{r_{\mathbf{t}}+|\mathbf{t}|}h_{r_{\mathbf{s}}+|\mathbf{s}|} \neq 0$. We have that $W_{\mathbf{t},\mathbf{s}}^{(0,0)}$ is a parabolic (resp. hyperbolic) sector of the origin if and only if $\beta_V < 0$ (resp. $\beta_V > 0$).*

We denote by $\beta_V^{(\sigma_1,\sigma_2)}$ the value of β_V after making the change $\bar{x} = (-1)^{\sigma_1}x$, $\bar{y} = (-1)^{\sigma_2}y$, $\sigma_1, \sigma_2 \in \{0, 1\}$.

It is easy to prove that the vector field $(ax^m y^{n-1}, bx^{m-1} y^n)^T$ is transformed, by means of the quoted change, in

$$(-1)^{(m-1)\sigma_1+(n-1)\sigma_2} (a\bar{x}^m \bar{y}^{n-1}, b\bar{x}^{m-1} \bar{y}^n)^T,$$

thus

$$\begin{aligned} \beta_V^{(\sigma_1,\sigma_2)} &= (-1)^{(m+j_0 s_2-1)\sigma_1+(n-j_0 s_1-1)\sigma_2} \tilde{c}_{j_0} (-1)^{(m-i_0 t_2-1)\sigma_1+(n+i_0 t_1-1)\sigma_2} c_{i_0} \\ &= (-1)^{j_0(s_1\sigma_2+s_2\sigma_1)+i_0(t_1\sigma_2+t_2\sigma_1)} \beta_V. \end{aligned} \tag{8}$$

From Proposition 11 and (8), it has the following result which shows when there exist characteristic orbits associated to an inner vertex defined on quadrants different from the first one.

Proposition 12. *Let V be an inner vertex of the Newton diagram of system (1) with $h_{r_{\mathbf{t}}+|\mathbf{t}|}h_{r_{\mathbf{s}}+|\mathbf{s}|} \neq 0$. It has that $W_{\mathbf{t},\mathbf{s}}^{(\sigma_1,\sigma_2)}$ is a parabolic (resp. hyperbolic) sector of the origin if and only if $(-1)^{j_0(s_1\sigma_2+s_2\sigma_1)+i_0(t_1\sigma_2+t_2\sigma_1)} \beta_V$ is positive (resp. negative).*

We note that if the Newton diagram of \mathbf{F} has a vertex V with $\beta_V < 0$, then system (1) is not locally integrable.

The following result facilitates the study of the existence of characteristic orbits defined in the remaining quadrants.

Proposition 13. *Let system (1) be with the axis $y = 0$ non-invariant. We assume that its Newton diagram has an inner vertex $V = (m, n)$ with $\beta_V > 0$ and $h_{r_t+|t|}h_{r_s+|s|} \neq 0$.*

If the axis $x = 0$ is not invariant, then there exist characteristic orbits associated to V in a quadrant different from the first one if it satisfies at least one of the following properties:

- (a) n or m are odd,
- (b) $h_{r_t+|t|}$ or $h_{r_s+|s|}$ have a factor of the form x , y , $y^{t_1} - \lambda x^{t_2}$ or $y^{s_1} - \lambda x^{s_2}$, $\lambda \neq 0$, which has odd multiplicity order.

If the axis $x = 0$ is invariant, then there exist characteristic orbits associated to V in the fourth quadrant, if n is odd or there is a factor different from x satisfying (b).

4. Examples

First, we analyze the monodromy for the linear cases. Later, we give two examples of degenerated systems (without linear terms) already showed by García *et al.* [10] and finally we consider an unsuccessfully system already studied. The monodromy or not of all of them is obtained straightforwardly from Theorem 3 and Theorem 4.

Example 1. *Linear center, focus and saddle.*

Newton diagram of the systems

$$\begin{aligned} \dot{x} &= -y, & \dot{y} &= x, \\ \dot{x} &= -y + \lambda x, & \dot{y} &= x + \lambda y, \quad \lambda \neq 0, \\ \dot{x} &= x + y, & \dot{y} &= x - y, \end{aligned}$$

named linear center, focus and saddle, respectively, contain two exterior vertices and an edge only of type $(1, 1)$. For two first cases, the associated Hamiltonian is $h_2(x, y) = x^2 + y^2$, which does not have any real factors, thus they are monodromic. For saddle case, the coefficients of the vector coefficients of the exterior vertices have the same sign, therefore the singular point is non-monodromic.

Example 2. We consider the system $\dot{x} = bx^2 + axy^2 - by^3 - x^4$, $\dot{y} = 4bxy^2 - ay^3 + 2x^5$ with $b \neq 0$. Its Newton diagram has two exterior vertices $V_0 = (0, 4)$ and $V_2 = (6, 0)$ whose associated vector fields are $(-by^3, 0)^T$ and $(0, 2x^5)^T$. For the origin to be monodromic, it must hold that $b > 0$ and, in such a case, $V_1 = (2, 1)$ is an inner vertex, which has odd coordinates. Therefore, the origin is not monodromic.

Example 3. Let the four-parametric system $\dot{x} = y(ax^2 + bxy + cy^2)$, $\dot{y} = y^2(ax + by) + dx^5$. If $cd = 0$, the system has an invariant axis. Otherwise, its Newton diagram has two exterior vertices $V_0 = (0, 4)$ and $V_2 = (6, 0)$ whose associated vector fields are $(cy^3, 0)^T$ and $(0, dx^5)^T$. So, the origin is monodromic if $dc < 0$.

If $a \neq 0$, $V = (2, 2)$ is the unique inner vertex, associated to the vector field $(ax^2y, axy^2)^T$. The lowest degree Hamiltonians associated to the edges of type $(1, 1)$ and $(1, 2)$ are $h_4^{(1,1)} = -cy^4$ and $h_6^{(1,2)} = x^2(dx^4 - ay^2)$, respectively. Thus, we have that $\beta_V = ac$, that is, it must hold $ac > 0$ for the origin to be monodromic. As $ac > 0$ and $cd < 0$, we have that $ad < 0$, that is, the unique real factors of both polynomials are x and y , therefore, the origin is monodromic.

If $a = 0$ and $b \neq 0$, the Newton diagram contains the vertex $V = (1, 3)$, support of $(bxy^2, by^3)^T$, which has odd coordinates. Therefore, the origin is not monodromic.

If $a = b = 0$, its Newton diagram only has a bounded edge of type $(2, 3)$ whose associated Hamiltonian is $h_{12}^{(2,3)} = dx^6 - cy^4$. The origin is monodromic since $h_{12}^{(2,3)}$ does not have any real factor.

