Quasi-homogeneous linearization of degenerate vector fields

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Abstract

We characterize the analytic planar vector fields orbitally equivalent to its quasi-homogeneous leader term, by means the existence of a class of inverse integrating factors. Such a class of inverse integrating factors is determined by providing a normal form of two-dimensional scalar functions. This fact allows us to give some relevant criteria on $\mathbf{t}$-linearization, analytical integrability and characterization of the centers for several families of planar vector fields. We study the systems whose leader term is quadratic or cubic and we analize also a class of nilpotent systems.

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1. Introduction

A method that seeks to explore the dynamics of a differential system near a fixed point, lies in transforming the system into another whose orbital structure is known. Usually, a technique is to express the vector field as a perturbation of another given and to analyze when both vector fields have the same dynamics. This method has been used often for vector fields with non-null linear part.

In this paper, we deal with analytic or polynomial vector fields in general, i.e. within or without non-null linear part. We apply our results to three wide families of analytic vector fields:
- $(P_n+\cdots)\partial_x+(Q_n+\cdots)\partial_y$, with P_n, Q_n homogeneous polynomials of degree $n\in\{1,2,3\}$ and xQ_n-yP_n having only simple factors on $\mathbb{C}[x,y]$.
- $(-y+dx^m+\cdots)\partial_x+(m\lambda x^{2m-1}+mdx^{m-1}y+\cdots)\partial_y$, with $m>1$ and $\lambda\neq 0$.
- $(-2y+\cdots)\partial_x+((2m+1)\lambda x^{2m}+\cdots)\partial_y$, with m a natural number and $\lambda\neq 0$.

We characterize the systems of these three families orbitally equivalent to their leader vector fields and we give necessary and sufficient conditions of analytical integrability.

We need to recall the concepts of quasi-homogeneous functions, quasi-homogeneous vector fields and inverse integrating factor in order to state our results.

Given $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$, a two-dimensional polynomial f is *a quasi-homogeneous polynomial* of degree j with respect to the type $\mathbf{t}$, it is also named $\mathbf{t}$*-homogeneous polynomial*, if $f(E\mathbf{x}) = \epsilon^j f(\mathbf{x})$, being $E = \operatorname{diag}(\epsilon^{t_1}, \epsilon^{t_2})$ and $\mathbf{x} = (x, y)^T$. We denote by $\mathcal{P}_j^{\mathbf{t}}$ the vector space of all $\mathbf{t}$-homogeneous polynomials.

For any formal scalar function $f \in \mathbb{C}[[x, y]]$ (algebra of the power series in x, y with coefficients in $\mathbb{C}$), by expanding in ϵ, it has that $f(E\mathbf{x}) = \sum_{j \geq s} f_j(\mathbf{x})\epsilon^j$. So, for $\epsilon = 1$ in the above expression, f can be written as

$$f(\mathbf{x}) = f_s(\mathbf{x}) + f_{s+1}(\mathbf{x}) + \cdots, \quad f_j \in \mathcal{P}_j^{\mathbf{t}},$$

the $\mathbf{t}$-homogeneous expansion of f. Such expansions are expressed as $f = f_s +$ q-h.h.o.t., where "q-h.h.o.t." means "$\mathbf{t}$-homogeneous higher order terms". If $\mathbf{t} = (1, 1)$, it is a Taylor expansion. and we write $f = f_s +$ h.o.t..

The use of $\mathbf{t}$-homogeneous expansions of functions allows the management at once of linear and nonlinear terms (monomials with different degrees could have the same $\mathbf{t}$-homogeneous degree). In some sense, the $\mathbf{t}$-homogeneous expansion of a function f suggests that the function can be understood as a perturbation of f_s with higher-degree $\mathbf{t}$-homogeneous terms.

A polynomial vector field $\mathbf{F} = (P, Q)^T$ is a $\mathbf{t}$*-homogeneous vector field* of degree j if $\mathbf{F}(E\mathbf{x}) = \epsilon^j E\mathbf{F}(\mathbf{x})$, or equivalently $P \in \mathcal{P}_{j+t_1}^{\mathbf{t}}$, $Q \in \mathcal{P}_{j+t_2}^{\mathbf{t}}$. We denote by $\mathcal{Q}_j^{\mathbf{t}}$ to the vector space of this class of vector fields.

In the present paper, the sub-index will denote the corresponding quasi-homogeneous degree.

Every $\mathbf{t}$-homogeneus vector field $\mathbf{F}_r = (P_{r+t_1}, Q_{r+t_2})^T \in \mathcal{Q}_r^{\mathbf{t}}$ can be expressed as $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0$ with

$$h_{r+|\mathbf{t}|} = \frac{1}{r + |\mathbf{t}|}(t_1 x Q_{r+t_2} - t_2 y P_{r+t_1}) \text{ and } \mu_r = \frac{1}{r + |\mathbf{t}|}\operatorname{div}(\mathbf{F}_r),$$

where $\mathbf{X}_h$ denotes the Hamiltonian vector field whose Hamiltonian function is $h, \mathbf{X}_h := (-\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x})^T$, $\mathbf{D}_0 := (t_1 x, t_2 y)^T$ a $\mathbf{t}$-homogeneous vector field of zero quasi-homogeneous degree and $|\mathbf{t}| = t_1 + t_2$. This expression is known as *the conservative-dissipative splitting* of a quasi-homogeneous vector field. For more details, we refer the reader to [6].

Two analytic vector fields $\mathbf{F}$ and $\mathbf{G}$ are *orbitally equivalent* if there exist a near-identity formal change of variables and a formal re-parameterization of the time-variable such that one is transformed into the other, i.e. there exist a formal diffeomorphism ϕ and a formal scalar function η, with $\eta(\mathbf{0}) = 1$, such that $\mathbf{G} = \phi_*(\eta\mathbf{F})$, where we denote by ϕ_* the push-forward defined by the diffeomorphism ϕ, $\phi_*\mathbf{F}(\mathbf{y}) := (D\phi(\mathbf{y}))^{-1}\mathbf{F}(\phi(\mathbf{y}))$, see [20].

Definition 1. We say that an analytic vector field is $\mathbf{t}$*-linearizable* if it is orbitally equivalent to its lowest-degree $\mathbf{t}$-homogeneous vector field.

If an analytic vector field has this property, the study of its $\mathbf{t}$-homogeneous leader vector field provides information about the vector field, as for example, the analytic integrability at origin. So, by assuming the $\mathbf{t}$-linearization, if its

$\mathbf{t}$-homogeneous leader vector field is analytically integrable, then the vector field is formally integrable and from Mattei and Moussu [21, Theorem A], it has also an analytic first integral. Concretely, the formal and analytic integrability are equivalent in two-dimension, see [21].

There are some results for specific families. For example, the following systems with non-null linear part are linearizable (orbitally equivalent to its linear part; in this case, we consider $\mathbf{t} = (1, 1)$):
Systems $(-y+\text{h.o.t.})\partial_x + (x+\text{h.o.t.})\partial_y$ which have a center at origin, Poincaré [22].
Systems $(-y + \lambda x + \text{h.o.t.})\partial_x + (x + \lambda y + \text{h.o.t.})\partial_y,\ \lambda \neq 0$, the origin is a strong focus, Sternberg [24].
Analytically integrable systems $(px + \text{h.o.t.})\partial_x + (-qy + \text{h.o.t.})\partial_y,\ p, q$ natural numbers, the origin is a resonant saddle point, Christopher & Rousseau [13], Hang & Jiang [19].

There are results for vector fields with null linear part:
Analytically integrable systems $(xP_2 + \text{h.o.t.})\partial_x + (yQ_2 + \text{h.o.t.})\partial_y$, with P_2, Q_2 quadratic polynomials, perturbations of Kolmogorov cubic systems, Algaba *et al.* [4].

There are also results on $\mathbf{t}$-linearization with $\mathbf{t} \neq (1, 1)$. For example:
Systems $(y + \text{q-h.h.o.t.})\partial_x + (-x^{2m-1} + \text{q-h.h.o.t.})\partial_y$, $\mathbf{t} = (1, m)$, which have a center at origin and they are analytically integrable, Strozyna & Zoladek [25].

