

Functional forms for approximating the relative optical air mass

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[1] This article constitutes a review and systematic comparison of functional forms for approximating the air mass from the zenith to the horizon. Among them, we find the most meaningful forms in atmospheric optics, geophysics, meteorology, and solar energy science, as well as several forms arising from the study of the atmospheric delay of electromagnetic signals, whose relationship with the air mass was recently proved by the authors. In total, we have compared 26 functional forms, and the fits have been done for three atmospheric profiles, an observer at sea level, and the median wavelength of the Sun's spectral irradiance (0.7274 μm). As a result, the best of the uniparametric forms has more than three centuries of history; the best of the biparametric forms was recently introduced by one of the authors; the best of the tri- and tetraparametric forms were originally proposed for modeling the atmospheric delay of radio signals; and the best of the forms with more than four parameters is used here for the first time. On the basis of these, for the 1976 U.S. Standard Atmosphere (USSA-76), we provide one-, two-, three-, four-, and five-parameter formulas whose maximum deviations are 1.70, 2.91×10^{-1} , 3.28×10^{-2} , 2.49×10^{-3} , and 3.24×10^{-4} , respectively.

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1. Introduction

[2] Extraterrestrial radiation undergoes attenuation as it travels through the set of layers composing the atmosphere. Thus, for clean dry air (Rayleigh atmosphere) and a given wavelength, the normal irradiance I for an observer on the Earth's surface is given by *Bouguer's* [1729] law,

$$I = I_0 e^{-\beta M}, \quad (1)$$

where I_0 is the normal irradiance at the top of the atmosphere, β is the mass extinction coefficient of air, and M is the absolute optical air mass. According to *Bemporad* [1904, chapter 1], the former magnitude is defined as

$$M = \int_S \rho(s) ds, \quad (2)$$

where S represents the path of the radiation and $\rho(s)$ is the air density along S .

[3] In general, the integrand of the latter equation has no elementary primitive function. Hence, in the canonical hypothesis of an atmosphere with spherical symmetry, the absolute optical air mass is usually estimated through an obliquity function $f(z, \mathbf{a})$:

$$M = M(z) \approx M_0 f(z, \mathbf{a}), \quad (3)$$

where z is the zenith distance of the radiant source, $M_0 = M(0)$ is the absolute optical air mass for the vertical incidence of the rays, and \mathbf{a} is a vector representing a set of parameters a_i ($i = 1, 2, \dots, p$). In the most ambitious obliquity functions [e.g., *Maurer*, 1882, part 2; *Chapman*, 1931; *Saar*, 1973; *Schaefer*, 1993], the value of these parameters must be determined from a set of physical variables v_j ($j = 1, 2, \dots, q$), among which *Young* [1994] suggests including height, temperature, and pressure at the observation point (h_A , T_A , and P_A , respectively), as well as the incident radiation wavelength (λ). Therefore, as an example, if we were to decide to adopt the four variables just mentioned, we would get

$$a_i = a_i(h_A, T_A, P_A, \lambda). \quad (4)$$

[4] When the absolute optical air mass is estimated through an obliquity function $f(z, \mathbf{a})$, the same two error sources that were pointed out by *Yan et al.* [2002] for the mapping functions of the atmospheric delay of radio signals concur. Thus, while the first error source lies in the mathematical form adopted by the function $f(z, \mathbf{a})$, including the choice of the number p of components and a_i of vector \mathbf{a} ($i = 1, 2, \dots, p$), the second error source lies in the nature of the expressions a_i (v_1, v_2, \dots, v_q), commencing with the selection of physical variables v_j ($j = 1, 2, \dots, q$).

[5] For centuries, astronomers, actinometrists, and meteorologists have tried to quantify the atmospheric attenuation of extraterrestrial radiation. As a result of this effort, nowadays, we have a number of obliquity functions, as well as tables and graphics comparing their degree of precision [*Forbes*, 1842; *Radau*, 1877, chapter 3; *Müller*, 1897, chapter 3; *Bemporad*,

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1906, 1908; *Robinson et al.*, 1966; *Swider and Gardner*, 1967; *Sivkov*, 1971, chapter 3; *Young*, 1974; *Nagel*, 1974; *Thomason et al.*, 1983; *Kittler and Mikler*, 1986, chapter 1; *Piršel*, 1991; *Gueymard*, 1993, 2003; *Kasten*, 1993; *Yin*, 1997; *Huestis*, 2001; *Li and Shibata*, 2006; *Rapp-Arrarás and Domingo-Santos*, 2008; *Rapp-Arrarás*, 2009, chapter 4]. Nevertheless, generally these tables and graphics have the disadvantage of not considering the contribution of each error source mentioned above to the total error of approximation. Thus, for instance, apart from the research article by *Rapp-Arrarás and Domingo-Santos* [2008], there are no published works specifically examining the first error source.

[6] This article constitutes a review and systematic comparison of functional forms for approximating the relative optical air mass. Among them, we have considered the most meaningful forms in atmospheric optics, geophysics, meteorology, and solar energy science, as well as several forms arising from the study of the atmospheric delay of electromagnetic signals, whose relationship with the optical air mass was shown by *Rapp-Arrarás and Domingo-Santos* [2008].

2. Preliminary Considerations

2.1. Air Mass

[7] The relative optical air mass or, in short, air mass is defined as the obliquity ratio of the absolute optical air mass, that is,

$$m(z) = \frac{M(z)}{M_0}, \quad (5)$$

which implies that

$$M(z) = M_0 m(z). \quad (6)$$

[8] Recalling equation (3), it becomes clear that the search for an appropriate obliquity function $f(z, \mathbf{a})$ is just a problem of approximation of functions, where the air mass $m(z)$ is the target function.

2.2. Apparent Zenith Distance Versus True Zenith Distance

[9] In astrometry, a distinction is established between the true zenith distance z_t and the apparent zenith distance z_a . Thus, z_t refers to the direction of the straight line connecting the radiant source to the observer, whereas z_a refers to the direction of the rays reaching the observer once refracted by the atmosphere (evidently, without atmosphere and, consequently, with no refraction, both magnitudes would coincide).

[10] Therefore, obliquity functions of the absolute optical air mass can be developed either in terms of z_t or in terms of z_a . In fact, the choice as an argument of one variable or the other will depend on the context in which those functions will be applied, that is, on the zenith distance available in each case [*Young*, 1994]. Hence, obliquity functions are sometimes defined from the true zenith distance (or its complement, the true altitude) [*Young and Irvine*, 1967; *Young*, 1994], and in other cases, the most frequent, from the apparent zenith distance (or its complement, the apparent altitude) [*Forbes*, 1842; *Trépid*, 1876; *Maurer*, 1882, part 2; *Bemporad*, 1908; *Makhotkin*,

1960; *Hardie*, 1962; *Rozenberg*, 1963; *Kasten*, 1965, 1993; *Saunders*, 1968; *Saar*, 1973; *Young*, 1974; *Nagel*, 1974; *Heindl and Koch*, 1976; *Wittmann*, 1980; *Bruin*, 1981; *Rawlins*, 1982, 1992; *Gushchin and Vinogradova*, 1983, chapter 2; *Rohlf*, 1986, chapter 7; *Garstang*, 1986; *Kasten and Young*, 1989; *Gueymard*, 1993, 1995, 2003; *Schaefer*, 1993; *Kocifaj*, 1996; *Yin*, 1997; *Kristensen*, 1998; *Pickering*, 2002; *Li and Shibata*, 2006; *Vollmer and Gedzelman*, 2006; *Rapp-Arrarás and Domingo-Santos*, 2008; *Rapp-Arrarás*, 2009, chapter 4]. Notwithstanding, there are authors who, seeking simplicity, have opted for neglecting the effect of the atmospheric refraction, so the distinction between true and apparent angles makes no sense in their formulas [*Radau*, 1877, chapter 3; *Chapman*, 1931; *Fesenkov*, 1958; *Green and Barnum*, 1963; *Fitzmaurice*, 1964; *Swider*, 1964; *Robinson et al.*, 1966, equation (3.12); *Rodgers*, 1967, chapter 5; *Young*, 1969; *Sivkov*, 1971, equation (3.7); *Smith and Smith*, 1972; *Kreith and Kreider*, 1978, chapter 2; *Titheridge*, 1988; *Nijegorodov et al.*, 1994; *Huestis*, 2001].