Summarizing, the origin is monodromic if and only if $cd < 0$ and, either $a = b = 0$ or $ac > 0$.

Example 4. We consider system (5), see page 4. If $a(h-g) > 0$ or $b(h-3g) < 0$, system (5) has strong factors associated to $h_4^{(1,1)}$ or to $h_8^{(1,3)}$. Otherwise, if $a(h-g) < 0$ and $b(h-3g) > 0$, then system (5) does not have strong factors, but $\beta_{V_C} = \frac{1}{32}(h-g)(h-3g) < 0$, since $ab < 0$. Thus, the origin is not monodromic. If $h = g$ and $bg < 0$ then $h_4^{(1,1)} = -\frac{a}{4}y^4$ does not have non-trivial factors and $\beta_{V_C} > 0$, that is, origin is monodromic. If $h = 3g$ and $ag < 0$ then $h_8^{(1,3)} = \frac{b}{8}x^8$ does not have non-trivial factors and $\beta_{V_C} > 0$, that is, origin is monodromic.

5. Proofs

PROOF OF THEOREM 1. Let $\gamma(t)$ be the orbit $\mathbf{x}(t)$ of (1) transformed by (6). We first prove that there is no pair $\theta_1, \theta_2 \in [0, T_{\mathbf{t}}]$, $\theta_1 < \theta_2$, such that $\theta_1 = \lim_{\tau \rightarrow \infty} \inf(\theta(\tau))$, $\theta_2 = \lim_{\tau \rightarrow \infty} \sup(\theta(\tau))$. In fact, if they existed then there would be an infinity of equilibria of (7) in the segment $u = 0$, $\theta \in (\theta_1, \theta_2)$. In such a case, as \mathbf{F} is analytic, $h_{r+|\mathbf{t}|}$ must be identically null. Thus, reparameterizing the time in (7), we have the system

$$\begin{aligned} \dot{u} &= 2t_1 t_2 \mu_r(\theta) + u \sum_{j=1}^{\infty} \left[-h'_{r+j+|\mathbf{t}|}(\theta) + 2t_1 t_2 \mu_{r+j}(\theta) \right] u^{j-1}, \\ \dot{\theta} &= \sum_{j=1}^{\infty} (r+j+|\mathbf{t}|) h_{r+j+|\mathbf{t}|}(\theta) u^{j-1}, \end{aligned} \tag{9}$$

with $\mu_r \neq 0$. Let $\theta^* \in [0, T_{\mathbf{t}}]$ be with $\theta_1 < \theta^* < \theta_2$ and $\mu_r(\theta^*) \neq 0$. The point $(u, \theta) = (0, \theta^*)$ is a regular point of (9). Therefore, the orbits cross the axis $u = 0$ transversely in a neighborhood of $(u, \theta) = (0, \theta^*)$. This fact leads to a contradiction since it would have several solutions of (1) of the same problem with initial conditions.

Summarizing, the only possible cases are either $\lim_{\tau \rightarrow \infty} \theta(\tau) = \theta^* \in [0, T_{\mathbf{t}})$, i.e. the orbit is a \mathbf{t} -characteristic orbit of system (1) for the arbitrary type \mathbf{t} , or $\lim_{\tau \rightarrow \infty} \theta(\tau) = \pm\infty$, in such a case there is no periodic orbit $u = u(\theta)$ of the system (7) which tends to a point of the segment $u = 0, 0 \leq \theta < T_{\mathbf{t}}$. If there is a finite number of orbits verifying $u(0) = u(T_{\mathbf{t}})$ in W , then for ϵ small enough, the orbits hold $u(0) \neq u(T_{\mathbf{t}})$ in W . For the uniqueness of the solutions, it must hold either that $u(0) < u(T_{\mathbf{t}})$ (repulsive focus) or $u(0) > u(T_{\mathbf{t}})$ (attractive focus). Otherwise, if there are an infinite number of orbits with $u(0) = u(T_{\mathbf{t}})$ in W , the origin is a center (it could not be a center-focus, since \mathbf{F} is analytic, see Il'yashenko [12]). \square

PROOF OF PROPOSITION 5. If $h_{r+|\mathbf{t}|} \equiv 0$, we can transform system (7) into (9). For any θ with $\mu_r(\theta) \neq 0$, the system is a flow-box in a neighborhood of $(u, \theta) = (0, \theta)$, hence the origin of system (1) is a node. \square

PROOF OF PROPOSITION 6. We assume that $h_{r+|\mathbf{t}|}(\theta) \neq 0$, for all θ . Taking the reparametrization of the time $dt = (r + |\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)d\tau$, system (7) becomes

$$\begin{aligned} \dot{u} &= u \sum_{j=0}^{\infty} u^j \left[-\frac{h'_{r+j+|\mathbf{t}|}(\theta)}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)} + 2t_1 t_2 \frac{\mu_{r+j}(\theta)}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)} \right], \\ \dot{\theta} &= 1 + \sum_{j=1}^{\infty} \frac{(r+j+|\mathbf{t}|)h_{r+j+|\mathbf{t}|}(\theta)}{(r+|\mathbf{t}|)h_{r+|\mathbf{t}|}(\theta)} u^j. \end{aligned}$$

It is easy to check that the system has non-equilibrium in the axis $u = 0$, hence the origin of (1) is monodromic. \square

PROOF OF PROPOSITION 7. As $\mathbf{x}(t)$ is a \mathbf{t} -characteristic orbit of system (1), then $\lim_{t \rightarrow \infty} \gamma(t) = (0, \theta^*)$ with $\theta^* \in [0, T_{\mathbf{t}})$. Hence, the orbit $\gamma(t)$ tends to a equilibrium point lying on the invariant axis, that is $h_{r+|\mathbf{t}|}(\theta^*) = 0$ since $h_{r+|\mathbf{t}|} \not\equiv 0$.

If $\theta^* = T_{\mathbf{t}}/4, 3T_{\mathbf{t}}/4$, then $Cs(\theta^*) = 0$ and thus x is a factor of $h_{r+|\mathbf{t}|}(x, y)$. Also we have that $\lim_{t \rightarrow +\infty} \frac{y(t)^{t_1}}{x(t)^{t_2}} = \pm\infty$.

The remaining situations are obtained easily. \square

PROOF OF PROPOSITION 8. We observe that each real root of $h_{r+|\mathbf{t}|}(\theta)$ corresponds an irreducible factor x, y or $y^{t_1} - \lambda_j x^{t_2}$, $\lambda_j \in \mathbb{R}$, $\lambda_j \neq 0$, with the same multiplicity order as the order of the root.

