A note on analytic integrability of planar vector fields

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We give a new characterization of integrability of a planar vector field at the origin. This allows us to prove that the analytic systems

$$\dot{x} = (\frac{\partial h}{\partial y}(x,y)K(h,y^n)+y^{n-1}\Psi(h,y^n))\xi(x,y), \qquad \dot{y} = -\frac{\partial h}{\partial x}(x,y)K(h,y^n)\xi(x,y),$$

where h, K, Ψ and ξ are analytic functions defined in a neighbourhood of O with $K(O) \neq 0$ or $\Psi(O) \neq 0$ and $n \geq 1$, have a local analytic first integral at the origin.

We show new families of analytically integrable systems which are held in the above class. In particular, this class includes all the nilpotent and generalized nilpotent integrable centres which we know.

Key Words: First integral, centre, saddle, monodromic.

1. INTRODUCTION.

One of the most important problems related to the analytic planar systems differential equations

$$(\dot{x}, \dot{y})^T = \mathbf{X}(x,y) = (P(x,y), Q(x,y))^T, \tag{1}$$

is determining when the system has an analytic first integral defined in a neighbourhood of a singular point (we can assume that the origin is the singular point).

We recall that a function H is said to be an analytic first integral of (1) in an open subset U of $\mathbb{R}^2$ if H is a non-constant analytic function in U which is constant on each solution curve of (1) in U. Clearly, in this case, such condition is equivalent to $\nabla H \cdot \mathbf{X} \equiv 0$ in U. We say that a non-zero formal power series $H = a_{00} + a_{10}x + a_{01}y + \cdots$ is a formal first integral of (1) if it satisfies formally that $\partial H/\partial x P(x,y) + \partial H/\partial y Q(x,y) \equiv 0$. If there exists a formal (analytic) first integral of (1) in a neighbourhood of the origin, it says (1) is formally (analytically) integrable. The problem of determining the existence of a first integral is known as integrability problem. For isolated singularities, in the analytic case, according to the result of Mattei & Moussu [20], under generic conditions, both formal and analytic integrability are equivalent.

The existence of a first integral can be used to determine the local phase portrait at an isolated singular point, and in particular to characterize when the monodromic singular point (the orbits of the system close to the isolated singular point turn around it) is a centre or a focus (centre problem). Hence it will be interesting to know when a monodromic point has a first integral around it, since in such a case it is a centre.

It is known that any analytic differential system with a centre at the origin has an analytic first integral defined in a punctured neighbourhood of it, see Li *et al.* [18]. Also, Mazzi & Sabatini [21] prove that any analytic differential system with a centre in a given singular point has a smooth first integral around it. Therefore, another interesting problem will be to recognize when this first integral is, or is not, analytic.

Let us do now a more detailed review of the known results related to both problems. We know that a system can have a monodromic point at the origin only if it has either linear part of centre type, i.e. with imaginary eigenvalues (non-degenerate point), or nilpotent linear part (nilpotent point) or null linear part (degenerate point).
When the system has linear part of centre type, it follows the classical method of seeking a Liapunov function V gives $V = x^2 + y^2 + O(|x, y|^2)$ defined in a neighbourhood of the origin. It is known, see [23], that the function V can be constructed such that $\dot{V}$, its rate of change along trajectories, is of the form $\dot{V} = \eta_2(x^2 + y^2) + \eta_4(x^2 + y^2)^2 + ...$ where η_j are polynomials in the coefficients of the system. The critical point is a centre if and only if all the focal values η_{2j} are zero; in such a case, there exists an analytic first integral defined at the origin (Non-degenerate centre Theorem, due to Poincaré [24] and Liapunov [19]).

If a nilpotent or degenerate monodromic singular point has an analytic first integral then is a centre. However, there are some nilpotent and degenerate centres which are non-integrable. If the linear part of the system is nilpotent, theoretically, Moussu [22] characterizes the centres and Strozyna and Zoladek [25] obtain the orbital normal form of the centres with analytic first integral. Chavarriga *et al.* [6] prove that if a nilpotent centre of an analytic system has a formal (analytic) first integral, then it has a formal (analytic) first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two. This result is used as a tool in order to detect if an analytic nilpotent centre has an analytic first integral or not. Thus, for instance, the system $\dot{x} = y + x^2,\ \dot{y} = -x^3$, has a centre at the origin, but there exists neither an analytic first integral defined at the origin, nor a formal first integral. In that paper, the authors also prove that the monodromic nilpotent systems time reversible under the change of variables $(x, y, t) \to (x, -y, -t)$ have a local analytic first integral at the

origin, see [17] for a short proof of the same result.
Giacomini et al. [12, 13] prove that the analytic nilpotent systems with a centre can be expressed as limit of non-degenerate systems with a centre, and consequently the Poincaré-Liapunov method can be used to find the nilpotent centres.
Others papers related to the analytic integrability problem of nilpotent centre are [7, 8, 9, 15, 10] and related to the nilpotent centres are [3, 11].