Algaba et al. [10] have characterized whether a vector field is $\mathbf{t}$-linearizable by means of the existence of a normalizator of the vector field. We recall that a vector field $\mathbf{G}$ is a normalizator of a vector field $\mathbf{F}$ if there exists a formal scalar function λ such that $[\mathbf{F}, \mathbf{G}] = \lambda\mathbf{F}$, where $[\mathbf{F}, \mathbf{G}] := (D\mathbf{F})\mathbf{G} - (D\mathbf{G})\mathbf{F}$ is the Lie bracket of both vector fields.

For completeness, we show the result.

Theorem 1. *[10, Theorem 1.3] Let $\mathbf{F} = \mathbf{F}_r +$ q-h.h.o.t. be an analytic vector field with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. Then, $\mathbf{F}$ and $\mathbf{F}_r$ are orbitally equivalent if and only if there exists an analytic vector field $\mathbf{G} = \mathbf{D}_0 +$ q-h.h.o.t. and an analytic scalar function μ with $\mu(\mathbf{0}) = r$ verifying $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$.*

Here, we are interested in solving the problem of $\mathbf{t}$-linearization through the existence of a formal inverse integrating factor.

We recall the concepts of invariant curve and inverse integrating factor of a vector field. A formal function f with $f \in \mathbb{C}[[x, y]]$ non-null, is a formal Darboux function or semi-invariant of the vector field $\mathbf{F}$, if there is $K \in \mathbb{C}[[x, y]]$ such that $L_{\mathbf{F}}f = Kf$, being $L_{\mathbf{F}}f := \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q$. A function K is named the *cofactor* of the Darboux function f.
If f is a semi-invariant, we say that $f(x, y) = 0$ is an invariant curve of $\mathbf{F}$.

A non-null function $V \in \mathbb{C}[[x, y]]$ is an *inverse integrating factor* of $\mathbf{F}$ if $V = 0$ is an invariant curve of $\mathbf{F}$ whose cofactor is the divergence of the vector field. This name for V comes from the fact that V^{-1} defines on the set $\{V \neq 0\}$ an integrating factor of $\mathbf{F}$, i.e. $\mathbf{F}/V$ is divergence-free.

The presence of inverse integrating factors of a vector field makes it possible to determine some aspects about the dynamics of the vector field: integrability, existence of limit cycles, among other properties. For more details, see [7, 15, 16, 17, 18].

Algaba et al. [8] study a class of vector fields $\mathbf{F} = \mathbf{X}_h$ + q-h.h.o.t. with $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ whose factorization on $\mathbb{C}[x, y]$ has only simple factors. This class has in it, among others, systems whose origin is a singular point type center-focus. They prove that $\mathbf{F}$ and $\mathbf{X}_h$ are orbitally equivalent if and only if there exists a formal inverse integrating factor $V = h +$ q-h.h.o.t.
In [7], the same result on orbital equivalence is obtained for nilpotent systems of the form $(y +$ q-h.h.o.t$)\partial_x + (\sigma x^n +$ q-h.h.o.t$)\partial_y$. Here, $h = \frac{1}{2}y^2 + \frac{\sigma}{n+1}x^{n+1}$.
In [4], the authors prove that the perturbations of the integrable Kolmogorov cubic systems are **t**-linearizable if there exists a formal inverse integrating factor starting by h.

Recently, Algaba *et al.* [3] have provided the following result which characterizes the **t**-linearizable analytic vector fields, through the existence of a formal inverse integrating factor.

Theorem 2. *[3, Proposition 2.15] Fixed a type* $\mathbf{t} = (t_1, t_2)$, *consider* $\mathbf{F} = \mathbf{F}_r +$ *q-h.h.o.t., with* $\mathbf{F}_r = (P_{r+t_1}, Q_{r+t_2})^T \in \mathcal{Q}^{\mathbf{t}}_r$ *and* $h := t_1 x Q_{r+t_2} - t_2 y P_{r+t_1}$. $\mathbf{F}$ *and* $\mathbf{F}_r$ *are orbitally equivalent if, and only if, there exists* $V = h +$ *q-h.h.o.t. a formal inverse integrating factor of* $\mathbf{F}$ *and there exists a change of variables* ϕ *such that* $V \circ \phi = h$.

The following remark reveals that the existence of a formal inverse integrating factor starting by h does not imply that the vector field is **t**-linearizable, in general.

Remark 1. *Consider the type* $\mathbf{t} = (2, 3)$ *and the functions* $f := y^4 + x^6 + 4cx^2y^3 + 6x^4y^2$ *and* $h := y^4 + x^6$, *i.e.* $f = h +$ *q-h.h.o.t.. The vector field* $(-4y^3 - 12cx^2y^2 - 12x^4y)\partial_x + (6x^5 + 8cxy^3 + 24x^3y^2)\partial_y$ *is Hamiltonian whose Hamiltonian function* f *is an inverse integrating factor. Nevertheless, for* $c \neq \pm 1$, $\mathbf{X}_f$ *and* $\mathbf{X}_h$ *are not orbitally equivalent. Indeed, from [10, Theorem 1.3], if* $\mathbf{X}_f$ *would be orbitally equivalent to* $\mathbf{X}_h$, *there would exist a formal vector field* $\mathbf{G} = (2x + u_{01}y + u_{20}x^2 + u_{11}xy +$ *q-h.h.o.t.*$, 3y + v_{20}x^2 + v_{11}xy + v_{30}x^3 + v_{02}y^2 +$ *q-h.h.o.t.*$)^T$ *and a formal scalar function* $\nu = 7 + d_{10}x + d_{01}y + d_{20}x^2 +$ *q-h.h.o.t. such that* $[\mathbf{X}_f, \mathbf{G}] = \nu \mathbf{X}_f$, *i.e.,*

$$(D\mathbf{X}_f)\mathbf{G} - (D\mathbf{G})\mathbf{X}_f - \nu \mathbf{X}_f = 0. \tag{1}$$

the coefficient of x^2y^2 *of the first component of (1) is* $-12(c + v_{20})$, *therefore equation (1) holds if* $v_{20} = -c$. *In such a case, the coefficient of* x^3y^2 *of the second component is* $-48(c - 1)(c + 1)$. *So, equation (1) is not satisfied for* $c \neq \pm 1$, *i.e.* $\mathbf{X}_f$ *is not orbitally equivalent to* $\mathbf{X}_h$, *for* $c \neq \pm 1$.
From Theorem 2, we can conclude that there are no changes of variables which

transform f into h and, therefore, the existence of a formal inverse integrating factor $V = h +$ q-h.h.o.t. is a necessary condition but it is not sufficient to be the vector field a **t***-linearizable vector field.*

So, in view of Theorem 2, given h a $\mathbf{t}$-homogeneous function, our task is now to determine when it is possible to transform a function $f = h +$ q-h.h.o.t., by means of a change of variables, into h.

Definition 2. We say that two formal (analytic) scalar functions f and g are *conjugated* if there exists a formal diffeomorphism ϕ such that $f \circ \phi = g$, i.e. there exists a change of variables such that f is transformed into g.
Moreover, we say that a function $f = h +$ q-h.h.o.t., with h a $\mathbf{t}$-homogeneous polynomial, is **t**-*linearizable* if f and h are conjugated.

The following result provides a necessary and sufficient condition, by means of Poisson symmetries, so that a scalar function can be transformed into its lowest-degree $\mathbf{t}$-homogeneous term. We note that this result is true for n-dimensional scalar function.

Theorem 3. *Assume that $f = f_s +$ q-h.h.o.t. is a formal (analytic) function with $f_s \in \mathcal{P}_s^{\mathbf{t}}$. Then, f and f_s are conjugated if and only if there exists a formal (analytic) vector field $\mathbf{G} = \mathbf{D}_0 +$ q-h.h.o.t. such that $L_{\mathbf{G}} f = sf$.*

The proof of Theorem 3 is given in Section 5.