If θ_0 is a root of $h_{r+|\mathbf{t}|}(\theta)$, the change $\alpha = \theta - \theta_0$ takes the singular point $(u, \theta) = (0, \theta_0)$ of the system (7) to the origin.

For all $j \geq 0$, we can write

$$\begin{aligned} (r + |\mathbf{t}| + j)h_{r+|\mathbf{t}|+j}(\alpha) &= \sum_{i \geq 0} c_i^{(j)} \alpha^i, \quad c_0^{(0)} = 0, \\ 2t_1 t_2 \mu_{r+j}(\alpha) &= \sum_{i \geq 0} d_i^{(j)} \alpha^i. \end{aligned}$$

System (7), in the new coordinates (u, α) , becomes

$$\begin{aligned}\dot{u} &= u \sum_{j \geq 0} \left[\sum_{i \geq 0} (d_i^{(j)} - \frac{i+1}{r+|t|+j} c_{i+1}^{(j)}) \alpha^i \right] u^j, \\ \dot{\alpha} &= \sum_{i \geq 0} c_{i+1}^{(0)} \alpha^{i+1} + \sum_{j \geq 1} \left[\sum_{i \geq 0} c_i^{(j)} \alpha^i \right] u^j.\end{aligned}\tag{10}$$

We analyze the following cases separately:

If there is a factor of $h_{r+|t|}$ of odd multiplicity order, then we have that $c_{2m+1}^{(0)} \neq 0$ and $c_i^{(0)} = 0$, for $0 \leq i \leq 2m$. The system (10) has the form

$$\begin{aligned}\dot{u} &= u\Psi(u, \alpha), \\ \dot{\alpha} &= c_{2m+1}^{(0)} \alpha^{2m+1} + \alpha^{2m+2} \Phi_1(\alpha) + u\Phi_2(u, \alpha),\end{aligned}$$

with Ψ, Φ_1, Φ_2 analytic functions.

Next, we prove that this system has orbits which tend to the origin when $t \rightarrow \pm\infty$ and therefore the system (1) has characteristic orbits. Actually, the axis $u = 0$ is invariant and its dynamics are determined in a neighborhood of the origin, repulsive if $c_{2m+1}^{(0)} > 0$ or attractive if $c_{2m+1}^{(0)} < 0$. Also, there are at least two vertical isoclines (the axis $u = 0$ for $u > 0$ and for $u < 0$).

From Malgrange theorem, as the lowest power in α of $\dot{\alpha}$ has an exponent odd then there will be at least one curve of horizontal isoclines with odd multiplicity order. Therefore, the vertical components of the field on both sides of the horizontal isocline have opposite directions.

We can assume, by means of a change, that the horizontal isocline is $\alpha = u$ and, by clarity and simplicity we suppose that the vertical isoclines closer to $\alpha = u$ on both sides are $\alpha = 1/2u$ and $\alpha = 2u$. We can also assume (by changing the sign of the time, if necessary) that the direction of the field is $(0, 1)$ on the vertical isocline $\alpha = 1/2u$ and $(0, -1)$ on $\alpha = 2u$.

If the direction of the vector field on $\alpha = u$ is $(1, 0)$, then there is a characteristic orbit in the sector $\frac{1}{2} < \alpha < u$ which tends to the origin when $t \rightarrow -\infty$, see Zhang *et al.* [17], pages 68–69. Otherwise, if the direction of the vector field on $\alpha = u$ is $(-1, 0)$ then all the orbits lying in the sector $u < \alpha < 2u$ tend to the origin when $t \rightarrow +\infty$.

If there is a factor of $h_{r+|t|}$ of even multiplicity order ($2m$) and it is a factor of μ_r with even multiplicity order ($2n$) with $0 \leq n < m$, we have that the polynomial $h_{r+|t|}$ verifies $c_j^{(0)} = 0$ for $0 \leq j \leq 2m - 1$, $c_{2m}^{(0)} \neq 0$ and μ_r verifies $d_j^{(0)} = 0$ for $0 \leq j \leq 2n - 1$, $d_{2n}^{(0)} \neq 0$. Now, the system (10) has the form

$$\begin{aligned}\dot{u} &= d_{2n}^{(0)} \alpha^{2n} u + \alpha^{2n+1} u \Psi_1(\alpha) + u^2 \Psi_2(u, \alpha), \\ \dot{\alpha} &= \alpha^{2n+2} \Phi_1(\alpha) + u \Phi_2(u, \alpha).\end{aligned}$$

Taking $\mathbf{t} = (2n + 1, 1)$, the lowest-degree quasi-homogeneous term of \mathbf{F} is

$$\mathbf{F}_{2n}(u, \alpha) = (d_{2n}^{(0)} \alpha^{2n} u, 0)^T,$$

with $h_{4n+2}(u, \alpha) = -\frac{1}{4n+2} d_{2n}^{(0)} \alpha^{2n+1} u$ which has a factor with odd multiplicity order and thus the result follows. \square

PROOF OF PROPOSITION 9. As V is a vertex (inner or exterior) of the Newton diagram, it is always possible to choose a type $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 non-zero such that $t_1 b - t_2 a \neq 0$, (a, b) being the vector coefficient of the vertex. Thus, $\mathbf{F}_r(x, y) = (ax^m y^{n-1}, bx^{m-1} y^n)^T$ and $h_{r+|\mathbf{t}|}(x, y) = (t_1 b - t_2 a) x^m y^n \neq 0$. If, for example, the axis $x = 0$ is not invariant and m is odd, the polynomial x is a strong factor of \mathbf{F} associated to the type \mathbf{t} since it is a factor of $h_{r+|\mathbf{t}|}$ with odd multiplicity order. From Proposition 8, there exists a characteristic orbit associated to the strong factor x . Specifically, there is a θ^* with $\text{Cs}(\theta^*) = 0$, $(\text{Sn}(\theta^*) = 1)$, that is the characteristic orbit verifies $\lim_{t \rightarrow \infty} \frac{y^{t_1}}{x^{t_2}} = \infty$. Thus, the orbit is different from $y = 0$ and hence it is different from the axes. \square

We start giving a type of blow-up, which we name *blow-up of vertices*, and it will play the main role in our study.