[11] Moreover, the rigorous calculation of air mass for an arbitrary value of z_t has a considerably higher computational cost than when performing it in terms of z_a . The reason is that in the second case, the problem is reduced to evaluate a definite integral whose integrand, by virtue of Snell's law, explicitly depends on z_a [see, e.g., *Bemporad*, 1906; *Kasten*, 1965]. Taking into account that our main objective is to identify the best forms for obtaining obliquity functions and that the difference between z_a and z_t is always small, we have chosen z_a as the independent variable in the fits. Besides, as we will see in section 6, this choice will allow us to pose the approximations in terms of a continuous variable.

[12] From this point onward, we will only deal with the apparent zenith distance, which will be symbolized by z .

3. Functional Forms

[13] In sections 3.1–3.5, we will introduce the different functional forms that we have compared. In order to confine the scope of our study, we have discarded the piecewise discontinuous forms [*Chapman*, 1931, equations (22) and (38); *Fesenkov*, 1958, equations (30) and (31); *Smith and Smith*, 1972, equation (13)], as well as the nonelementary forms [*Fitzmaurice*, 1964, equation (3); *Swider*, 1964, equation (52); *Saar*, 1973, equation (24); *Kocifaj*, 1996; *Huestis*, 2001, section 3], and the forms where the refraction angle appears as an argument [*Forbes*, 1842, section 3; *Maurer*, 1882, part 2]. In addition, as clarified in section 6, we have put aside the functional forms that are divergent for a horizontal incidence of rays. In order to facilitate the exposition and analysis of the results, the various forms have been grouped regarding the number of their parameters. Moreover, for a given number of parameters, the forms are presented in chronological order of their use for approximating the air mass.

[14] Before presenting the functional forms, we would like to make a semantic annotation. In the following paragraphs of this section, when we affirm that an author used or applied a specific form, it must be understood that after adopting a preexistent mathematical expression, the author has assigned new values to the parameters composing it (either by analytical deduction or by fitting techniques).

3.1. Uniparametric Functional Forms

3.1.1. The One-Parameter Form of De Mairan and Radau (DR-1)

[15] Formally introduced by *Radau* [1877, chapter 3] and applied, or newly deduced, by a number of authors since then [*Bemporad*, 1908, section 17; *Makhotkin*, 1960; *Robinson et al.*, 1966; *Saunders*, 1968; *Sivkov*, 1971, chapter 3; *Kreith and Kreider*, 1978, chapter 2; *Piršel*, 1991; *Kasten*, 1993; *Nijegorodov et al.*, 1994; *Vollmer and Gedzelman*, 2006; *Rapp-Arrarás*, 2009, chapter 4], this functional form is ultimately based on the geometric approach developed by *De Mairan* [1723]. We will refer to it as DR-1 and its mathematical expression is

$$f(z, \mathbf{a}) = \sqrt{1 + 2a_1 + a_1^2 \cos^2 z} - a_1 \cos z. \quad (7)$$

[16] It should be noted that *Vollmer and Gedzelman* [2006] also employed this pattern for estimating the optical mass of atmospheric aerosols. Moreover, as proven in Appendix A, *Spilker* [1996] adopted an equivalent functional form in order to define an obliquity function for atmospheric attenuation of GPS signals.

3.1.2. The One-Parameter Form of Rodgers (Ro-1)

[17] Proposed by *Rodgers* [1967, section 5.2.4], this functional form was used by *Wittmann* [1980], albeit under a different appearance (see Appendix B), and by *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as Ro-1 and its mathematical expression is

$$f(z, \mathbf{a}) = \frac{a_1 \sec z}{\sqrt{a_1^2 - 1 + \sec^2 z}}. \quad (8)$$

[18] Note that, although it converges to the correct value at the zenith, this functional form is not strictly defined at that position. Nevertheless, this discontinuity is easily avoided by assuming that $f(0, \mathbf{a}) = 1$.

[19] It is a fact that decades ahead, *Cabannes and Dufay* [1925] proposed a model for estimating the relative optical ozone mass that, from a mathematical point of view, is equivalent to form Ro-1. Indeed, the model by *Cabannes and Dufay* [1925] was used (or newly deduced) by a number of researchers on the absorption of solar radiation by the atmospheric ozone (see Appendix B).

3.1.3. The One-Parameter Form of Heindl and Koch (HK-1)

[20] Introduced by *Heindl and Koch* [1976], this functional form was applied by *Gushchin and Vinogradova* [1983, chapter 2] and *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as HK-1 and its mathematical expression, in terms of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{2 + a_1}{\cos z + \sqrt{2a_1 + \cos^2 z}}. \quad (9)$$

[21] As proven in Appendix C, it can be regarded as a simplification of form DR-1.

3.1.4. The One-Parameter Form of Bruin (Br-1)

[22] Implicit in the second equation proposed by *Bruin* [1981, equation (5.10)] for estimating atmospheric extinction of stellar light, this functional form was used by *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as Br-1, and

its mathematical expression, as a function of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1}{\cos(z - a_1)}. \quad (10)$$

[23] It should be noted that this form arose as a simplification of form Br-2 (section 3.2.3).

3.1.5. The One-Parameter Form of Li and Shibata (LS-1)

[24] Introduced by *Li and Shibata* [2006, equation (9)] and applied by *Rapp-Arrarás* [2009, chapter 4], we will refer to this functional form as LS-1. Its mathematical expression is

$$f(z, \mathbf{a}) = \frac{2 + a_1}{2 \cos z + \sqrt{(a_1^2 - 1) \cos^2 z + 1}}. \quad (11)$$

3.1.6. The One-Parameter Form of Linke (Li-1)

[25] Adopted, for instance, by *Linke* [1943, equation (12)] and *Robinson et al.* [1966] for modeling the optical mass of atmospheric ozone, this functional form was used by *Rapp-Arrarás* [2009, chapter 4] to approximate air mass. We will refer to it as Li-1 and its mathematical expression is

$$f(z, \mathbf{a}) = \frac{1 + a_1}{\sqrt{\cos^2 z + 2a_1}}. \quad (12)$$

[26] As shown in Appendix D, it can be considered as a simplification of form Ro-1.

3.1.7. The One-Parameter Form of Green and Martin (GM-1)

[27] Developed by *Green and Martin* [1966, equation (61)] for estimating the optical mass of atmospheric ozone, this functional form was also applied by *Rapp-Arrarás* [2009, chapter 4] to approximate air mass. We will refer to it as GM-1, and its mathematical expression is

$$f(z, \mathbf{a}) = \frac{1}{\sqrt{\cos^2 z + a_1}}. \quad (13)$$

[28] As the preceding one, this functional form can be regarded as a simplification of form Ro-1 (see Appendix D).

3.2. Biparametric Functional Forms

3.2.1. The Two-Parameter Form of Rozenberg (Rz-2)

[29] Proposed by *Rozenberg* [1963, section 5], this functional form was used by *Schaefer* [1993] and *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as Rz-2, and its mathematical expression is

$$f(z, \mathbf{a}) = \frac{1}{\cos z + a_1 e^{-a_2 \cos z}}. \quad (14)$$

[30] *Schaefer* [1993] also provided an obliquity function for the optical mass of atmospheric aerosols according to this pattern.

3.2.2. The Two-Parameter Form of Green and Barnum (GB-2)

[31] Originally presented in an internal document of the Space Science Laboratory, the approximation formula for the Chapman function developed by *Green and Barnum*

[1963] became known by the scientific community after the work of *Green et al.* [1964]. For a fixed value of its linear argument, it is parameterized as

$$f(z, \mathbf{a}) = \exp\left[\frac{1}{2}\left(\frac{z^2}{1 + a_1 z^2 + a_2 z^4}\right)\right], \quad (15)$$

where z must be introduced in radians. We will refer to this functional form as GB-2.