The following result provides a polynomial normal form of two-dimensional functions whose lowest-degree $\mathbf{t}$-homogeneous term has only simple factors.

Theorem 4 (Polynomial Normal Form of a scalar two-dimensional function)**.** *Let $f_s \in \mathcal{P}_s^{\mathbf{t}}$ be a quasi-homogeneous polynomial whose factorization on $\mathbb{C}[x, y]$ has only simple factors. Then, given a formal function $f = f_s +$ q-h.h.o.t., there exists a diffeomorphism ϕ such that $f \circ \phi$ is a polynomial and it is given by*

$$f \circ \phi = f_s + \sum_{k=1}^{m_0+s-|\mathbf{t}|} g_{s+k}, \tag{2}$$

with $g_{s+k} \in Cor(\tilde{\ell}_{s+k})$, a complementary subspace to the range of the linear operator $\tilde{\ell}_{s+k}$. The definition of the operator $\tilde{\ell}_{s+k}$ can be seen in Definitions 4 and 5, and $m_0 \in \mathbb{N}$ is established by Lemma 17.

The proof of Theorem 4 is given in Section 6.

Next two sections contain some relevant results on $\mathbf{t}$-linearization, the center problem and integrability. The joint application of Theorems 2 and 4 will allow us, by means of the existence of a formal inverse integrating factor, to characterize the $\mathbf{t}$-linearization, the analytic integrability and to study the center problem for a wide family of differential systems. This class contains to the families studied in [4, 13, 19, 22, 24, 25], cited in the page 3.

2. Perturbations of a class of linear, quadratic and cubic homogeneous systems

We consider the analytic vector fields

$$(P_n + \text{h.o.t.})\partial_x + (Q_n + \text{h.o.t.})\partial_y, \tag{3}$$

with P_n, Q_n homogeneous polynomials of degree $n \in \{1, 2, 3\}$ and $xQ_n - yP_n$ having only simple factors on $\mathbb{C}[x, y]$. Note that P_n or Q_n can be identically zero.

The following result gives a normal form of the perturbations of a homogeneous polynomial of degree less than or equal to 4 and whose factorization on $\mathbb{C}[x, y]$ has only simple factors.

Theorem 5. *Let h be a homogeneous polynomial of degree $n \leq 4$ and having only simple factors. A normal form of $f = h + h.o.t.$, a perturbation of h, is h.*

Proof. We consider a formal function $f = h + \text{h.o.t.}$. For all $k \in \mathbb{N}_0$, the sets $\mathcal{P}_k^{(1,1)}$ are non-trivial sets, therefore $m_0 = -1$ and $m_0 + s - |\mathbf{t}| = n - 3$.
For $n \in \{1, 2, 3\}$, we have that $m_0 + s - |\mathbf{t}| \leq 0$, and by Theorem 4, as h has only simple factors, a normal form of f is h.
For $n = 4$, we have that $m_0 + s - |\mathbf{t}| = 1$. By Theorem 4, a normal form of h+h.o.t. is $h + q_5(x, y)$ with $q_5(x, y) = c_{50}x^5 + c_{41}x^4y + c_{32}x^3y^2 + c_{23}x^2y^3 \in \operatorname{Cor}(\tilde{\ell}_5)$, a complementary subspace to the range of the operator linear $\tilde{\ell}_5$ defined by

$$\begin{aligned} \tilde{\ell}_5 \quad : \quad & \Delta_3^{\mathbf{t}} \longrightarrow \Delta_5^{\mathbf{t}} \\ & g \to \tilde{\ell}_5(g) := \operatorname{Proj}_{\Delta_5^{\mathbf{t}}}(P_3 g_x + Q_3 g_y), \end{aligned}$$

with $\Delta_3^{\mathbf{t}} = \mathcal{P}_3^{\mathbf{t}} = \langle x^3, x^2y, xy^2, y^3 \rangle$ and $\Delta_5^{\mathbf{t}}$ a complementary subspace to $h\mathcal{P}_2^{\mathbf{t}}$. By means of a linear change of variables, we can assume that h is one of the following polynomials:
- $xy(y - x)(y - cx)$, with $c \neq 0, c \neq 1$,
- $xy(y^2 + d^2x^2)$, with $d \neq 0$,
- $(y^2 + x^2)(y^2 + d^2x^2)$, with $d \neq 0, d \neq \pm 1$.

We analyze each case separately:
- $h = xy(y - x)(y - cx)$, with $c \neq 0, c \neq 1$.

We choose $\Delta_5^{\mathbf{t}} = \langle x^5, x^4y, x^3y^2, y^5 \rangle$. The transformed by $\tilde{\ell}_5$ of the base are:

$$\begin{aligned} &\tilde{\ell}_5(x^3) = -3cx^5 + 6(c + 1)x^4y - 9x^3y^2, \\ &\tilde{\ell}_5(x^2y) = 6cx^4y - 3(c + 1)x^3y^2, \\ &\tilde{\ell}_5(xy^2) = 3(c + 1)cx^4y - 3(c^2 + 1)x^3y^2, \\ &\tilde{\ell}_5(y^3) = 3(2c^2 + c + 2)cx^4y - 3(c + 1)(2c^2 - c + 2)x^3y^2 + 3y^5. \end{aligned}$$

The determinant of the matrix whose coefficients are the coefficients of the transformed of the base is not zero for $c \neq 0$ and $c \neq 1$, i.e. $\tilde{\ell}_5(\Delta_3^{\mathbf{t}})$ has full range and $\operatorname{Cor}(\tilde{\ell}_5) = \{0\}$. So, by Theorem 4, h is a normal form of any function $h + \text{h.o.t.}$.

• $h = xy(y^2 + d^2x^2)$, with $d \neq 0$.
In this case, a base of $\Delta_5^{\mathbf{t}}$ is $\{x^5, x^4y, x^3y^2, y^5\}$. The transformed by $\tilde{\ell}_5$ of the base are:

$$\begin{array}{ll} \tilde{\ell}_5(x^3) = -3d^2x^5 - 9x^3y^2, & \tilde{\ell}_5(x^2y) = 6d^2x^4y, \\ \tilde{\ell}_5(xy^2) = 6d^2x^3y^2, & \tilde{\ell}_5(y^3) = -9d^4x^4y + 3y^5. \end{array}$$

Again, $\tilde{\ell}_5(\Delta_3^{\mathbf{t}})$ has full range for $d \neq 0$. Therefore, we conclude that $\mathrm{Cor}(\tilde{\ell}_5) = \{0\}$.
• $h = (y^2 + x^2)(y^2 + d^2x^2)$, with $d \neq 0, d \neq \pm 1$.
In this case, we choose $\Delta_5^{\mathbf{t}} = \langle x^5, x^4y, x^3y^2, x^2y^3 \rangle$. The transformed by $\tilde{\ell}_5$ of the base are:

$$\begin{array}{l} \tilde{\ell}_5(x^3) = -6(d^2+1)x^4y - 12x^2y^3, \\ \tilde{\ell}_5(x^2y) = 12d^2x^5 + 6(d^2+1)x^3y^2, \\ \tilde{\ell}_5(xy^2) = 12d^2x^4y + 6(d^2+1)x^2y^3, \\ \tilde{\ell}_5(y^3) = -6(d^2+1)d^2x^5 - 6(d^4+1)x^3y^2. \end{array}$$

We have that $\tilde{\ell}_5(\Delta_3^{\mathbf{t}})$ has full range if $d \neq \pm 1$.
Consequently, by Theorem 4, if h is a homogeneous polynomial of degree 4 and having only simple factors, we have that h is a normal form of any perturbations of h. ♢

The following result characterizes the $\mathbf{t}$-linearizable systems (3) through the existence of a formal inverse integrating factor starting by $xQ_n - yP_n$.