Analogously, in general, if the linear part of the system is null, we cannot expect to have integrability at a centre. An example of a degenerate centre (null linear part) which does not have an analytic first integral at the centre is given in [23] p.122. The proof of this fact can be seen in [14]. The above example is a reversible system. Recently, Algaba *et al.*, [1], p.426, Proposition 12, gave a family of systems with a degenerate singular point. This family contains, in particular, non-integrable, non-hamiltonian and non-reversible centres.

On the other hand, there are also integrable systems with a non-monodromic critical point. In this sense, Algaba *et al.* [2], using normal form theory, solve the integrability problem of a family of planar systems which can be expressed as a perturbation of a hamiltonian quasi-homogeneous system with high-degree quasi-homogeneous terms and whose Hamiltonian satisfies some generic conditions.
Others papers related to integrability of degenerate systems are [14, 16].

The next section contains our contribution. We give a new characterization of analytically integrable systems (Theorem 2.1). This fact allows us to find a wide family of systems (Theorem 2.2) which includes, among others, to the above mentioned vector fields. Furthermore, we give some new families of analytically integrable non-degenerate, nilpotent and degenerate systems (Theorem 2.3). We conclude that all the integrable nilpotent or generalized nilpotent centres which we know, belong to the family (3).

2. SOME FAMILIES OF ANALYTICALLY INTEGRABLE PLANAR SYSTEMS.

Now, we prove the main results obtained. The first is a new characterization of analytically integrable systems.

THEOREM 2.1. *System (1) has a local analytic first integral defined in a neighbourhood of O if and only if there exist two analytic functions u and v with $\frac{\partial(u,v)}{\partial(x,y)}(x,y) \neq 0$ in a neighbourhood of O, except possibly in a null Lebesgue measure set, such that $\nabla u \cdot \mathbf{X} = F(u,v)\nabla v \cdot \mathbf{X}$ where F is an analytic function in a neighbourhood of $(u(O), v(O))$.*

Proof. We assume that (1) has an analytic first integral H. Choosing $u = H(x,y)$, v any analytic function defined in a neighbourhood of the origin satisfying $\frac{\partial(u,v)}{\partial(x,y)}(x,y) \neq 0$ except perhaps in a null Lebesgue measure

set, and $F(u,v)\equiv 0$, The result follows.
Conversely, making the change of variables (possibly singular), $u=u(x,y)$, $v=v(x,y)$, the system (1) becomes

$$\dot{u} = \nabla u\cdot \mathbf{X} = F(u,v)\nabla v\cdot \mathbf{X},$$
$$\dot{v} = \nabla v\cdot \mathbf{X}.$$

If $\nabla v\cdot \mathbf{X}\equiv 0$, then $v(x,y)$ is a first integral of $\mathbf{X}$, otherwise, by redefining the time variable by $d\tau = \nabla v\cdot \mathbf{X}dt$ and by denoting $\frac{d}{d\tau} ='$ it turns out that

$$u' = F(u,v), \quad v' = 1. \tag{2}$$

From Cauchy-Arnold's Theorem (see Bruno [5], page 98), the system (2) has an analytic first integral $H(u,v)$ defined in a neighbourhood of $(u(O),v(O))$, that is, $\nabla H\cdot(F,1)(u,v) = \frac{\partial H}{\partial u}F+\frac{\partial H}{\partial v} = 0$. We define $\tilde{H}(x,y) := H(u(x,y),v(x,y))$ which is an analytic function in a neighbourhood N of the origin, since it is a composition of analytic functions. Also, $\nabla\tilde{H}\cdot\mathbf{X}(x,y) = \frac{\partial H}{\partial u}(\nabla u\cdot \mathbf{X}) + \frac{\partial H}{\partial v}(\nabla v\cdot \mathbf{X}) = \nabla v\cdot \mathbf{X}(x,y)[\nabla H\cdot(F,1)(u,v)] = 0$, for all $(x,y)\in N$ (by reducing N if necessary). Moreover, $\tilde{H}$ is non-constant since H is non-constant and $\frac{\partial(u,v)}{\partial(x,y)}(x,y) = 0$ except possibly in a null Lebesgue meausure set. Therefore, $\tilde{H}$ is an analytic first integral of (1) in N. ∎

The next result provides a new class of systems with a local analytic first integral.