Lemma 14. *Let V be an inner vertex of the Newton diagram of system (1) with $\alpha_{\bar{\ell}}$ and α_{ℓ} the exponents of the adjacent edges, lower and upper, respectively. Let $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$, with $\alpha_{\ell} \leq t_2/t_1 < s_2/s_1 \leq \alpha_{\bar{\ell}}$ and $h_{r_{\mathbf{t}}+|\mathbf{t}|}$, $h_{r_{\mathbf{s}}+|\mathbf{s}|}$ the Hamiltonians associated to the lowest-degree quasi-homogeneous terms of \mathbf{F} of type \mathbf{t} and \mathbf{s} , respectively.*

If $h_{r_{\mathbf{t}}+|\mathbf{t}|} h_{r_{\mathbf{s}}+|\mathbf{s}|} \neq 0$, the blow-up

$$x = u^{t_1 s_2} v^{t_1 s_1}, \quad y = u^{t_2 s_2} v^{t_1 s_2}$$

and a reparametrization of the time variable, transforms system (1) into

$$\begin{aligned} u' &= -u \left[\frac{t_1(r_{\mathbf{s}}+|\mathbf{s}|)}{s_2(r_{\mathbf{t}}+|\mathbf{t}|)} \frac{\tilde{c}_{j_0}}{c_{i_0}} u^{j_0 d s_2} + u^{(j_0+1) d s_2} \Phi_1(u^{d s_2}) + v^{t_1} \Phi_2(u^{s_2}, v^{t_1}) \right], \\ v' &= v \left[v^{i_0 d t_1} + v^{(i_0+1) d t_1} \Psi_1(v^{d t_1}) + u^{s_2} \Psi_2(u^{s_2}, v^{t_1}) \right] \end{aligned} \quad (11)$$

where $d = t_1 s_2 - t_2 s_1 \in \mathbb{N}$, Φ_i, Ψ_i analytic functions, $i_0 = \min \{i \geq 0 \mid c_i \neq 0\}$, $j_0 = \min \{j \geq 0 \mid \tilde{c}_j \neq 0\}$ c_i and \tilde{c}_j being the coefficients of the polynomials $h_{r_{\mathbf{t}}+|\mathbf{t}|}$ and $h_{r_{\mathbf{s}}+|\mathbf{s}|}$, ordered from the highest to the lowest exponent in x and y , respectively.

We make the following considerations:

- i) the above change of variables transforms the rectangle of the (u, v) -plane $\{0 \leq u \leq \delta_1, 0 \leq v \leq \delta_2\}$ in the region of the first quadrant $W_{\mathbf{t}, \mathbf{s}}^{(0,0)}$ with $\epsilon \delta_1^{t_1 s_2 (s_2/s_1 - t_2/t_1)} = \epsilon \delta_2^{t_1 s_1 (s_2/s_1 - t_2/t_1)} = 1$, $\epsilon > 0$.

- ii) Lemma 14 is also true when an adjacent edge to the inner vertex is not bounded. In such a case, it is either $\alpha_\ell = t_2/t_1 = 0$ with $(t_1, t_2) = (1, 0)$, or $\alpha_{\tilde{\ell}} = s_2/s_1 = +\infty$ with $(s_1, s_2) = (0, 1)$. Nevertheless, it is not true for any vertex of the Newton diagram on the axis (exterior vertex) since, for instance, if we consider system (1) with $\mathbf{F} = (y^n, 0)^T$ with support $V = (0, n+1)$, its adjacent edge would be the axis $x = 0$ whose type is $(t_1, t_2) = (1, 0)$ and thus $h_{r_{\mathbf{t}}+|\mathbf{t}|} \equiv 0$.
- iii) If $(m - it_2, n + it_1)$ ($(m + js_2, n - js_1)$), $i = 0, 1, \dots, k$ ($j = 0, 1, \dots, \tilde{k}$) are the support points of the upper (lower) adjacent edge to V ordered from higher to lower abscissa (ordinate), and (a_i, b_i) , $i = 0, 1, \dots, k$, $(\tilde{a}_j, \tilde{b}_j)$, $j = 0, 1, \dots, \tilde{k}$ the vectors of the support points (in particular, $a = a_0 = \tilde{a}_0$, $b = b_0 = \tilde{b}_0$), then we have that

$$\begin{aligned} h_{r_{\mathbf{t}}+|\mathbf{t}|}(x, y) &= \sum_{i=0}^k c_i x^{m-it_2} y^{n+it_1}, \\ h_{r_{\mathbf{s}}+|\mathbf{s}|}(x, y) &= \sum_{j=0}^{\tilde{k}} \tilde{c}_j x^{m+js_2} y^{n-js_1}, \end{aligned} \quad (12)$$

where $c_i = t_1 b_i - t_2 a_i$, $i = 0, 1, \dots, k$ and $\tilde{c}_j = s_1 \tilde{b}_j - s_2 \tilde{a}_j$, $j = 0, 1, \dots, \tilde{k}$, and, therefore, $i_0 j_0 = 0$ since $c_0 = t_1 b - t_2 a \neq s_1 \tilde{b} - s_2 \tilde{a} = \tilde{c}_0$.

PROOF. Making the blow-up $x = x(u, v) = u^{t_1 s_2} v^{t_1 s_1}$, $y = y(u, v) = u^{t_2 s_2} v^{t_2 s_1}$, system (1) becomes

$$\begin{aligned} \dot{u} &= -u \sum_{j=0}^{\infty} \frac{r_{\mathbf{s}}+|\mathbf{s}|+j}{s_2} h_{r_{\mathbf{s}}+|\mathbf{s}|+j}(x(u, v), y(u, v)), \\ \dot{v} &= v \sum_{j=0}^{\infty} \frac{r_{\mathbf{t}}+|\mathbf{t}|+j}{t_1} h_{r_{\mathbf{t}}+|\mathbf{t}|+j}(x(u, v), y(u, v)). \end{aligned} \quad (13)$$

We see the expression of $h_{r_{\mathbf{s}}+|\mathbf{s}|+j}(x(u, v), y(u, v))$. As $h_{r_{\mathbf{s}}+|\mathbf{s}|+j} \in \mathcal{P}_{r_{\mathbf{s}}+|\mathbf{s}|+j}^{\mathbf{s}}$ and $d = t_1 s_2 - t_2 s_1 \in \mathbb{N}$, since $t_2/t_1 < s_2/s_1$, we have that

$$h_{r_{\mathbf{s}}+|\mathbf{s}|+j}(x(u, v), y(u, v)) = (u^{t_2} v^{t_1})^{r_{\mathbf{s}}+|\mathbf{s}|+j} h_{r_{\mathbf{s}}+|\mathbf{s}|+j}(u^d, 1).$$