Theorem 6. *The analytic vector field* $(P_n + h.o.t.)\partial_x + (Q_n + h.o.t.)\partial_y$*, with* P_n, Q_n *homogeneous polynomials of degree* $n \in \{1, 2, 3\}$ *with* $h = xQ_n - yP_n$ *having only simple factors on* $\mathbb{C}[x, y]$*, is orbitally equivalent to* $P_n\partial_x + Q_n\partial_y$ *if, and only if, it has a formal inverse integrating factor* $h + h.o.t.$*.*

Proof. On the one hand, by Theorem 2, the vector field is $(1, 1)$-linearizable, if, and only if, there exists a formal inverse integrating factor $V = h +$ h.o.t. such that V and h are conjugated. On the other hand, by Theorem 5, any perturbation of h is conjugated to h. So, the result follows. ♢

We give the following criterium of linearization and analytical integrability of resonant saddle, in this case $(1, 1)$-linearization, by using an inverse integrating factor.

Theorem 7. *Consider the analytic vector field* $(-px + h.o.t.)\partial_x + (qy + h.o.t.)\partial_y$*, with* p, q *natural numbers. The following statements are satisfied:*
(i) The vector field is linearizable if, and only if, it has a formal inverse integrating factor $xy + h.o.t.$*.*
(ii) The vector field is analytically integrable at the origin if, and only if, it has a formal inverse integrating factor $xy + h.o.t.$*.*

Proof. The vector field is a vector field (3) with $n = 1$ and the Hamiltonian part of the lowest-degree term of the vector field is $h = (p + q)xy$. As h has only simple factors, by applying Theorem 6, item *(i)* follows.
We prove the second part. We see the necessary condition. By Christopher &

Rousseau [13], a resonant saddle is analytically integrable at the origin if, and only if, it is linearizable. Moreover, it has a first integral $x^q y^p$ + h.o.t.. The result follows by applying item *(i)*.
We prove the sufficiency. We assume that the vector field has a formal inverse integrating factor starting with xy. From item *(i)*, it is orbitally equivalent to its leader vector field $-px\partial_x + qy\partial_y$, it which has the first integral $x^q y^p$. Therefore, we can assure that the vector field is formally integrable. The analytic integrability follows by applying [21, Theorem A].

♢

As a straight consequence of Theorem 6, we obtain the following criterium of linearization of vector fields whose origin is an isolated singular point type center-focus or non-degenerate saddle. We give also a criterium to solving the non-degenerate center problem.

Theorem 8. *Consider the analytic vector field* $(-y + dx + h.o.t.)\partial_x + (\lambda x + dy + h.o.t.)\partial_y$*, with* $\lambda \neq 0$*. The following statements are satisfied:*
(i) The vector field is linearizable if, and only if, it has a formal inverse integrating factor $y^2 + \lambda x^2 + h.o.t.$*.*
(ii) The origin of vector field is a center if, and only if, $d = 0$, $\lambda > 0$ *and it has a formal inverse integrating factor* $y^2 + \lambda x^2 + h.o.t.$*.*

Proof. For $\mathbf{t} = (1, 1)$, the vector field is $\mathbf{F}_0$ + h.o.t. with $h = y^2 + \lambda x^2$. Item *(i)* follows from Theorem 6 for $n = 1$.
We prove item *(ii)*. We see the necessity. It is well-known, that if $d \neq 0$ the origin is a strong focus, and if $\lambda < 0$ the origin is not monodromic. Otherwise, by Poincaré [22], the origin is a non-degenerate center if, and only if, the vector field is linearizable. Therefore, by applying item *(i)*, the result follows.
The sufficiency follows from item *(i)* and [21, Theorem A]. ♢

From Theorem 7 and Theorem 8, if we want to study the analytic integrability of a resonant saddle point or a center-focus point, we can replace the existence of a first integral by the existence of a formal inverse integrating factor xy + h.o.t. or $x^2 + y^2$ + h.o.t.. That is an improvement since the first integral is a particular class of inverse integrating factor (when its cofactor is zero). Indeed, the domain of definition and the regularity of an inverse integrating factor usually are larger than that of the first integral, see [12, 23].

For analytic vector fields $(-y + \text{h.o.t.})\partial_x + (x + \text{h.o.t.})\partial_y$, i.e $h = x^2 + y^2$, Poincaré [22] proved that the vector field is analytically integrable if, and only if, it is linearizable. We generalize this result for analytic vector fields with h homogeneous polynomial of degree $n \leq 4$ and having only simple factors on $\mathbb{C}[x, y]$.

Theorem 9. *Consider the analytic vector field* $(-h_y + h.o.t.)\partial_x + (h_x + h.o.t.)\partial_y$ *with* h *a homogeneous polynomial of degree* n *having only simple factors and* $n \in \{3, 4\}$*. The origin is analytically integrable if, and only if, the vector field is orbitally equivalent to* $-h_y\partial_x + h_x\partial_y$*, or equivalently, it has a formal inverse integrating factor* $V = h + h.o.t.$*.*

Proof. The vector field is $\mathbf{F} = \mathbf{X}_h + \text{h.o.t.}$, i.e. it is a perturbation of a Hamiltonian vector field whose Hamiltonian function is a quasi-homogeneous polynomial h having only simple factors.

Let us prove the necessary condition. Assume that the origin is analytically integrable. From [2, Theorem 3.19], $\mathbf{F}$ is orbitally equivalent to a Hamiltonian vector field $\mathbf{X}_H$ with $H = h+$q-h.h.o.t.. Moreover, by Theorem 5, we have that H and h are conjugated, i.e. there exists a near-identity change of variables $\mathbf{x} = \phi(\mathbf{u})$ such that $H(\mathbf{u}) = h(\phi(\mathbf{u}))$. This change transforms $\mathbf{X}_{h\circ\phi}(\mathbf{u}) = (\det D\phi(\mathbf{u}))\phi_*\mathbf{X}_h(\mathbf{u})$, i.e., $\mathbf{X}_H$ and $\mathbf{X}_h$ are orbitally equivalent.

We prove the the sufficient condition. The vector field $\mathbf{F}$ is orbitally equivalent to $\mathbf{X}_h$ whose origin is analytically integrable (h is a first integral). Therefore, the origin of $\mathbf{F}$ is formally integrable and from [21, Theorem A] $\mathbf{F}$ is analytically integrable also.

The existence of a formal inverse integrating factor $V = h + \text{h.o.t.}$ is given by Theorem 6. $\diamondsuit$

The following remark shows that Theorem 6 is not verified for $n = 4$ and therefore, it reveals the importance of the above results and the complexity of the problem we are discussing.

Remark 2. *For $n \geq 4$, Theorem 6 is not satisfied. For example, the vector field $(-y^4 - x^3y^2)\partial_x + (x^4 + x^2y^3)\partial_y$ has the polynomial inverse integrating factor $V = y^5 + x^5 + \frac{5}{3}x^3y^3$. However, it is not linearizable. Indeed, from [10, Theorem 1.3], if the vector field would be linearizable, there would exist a formal vector field $\mathbf{G} = (x + u_{20}x^2 + u_{11}xy + u_{02}y^2 +$ q-h.h.o.t.$, y + v_{20}x^2 + v_{11}xy + v_{02}y^2 +$ q-h.h.o.t.$)^T$ and a formal scalar function $\nu = 3 + d_{10}x + d_{01}y +$ q-h.h.o.t. such that $[(-y^4 - x^3y^2, x^4 + x^2y^3)^T, \mathbf{G}] = \nu(-y^4, x^4)^T$. It is enough to check that the coefficients of x^3y^2 of the first component of each side are not equal. Therefore, the vector field is not orbitally equivalent to its lowest-degree term.*

3. Systems whose quasi-homogeneous leader term is $\mathbf{X}_h + \mu\mathbf{D}_0$ with $h = y^2 + \lambda x^n$, with $n > 2$ and $\lambda \neq 0$.

Next, we study the problem for the analytic vector fields whose lowest-degree quasi-homogeneous vector field has the associated Hamiltonian function $h = y^2 + \lambda x^n$ with $\lambda \neq 0$. We note that for n even and $\lambda > 0$, the origin is a monodromic point. Thus, there are families of this class of vector fields whose origin is a center.