3.2.3. The Two-Parameter Form of Bruin (Br-2)

[32] Implicit in the first equation proposed by *Bruin* [1981, equation (5.7)] for estimating atmospheric extinction of stellar light, we will refer to this functional form as Br-2. Its mathematical expression, as a function of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{a_1}{\cos(z - a_2)}. \quad (16)$$

3.2.4. The Two-Parameter Form of Green and Martin (GM-2)

[33] Developed by *Green and Martin* [1966, equation (17)] for modeling the optical mass of atmospheric ozone, this functional form was adopted by *Rapp-Arrarás* [2009, chapter 4] to approximate air mass. We will refer to it as GM-2, and its mathematical expression is

$$f(z, \mathbf{a}) = a(z, a_1) + a_2 \sin^2 z a^5(z, a_1), \quad (17)$$

where $a(z, a_1) = (1 - a_1 \sin^2 z)^{-1/2}$.

3.2.5. The Two-Parameter Form of Rapp-Arrarás (Ra-2)

[34] Proposed by *Rapp-Arrarás* [2009, equation (4.17)], we will refer to this functional form as Ra-2. Its mathematical expression is

$$f(z, \mathbf{a}) = \frac{1}{\cos z + a_1(1 - \cos z)/(\cos z + a_2)}. \quad (18)$$

3.2.6. The Two-Parameter Form of Herring (He-2)

[35] Adopted by *Guo and Langley* [2003] for modeling the atmospheric delay of radio signals, the biparametric version of form He-3 (section 3.3.6) was used by *Rapp-Arrarás* [2009, chapter 4] to approximate air mass. We will refer to it as He-2, and its mathematical expression, in terms of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1 + a_1/(1 + a_2)}{\cos z + a_1/(\cos z + a_2)}. \quad (19)$$

3.3. Triparametric Functional Forms

3.3.1. The Three-Parameter Form of Kasten (Ka-3)

[36] Introduced by *Kasten* [1965], this functional form was applied by *Kasten and Young* [1989], *Rapp-Arrarás and Domingo-Santos* [2008], and *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as Ka-3, and its mathematical expression, as a function of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1}{\cos z + a_1/(a_2 - z)^{a_3}}. \quad (20)$$

Kasten [1965] also provided an obliquity function for the optical mass of atmospheric water vapor following this model.

3.3.2. The Three-Parameter Form of Rawlins (Rw-3)

[37] Proposed by *Rawlins* [1982], this functional form was afterward used by himself [*Rawlins*, 1992, footnote 63] and *Pickering* [2002, footnote 39]. We will refer to it as Rw-3, and its mathematical expression, in terms of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1}{\cos\{z + 1/[a_1 + a_2(90 - z)^{a_3}]\}}, \quad (21)$$

where z must be introduced in degrees. Note that the model in question can be considered as an enhanced version of form Br-1.

[38] *Rawlins* [1992, footnote 16] and *Pickering* [2002, footnote 39] also adopted this form for their respective obliquity functions for the optical mass of atmospheric dust.

3.3.3. The Three-Parameter Form of Titheridge (Ti-3)

[39] For a fixed value of its linear argument, the approximation formula for the Chapman function developed by *Titheridge* [1988, equation (5)] is parameterized as

$$f(z, \mathbf{a}) = \frac{1}{\cos[z - a(z, \mathbf{a})]}, \quad (22)$$

where $a(z, \mathbf{a}) = a_1[\sec(a_2 z) + a_3]$. Applied by *Rapp-Arrarás* [2009, chapter 4], we will refer to this functional form as Ti-3. As it can be noticed at a glance, this is a refined version of form Br-1.

3.3.4. The Three-Parameter Form of Gueymard (Gu-3)

[40] Introduced by *Gueymard* [1993], this functional form was used by *Rapp-Arrarás and Domingo-Santos* [2008] and *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as Gu-3, and its mathematical expression, in terms of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1}{\cos z + a_1 z / (a_2 - z)^{a_3}}. \quad (23)$$

[41] *Gueymard* [1993] also adopted this form for proposing an obliquity function for the optical mass of atmospheric ozone and another one for the optical mass of atmospheric water vapor.

3.3.5. The Three-Parameter Form of Marini (Ma-3)

[42] Adopted by *Ifadis* [1986, 2000] for modeling the atmospheric delay of radio signals, the triparametric version of the truncated continued fraction by *Marini* [1972] was applied by *Rapp-Arrarás and Domingo-Santos* [2008] to approximate air mass. We will refer to it as Ma-3, and its mathematical expression, as a function of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1}{\cos z + a_1/[\cos z + a_2/(\cos z + a_3)]}. \quad (24)$$

3.3.6. The Three-Parameter Form of Herring (He-3)

[43] Proposed by *Herring* [1992] for estimating the atmospheric delay of electromagnetic signals, this functional form was adopted by *Rapp-Arrarás and Domingo-Santos*

[2008] and *Rapp-Arrarás* [2009, chapter 4] to approximate air mass. We will refer to it as He-3, and its mathematical expression, in terms of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1 + a_1/[1 + a_2/(1 + a_3)]}{\cos z + a_1/[\cos z + a_2/(\cos z + a_3)]}. \quad (25)$$

[44] It is worthy of remark that this form has become the reference model in the development of mapping functions for the atmospheric delay of electromagnetic signals [e.g., *Niell*, 1996, 2000; *Mendes et al.*, 2002; *Boehm and Schuh*, 2004; *Boehm et al.*, 2006; *Eresmaa et al.*, 2008].

3.4. Tetraparametric Functional Forms

3.4.1. The Four-Parameter Form of Young (Yo-4)

[45] Introduced by *Young* [1994, equation (6)] as a simplification of form Yo-6 (section 3.5.3), this functional form was applied by *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as Yo-4, and its mathematical expression, as a function of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{a_1 \cos z + a_2}{\cos^2 z + a_3 \cos z + a_4}. \quad (26)$$

[46] It should be noted that, for a fixed value of its linear argument, the first piece of the approximation formula for the Chapman function developed by *Smith and Smith* [1972, equation (13)] already corresponded to this pattern.

3.4.2. The Four-Parameter Form of Gueymard (Gu-4)

[47] Proposed by *Gueymard* [1995, section 4.1], this functional form was afterward used by himself [*Gueymard*, 2003] and *Rapp-Arrarás* [2009, chapter 4]. We will refer to it as Gu-4, and its mathematical expression is

$$f(z, \mathbf{a}) = \frac{1}{\cos z + a_1 z^{a_2} / (a_3 - z)^{a_4}}, \quad (27)$$

where, in principle, a_2 must be positive (notice that, at the limit, as a_2 tends to zero from the right, this model degenerates to form Ka-3).

[48] *Gueymard* [1995, 2003] also provided formulas for estimating the optical mass of atmospheric ozone, water vapor, nitrogen dioxide, and aerosols under this functional form. *Gueymard and Kambezidis* [2004] increased the list of such formulas for other atmospheric constituents causing attenuation of solar radiation.

3.4.3. The Four-Parameter Form of Kristensen (Kr-4)

[49] Introduced by *Kristensen* [1998] and applied by *Rapp-Arrarás* [2009, chapter 4], we will refer to this functional form as Kr-4. Its mathematical expression is

$$f(z, \mathbf{a}) = \sqrt{a^2(z, \mathbf{a}) \cos^2 z + 2a(z, \mathbf{a}) + 1} - a(z, \mathbf{a}) \cos z, \quad (28)$$

where $a(z, \mathbf{a}) = a_1 + a_2/[(\cos^2 z + a_3 \cos z + a_4)]$. As can be noted, it is an improved version of form DR-1.