Let $(\alpha_1^{(j)}, \alpha_2^{(j)})$ be the support point of the vector field $\mathbf{F}_{r_{\mathbf{s}}+|\mathbf{s}|+j}$ with the lowest abscissa and $(\alpha_1^{(j)} + i s_2, \alpha_2^{(j)} - i s_1)$, $i = 1, \dots, \tilde{k}_j$ for some $\tilde{k}_j \in \mathbb{N}$, the remainder support points. And let $(\tilde{a}_i^{(j)}, \tilde{b}_i^{(j)})$, $i = 0, 1, \dots, \tilde{k}_j$ the vector coefficients associated to the above support points. Using the factorization of $h_{r_{\mathbf{s}}+|\mathbf{s}|+j}$ on $\mathbb{R}[x, y]$, see Algaba *et al.* [3], we have that $h_{r_{\mathbf{s}}+|\mathbf{s}|+j}(u^d, 1) = u^{d\alpha_1^{(j)}} \sum_{i=0}^{\tilde{k}_j} \tilde{c}_i^{(j)} u^{ids_2}$, with $\tilde{c}_i^{(j)} = s_1 \tilde{b}_i^{(j)} - s_2 \tilde{a}_i^{(j)}$ for each i . Moreover, $r_{\mathbf{s}}+|\mathbf{s}|+j = s_1 \alpha_1^{(j)} + s_2 \alpha_2^{(j)}$ and we can also define $\tilde{\gamma}_j = t_1 \alpha_1^{(j)} + t_2 \alpha_2^{(j)} - (r_{\mathbf{t}}+|\mathbf{t}|) \in \mathbb{N}_0$ since the support point $(\alpha_1^{(j)}, \alpha_2^{(j)})$ is in the Newton diagram, for $j \geq 0$. So, we have that $h_{r_{\mathbf{s}}+|\mathbf{s}|+j}((x(u, v), y(u, v)))$ is

$$\begin{aligned} &u^{(r_{\mathbf{s}}+|\mathbf{s}|+j)t_2+(t_1 s_2-t_2 s_1)\alpha_1^{(j)}} v^{(r_{\mathbf{s}}+|\mathbf{s}|+j)t_1} \sum_{i=0}^{\tilde{k}_j} \tilde{c}_i^{(j)} u^{ids_2} \\ &= u^{(t_1 \alpha_1^{(j)}+t_2 \alpha_2^{(j)})s_2} v^{(r_{\mathbf{s}}+|\mathbf{s}|+j)t_1} \sum_{i=0}^{\tilde{k}_j} \tilde{c}_i^{(j)} u^{ids_2} \\ &= u^{(r_{\mathbf{t}}+|\mathbf{t}|)s_2} v^{(r_{\mathbf{s}}+|\mathbf{s}|)t_1} u^{\tilde{\gamma}_j s_2} v^{j t_1} \sum_{i=0}^{\tilde{k}_j} \tilde{c}_i^{(j)} u^{ids_2}. \end{aligned}$$

Analogously, let $(\beta_1^{(j)}, \beta_2^{(j)})$ be the support point of the vector field $\mathbf{F}_{r_{\mathbf{t}}+|\mathbf{t}|+j}$ with lowest ordinate. The remaining support points will be of the form $(\beta_1^{(j)} - it_2, \beta_2^{(j)} + it_1)$, $i = 1, \dots, k_j$ for some $k_j \in \mathbb{N}$. Let $(a_i^{(j)}, b_i^{(j)})$, $i = 0, 1, \dots, k_j$ the vector coefficients associated to the above support points. And we also have that $r_{\mathbf{t}} + |\mathbf{t}| + j = t_1\beta_1^{(j)} + t_2\beta_2^{(j)}$ and $\gamma_j = s_1\beta_1^{(j)} + s_2\beta_2^{(j)} - (r_{\mathbf{s}} + |\mathbf{s}|) \in \mathbb{N}_0$. Reasoning in a similar way, we have that

$$h_{r_{\mathbf{t}}+|\mathbf{t}|+j}((x(u, v), y(u, v))) = u^{(r_{\mathbf{t}}+|\mathbf{t}|)s_2v^{(r_{\mathbf{s}}+|\mathbf{s}|)t_1}} u^{js_2} v^{\gamma_j t_1} \sum_{i=0}^{k_j} c_i^{(j)} v^{idt_1}.$$

It is easy to see that if (a, b) is the vector coefficient of the vertex V and (m, n) is its support point, then for $j = 0$ is $\alpha_1^{(0)} = m$, $\beta_2^{(0)} = n$ and thus $\gamma_0 = \tilde{\gamma}_0 = 0$, $\tilde{a}_0^{(0)} = a_0^{(0)} = a$, $\tilde{b}_0^{(0)} = b_0^{(0)} = b$, $r_{\mathbf{t}} + |\mathbf{t}| = t_1m + t_2n$ and $r_{\mathbf{s}} + |\mathbf{s}| = s_1m + s_2n$. So, the system (13) after applying the reparametrization of the time $d\tau/dt = u^{(r_{\mathbf{t}}+|\mathbf{t}|)s_2}v^{(r_{\mathbf{s}}+|\mathbf{s}|)t_1}$, becomes

$$\begin{aligned} \dot{u} &= -u \left[\frac{r_{\mathbf{s}}+|\mathbf{s}|}{s_2} \sum_{i=0}^{\tilde{k}_0} \tilde{c}_i^{(0)} u^{ids_2} + v^{t_1} \sum_{j=1}^{\infty} \frac{r_{\mathbf{s}}+|\mathbf{s}|+j}{s_2} u^{\tilde{\gamma}_j s_2} v^{(j-1)t_1} \sum_{i=0}^{\tilde{k}_j} \tilde{c}_i^{(j)} u^{ids_2} \right], \\ \dot{v} &= v \left[\frac{r_{\mathbf{t}}+|\mathbf{t}|}{t_1} \sum_{i=0}^{k_0} c_i^{(0)} v^{idt_1} + u^{s_2} \sum_{j=1}^{\infty} \frac{r_{\mathbf{t}}+|\mathbf{t}|+j}{t_1} u^{(j-1)s_2} v^{\gamma_j t_1} \sum_{i=0}^{k_j} c_i^{(j)} v^{idt_1} \right]. \end{aligned}$$

As $h_{r_{\mathbf{t}}+|\mathbf{t}|}h_{r_{\mathbf{s}}+|\mathbf{s}|} \neq 0$ then there exist $i_0 = \min \left\{ 0 \leq i \leq k_0 \mid c_i^{(0)} \neq 0 \right\}$ and $j_0 = \min \left\{ 0 \leq j \leq \tilde{k}_0 \mid \tilde{c}_j^{(0)} \neq 0 \right\}$. Denoting $c_{i_0} = c_{i_0}^{(0)}$, $\tilde{c}_{j_0} = \tilde{c}_{j_0}^{(0)}$ and applying the reparametrization of the time $dt/d\tau = \frac{t_1}{(r_{\mathbf{t}}+|\mathbf{t}|)c_{i_0}}$, the proof is completed. \square