We compute the normal form of a perturbation of $h = y^2 + \lambda x^n$ with $\lambda \neq 0$.

Lemma 10. *Let $f = h +$ q-h.h.o.t. with $h = y^2 + \lambda x^n$ with $\lambda \neq 0$ and $n > 2$. A normal form of f is h.*

Proof. According to the evenness of n, we distinguish two cases:

Case n even. So, $h = y^2 + \lambda x^{2m} \in \mathcal{P}_{2m}^{\mathbf{t}}$ with $\mathbf{t} = (1, m)$.

As the factorization of h on $\mathbb{C}[x, y]$ has only simple factors, by applying Theorem 4, where $m_0 = -1, |\mathbf{t}| = 1 + m$ and $s = 2m$, we have that a normal form of

$f = h +$ q-h.h.o.t. is of the form $h + \sum_{j=1}^{m-2} g_{2m+j}$, with $g_{2m+j} \in \mathrm{Cor}(\tilde{\ell}_{2m+j})$, a complementary subspace to the range of the linear operator $\tilde{\ell}_{2m+j}$ defined by

$$\begin{aligned} \tilde{\ell}_{2m+j} \quad &: \quad \Delta^{\mathbf{t}}_{1+m+j} \longrightarrow \Delta^{\mathbf{t}}_{2m+j} \\ &\qquad g_{1+m+j} \to \tilde{\ell}_{2m+j}(g_{1+m+j}) := \mathrm{Proj}_{\Delta^{\mathbf{t}}_{2m+j}} L_{\mathbf{X}_h} g_{1+m+j}. \end{aligned}$$

For $j = 1, \ldots, m-2$, we have that $\Delta^{\mathbf{t}}_{1+m+j} = \mathrm{span}\{x^{1+m+j}, x^{1+j}y\}$. The transformed of the basis are

$$\tilde{\ell}_{2m+j}(x^{1+m+j}) = -(1+m+j)x^{m+j}y, \;\; \tilde{\ell}_{2m+j}(x^{1+j}y) = (1+m+j)\lambda x^{2m+j}.$$

Therefore, $\mathrm{Cor}(\tilde{\ell}_{2m+j}) = \{0\}$ and h is a normal form of f.
Case n odd, that is, $h = y^2 + \lambda x^{2m+1} \in \mathcal{P}^{\mathbf{t}}_{4m+2}$ with $\mathbf{t} = (2, 2m+1)$.
We compute a normal form of a perturbation of h, $f = h+$q-h.h.o.t.. In this case, $m_0 = 2m-1, |\mathbf{t}| = 2m+3$ and $s = 4m+2$, thus, by Theorem 4, a polynomial normal form is $y^2 + \lambda x^{2m+1} + \sum_{j=1}^{4m-2} g_{4m+2+j}$, with $g_{4m+2+j} \in \mathrm{Cor}(\tilde{\ell}_{4m+2+j})$.
We distinguish two cases:
If $j = 2k,\ 0 \leq k \leq m-1$, we have that $\Delta^{\mathbf{t}}_{2m+1+2(k+1)} = \mathrm{span}\{x^{1+k}y\}$. The transformed of the basis is $\tilde{\ell}_{4m+2+j}(x^{1+k}y) = (3+2m+2k)x^{2m+1+k}$.
If $j = 2k-1,\ 0 \leq k \leq m-1$, we have that $\Delta^{\mathbf{t}}_{2m+2+2k} = \mathrm{span}\{x^{1+m+k}\}$. The transformed of the basis is $\tilde{\ell}_{4m+2+j}(x^{1+m+k}) = -2(1+m+k)x^{m+k}y$.
Therefore, $\mathrm{Cor}(\tilde{\ell}_{4m+2+j}) = \{0\}$ and h is a normal form of f. $\diamondsuit$

We provide a criterium on linearization for this class of nilpotent vector fields. This result generalizes to [7, Theorem 4].

Theorem 11. *The following statements are satisfied:*
(i) Fixed the type $(1, m)$, *with* m *a natural number* $m > 1$, *the analytic vector field* $(-y + dx^m +$ *q-h.h.o.t.*$)\partial_x + (m\lambda x^{2m-1} + mdx^{m-1}y +$ *q-h.h.o.t.*$)\partial_y$, *with* $\lambda \neq 0$, *is* $\mathbf{t}$*-linearizable if, and only if, it has a formal inverse integrating factor* $y^2 + \lambda x^{2m} +$ *q-h.h.o.t..*
(ii) Fixed the type $(2, 2m+1)$, *with* m *a natural number, the analytic vector field* $(-2y +$ *q-h.h.o.t.*$)\partial_x + ((2m+1)\lambda x^{2m} +$ *q-h.h.o.t.*$)\partial_y$, *with* $\lambda \neq 0$, *is* $\mathbf{t}$*-linearizable if, and only if, it has a formal inverse integrating factor* $y^2 + \lambda x^{2m+1} +$ *q-h.h.o.t..*

Proof. We prove statement *(i)*. The analytic vector field, for the type $\mathbf{t} = (1, m)$, is $\mathbf{F}_{m-1} +$ q-h.h.o.t. with $\mathbf{F}_{m-1} = (-y + dx^m, m\lambda x^{2m-1} + mdx^{m-1}y)^T$, i.e. $h = y^2 + \lambda x^{2m} \in \mathcal{P}^{\mathbf{t}}_{2m}$.
By Lemma 10 and Theorem 2, we have the first part.

We prove *(ii)*. For the type $\mathbf{t} = (2, 2m+1)$, the analytic vector field is $\mathbf{F}_{2m-1} +$ q-h.h.o.t.. with $\mathbf{F}_{2m-1} = (-2y, (2m+1)\lambda x^{2m})^T$, that is, $h = y^2 + \lambda x^{2m+1} \in \mathcal{P}^{\mathbf{t}}_{4m+2}$. From Lemma 10 and Theorem 2, the resul follows. $\diamondsuit$

Strozyna & Zoladek [25] have proved that analytically integrable centers of $(-y +$ q-h.h.o.t.$)\partial_x + (x^{2m-1} +$ q-h.h.o.t.$)\partial_y$ are $\mathbf{t}$-linearizable. Thus, as a consequence of Theorem 11 and the work of Strozyna & Zoladek, we have the following result.

Theorem 12. *Fixed the type* $\mathbf{t} = (1, m), m > 1$, *the origin of* $(-y + q\text{-}h.h.o.t.)\partial_x + (x^{2m-1} + q\text{-}h.h.o.t.)\partial_y$ *is an analytically integrable center if, and only if, it has an inverse integrating factor* $my^2 + x^{2m} + q\text{-}h.h.o.t.$.

4. Applications

The existence of a formal inverse integrating factor $h \circ \phi$ reduces the study of the analytic integrability and the characterization of the centers of a family of analytic vector fields to the study of both problems for the $\mathbf{t}$-homogeneous leader vector field.

Consider the following six-parameter family

$$x(2y - x + a_{20}x^2 + a_{11}xy + a_{02}y^2)\partial_x + y(2x - y + b_{20}x^2 + b_{11}xy + b_{02}y^2)\partial_y. \quad (4)$$

This family is a perturbation of a Lotka-Volterra system and the edges $x = 0$ and $y = 0$ are invariant curves.

We fix the type $\mathbf{t} = (1, 1)$, i.e. we consider Taylor expansions. The first homogeneous component of the vector field is $(x(2y-x),\ y(2x-y))^T$. Therefore, $h = xy(x - y)$. The vector field is polynomially integrable and a primitive first integral is h.

The following result solves the analytical integrability problem at the origin for this family.