THEOREM 2.2. *The family of systems of differential equations*

$$\begin{aligned}\dot{x} &= (\tfrac{\partial h}{\partial y}(x,y)K(h,y^n) + y^{n-1}\Psi(h,y^n))\xi(x,y),\\ \dot{y} &= -\tfrac{\partial h}{\partial x}(x,y)K(h,y^n)\xi(x,y),\end{aligned} \tag{3}$$

where h, K, Ψ and ξ are analytic functions defined in a neighbourhood of O with $n\geq 1$, $\frac{\partial h}{\partial x}(x,y)\not\equiv 0$ and $K(O)\neq 0$ or $\Psi(O)\neq 0$, is analytically integrable at the origin.

Proof. If $\Psi(O)\neq 0$, by taking

$$u = y^n,\ v = h,\ F(u,v) = \frac{-nK(u,v)}{\Psi(u,v)}$$

and by applying Theorem 2.1 the result follows. In the case $\Psi(O) = 0$, then $K(O)\neq 0$, and we can choose

$$u = h,\ v = y^n,\ F(u,v) = \frac{-\Psi(u,v)}{nK(u,v)}.$$

∎

This class is wide enough and includes some systems and families of interest. We now give several subfamilies of (3).

THEOREM 2.3. *The following systems have a local analytic first integral at the origin:*
(a)

$$\begin{aligned}\dot{x} &= x + a_1 y + a_2 xy + a_3 y^2 + a_4 x^2 y + a_5 xy^2 + a_6 y^3,\\ \dot{y} &= -y + (b_1 + b_2 x + b_3 y) y^2,\end{aligned} \tag{4}$$

with a_k and b_k arbitrary constants such that $a_4 + b_2 = 0$.
(b)

$$\begin{aligned}\dot{x} &= y^3 + a_1 x^2 + a_1(4a_4 - b_2) x^2 y^2 + a_3 y^5 + a_4 x^4 y,\\ \dot{y} &= -x^3 - 2a_1 xy + b_2 x^3 y^2 + 2a_1 b_2 xy^3,\end{aligned} \tag{5}$$

with a_1, a_3, a_4, b_2 arbitrary constants.
(c)

$$\dot{x} = y^{n-1}\bar{\Psi}(x^m, y^n), \quad \dot{y} = x^{m-1}\bar{K}(x^m, y^n), \tag{6}$$

with $\bar{\Psi}$ and $\bar{K}$ analytic functions such that $\bar{\Psi}(O) \neq 0$ or $\bar{K}(O) \neq 0$ and $n, m \geq 1$.
(d)

$$\dot{x} = y + \frac{\partial h_{2m}}{\partial y} h_{2m}^p + \sum_{k=0}^{p} a_k h_{2m}^{p-k} y^{2m(k+1)-1}, \quad \dot{y} = -\frac{\partial h_{2m}}{\partial x} h_{2m}^p, \tag{7}$$

where $p \geq 0$, h_{2m} is a homogeneous polynomial of degree $2m$ and a_k arbitrary constants.
(e)

$$\dot{x} = y^{l-1} + \Omega(x^m), \quad \dot{y} = -x^{m-1}, \tag{8}$$

where Ω is an analytic function and $m, l \geq 1$.

Proof.
(a) For $n = 1, \xi \equiv 1,\ h(x, y) = xy$ and

$$\begin{aligned}K(h, y) &= 1 + a_4 h - b_1 y - b_3 y^2,\\ \Psi(h, y) &= a_1 + (a_2 + b_1) h + a_3 y^2 + (a_5 + b_3) hy + a_6 y^3,\end{aligned}$$

in (3) becomes (4). Therefore, the origin of these systems is an integrable saddle point.
(b) Taking $n = 2, \xi \equiv 1$ and

$$\begin{aligned}h(x, y) &= \tfrac{1}{4}x^4 + \tfrac{1}{4}y^4 + a_1 x^2 y,\\ K(h, y^2) &= 1 - b_2 y^2,\\ \Psi(h, y^2) &= 4a_4 h + (a_3 + b_2 - a_4) y^4\end{aligned}$$

in (3), becomes (5).
We note that if $a_1 = 0$ the origin is a monodromic point and therefore it is a centre. Otherwise, O is a saddle point.