PROOF OF PROPOSITION 10. Let (a, b) be the vector coefficient of V . The constant $\alpha_V \in (t_2/t_1, s_2/s_1)$ if and only if $ab \neq 0$ and $\frac{s_1b-s_2a}{t_1b-t_2a} < 0$. Thus, applying the Lemma 14 for \mathbf{t} and \mathbf{s} , the system (1) is transformed in (11) with $i_0 = 0$, $j_0 = 0$, $\tilde{c}_0 = s_1b - s_2a$, $c_0 = t_1b - t_2a$ and $\frac{t_1(r_{\mathbf{s}}+|\mathbf{s}|)}{s_2(r_{\mathbf{t}}+|\mathbf{t}|)} \frac{\tilde{c}_{j_0}}{c_{i_0}} < 0$. So, if $\alpha_V \in (t_2/t_1, s_2/s_1)$, the origin is a node which has infinite characteristic orbits defined on the first quadrant. These orbits are transformed, by means of the corresponding blow-down, into characteristic orbits of system (1) which is contained in the region $\epsilon x^{s_2/s_1} \leq y \leq \frac{1}{\epsilon} x^{t_2/t_1}$, hence $W_{\mathbf{t}, \mathbf{s}}^{(0,0)}$ is a parabolic sector.

If $\alpha_V \notin [t_2/t_1, s_2/s_1]$ the origin of (11) is a saddle point and the invariant axes are its characteristic orbits, that is, $W_{\mathbf{t}, \mathbf{s}}^{(0,0)}$ is a hyperbolic sector. \square

PROOF OF PROPOSITION 11. Applying the blow-up of the Lemma 14 for the types \mathbf{t} and \mathbf{s} associated to the adjacent edges of V , we obtain the system (11) with $i_0j_0 = 0$, which is orbitally equivalent to the following systems:

i) For $i_0 > 0, j_0 = 0$,

$$\begin{aligned} u' &= -u, \\ v' &= v \left[\frac{s_2(t_1 m + t_2 n)}{t_1(s_1 m + s_2 n)} \frac{\beta_V}{c_{j_0}^2} v^{i_0 dt_1} + v^{(i_0+1)dt_1} \bar{\Psi}_1(v) + u \bar{\Psi}_2(u^{s_2}, v^{t_1}) \right]. \end{aligned}$$

Next, we prove that if $\beta_V < 0$ then the above system has characteristic orbits in the first quadrant different from the coordinates axes, and if $\beta_V > 0$ then the unique characteristic orbits are the invariant axes.

In fact, this system has a unique vertical isocline $u = 0$ which has an attractive dynamic. The horizontal isocline $v = 0$ has attractive dynamic for $\beta_V < 0$ and repulsive for $\beta_V > 0$. If the system has no other horizontal isocline, then it follows the result since the first quadrant will be the unique sector. Otherwise, if there exist two horizontal isoclines in the first quadrant, both have the same dynamic as the isocline $v = 0$, since there is only one vertical isocline. So, the horizontal dynamic is constant, towards the right if $\beta_V > 0$ and towards the left if $\beta_V < 0$, thus, we obtain the result.

ii) For $j_0 > 0, i_0 = 0$,

$$\begin{aligned} u' &= -u \left[\frac{t_1(s_1 m + s_2 n)}{s_2(t_1 m + t_2 n)} \frac{\beta_V}{c_{i_0}^2} u^{j_0 ds_2} + u^{(j_0+1)ds_2} \bar{\Phi}_1(u) + v^{t_1} \bar{\Phi}_2(u^{s_2}, v^{t_1}) \right], \\ v' &= v. \end{aligned}$$

Reasoning in an analogous way, interchanging x and y , we obtain a similar result to the above one.

iii) For $i_0 = j_0 = 0$,

$$\begin{aligned} u' &= -u \left[\frac{t_1(s_1 m + s_2 n)}{s_2(t_1 m + t_2 n)} \frac{\beta_V}{c_{i_0}^2} + u \bar{\Phi}_3(u) + v \bar{\Phi}_4(u, v) \right], \\ v' &= v. \end{aligned}$$

This system is a node for $\beta_V < 0$ and a saddle for $\beta_V > 0$.

Summarizing, if $\beta_V < 0$ the system (11) is orbitally equivalent to another system which has got a parabolic sector in the first quadrant. Nevertheless, if $\beta_V > 0$ by means of the corresponding blow-down, we have that $W_{\mathbf{t}, \mathbf{s}}^{(0,0)}$ is a hyperbolic sector of (1). \square

PROOF OF PROPOSITION 13. From Proposition 11, if there were characteristic orbits associated to V , as $\beta_V > 0$ these orbits should be in another quadrant different from the first one. Thus, for some $\sigma_1, \sigma_2 \in \{0, 1\}$, $\sigma_1 + \sigma_2 > 0$, it would be $\beta_V^{(\sigma_1, \sigma_2)} < 0$. From (8), $j_0(s_1 \sigma_2 + s_2 \sigma_1) + i_0(t_1 \sigma_2 + t_2 \sigma_1)$ must be odd. In particular, $i_0 + j_0$ is non-zero. We suppose $i_0 = 0$ and $j_0 > 0$ (otherwise, $i_0 > 0$ and $j_0 = 0$ is analogue). So, we have that $j_0(s_1 \sigma_2 + s_2 \sigma_1)$ must be odd and thus both j_0 and $s_1 \sigma_2 + s_2 \sigma_1$ are odd.

Also, as the axis $y = 0$ is not invariant and $h_{r_s + |s|} \neq 0$, we can write $h_{r_s + |s|}(x, y) =$

$\sum_{j=j_0}^{\tilde{k}} \tilde{c}_j x^{m+js_2} y^{n-js_1}$, with $j_0 \leq \tilde{k}$, $\tilde{c}_{j_0} \neq 0$. If we denote $l = \max \{j_0 \leq j \leq \tilde{k} : \tilde{c}_l \neq 0\}$, the factorization of $h_{r_s+|s|}(x, y)$ on $\mathbb{R}[x, y]$ becomes

$$h_{r_s+|s|}(x, y) = \tilde{c}_l x^{m+j_0 s_2} y^{n-l s_1} \prod_{j=0}^M f_j^{m_j} \prod_{i=0}^N g_i^{n_i},$$

where $f_j = y^{s_1} - \lambda_j x^{s_2}$, $g_i = (y^{s_1} - a_i x^{s_2})^2 + b_i^2 x^{2s_2}$.

We assume that n is even; otherwise, it follows the result.