Theorem 13. *System (4) is analytically integrable if, and only if, one of the following conditions holds:*
(i) $a_{02} = b_{02} = a_{11} - 2b_{11} - 2b_{20} = a_{20} + b_{20} + b_{11} = 0$,
(ii) $a_{02} = b_{02} = a_{11} + b_{11} = b_{20} + 3a_{20} = 0$,
(iii) $b_{11} = a_{11} + a_{02} - 2b_{20} - 3b_{02} = a_{20} + b_{20} + 2b_{02} = a_{02}b_{20} + 2a_{02}b_{02} + b_{20}b_{02} = 0$,
(iv) $b_{11} = a_{11} = a_{02} + 3b_{02} = 3a_{20} + b_{20} = 0$,
(v) $a_{02} = b_{11} + 3b_{02} = b_{20} + b_{02} = a_{11} + 2b_{02} = a_{20} - b_{02} = 0$,
(vi) $b_{20} = a_{02} = a_{11} - 2b_{11} - 3b_{02} = a_{20} + b_{11} + 2b_{02} = 0$,
(vii) $b_{20} = a_{02} = a_{11} + b_{11} = a_{20} + b_{02} = 0$.

Proof. We prove the necessary condition. A Taylor expansion of a first integral of system (4) is $I = \sum_{j\geq 3} I_j$ with $I_3 := h = xy(x - y)$. By imposing that $F(I) = 0$ up to order 10, we obtain six conditions on the parameters $a_{20}, a_{11}, a_{02}, b_{20}, b_{11}$ and b_{02}. They are polynomials too long, so we do not given them here. Moreover, the irreducible decomposition of the variety of the ideal generated in assuming the conditions has been obtained using the computer algebra system SINGULAR [14]. This decomposition leads us to the systems (4) for the cases *(i)-(vii)*.

We prove the sufficiency. The leader term is integrable. Therefore, in order to prove the integrability of the vector field, it is enough to find an inverse integrating factor starting by $h = xy(x - y)$ since, by applying Theorem 6, the vector field is linearizable.

We compute an inverse integrating factor for each case:
The polynomial $V = xy(x-y)(1+b_{20}x+b_{11}x)$ is an inverse integrating factor of system (4) for the case *(i)*.
System (4) for the case *(ii)* is an Hamiltonian vector field. Therefore, the Hamiltonian function $V = xy(x-y-a_{20}x^2+\frac{b_{11}}{2}xy)$ is a first integral and also is an inverse integrating factor.
The polynomial $V = xy(x-y)(1+2b_{02}x+b_{20}x-b_{02}y)$ is an inverse integrating factor of system (4) for the case *(iii)*.
System (4) for the case *(iv)* is an Hamiltonian vector field. Therefore, the Hamiltonian function $V = xy(x-y-a_{20}x^2+b_{02}y^2)$ is a first integral and also is an inverse integrating factor.
The function

$$V = xy(x-y-b_{02}x^2+\frac{b_{02}}{2}xy)(1-2b_{02}y-2b_{02}^2xy+b_{02}^2y^2)^{1/2}$$

is an inverse integrating factor of system (4) for the case *(v)*.
The polynomial $V = xy(x-y)(1+b_{11}x+2b_{02}x-b_{02}y)$ is an inverse integrating factor of system (4) for the case *(vi)*.
The polynomial

$$V = xy(x-y+\frac{b_{11}}{2}xy+\frac{b_{02}}{2}xy)(1+b_{02}x-b_{02}y+b_{02}b_{11}xy+b_{02}^2xy)$$

is an inverse integrating factor of system (4) for the case *(vii)*.
This completes the proof. $\diamondsuit$

Last on, we apply our results for solving the center problem of the family

$$y\partial_x + (-x^3+xy+b_{40}x^4+b_{21}x^2y+b_{50}x^5)\partial_y. \tag{5}$$

The quasi-homogeneous expansion of the associated vector field to system (5) with respect to the type $\mathbf{t}=(1,2)$ is $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3$, with $\mathbf{F}_1=(y,-x^3+xy)^T$, $\mathbf{F}_2=(0,b_{40}x^4+b_{21}x^2y)^T$ and $\mathbf{F}_3=(0,b_{50}x^5)^T$.

Notice that the lowest-degree term of $\mathbf{F}$ is not Hamiltonian. Moreover, from Algaba et al. [9], the origin of (5) is a monodromic singular point (a focus or a center) since $h_2=-2y^2+x^2y-x^4=-2[(y-\frac{1}{4}x^2)^2+\frac{7}{16}x^4]$ is defined at the origin.

Next, we characterize the centers of system (5).

Theorem 14. *The origin of system (5) is a center if, and only if, one of the following series of conditions is satisfied:*
(i) $b_{40}=b_{21}=0$,
(ii) $3b_{40}+5b_{21}=3b_{50}+2b_{21}^2=0$.

Proof. We prove the necessary condition. An orbitally equivalent normal form of the vector field (5), up to order 6, is given by

$$y\partial_x+(-x^3+xy+b_3^*x^2y+b_5^*x^4y)\partial_y.$$

Moreover, the coefficients b_3^* and b_5^* cannot be eliminated, see page 423 of Algaba et al. [1].

From Berthier and Moussu [11], if the origin is a center of $\mathbf{F}$ then the two first coefficients are zero, $b_3^* = b_5^* = 0$.

The first coefficient is $b_3^* = 5b_{21}+3b_{40}$. For $b_{40} = -\frac{5}{3}b_{21}$, the second coefficient is $b_5^* = -\frac{5}{9}b_{21}(2b_{21}^2 + 3b_{50})$. Therefore, if the origin of (5) is a center then *(i)* or *(ii)* are satisfied.

We see the sufficient condition. The origin of (5) under the condition *(i)* is a center since it is a monodromic singular point and (5) is time-reversible.

The vector field (5) with $b_{40} = -\frac{5}{3}b_{21},\ b_{50} = -\frac{2}{3}b_{21}^2$ has the inverse integrating factor $V = h_2 + \frac{2}{3}b_{21}x^3y - \frac{4}{3}b_{21}x^5 - \frac{4}{9}b_{21}^2x^6$.

From Theorem 6, $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_1$, and the origin of $\mathbf{F}_1$ is a center because it is a monodromic singular point and the vector field $\mathbf{F}_1$ is time-reversible. Thus, system (5) has a center at the origin. ♢

5. Proof of Theorem 3

Proof of Theorem 3. We prove the necessary condition. We assume that f and f_s are conjugated, that is, there exists a formal (analytic) near-identity ϕ such that $f = f_s \circ \phi$. Consider the vector field $\mathbf{G} = \phi_* \mathbf{D}_0$. It is of the form $\mathbf{G} = \mathbf{D}_0 +$ q-h.h.o.t.. We have that

$$\begin{aligned} L_{\mathbf{G}} f(\mathbf{x}) &= L_{\phi_* \mathbf{D}_0}(f_s(\phi))(\mathbf{x}) \\ &= \nabla f_s(\phi(\mathbf{x})) D\phi(\mathbf{x}) \cdot (D\phi)^{-1}(\mathbf{x}) \mathbf{D}_0(\phi(\mathbf{x})) \\ &= L_{\mathbf{D}_0} f_s(\phi(\mathbf{x})). \end{aligned} \tag{6}$$

By the Euler Theorem for quasi-homogeneous function, $L_{\mathbf{D}_0} f_s = s f_s$. Therefore, $L_{\mathbf{G}} f(\mathbf{x}) = s f_s(\phi(\mathbf{x})) = s f(\mathbf{x})$.