3.4.4. The Four-Parameter Form of Herring (He-4)

[50] Adopted by *Ifadis and Savvaidis* [2001] for modeling the atmospheric delay of radio signals, the tetraparametric version of form He-3 was used by *Rapp-Arrarás and Domingo-Santos* [2008] and *Rapp-Arrarás* [2009, chapter 4]

to approximate air mass. We will refer to it as He-4, and its mathematical expression, in terms of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1 + a_1/\{1 + a_2/[1 + a_3/(1 + a_4)]\}}{\cos z + a_1/\{\cos z + a_2/[\cos z + a_3/(\cos z + a_4)]\}}. \quad (29)$$

3.5. Functional Forms With More Than Four Parameters

3.5.1. The Five-Parameter Form of Dogniaux (Do-5)

[51] The obliquity function developed by *Dogniaux* [1975, section 3.1.1] can be parametrized as

$$f(z, \mathbf{a}) = \frac{a_1 z^2 + a_2 z + a_3}{z^2 + a_4 z + a_5}. \quad (30)$$

[52] We will refer to this functional form as Do-5.

3.5.2. The Five-Parameter Form of Herring (He-5)

[53] Adopted here for the first time, the pentaparametric version of form He-3 will be referred as He-5. Its mathematical expression, as a function of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{1 + a_1/(1 + a_2/\{1 + a_3/[1 + a_4/(1 + a_5)]\})}{\cos z + a_1/(\cos z + a_2/\{\cos z + a_3/[\cos z + a_4/(\cos z + a_5)]\})}. \quad (31)$$

3.5.3. The Six-Parameter Form of Young (Yo-6)

[54] Introduced by *Young* [1994, equation (5)], we will refer to this functional form as Yo-6. Its mathematical expression, in terms of the apparent zenith distance, is

$$f(z, \mathbf{a}) = \frac{a_1 \cos^2 z + a_2 \cos z + a_3}{\cos^3 z + a_4 \cos^2 z + a_5 \cos z + a_6}. \quad (32)$$

4. Atmospheric Models

[55] In order to test the suitability of each of the functional forms presented in sections 3.1–3.5, we have adopted the USSA-76 as the main atmospheric model [*Committee on Extension to the Standard Atmosphere*, 1976]. Nonetheless, for a broader generality and as a contrast to the multi-stratified USSA-76, calculations have also been made for two elementary models: the exponential density profile and the quartic density profile.

4.1. The USSA-76

[56] The USSA-76 is an idealized, steady-state representation of the mean annual conditions of the atmosphere for northern midlatitudes [*Champion et al.*, 1985]. Following *Kivalov* [2007], we have taken into account the lower seven layers of USSA-76, that is, the first 86 km from sea level. In that portion of the USSA-76, it is assumed that the air has a uniform composition (e.g., 314 ppm of CO₂ and no H₂O) and behaves as an ideal gas in hydrostatic equilibrium. Furthermore, each of these layers is characterized by a constant temperature gradient in terms of geopotential height that contradistinguishes it from contiguous layers [*Committee on Extension to the Standard Atmosphere*, 1976].

4.2. The Exponential Profile

[57] The exponential density profile has been adopted by a number of researchers on atmospheric extinction of radiation [e.g., *Bouguer*, 1729, chapter 5; *Forbes*, 1842; *Chapman*, 1931; *Young*, 1969; *Kristensen*, 1998]. Its mathematical expression is

$$\rho(h) = \rho_0 \exp\left(-\frac{h}{H}\right), \quad (33)$$

where $\rho(h)$ is the air density at height h , $\rho_0 = \rho(0)$, and H is the density scale height. Regarding the specific values of ρ_0 and H , we have adopted the density and the pressure scale height of USSA-76 at sea level, namely, $\rho_0 = 1.22500 \text{ kg m}^{-3}$ and $H = 8434.52 \text{ m}$.

4.3. The Quartic Profile

[58] Frequently adopted in the study of the atmospheric delay of radio signals [e.g., *Hopfield*, 1969; *Black*, 1978; *Black and Eisner*, 1984; *Yan and Ping*, 1995], in the context of atmospheric extinction of radiation, the quartic profile has only been employed by *Rapp-Arrarás* [2009, chapter 4]. Its mathematical expression, in terms of the air density, is

$$\rho(h) = \rho_0 \left(1 - \frac{h}{h_B}\right)^4, \quad (34)$$

where h_B is the height of the upper limit of the profile, while the rest of the variables are known (see section 4.2). We have assumed the value of h_B such that the effective height of the quartic profile will be coincident with that of the exponential profile. After *Yan and Ping* [1995], this requirement is satisfied when $h_B = 5 \cdot H$, so $h_B = 42,172.6 \text{ m}$.

5. Calculation of Air Mass

[59] According to *Kivalov* [2007], the absolute optical air mass for a point at height h_A is defined as

$$M(z) = \int_{h_A}^{h_B} \frac{\rho(h)n(h)(R+h)dh}{\sqrt{n^2(h)(R+h)^2 - n^2(h_A)(R+h_A)^2 \sin^2 z}}, \quad (35)$$

where $n(h)$ is the refractive index of air at height h , R is the terrestrial radius, h_B is the height where air density vanishes, and z must be in the range of 0° – 90° (both inclusive). When the absolute optical air mass is evaluated at the zenith ($z = 0^\circ$), the former integral is reduced to

$$M_0 = M(0^\circ) = \int_{h_A}^{h_B} \rho(h)dh. \quad (36)$$

[60] Note that, in general, even assuming the simple Gladstone-Dale relationship (as done, e.g., by *Link and Neužil* [1958], *Kasten* [1965], *Saar* [1973], *Kasten and Young* [1989], *Kristensen* [1998], and *Kivalov* [2007]),

$$n(h) - 1 = \alpha\rho(h), \quad (37)$$

where α , the specific refractivity of the air, is considered a constant, the integrand of equation (35) has no elementary primitive for the three atmospheric models adopted here. Therefore, apart from (eventually) the singular cases provided by equation (36), we have always been forced to a numerical evaluation of the integral given by equation (35).

5.1. Limit Heights

[61] It is well known that air mass depends, to some extent, on the observer's height h_A [see e.g., *Pressly*, 1953; *Link and Neužil*, 1958]. In fact, in the literature, we can find several attempts to reflect this dependence analytically, although their success, always relative, is limited to unrealistic initial hypotheses, such as the assumption of an exponential atmosphere where refraction effects are neglected [e.g., *Chapman*, 1931; *Huestis*, 2001]. In any case, the precise value of h_A is not relevant in the scope of our study, so we have just assumed that the observer is at sea level, $h_A = 0$ (as adopted, among others, by *Kasten* [1965], *Kasten and Young* [1989], and *Kivalov* [2007]).

[62] Regarding the conventional upper limit of the atmosphere h_B , its influence in the computation of air mass will be insignificant over a given value. Indeed, most authors do not specify it, and those doing so are not necessarily coincident at their choice. Thus, for example, while *Pressly* [1953] adopted a value of 220 km, *Kasten* [1965] lessened it to 84 km, and *Link and Neužil* [1969] down to 40 km for an observer at sea level.

[63] In this study, the value of h_B for the main atmospheric model, namely, USSA-76, is determined by the layers that we have considered (see section 4.1), thus $h_B = 86,000 \text{ m}$. For the exponential atmospheric profile, we have chosen the usual value $h_B = \infty$, whereas for the quartic profile, as explained in section 4.3, $h_B = 42,172.6 \text{ m}$.

5.2. Wavelength and Terrestrial Radius

[64] The specific refractivity of air and, consequently, its refractive index, depend on the frequency of the radiation that propagates through it [*Garfinkel*, 1967]. For our computations, after the criterion established by *Kasten* [1965], we have taken the median wavelength of the Sun's spectral irradiance, namely, $\lambda = 0.7274 \mu\text{m}$ [see *Rapp-Arrarás*, 2009, Appendix 4E]. Following the procedure put forward by *Ciddor* [1996, Appendix B] and in accordance with the content of CO_2 and H_2O of the layers considered for the USSA-76, the specific refractivity of the air for the wavelength just mentioned is $\alpha = 2.24863 \times 10^{-4} \text{ m}^3 \text{ kg}^{-1}$.

[65] Regarding the terrestrial radius, we have adopted the mean Earth's radius of curvature implicit in USSA-76 for the three atmospheric models, namely, $R = 6,378,759 \text{ m}$.