We distinguish the following cases:

- 1) We suppose that l is even. By considering the greatest degree of x in $h_{r_s+|s|}$, we have that $m+l s_2 = m+j_0 s_2 + \sum_{j=0}^M m_j s_2 + \sum_{i=0}^N 2s_2 n_i$, and since $s_2 > 0$, it follows that $l-j_0 = \sum_{j=0}^M m_j + \sum_{i=0}^N 2n_i$. As $l-j_0$ is odd, there would be m_j odd and, therefore, **(b)** holds for the factor $(y^{s_1} - \lambda_j x^{s_2})^{m_j}$.
- 2) We suppose that l is odd. If s_1 is odd, then it holds **(b)** since $h_{r_s+|s|}$ has the factor $y^{n-l s_1}$ with odd multiplicity order, since n is even. Otherwise, s_1 even (so, s_2 odd). If the axis $x = 0$ is not invariant and m even, it has **(b)** since $x^{m+j_0 s_2}$ is a factor of $h_{r_t+|t|}$ with odd multiplicity order. On the other hand, if the axis $x = 0$ is invariant, as $s_2 \sigma_1 + s_1 \sigma_2$ must be odd, it follows that $\sigma_1 = 1$, i.e. the characteristic orbits would be lying on the second or third quadrant. \square

PROOF OF THEOREM 2. Let $(x(t), y(t))$ be a characteristic orbit of system (1) lies on the first quadrant and it is not of the form $x = 0$ or $y = 0$. Without loss of generality we can assume that the orbit arrives at the origin when $t \rightarrow +\infty$. We define the sets

$$\begin{aligned} \Omega &= \left\{ 0 \leq t_2/t_1 \in \mathbb{Q} : \lim_{t \rightarrow +\infty} \frac{y^{t_1}}{x^{t_2}} = 0 \right\}, \\ \Theta &= \left\{ 0 \leq t_2/t_1 \in \mathbb{Q} \cup \{1/0\} : \lim_{t \rightarrow +\infty} \frac{y^{t_1}}{x^{t_2}} = +\infty \right\}. \end{aligned}$$

We observe that both sets are disjoint and non-empty since $0 \in \Omega$ and $1/0 \in \Theta$ because $\lim_{t \rightarrow +\infty} y(t) = 0$ and $\lim_{t \rightarrow +\infty} \frac{1}{x(t)} = +\infty$. The set Θ is lower bounded, therefore there exists $\beta_* = \inf(\Theta)$.

We denote by V_j , $j = 1, \dots, p-1$, the vertex of the Newton diagram of (1) ordered according their abscissas and denote by ℓ_i , $j = 0, \dots, p$, the edges, that is, the exponents satisfy $0 = \alpha_{\ell_0} < \dots < \alpha_{\ell_p} = 1/0$.

First, we assume that β_* is the exponent α_{ℓ} of an edge of the Newton diagram of (1), $\beta_* = t_2/t_1$. We differentiate the following cases: $\beta_* \notin \Omega \cup \Theta$, $\beta_* \in \Omega$ or $\beta_* \in \Theta$.

- If $\beta_* \notin \Omega \cup \Theta$, we have that β_* is a real number non-zero, that is, the edge is bounded.

Applying the change (6), the characteristic orbit is transformed in $(u(t), \theta(t))$

with $\lim_{t \rightarrow \infty} (u(t), \theta(t)) = (0, \theta_*)$, $\theta_* \in [0, T_{\mathbf{t}})$. Therefore, this orbit tends to a equilibrium in $u = 0$.

If $h_{r+|\mathbf{t}|} \equiv 0$, from Proposition 5 the origin is a node; thus, we have **(e1)**.

If $h_{r+|\mathbf{t}|} \not\equiv 0$, from Proposition 7, we have that θ_* is a root of $h_{r+|\mathbf{t}|}(\theta)$ and $y^{t_1} - \tilde{a}x^{t_2}$, with $\tilde{a} = \frac{\text{Sn}^{t_1}(\theta_*)}{\text{Cs}^{t_2}(\theta_*)}$, is a factor of $h_{r+|\mathbf{t}|}(x, y)$. Thus, we have **(e2)**.

• We assume that $\beta_* \in \Omega$. Applying the directional blow-up $x = u^{t_1}$, $y = u^{t_2}\bar{y}$, and rescaling the time by $dt = \frac{t_1}{u^r}d\tau$, the system (1) becomes

$$\begin{aligned} \dot{u} &= \frac{du}{d\tau} = u \sum_{j=0}^{\infty} P_{r+j+t_1}(1, \bar{y})u^j, \\ \dot{\bar{y}} &= \frac{d\bar{y}}{d\tau} = \sum_{j=0}^{\infty} (r + |\mathbf{t}| + j)h_{r+j+|\mathbf{t}|}(1, \bar{y})u^j. \end{aligned} \tag{14}$$

The characteristic orbit $(x(t), y(t))$ of (1) is transformed into the orbit $(u(\tau), \bar{y}(\tau))$ which is a characteristic orbit of system (14) in the positive quadrant, hence

$$\lim_{t \rightarrow +\infty} \frac{y^{t_1}}{x^{t_2}} = \lim_{t \rightarrow +\infty} \frac{u^{t_1 t_2} \bar{y}^{t_1}}{u^{t_1 t_2}} = 0,$$

thus $\lim_{\tau \rightarrow +\infty} (u, \bar{y}) = (0, 0)$. As $\beta_* = \inf(\Theta)$, for all $n \in \mathbb{N}$ it holds that $\lim_{t \rightarrow +\infty} \frac{y^{t_1}}{x^{t_2+1/n}} = +\infty$. In the new variables, $\lim_{\tau \rightarrow +\infty} \frac{\bar{y}}{u^{1/n}} = +\infty$. Thus, there exists a solution of the system (14), different from the trivial solution $u = 0$, that verifies $\bar{y} > u^{1/n}$ in a neighborhood of the origin. Therefore, the solution of the system (1) verifies $y > x^{\frac{t_2}{t_1} + \frac{1}{nt_1}}$ for all $n \in \mathbb{N}$, hence it has the form $y = x^{\frac{t_2}{t_1}} \tau(x)$ with $\lim_{x \rightarrow 0} \tau(x) = 0$ and $\lim_{x \rightarrow 0} \frac{\tau(x)}{x^{1/n}} = +\infty$ for all $n \in \mathbb{N}$.