Conversely, let $\mathbf{G}$ be formal (analytic) vector field with $\mathbf{G} = \sum_{j\geq 0} \mathbf{G}_j$, $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{G}_0 = \mathbf{D}_0$, such that $L_{\mathbf{G}} f(\mathbf{x}) = s f(\mathbf{x})$. From [5, Proposition 2.2], we can assert that there exists a change of variables $\mathbf{x} = \Phi(\mathbf{y})$ with Φ formal (analytic) and $\Phi(\mathbf{y}) = \Phi_0(\mathbf{y}) + \Phi_i(\mathbf{y}) +$ q-h.h.o.t. being $\Phi_0(\mathbf{y}) = \mathbf{y}$, such that $\Phi_* \mathbf{G} = \mathbf{D}_0$. We consider $\Phi(f) = \overline{f} = \sum_{j\geq 0} \overline{f}_{s+j}$ where $\overline{f}_s = f_s$. By reasoning as in (6), we have that $L_{\mathbf{D}_0} \overline{f}(\mathbf{y}) = L_{\mathbf{G}} f(\Phi(\mathbf{y}))$, and by hypothesis, $L_{\mathbf{G}} f(\Phi(\mathbf{y})) = s f(\Phi(\mathbf{y}))$, that is $L_{\mathbf{D}_0} \overline{f} = s\overline{f}$. Equaling both $\mathbf{t}$-homogeneous expansions,

$$\sum_{j\geq 0} s\overline{f}_{s+j} = s\overline{f} = L_{\mathbf{D}_0}\overline{f} = \sum_{j\geq 0} (s+j)\overline{f}_{s+j}.$$

Therefore, $\overline{f} = f_s$. ♢

6. Proof of Theorem 4

We next describe how we can proceed to obtain a normal form of a function f whose $\mathbf{t}$-homogeneous expansion is $f = f_s +$ q-h.h.o.t..

Firstly, we fix the type $\mathbf{t}$. Next on stage, we will perform the near-identity transformation $\mathbf{x} = \phi(\mathbf{y}) = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$ where $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ with $k \geq 1$. The transformed function is

$$g(\mathbf{y}) := f(\phi(\mathbf{y})) = f(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) = g_s(\mathbf{y}) + g_{s+1}(\mathbf{y}) + \cdots,$$

being $g_j \in \mathcal{P}_j^{\mathbf{t}},\ j \geq s$.

The following result shows how the transformations identity plus a $\mathbf{t}$-homogeneous term affect the expression of the function.

Proposition 15. *With the above notation, it has that,*
(i) $g_j(\mathbf{y}) = f_j(\mathbf{y})$ *for* $j = s, s+1, \ldots, s+k-1$,

(ii) $g_{s+k}(\mathbf{y}) = f_{s+k}(\mathbf{y}) + L_{\mathbf{P}_k} f_s(\mathbf{y})$.

Proof. Indeed, if $E = \operatorname{diag}(\epsilon^{t_1}, \epsilon^{t_2})$, then

$$g(E\mathbf{y}) = \sum_{j \geq s} f_j(E\mathbf{y} + \mathbf{P}_k(E\mathbf{y})) = \sum_{j \geq s} \epsilon^j f_j(\mathbf{y} + \epsilon^k \mathbf{P}_k(\mathbf{y})).$$

Moreover, $f_j(\mathbf{y} + \epsilon^k \mathbf{P}_k(\mathbf{y})) = f_j(\mathbf{y}) + L_{\mathbf{P}_k} f_j(\mathbf{y})\epsilon^k + \mathcal{O}(\epsilon^{k+1})$. Thus,

$$\begin{aligned} g(E\mathbf{y}) &= \sum_{j \geq s} \left(\epsilon^j (f_j(\mathbf{y}) + L_{\mathbf{P}_k} f_j(\mathbf{y})\epsilon^k) + \mathcal{O}(\epsilon^{j+k+1})\right) \\ &= f_s(\mathbf{y})\epsilon^s + f_{s+1}(\mathbf{y})\epsilon^{s+1} + \cdots + f_{s+k-1}(\mathbf{y})\epsilon^{s+k-1} \\ &\quad + (f_{s+k}(\mathbf{y}) + L_{\mathbf{P}_k} f_s(\mathbf{y}))\epsilon^{s+k} + \mathcal{O}(\epsilon^{s+k+1}). \end{aligned}$$

$\diamondsuit$

The above result suggests to define the following homological operator.

Definition 3. For each $k \geq 1$, it is defined the operator

$$\begin{aligned} \mathcal{E}_{s+k} \quad : \quad & \mathcal{Q}_k^{\mathbf{t}} \longrightarrow \mathcal{P}_{s+k}^{\mathbf{t}} \\ & \mathbf{P}_k \to \mathcal{E}_{s+k}(\mathbf{P}_k) := -L_{\mathbf{P}_k} f_s. \end{aligned} \tag{7}$$

Notice that the above operator only depends on the lowest-degree $\mathbf{t}$-homogeneous term of f, and it is linear.

Proposition 15 assures that the $\mathbf{t}$-homogeneous terms up to order $s+k-1$ do not change, and the term of degree $s+k$ of the transformed function is

$$g_{s+k} = f_{s+k} - \mathcal{E}_{s+k}(\mathbf{P}_k).$$

Following the same ideas that in the conventional normal form theory, it is possible to annihilate in these terms the part belonging to the range of the linear operator $\mathcal{E}_{s+k}$, by selecting $\mathbf{P}_k$ adequately. Next result provides a normal form of a formal scalar function.

Proposition 16. *Assume that f is a formal function. Then, there exists a near-identity transformation $\mathbf{x} = \xi(\mathbf{y})$ such that $f(\xi(\mathbf{y})) = g_s(\mathbf{y}) + g_{s+1}(\mathbf{y}) + \cdots$ with $g_s = f_s$ and g_{s+k} belongs to a complementary subspace to the range of the linear operator $\mathcal{E}_{s+k}$, for all $k \geq 1$. (g is named a normal form of f).*

From now, we focus on getting a normal form of a scalar function on $\mathbf{C}[[x, y]]$.
First, we analyze in detail the sets $\mathcal{P}_k^{\mathbf{t}} \subset \mathbb{C}[x, y]$.

Lemma 17. *Fixed $\mathbf{t} = (t_1, t_2)$, we define the set $\mathcal{I}^{\mathbf{t}} := \{k = k_1t_1 + k_2t_2 + k_3t_1t_2 \in \mathbb{N} : k_1, k_2, k_3 \in \mathbb{N}_0, k_1 < t_2, k_2 < t_1\}$. It has that:*
(i) $\mathcal{P}_k^{\mathbf{t}}$ is a non-trivial set if and only if $k \in \mathcal{I}^{\mathbf{t}}$. Moreover, in such a case, $\mathcal{P}_k^{\mathbf{t}} = \langle x^{k_1+t_2(k_3-j)} y^{k_2+t_1 j},\ j = 0, \ldots, k_3 \rangle$.
(ii) $\mathcal{P}_{-1}^{\mathbf{t}} = \{0\}$ and $\mathcal{P}_k^{\mathbf{t}} \neq \{0\}$ for all $k > t_1t_2 - |\mathbf{t}|$, i.e. there always exists $m_0 := max\{k \in \mathbb{Z}, \mathcal{P}_k^{\mathbf{t}} = \{0\}\}$ and $-1 \leq m_0 \leq t_1t_2 - |\mathbf{t}|$.

Its proof can be seen in [6].
We introduce the following subspaces of $\mathcal{Q}_k^{\mathbf{t}}$,

$$\mathcal{C}_k^{\mathbf{t}} := \{\mathbf{X}_{h_{k+|\mathbf{t}|}} \in \mathcal{Q}_k^{\mathbf{t}} :\ h_{k+|\mathbf{t}|} \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}\}, \quad \mathcal{D}_k^{\mathbf{t}} := \{\mu_k \mathbf{D}_0 \in \mathcal{Q}_k^{\mathbf{t}} : \mu_k \in \mathcal{P}_k^{\mathbf{t}}\}.$$

By the conservative-dissipative splitting of a quasi-homogeneous vector field, $\mathcal{Q}_k^{\mathbf{t}}$ can be represented uniquely as $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{C}_k^{\mathbf{t}} \oplus \mathcal{D}_k^{\mathbf{t}}$, for any k natural number.

Definition 4. Consider $f_s \in \mathcal{P}_s^{\mathbf{t}} \setminus \{0\}$. We define the linear operator

$$\begin{aligned} \ell_{s+k} \quad : \quad & \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}} \longrightarrow \mathcal{P}_{s+k}^{\mathbf{t}} \\ & h_{k+|\mathbf{t}|} \to \ell_{s+k}(h_{k+|\mathbf{t}|}) := L_{\mathbf{X}_{f_s}} h_{k+|\mathbf{t}|}. \end{aligned}$$

This operator is defined in terms of the Poisson bracket, namely, $\ell_{s+k}(h_{k+|\mathbf{t}|}) = \{f_s, h_{k+|\mathbf{t}|}\}$.