5.3. Sample Values of Air Mass

[66] Table 1 gives the results of computing air mass for a small set of apparent zenith distances and each of the atmospheric profiles adopted. Naturally, these values have been obtained by application of equation (5) after evaluating the integrals of equations (35) and (36). As it can be seen, there are no large differences among the atmospheric profiles, and this is true even at the horizon ($z = 90^\circ$).

6. Degree of Approximation

[67] The most cited approximation formulas for air mass, namely, those of *Kasten* [1965] and *Kasten and Young* [1989], were developed following three steps: (1) Within the range of validity of the function, defined by the end-points z_m and z_M , we chose a set of apparent zenith distances z_k ($k = 1, 2, \dots, N$). (2) We obtain the corresponding values of air mass $m_k = m(z_k)$. (3) After adopting a specific

Table 1. Relative Optical Air Mass m , as a Function of the Zenith Distance z , for the Reference Atmospheres Considered

z (deg)	m		
	USSA-76 Profile	Exponential Profile	Quartic Profile
0	1.0000	1.0000	1.0000
18	1.0513	1.0513	1.0514
34	1.2057	1.2056	1.2057
48	1.4926	1.4923	1.4927
60	1.9939	1.9930	1.9942
70	2.9017	2.8985	2.9027
78	4.7071	4.6933	4.7113
84	8.8497	8.7749	8.8704
88	19.4471	19.0912	19.5298
90	38.1581	37.9371	38.2903

functional form $f(z, \mathbf{a})$, we find the value of the components of \mathbf{a} minimizing the approximation distance given by

$$\delta(\mathbf{a}) = \sqrt{\frac{\sum_{k=1}^N e_k(\mathbf{a})^2}{N}}, \tag{38}$$

where $e_k(\mathbf{a}) = [m_k - f(z_k, \mathbf{a})]/m_k$. Thus, we are in a context of discrete approximation where fitting, based on the relative error and of nonlinear type, is performed by means of the least squares method.

[68] Nevertheless, the relative error of solar irradiance for a terrestrial observer is directly proportional to the absolute error (but not to the relative error) of air mass [see *Rapp-Arrarás*, 2009, Appendix 4F]. Furthermore, the mathematical nature of the problem allows for a continuous variable approach, which avoids discarding most of the information. Taking into consideration both circumstances and relying on *Bronshtein and Semendyayev* [1964, part 6], we have minimized the approximation distance defined by

$$\delta'(\mathbf{a}) = \sqrt{\frac{V(\mathbf{a})}{z_M - z_m}}, \tag{39}$$

where

$$V(\mathbf{a}) = \int_{z_m}^{z_M} E(z, \mathbf{a})^2 dz, \tag{40}$$

and, in turn, $E(z, \mathbf{a}) = m(z) - f(z, \mathbf{a})$. Therefore, for each combination of atmospheric model and functional form, we have obtained the value of parameters a_i ($i = 1, 2, \dots, p$) that minimize the function $\delta'(\mathbf{a})$.

[69] With respect to the fitting interval of the forms compared, we decided to cover the whole celestial vault, that is, from the zenith ($z_m = 0^\circ$) to the horizon ($z_M = 90^\circ$), in agreement with the range of validity of most obliquity functions [*Forbes*, 1842, section 3; *Radau*, 1877, chapter 3; *Maurer*, 1882, part 2; *Bemporad*, 1908, equation (63); *Fesenkov*, 1958, section 2; *Makhotkin*, 1960, equation (3a); *Rozenberg*, 1963, equation (25); *Swider*, 1964, equation (52); *Kasten*, 1965, 1993, equation (4.2); *Robinson et al.*, 1966, equation (3.12); *Rodgers*, 1967, section 5.2.4; *Sivkov*, 1971, equation (3.7); *Smith and Smith*, 1972, equation (13); *Heindl and Koch*, 1976; *Rawlins*, 1982; *Titheridge*, 1988, equation (5); *Kasten and Young*, 1989; *Piršcl*, 1991, equation (4); *Gueymard*, 1993, 1995, section 4.1, 2003, equation (B.8); *Schaefer*, 1993, equation (3a); *Nijegorodov et al.*, 1994; *Kocifaj*, 1996, equation (11b); *Yin*, 1997; *Kristensen*, 1998; *Huestis*, 2001; *Vollmer and Gedzelman*, 2006; *Rapp-Arrarás and Domingo-Santos*, 2008; *Rapp-Arrarás*, 2009; chapter 4]. In fact, that is the reason for discarding the functional forms diverging for horizontal incidence of radiation, among which we have the forms based on the series expansion adopted by *Lambert* [1760, section 880], such as that of *Trépiéd* [1876]; the families of polynomials in $\sec z$, such as those of *Hardie* [1962], *Fitzmaurice* [1964, equation (5)], *Snell and Heiser* [1968], and *Rohlf's* [1986, chapter 7]; and some other forms, such as those of *Linke* [1943, section 49] and *Nagel* [1974, equation (4)].

[70] If we recall that the evaluation of the integral given by equation (35) was only possible through numerical methods, we will understand that evaluating the integral in equation (40) and, consequently, minimizing the function of equation (39) must also be worked out numerically. Needless to say that, without appropriate computing tools, such a calculation task would, in practice, have been unaffordable.

7. Results

[71] Table 2 gives, for each of the one-parameter functional forms considered and each of the atmospheric models adopted, the essential features of the calibrations performed. Thus, while the third, sixth, and ninth columns contain the optimal vectors \mathbf{a}_o that minimize the value of the approximation distance $\delta'(\mathbf{a})$,

Table 2. Results of the Optimal Approximations to Air Mass for the Uniparametric Functional Forms and the Atmospheric Models Considered^a

Functional Form	USSA-76			Exponential Profile			Quartic Profile		
	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'
DR-1	1.21×10^{-1}	6.64210×10^2	1.70	1.58×10^{-1}	6.37650×10^2	2.21	1.14×10^{-1}	6.71715×10^2	1.62
Ro-1	4.00×10^{-1}	3.17861×10^1	6.37	4.33×10^{-1}	3.11347×10^1	6.80	3.94×10^{-1}	3.19657×10^1	6.32
HK-1	1.21×10^{-1}	1.50668×10^{-3}	1.70	1.58×10^{-1}	1.56949×10^{-3}	2.21	1.14×10^{-1}	1.48984×10^{-3}	1.62
Br-1	4.88×10^{-1}	1.30076	-5.89	4.47×10^{-1}	1.32576	-5.28	4.97×10^{-1}	1.29372	-6.00
LS-1	3.72×10^{-1}	3.02372×10^1	5.92	4.06×10^{-1}	2.95835×10^1	6.35	3.67×10^{-1}	3.04172×10^1	5.87
Li-1	4.00×10^{-1}	4.95365×10^{-4}	6.37	4.33×10^{-1}	5.16333×10^{-4}	6.80	3.94×10^{-1}	4.89811×10^{-4}	6.32
GM-1	3.99×10^{-1}	9.89464×10^{-4}	6.37	4.32×10^{-1}	1.03130×10^{-3}	6.80	3.93×10^{-1}	9.78382×10^{-4}	6.32

^aHere $\delta'_m = \delta'(\mathbf{a}_o)$ is the minimum value of the approximation distance $\delta'(\mathbf{a})$ provided by the optimal vector \mathbf{a}_o , and $E' = E(z_M, \mathbf{a}_o)$ is the error $E(z, \mathbf{a}_o)$ at the ending point of the range of validity of the resulting obliquity functions, namely, $z_M = 90^\circ$, where the deviations are maximal. The value of the component of \mathbf{a}_o for form Br-1 is only valid when the zenith distance z is introduced in degrees.