Let V be the vertex adjacent to the edge α_ℓ with minor ordinate and (a, b) its support. Now we prove that $\alpha_V = \alpha_\ell = t_2/t_1$. Otherwise, $\alpha_V \neq t_2/t_1$, that is $t_1 b - t_2 a \neq 0$. Applying the above directional blow-up, after a reparameterization of the variable time, the system (1) is transformed into the system (14) of the form

$$\begin{aligned} u' &= u[a\bar{y}^{n_0-1} + \dots], \\ \bar{y}' &= (bt_1 - at_2)\bar{y}^{n_0} + \dots \end{aligned}$$

Due to $t_1 b - t_2 a \neq 0$ we have that $V' = (1, n_0)$, the support of the vector field $(au\bar{y}^{n_0-1}, (bt_1 - at_2)\bar{y}^{n_0})^T$, is the upper vertex of the Newton diagram of the system (14), where $\alpha_{V'} = \frac{bt_1 - at_2}{a} \neq 0$. Therefore, there exists $n \in \mathbb{N}$ such that $\alpha_{V'} \notin [0, 1/n]$. From Proposition 10, there is no solution of the system (14) with $\frac{1}{\epsilon}u^{1/n} < \bar{y}$. This leads to a contradiction.

Furthermore, V is an inner vertex, since, otherwise, the support of V would be of the form $(a, 0)$, that is $x = 0$ would be an invariant axis. Therefore, $\alpha_V = \alpha_\ell = t_2/t_1 > 0$, hence $c_0 = t_1 b - t_2 a = 0$, i.e. $i_0 = 0$, $j_0 > 0$.

If $\beta_V > 0$, from Proposition 11, there would be no characteristic orbit in the first quadrant between the sectors $\epsilon x^{\alpha_\ell} \leq y \leq \frac{1}{\epsilon} x^{\alpha_\ell}$. This fact also arrives at a

contradiction and so we have **(v1)**.

- If $\beta_* = t_2/t_1 \in \Theta$, interchanging x and y we have **(v2)**.

Now we assume that β_* is different from all the exponents of the edges of the Newton diagram of system (1). In this case, β_* is a non-zero real number since, otherwise, if $\beta_* = 0$ then for all $n \in \mathbb{N}$ it holds $\lim_{t \rightarrow +\infty} \frac{y^n}{x} = +\infty$, thus $|x(t)| < |y(t)|^n$, i.e. the characteristic orbit is flat at $x = 0$. In such a case, it is easy to prove that $x = 0$ is an invariant axis. So, β_* is an exponent of an edge of the Newton diagram and thus it arrives at a contradiction. In a similar way, we prove that $\beta_* < +\infty$.

So, there is a j with $0 \leq j \leq p-1$ such that $\alpha_{\ell_j} < \beta_* < \alpha_{\ell_{j+1}}$. Next, we prove that $\beta_* = \alpha_{V_j}$. Otherwise, there would be two irreducible fractions $\alpha_{\ell_j} < t_2^{(0)}/t_1^{(0)} < \beta_* < s_2^{(0)}/s_1^{(0)} < \alpha_{\ell_{j+1}}$ with $\alpha_V \notin [t_2^{(0)}/t_1^{(0)}, s_2^{(0)}/s_1^{(0)}]$, and this cannot occur by Proposition 10.

Moreover, V_j is an inner vertex since $0 < \beta_* = \alpha_{V_j} < +\infty$. Therefore, if (a, b) is the vector coefficient of V_j , $\alpha_{\ell_j} = t_2/t_1$, $\alpha_{\ell_{j+1}} = s_2/s_1$, we have that $\beta_* = \alpha_{V_j} \in (\alpha_{\ell_j}, \alpha_{\ell_{j+1}})$. This implies that $t_2/t_1 < b/a < s_2/s_1$, and then $ab > 0$ and $\frac{t_1 b - t_2 a}{s_1 b - s_2 a} < 0$. That is, $ab \neq 0$, $i_0 = j_0 = 0$ and $\beta_V < 0$.

If $\beta_* \in \Theta$, or $\beta_* \in \Omega$ then $y = x^{b/a}\tau(x)$ or $x = y^{a/b}\tau(y)$, where $\lim_{x \rightarrow 0} \tau(x) = 0$ and $\lim_{x \rightarrow 0} \frac{\tau(x)}{x^{1/n}} = +\infty$ for all $n \in \mathbb{N}$, respectively. And if $\beta_* \notin \Omega \cup \Theta$ then $y = \lambda x^{b/a} + \mathcal{O}(x^{b/a})$ where $\lambda \in \mathbb{R} \setminus \{0\}$. So, we have the case **(v3)**. \square

PROOF OF THEOREM 3. Item 1 follows from Proposition 9, since, otherwise, the system would have a characteristic orbit.

We prove item 2. If system (1) does not have two points in the support lying on the axis, then there would be an invariant axis, thus the origin is not monodromic. Moreover, if a and b had the same sign, by parameterizing the time and rescaling the state variables, system (1) would become

$$\begin{aligned} \dot{x} &= y^{2m-1} + y^{2m}\Phi_1(y) + x\Phi_2(x, y), \\ \dot{y} &= x^{2n-1} + x^{2n}\Psi_1(x) + y\Psi_2(x, y), \end{aligned} \quad (15)$$

with $n, m \in \mathbb{N}$. We suppose that the origin of (15) is monodromic. We denote by V_i , $i = 0, \dots, p$, the vertex of the Newton diagram of (15) ordered according their abscissas, i.e. $V_0 = (0, 2m)$ and $V_p = (2n, 0)$, and denote by ℓ_i the edges, that is, the exponents satisfy $\alpha_{\ell_1} < \dots < \alpha_{\ell_p}$. Also, we denote by h_i to the lowest-degree Hamiltonian associated to the type $\mathbf{t}^{(i)} = (t_1^{(i)}, t_2^{(i)})$ of the edge ℓ_i . As the origin is monodromic, from Proposition 8, all the real factors of h_i have even multiplicity order; so, $h_i(x, y) = x^{2m_i}y^{2n_i}\tilde{h}_i(x, y)$, $m_i, n_i \in \mathbb{N}_0$.

On the one hand, both $\tilde{h}_i(1, 0)$ and $\tilde{h}_i(0, 1)$ have the same sign since otherwise there would be any factor with odd order multiplicity. On the other hand, from Proposition 11, $\beta_{V_i} = \tilde{h}_i(1, 0)\tilde{h}_{i+1}(0, 1)$ is positive, $i = 1, \dots, p-1$. Therefore, all coefficients of h_i with greater exponent in x and y have the same sign. But,

this leads to a contradiction since $\tilde{h}_1(0, 1) = -\frac{t_2^{(1)}}{r_1+|t^{(1)}|}$ and $\tilde{h}_p(1, 0) = \frac{t_1^{(p)}}{r_p+|t^{(p)}|}$ have different signs.

Finally, items 3 and 4 follow from Proposition 11 and Corollary 2. \square

PROOF OF THEOREM 4. If system (1) holds items 1-3 then it does not have any invariant axes. Applying Proposition 13, the result follows. \square

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