From the conservative-dissipative splitting of a quasi-homogeneous vector field and by using that $L_{\mathbf{X}_f} g = -L_{\mathbf{X}_g} f$ and $L_{\mathbf{D}_0} p_k = k p_k, p_k \in \mathcal{P}_k^{\mathbf{t}}$ (Euler's Theorem for quasi-homogeneous functions), we can re-write the homological operator $\mathcal{E}_{s+k}$ given in (7) as

$$\begin{aligned} \mathcal{E}_{s+k} \quad : \quad & \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{P}_{s+k}^{\mathbf{t}} \\ & (h_{k+|\mathbf{t}|}, \mu_k) \to \mathcal{E}_{s+k}(h_{k+|\mathbf{t}|}, \mu_k) := \ell_{s+k}(h_{k+|\mathbf{t}|}) - s\mu_k f_s. \end{aligned} \tag{8}$$

In what follows, for $k \geq s$, we denote by $\Delta_k^{\mathbf{t}}$ to a complementary subspace to $f_s \mathcal{P}_{k-s}^{\mathbf{t}}$ on $\mathcal{P}_k^{\mathbf{t}}$, i.e., $\mathcal{P}_k^{\mathbf{t}} = \Delta_k^{\mathbf{t}} \oplus f_s \mathcal{P}_{k-s}^{\mathbf{t}}$.

Definition 5. Fixed $\Delta_k^{\mathbf{t}}$ for all k, we define the linear operator

$$\begin{aligned} \tilde{\mathcal{E}}_{s+k} \quad : \quad & \Delta_{k+|\mathbf{t}|}^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{P}_{s+k}^{\mathbf{t}} \\ & (h_{k+|\mathbf{t}|}, \mu_k) \to \tilde{\mathcal{E}}_{s+k}(h_{k+|\mathbf{t}|}, \mu_k) := \tilde{\ell}_{s+k}(h_{k+|\mathbf{t}|}) - s\mu_k f_s. \end{aligned} \tag{9}$$

with $\tilde{\ell}_{s+k}$ the linear operator

$$\begin{aligned} \tilde{\ell}_{s+k} \quad : \quad & \Delta_{k+|\mathbf{t}|}^{\mathbf{t}} \longrightarrow \Delta_{s+k}^{\mathbf{t}} \\ & h_{k+|\mathbf{t}|} \to \tilde{\ell}_{s+k}(h_{k+|\mathbf{t}|}) := \mathrm{Proj}_{\Delta_{s+k}^{\mathbf{t}}} \ell_{s+k}(h_{k+|\mathbf{t}|}). \end{aligned} \tag{10}$$

We have the following result.

Proposition 18. *Fixed* $\Delta_k^{\mathbf{t}}$ *for all* k*, the following statements are satisfied:*
(i) Range$(\mathcal{E}_{s+k}) =$ *Range*$(\tilde{\mathcal{E}}_{s+k})$.
(ii) Cor$(\tilde{\mathcal{E}}_{s+k}) =$ *Cor*$(\tilde{\ell}_{s+k})$*, i.e. any complementary subspace to the range of the operator* $\tilde{\ell}_{s+k}$ *is also a complementary space to the range of the operator* $\tilde{\mathcal{E}}_{s+k}$.
(iii) Assume that the factorization of f_s *on* $\mathbb{C}[x,y]$ *has only simple factors and* $\mathcal{P}^{\mathbf{t}}_{k-s+|\mathbf{t}|} \neq \{0\}$*. Then, Cor*$(\tilde{\ell}_{s+k}) = \{0\}$.

Proof. We prove *(i)*. We write $h_{k+|\mathbf{t}|} = h_1 + h_2 f_s$ with $h_1 \in \Delta^{\mathbf{t}}_{k+|\mathbf{t}|}$ and $h_2 \in \mathcal{P}^{\mathbf{t}}_{k-s+|\mathbf{t}|}$. We have that

$$\begin{aligned}
\mathcal{E}_{s+k}(h_{k+|\mathbf{t}|}, \mu_k) &= \ell_{s+k}(h_{k+|\mathbf{t}|}) - s\mu_k f_s \\
&= \ell_{s+k}(h_1) + \ell_{s+k}(h_2 f_s) - s\mu_k f_s \\
&= \tilde{\ell}_{s+k}(h_1) + \mathrm{Proj}_{\mathcal{P}^{\mathbf{t}}_k f_s}(\ell_{s+k}(h_1)) + \{f_s, h_2\} f_s - s\mu_k f_s.
\end{aligned}$$

Denoting $a f_s := \mathrm{Proj}_{\mathcal{P}^{\mathbf{t}}_k f_s}(\ell_{s+k}(h_1))$ and $b := \{f_s, h_2\}$,

$$\mathcal{E}_{s+k}(h_{k+|\mathbf{t}|}, \mu_k) = \tilde{\mathcal{E}}_{s+k}(h_1, \mu_k - \frac{1}{s}(a+b)).$$

Thus, $\mathrm{Range}(\mathcal{E}_{s+k}) \subseteq \mathrm{Range}(\tilde{\mathcal{E}}_{s+k})$.
The other inclusion relation is trivial.

Statement *(ii)* follows from $\mathrm{Range}(\tilde{\mathcal{E}}_{s+k}) = \mathrm{Range}(\tilde{\ell}_{s+k}) \oplus f_s \mathcal{P}^{\mathbf{t}}_k$.

We see *(iii)*. Assume that f_s has only simple factors. It is easy to prove that $h_{k+|\mathbf{t}|} \in f_s \mathcal{P}^{\mathbf{t}}_{k+|\mathbf{t}|-s}$ if, and only if, $\ell_{s+k}(h_{k+|\mathbf{t}|}) \in f_s \mathcal{P}^{\mathbf{t}}_k$. Therefore, it is always possible to choose a complementary subspace to the range of the operator $\tilde{\ell}_{s+k}$ such that $\mathrm{Cor}(\tilde{\ell}_{s+k}) \subseteq \mathrm{Cor}(\ell_{s+k}) \cap \Delta^{\mathbf{t}}_{s+k}$. On the other hand, it is proved in [2] that if the factorization of f_s on $\mathbb{C}[x,y]$ has only simple factors and $\mathcal{P}^{\mathbf{t}}_{k-s+|\mathbf{t}|} \neq \{0\}$, then we can choose a complementary subspace to $\mathrm{Range}(\ell_{s+k})$ and to $\mathrm{Range}(\ell_k)$ such that $\mathrm{Cor}(\ell_{s+k}) = f_s \mathrm{Cor}(\ell_k)$. So, $\mathrm{Cor}(\tilde{\ell}_{s+k}) = \{0\}$. ♢

Proof of Theorem 4. Let g be a normal form of f. From Proposition 16, $g = f_s + \sum_{k>0} g_{s+k}$ with $g_{s+k} \in \mathrm{Cor}(\mathcal{E}_{s+k})$, a complementary subspace to $\mathrm{Range}(\mathcal{E}_{s+k})$.

From Proposition 18, $g_{s+k} \in \mathrm{Cor}(\tilde{\ell}_{s+k})$. For $k - s + |\mathbf{t}| > m_0$ it has that $k - s + |\mathbf{t}| \in \mathcal{I}^{\mathbf{t}}$ ($\mathcal{P}^{\mathbf{t}}_{k-s+|\mathbf{t}|}$ is a not trivial set) and, from Proposition 18, $g_{s+k} = 0$. ♢

Remark 3. *The normal form of Theorem 4 is polynomial. We need only to calculate the co-ranges of* $\tilde{\ell}_{s+k}$ *for* $k = 1, \ldots, m_0 + s - |\mathbf{t}|$.

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