Table 3. Results of the Optimal Approximations to Air Mass for the Biparametric Functional Forms and the Atmospheric Models Considered^a

Functional Form	USSA-76			Exponential Profile			Quartic Profile		
	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'
Rz-2	3.92×10^{-2}	2.59392×10^{-2} 1.26067×10^1	-3.94×10^{-1}	3.50×10^{-2}	2.61173×10^{-2} 1.11893×10^1	-3.52×10^{-1}	3.83×10^{-2}	2.58620×10^{-2} 1.29129×10^1	-3.77×10^{-1}
GB-2	1.14×10^{-1}	-2.44277×10^{-2} -9.83519×10^{-2}	8.81×10^{-1}	1.30×10^{-1}	-1.86530×10^{-2} -1.00462×10^{-1}	1.21	1.12×10^{-1}	-2.54789×10^{-2} -9.79932×10^{-2}	8.50×10^{-1}
Br-2	2.74×10^{-1}	1.15623 1.60372	-3.16	2.43×10^{-1}	1.14610 1.61503	-2.73	2.79×10^{-1}	1.15841 1.59919	-3.22
GM-2	3.04×10^{-1}	9.98304×10^{-1} 9.82622×10^{-7}	5.58	3.37×10^{-1}	9.98216×10^{-1} 1.10912×10^{-6}	6.01	2.99×10^{-1}	9.98329×10^{-1} 9.47796×10^{-7}	5.52
Ra-2	1.89×10^{-2}	1.71289×10^{-3} 6.48613×10^{-2}	2.92×10^{-1}	1.52×10^{-2}	1.99937×10^{-3} 7.53987×10^{-2}	2.26×10^{-1}	2.15×10^{-2}	1.65996×10^{-3} 6.30188×10^{-2}	3.26×10^{-1}
He-2	2.71×10^{-2}	1.56110×10^{-3} 5.88790×10^{-2}	3.86×10^{-1}	2.32×10^{-2}	1.80374×10^{-3} 6.77448×10^{-2}	3.16×10^{-1}	2.97×10^{-2}	1.51575×10^{-3} 5.73172×10^{-2}	4.22×10^{-1}

^aHere $\delta'_m = \delta'(\mathbf{a}_o)$ is the minimum value of the approximation distance $\delta'(\mathbf{a})$ provided by the optimal vector \mathbf{a}_o , and $E' = E(z_M, \mathbf{a}_o)$ is the error $E(z, \mathbf{a}_o)$ at the ending point of the range of validity of the resulting obliquity functions, namely, $z_M = 90^\circ$, where the deviations are maximal. The values of the components of \mathbf{a}_o for forms GB-2 and Br-2 are only valid when the zenith distance z is introduced in radians and degrees, respectively.

the corresponding values of that distance, $\delta'_m = \delta'(\mathbf{a}_o)$, are recorded in the second, fifth, and eighth columns. It must be noted that, for each functional form examined, once the optimal vector is adopted, the maximum deviation of the resulting obliquity function was always located at the end of its range of validity, that is, at the horizon ($z = 90^\circ$). The numeric values of the corresponding error, $E' = E(90^\circ, \mathbf{a}_o)$, are presented in the fourth, seventh, and tenth columns. Tables 3–6 are analogous to Table 2 but referring to the forms depending on two, three, four, and more than four parameters, respectively.

[72] Table 2 allows us to infer that functional forms DR-1 and HK-1 provide better approximations than forms LS-1, GM-1, Ro-1, Li-1, and Br-1. This is quite a surprising result, while form DR-1 arrived more than a century before forms Br-1 and LS-1. If we observe in detail the results of forms DR-1 and HK-1, we find that both provide the same accuracy,

at least for the number of significant figures displayed. This is not surprising since, as revealed in Appendix C, the latter is a slightly simplified version of the former. In any case, unlike form HK-1, form DR-1 has the advantage of providing the exact value, namely, one, at the zenith. Similarly, the results for forms Ro-1, Li-1, and GM-1 reflect the close relationship between them, as shown in Appendix D.

[73] With respect to the two-parameter functional forms, Table 3 reveals that form Ra-2 yields finer fits than forms He-2, Rz-2, and, particularly, than forms GB-2, Br-2, and GM-2. It is clear, at a glance, that the results achieved with forms GM-2 and Br-2 are downright poor. In their defense, it should be said, however, that *Green and Martin* [1966] focused the development of their obliquity functions to atmospheric species whose density profiles have a peak, such as ozone, and that *Bruin* [1981] derived his formulas to

Table 4. Results of the Optimal Approximations to Air Mass for the Triparametric Functional Forms and the Atmospheric Models Considered^a

Functional Form	USSA-76			Exponential Profile			Quartic Profile		
	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'
Ka-3	4.73×10^{-3}	1.08290 9.72125×10^1	6.90×10^{-2}	3.54×10^{-3}	1.32885 9.81402×10^1	4.91×10^{-2}	5.27×10^{-3}	1.42113 9.74792×10^1	7.40×10^{-2}
Rw-3	6.69×10^{-3}	1.88255 5.58731 3.74108	-2.10×10^{-1}	6.37×10^{-3}	1.86900 6.43844 4.28346	-1.86×10^{-1}	7.39×10^{-3}	1.98532 5.45923 3.66987	-2.30×10^{-1}
Ti-3	1.04×10^{-2}	1.12089 1.24711×10^{-1} 9.54603×10^{-1}	1.46×10^{-1}	8.40×10^{-3}	1.11149 1.42806×10^{-1} 9.48400×10^{-1}	1.12×10^{-1}	1.15×10^{-2}	1.13072 1.24181×10^{-1} 9.55065×10^{-1}	1.59×10^{-1}
Gu-3	5.35×10^{-3}	-1.94756 6.82815×10^{-3} 9.66433×10^1	7.36×10^{-2}	4.18×10^{-3}	-1.74266 7.56920×10^{-3} 9.74059×10^1	5.44×10^{-2}	5.85×10^{-3}	-2.07799 8.78040×10^{-3} 9.69125×10^1	7.89×10^{-2}
Ma-3	3.67×10^{-3}	1.66503 1.05940×10^{-3} 3.72465×10^{-3}	-4.43×10^{-2}	3.65×10^{-3}	1.62347 1.22016×10^{-3} 5.00721×10^{-3}	-3.94×10^{-2}	4.34×10^{-3}	1.76261 1.01940×10^{-3} 3.34877×10^{-3}	-5.59×10^{-2}
He-3	2.05×10^{-3}	9.20310×10^{-2} 1.07597×10^{-3} 3.93441×10^{-3} 9.58484×10^{-2}	-3.28×10^{-2}	1.83×10^{-3}	1.08060 1.24451×10^{-3} 5.37036×10^{-3} 1.13807×10^{-1}	-2.71×10^{-2}	2.82×10^{-3}	8.56681×10^{-2} 1.03401×10^{-3} 3.51756×10^{-3} 8.88322×10^{-2}	-4.46×10^{-2}

^aHere $\delta'_m = \delta'(\mathbf{a}_o)$ is the minimum value of the approximation distance $\delta'(\mathbf{a})$ provided by the optimal vector \mathbf{a}_o , and $E' = E(z_M, \mathbf{a}_o)$ is the error $E(z, \mathbf{a}_o)$ at the ending point of the range of validity of the resulting obliquity functions, namely, $z_M = 90^\circ$, where the deviations are maximal. The values of the components of \mathbf{a}_o for forms Ka-3, Rw-3, Ti-3, and Gu-3 are only valid when the zenith distance z is introduced in degrees.