(c) For $h(x,y) = x^m,\ \xi \equiv 1,\ \Psi(h,y^n) = \bar{\Psi}(mh,y^n)$ and $-mK(h,y^n) = \bar{K}(h,y^n)$ in (3), becomes (6).
(d) Taking $n = 2, \xi \equiv 1, h(x,y) = h_{2m}(x,y)$ and

$$K(y^2,h) = h_{2m}^p,\ \Psi(y^2,h) = 1 + \sum_{k=0}^{p} a_k h_{2m}^{p-k} y^{2m(k+1)-2}$$

in (3), becomes (7).
(e) Taking $K \equiv 1,\ n = 1,\ \xi \equiv 1$, and

$$h(x,y) = \frac{1}{l}y^l + \frac{1}{m}x^m,\ \Psi(h,y) = \Omega(mh - \frac{m}{l}y^l)$$

in (3), becomes (8). ■

3. CONCLUSIONS.

In the following remarks, we emphasize the wideness of the family (3).
• In [16] Giné studies the systems called generalized nilpotent systems, $(\dot{x},\dot{y})^T = (y^r + P_s, Q_s)^T$ with a formal first integral of the form $y^k + F(x,y)$, where P_s, Q_s are homogeneous polynomials of degree $s = 2,3,4,5,\ r = 1,2,3,\ r < s$ and F starts with terms of order higher than $k = 1,2,3,4,5,6,7$. All are included in the family (6).
In particular, if $m = 1,\ n = 2$ and $\bar{\Psi}(x,y^2) = 1 + f(x,y^2)$, the system (6) turns out to be

$$\dot{x} = y + yf(x,y^2), \quad \dot{y} = \bar{K}(x,y^2). \tag{9}$$

So, the nilpotent systems time reversible under the change of variables $(x,y,t) \to (x,-y,-t)$ are analytically integrable. In the case of a centre, this fact is proved in [6]. If the singular point is non-monodromic, Chavarriga's result is also true. For instance, system

$$\dot{x} = y + yf(x,y^2), \quad \dot{y} = x^2 + g(x,y^2),$$

where f, g are analytic function with $f(O) = g(O) = 0$, has an integrable cuspidal point, see [17].
Furthermore, the integrable systems $(\dot{x},\dot{y})^T = (y + P_n, Q_n)^T$ where P_n, Q_n are homogeneous polynomials of degree $n = 2,3,4,5$ studied in [15], also all belong to the family (9).
• Algaba et al. [3] study the centre problem of the analytic system of differential equations on the plane whose origin is a nilpotent singular point

$$\dot{x} = y + \sum_{i=1}^{\infty} P_{q-p+2is}(x,y), \qquad \dot{y} = \sum_{i=1}^{\infty} Q_{q-p+2is}(x,y), \tag{10}$$

where $p,q,n \in \mathbb{N},\ p \leq q,\ s = (n+1)p - q > 0$, and $\mathbf{F}_i = (P_i, Q_i)^T$ is a quasi-homogeneous vector field of type (p,q) and degree i with $Q_{(2n+1)p-q}(1,0) <$

0 (necessary condition of monodromy). In particular, the authors solve the centre problem for several subfamilies of (10) and show that the centres obtained have an analytic first integral at the origin. We note that these systems with a centre are Hamiltonian, time-reversible (of the form (9)) or can be expressed as systems (3) with K, ξ, Ψ and h polynomials.

• The systems (7) include the analytically integrable nilpotent family given in Lemma 2 of [4].
• The systems (8) are a family of generalized nilpotent systems. If $l = 2$ and $m = 2M$, we have the systems

$$\dot{x} = y + \Omega(x^{2M}), \quad \dot{y} = -x^{2M-1}. \tag{11}$$

Hence, nilpotent systems (11) are analytically integrable.

We conjecture that, in general, the integrable nilpotent systems and the integrable generalized nilpotent systems are included in the family (3).

• If a vector field has a local analytic inverse integrating factor V, i.e. $\nabla V \cdot \mathbf{X} = div(\mathbf{X}).V$, with $V(O) \neq 0$, its associated system can be written as

$$\dot{x} = \frac{\partial H}{\partial y}(x, y)V(x, y), \ \dot{y} = -\frac{\partial H}{\partial x}(x, y)V(x, y),$$

where $H = \int P/V dy + g(x)$ is an analytic first integral with $\frac{\partial g}{\partial x}(x, y) = -Q/V$. These systems are included in family (3), taking $K \equiv 1, \ \Psi \equiv 0, \ h = H$ and $V = \xi$.
Also, a normal form of a non-degenerate centre is

$$(\dot{x}, \dot{y})^T = (-y, x)^T (1 + \sum_{i=1}^{\infty} \beta_j (x^2 + y^2)^j), \tag{12}$$

where β_j are real numbers. This normal form is analytic, since it satisfies Brunos's conditions "A" and "ω", see [5]. Furthermore, it is included in the family (3); concretely, for the case $K \equiv 1, \ \Psi \equiv 0$ and

$$h(x, y) = -\frac{(x^2 + y^2)}{2}, \ \xi(x, y) = 1 + \sum_{i=1}^{\infty} \beta_j (x^2 + y^2)^j.$$

Therefore, the non-degenerate centres have a local analytic first integral (Non-degenerate centre Theorem, see [19, 24]).

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