Table 5. Results of the Optimal Approximations to Air Mass for the Tetraparametric Functional Forms and the Atmospheric Models Considered^a

Functional Form	USSA-76			Exponential Profile			Quartic Profile		
	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'
Yo-4	7.08×10^{-3}	9.91045×10^{-1} 8.16979×10^{-2} 7.51054×10^{-2} 2.14661×10^{-3}	9.91×10^{-2}	5.56×10^{-3}	9.91977×10^{-1} 9.33013×10^{-2} 8.66147×10^{-2} 2.46415×10^{-3}	7.36×10^{-2}	7.84×10^{-3}	9.90081×10^{-1} 8.13892×10^{-2} 7.43226×10^{-2} 2.13158×10^{-3}	1.08×10^{-1}
Gu-4	4.73×10^{-3}	1.08290 0.00000 9.72125×10^1 1.88255	6.90×10^{-2}	3.54×10^{-3}	1.32885 0.00000 9.81402×10^1 1.86900	4.91×10^{-2}	5.27×10^{-3}	1.42113 0.00000 9.74792×10^1 1.98532	7.40×10^{-2}
Kr-4	2.18×10^{-4}	4.81107×10^2 8.23426×10^{-1} 9.22258×10^{-2} 3.34340×10^{-3}	3.40×10^{-3}	2.76×10^{-4}	4.16371×10^2 1.08873 1.05403×10^{-1} 3.59815×10^{-3}	4.20×10^{-3}	2.88×10^{-4}	5.05194×10^2 6.09844×10^{-1} 7.77765×10^{-2} 2.68419×10^{-3}	4.68×10^{-3}
He-4	1.43×10^{-4}	1.03605×10^{-3} 2.19641×10^{-3} 7.90946×10^{-3} 1.42208×10^{-1}	2.49×10^{-3}	1.68×10^{-4}	1.19575×10^{-3} 2.94534×10^{-3} 1.07179×10^{-2} 1.64865×10^{-1}	2.84×10^{-3}	3.14×10^{-4}	9.89339×10^{-4} 1.87044×10^{-3} 5.84565×10^{-3} 1.18259×10^{-1}	5.25×10^{-3}

^aHere $\delta'_m = \delta'(\mathbf{a}_o)$ is the minimum value of the approximation distance $\delta'(\mathbf{a})$ provided by the optimal vector \mathbf{a}_o , and $E' = E(z_M, \mathbf{a}_o)$ is the error $E(z, \mathbf{a}_o)$ at the ending point of the range of validity of the resulting obliquity functions, namely, $z_M = 90^\circ$, where the deviations are maximal. The values of the components of \mathbf{a}_o for form Gu-4 are only valid when the zenith distance z is introduced in degrees.

estimate the atmospheric extinction of star light bearing in mind small altitudes (i.e., large zenith distances).

[74] Regarding the three-parameter functional forms, the figures of Table 4 show that form He-3 provides better adjustments than forms Ma-3, Ka-3, Gu-3, Rw-3, and, notably, than form Ti-3. Let us recall here that form He-3 arose in the context of modeling the atmospheric delay of electromagnetic signals.

[75] The results presented in Table 5 are not as conclusive as in the former cases. Thus, for the USSA-76 and the exponential profile, form He-4 provides better approximations than form Kr-4 and, particularly, than forms Gu-4 and Yo-4 (by the way, notice that form Gu-4 degenerated to the triparametric form Ka-3). In contrast, form Kr-4 showed the highest accuracy for the quartic profile. In summary, we

have found that form He-4 prevailed over form Kr-4 for two of the three atmospheric models adopted, including the most reliable of them, the USSA-76. Notice that, like form He-3, form He-4 was first adopted for modeling the atmospheric delay of radio signals.

[76] With regard to the functional forms with more than four parameters, Table 6 demonstrates that form He-5 provides much finer fits than form Yo-6 and, even more notably, than form Do-5. This is the first work in which form He-5 has been adopted.

[77] Taken together, Tables 2–6 indicate that accuracy of the best i -parameter functional form is roughly one order of magnitude higher than that of the best $(i - 1)$ -parameter functional form. Thus, the pentaparametric form He-5 is four

Table 6. Results of the Optimal Approximations to Air Mass for the Functional Forms With More Than Four Parameters and the Atmospheric Models Considered^a

Functional Form	USSA-76			Exponential Profile			Quartic Profile		
	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'	δ'_m	\mathbf{a}_o	E'
Do-5	3.04×10^{-2}	4.50136×10^{-1} -2.16910 2.48592 -3.28380 2.69572	-1.62×10^{-1}	2.82×10^{-2}	4.81606×10^{-1} -2.23685 2.57160 -3.33015 2.77005	-1.65×10^{-1}	3.03×10^{-2}	4.49112×10^{-1} -2.16618 2.48031 -3.28036 2.69021	-1.55×10^{-1}
He-5	1.52×10^{-5}	1.03146×10^{-3} 2.02572×10^{-3} 3.94105×10^{-3} 1.60400×10^{-2} 2.09689×10^{-1}	-3.24×10^{-4}	1.59×10^{-5}	1.18863×10^{-3} 2.66326×10^{-3} 4.91851×10^{-3} 1.80539×10^{-2} 2.17046×10^{-1}	-2.88×10^{-4}	3.82×10^{-5}	9.82347×10^{-4} 1.67577×10^{-3} 2.59409×10^{-3} 8.49924×10^{-3} 1.46108×10^{-1}	-7.05×10^{-4}
Yo-6	1.54×10^{-4}	1.00162 1.27857×10^{-1} 6.91864×10^{-3} 1.28435 7.67092×10^{-3} 1.81303×10^{-4}	-2.53×10^{-3}	1.08×10^{-4}	1.00179 1.57018×10^{-1} 9.87180×10^{-3} 1.57634×10^{-1} 1.06970×10^{-2} 2.60205×10^{-4}	-1.48×10^{-3}	2.25×10^{-4}	1.00184 1.26853×10^{-1} 7.13885×10^{-3} 1.27743×10^{-1} 7.74165×10^{-3} 1.86423×10^{-4}	-3.57×10^{-3}

^aHere $\delta'_m = \delta'(\mathbf{a}_o)$ is the minimums value of the approximation distance $\delta'(\mathbf{a})$ provided by the optimal vector \mathbf{a}_o , and $E' = E(z_M, \mathbf{a}_o)$ is the error $E(z, \mathbf{a}_o)$ at the ending point of the range of validity of the resulting obliquity functions, namely, $z_M = 90^\circ$, where the deviations are maximal. The values of the components of \mathbf{a}_o for form Do-5 are only valid when the zenith distance z is introduced in radians.

orders of magnitude more accurate than the uniparametric form DR-1.

8. Discussion

[78] As advanced in section 1, apart from the article by *Rapp-Arrarás and Domingo-Santos* [2008], there were no published works comparing functional forms to approximate the air mass with regard to the first error source established by *Yan et al.* [2002]. These researchers calibrated and compared five functional forms (by the way, all of them considered here) on the basis of the air mass table by *Kasten and Young* [1989], so in terms of discrete variable.

[79] For this review, we have compared 26 functional forms in combination with three atmospheric profiles. The air mass computations have been done for an observer at sea level and the median wavelength of the Sun's spectral irradiance ($0.7274 \mu\text{m}$), while the fits of the forms have been done in terms of continue variable and absolute error. As a result, the most suitable are among the uniparametric ones, form Dr-1; among the biparametric ones, form Ra-2; among the triparametric ones, form He-3; among the tetraparametric ones, form He-4; and among those with more than four parameters, form He-5. All these functional forms are exact at the zenith by default. On the basis of them, for the USSA-76, we have obtained a one-, a two-, a three-, a four-, and a five-parameter obliquity function whose maximum deviations are, respectively, 1.70 , 2.91×10^{-1} , 3.28×10^{-2} , 2.49×10^{-3} , and 3.24×10^{-4} , all of them occurring at the horizon. For the sake of comparison, consider that the most widely used obliquity function, the three-parameter formula by *Kasten and Young* [1989], yields a maximum deviation of 1.67×10^{-1} , also at the horizon.

[80] Although deviations smaller than 0.1 are negligible for most purposes, the accuracy of these formulas can be deceptive. In a pioneer work, *Link* [1937] calculated the air mass at large zenith angles for several latitudes, seasons, and meteorological conditions. From his results, one infers that formulas like the above mentioned may be in error by over 10% near the horizon if the actual atmosphere deviates markedly from the underlying model. Moreover, as we know (section 5.1), for a given atmospheric profile, the air mass varies slightly with the observer's height.

[81] *Link and Neužil* [1969] elaborated a family of detailed tables for estimating the air mass, as a function of the apparent zenith distance and the observer's height, for 15 climatic conditions corresponding to various latitudes and seasons. In principle, one may consider that these tables should serve to adjust the values provided by our formulas to other latitudes, seasons, and heights. However, the air mass tables of *Link and Neužil* [1969] are not widely available, are not user-friendly, and neglect the dependence on the wavelength of the radiation. All this leads to the following conclusion: After identifying the best functional forms for fit, the next challenge would be to choose a set of suitable atmospheric profiles and would be to obtain their respective obliquity functions, whose parameters would be themselves functions of certain variables (height, temperature, pressure, humidity, and wavelength). This would provide quite accurate approximation formulas when the local conditions are known [*Young*, 1994].

[82] After their limited comparison, *Rapp-Arrarás and Domingo-Santos* [2008] concluded that form He-3 provides

better approximations than forms Ka-3, Gu-3, and Ma-3. In addition, they found that form He-4 offers an accuracy of one order of magnitude higher than form He-3. Hence, although *Rapp-Arrarás and Domingo-Santos* [2008] performed their fits to the air mass data in terms of the relative error, their results are completely consistent with the results presented here.

Appendix A: Relationship Between the Form of Spilker and Form DR-1

[83] In order to estimate the atmospheric attenuation of GPS signals, *Spilker* [1996] proposed an obliquity function based on the model

$$f(\gamma) = \frac{2 + a}{\sqrt{\sin^2 \gamma + 2a + a^2 + \sin \gamma}}, \quad (\text{A1})$$

where γ is the satellite altitude (elevation angle) and a is the ratio between the atmospheric effective height and the Earth's radius. Introducing the change of variable $a = 1/b$, we obtain

$$\begin{aligned} f(\gamma) &= \frac{2 + 1/b}{\sqrt{\sin^2 \gamma + 2/b + 1/b^2 + \sin \gamma}} \\ &= \frac{(2b + 1)/b}{\sqrt{\sin^2 \gamma + (2b + 1)/b^2 + \sin \gamma}}, \end{aligned} \quad (\text{A2})$$

which, as pointed out by *Makhotkin* [1960, equation (3)], is equivalent to

$$f(\gamma) = b \left(\sqrt{\sin^2 \gamma + \frac{2b + 1}{b^2}} - \sin \gamma \right). \quad (\text{A3})$$

[84] At this point, if we adopt the zenith distance z as the independent variable ($\sin \gamma = \cos z$) and we operate, then

$$\begin{aligned} f(z) &= b \left(\sqrt{\cos^2 z + \frac{2b + 1}{b^2}} - \cos z \right) \\ &= \sqrt{b^2 \cos^2 z + 2b + 1} - b \cos z, \end{aligned} \quad (\text{A4})$$

which proves that the functional form by *Spilker* [1996] is equivalent to form DR-1.

Appendix B: Relationship Between the Form of Wittmann, the Model of Cabannes and Dufay, and Form Ro-1

[85] On the basis of the work of *Saar* [1973], *Wittmann* [1980] proposed an obliquity function following the pattern

$$f(z) = \frac{a}{\sqrt{\sin^2 z + a^2 \cos^2 z}}, \quad (\text{B1})$$

where a is a simple numeric parameter. If we consider that $\sin^2 z = 1 - \cos^2 z$ and that, by definition, $\sec z = 1/\cos z$, then

$$\begin{aligned} f(z) &= \frac{a}{\sqrt{1 - \cos^2 z + a^2 \cos^2 z}} = \frac{a}{\cos z \sqrt{\frac{1}{\cos^2 z} - 1 + a^2}} \\ &= \frac{a \sec z}{\sqrt{\sec^2 z - 1 + a^2}}, \end{aligned} \quad (\text{B2})$$

which means that the form by *Wittmann* [1980] is mathematically equivalent to form Ro-1.

[86] On the other hand, by an elementary geometric reasoning, *Cabannes and Dufay* [1925] deduced that the relative optical ozone mass can be expressed as

$$f(\zeta) = \sec \zeta, \quad (\text{B3})$$

where ζ ($0^\circ \leq \zeta < 90^\circ$) is an auxiliary angle that verifies that

$$\sin \zeta = b \sin z, \quad (\text{B4})$$

where b being a parameter whose value, slightly under 1, depends on the terrestrial radius and the effective height of the ozone layer. Now taking into account that $\sec \zeta = 1/\cos \zeta$ by definition and that $\cos \zeta = \sqrt{1 - \sin^2 \zeta}$, we will obtain the following expression:

$$f(\zeta) = \frac{1}{\cos \zeta} = \frac{1}{\sqrt{1 - \sin^2 \zeta}}, \quad (\text{B5})$$

which, combined with equation (B4) and the change of variable $c = b^2$, makes it possible to arrive at

$$f(z) = \frac{1}{\sqrt{1 - b^2 \sin^2 z}} = \frac{1}{\sqrt{1 - c \sin^2 z}}, \quad (\text{B6})$$

that is, the usual form of the approximation formulas for the relative optical ozone mass [e.g., *Green and Martin*, 1966, equation (62); *Young*, 1974; *Gushchin and Vinogradova*, 1983, chapter 2; *Schaefer*, 1993; *Bernhard et al.*, 2005], as well as an occasional one for the relative optical mass of nitrogen dioxide [*Tomasi et al.*, 1998]. Now if we introduce the change of variable $c = 1 - 1/a^2$, we can finally check that

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{1 - (1 - 1/a^2) \sin^2 z}} = \frac{1}{\sqrt{\cos^2 z + 1/a^2 \sin^2 z}} \\ &= \frac{a}{\sqrt{a^2 \cos^2 z + \sin^2 z}}, \end{aligned} \quad (\text{B7})$$

so the model for estimating the relative optical ozone mass by *Cabannes and Dufay* [1925] is formally equivalent to the functional form by *Wittmann* [1980] and, hence, to form Ro-1.

Appendix C: Relationship Between Forms HK-1 and DR-1

[87] Starting from the expression for atmospheric extinction of GPS signals proposed by *Spilker* [1996],

$$f(\gamma) = \frac{2 + a}{\sqrt{\sin^2 \gamma + 2a + a^2 + \sin \gamma}}, \quad (\text{C1})$$

and considering that the value of parameter a is much smaller than 2, we can conclude that the term “ a^2 ” is negligible in comparison with $2a$. Applying this property, we obtain

$$f(\gamma) = \frac{2 + a}{\sqrt{\sin^2 \gamma + 2a + \sin \gamma}}, \quad (\text{C2})$$

which proves that form HK-1 is a simplified version of the form by *Spilker* [1996] and, therefore (see Appendix A), of the form DR-1.

Appendix D: Relationship Between Forms GR-1, Li-1, and Ro-1

[88] Starting from the usual form of the approximation formulas for the relative optical ozone mass,

$$f(z) = \frac{1}{\sqrt{1 - a \sin^2 z}}, \quad (\text{D1})$$

where a is slightly less than 1 (see Appendix B), through the change of variable $a = 1/(1 + b)^2$, the trigonometric identity $\sin^2 z = 1 - \cos^2 z$, and some algebraic operations, we will arrive at

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{1 - 1/(1 + b)^2 \sin^2 z}} = \frac{1 + b}{\sqrt{(1 + b)^2 - \sin^2 z}} \\ &= \frac{1 + b}{\sqrt{1 + 2b + b^2 - (1 - \cos^2 z)}} = \frac{1 + b}{\sqrt{2b + b^2 + \cos^2 z}}, \end{aligned} \quad (\text{D2})$$

where b is slightly greater than 0. The small value of b implies that term b^2 can be neglected in comparison with term “ $2b$,” thus obtaining form Li-1,

$$f(z) = \frac{1 + b}{\sqrt{2b + \cos^2 z}}. \quad (\text{D3})$$

[89] Indeed, the parameter b can be neglected in comparison with 1. This fact, along with the change of variable $c = 2b$, allows us to arrive at form GM-1,

$$f(z) = \frac{1}{\sqrt{2b + \cos^2 z}} = \frac{1}{\sqrt{c + \cos^2 z}}. \quad (\text{D4})$$

[90] Therefore, either form Li-1 or form GM-1 can be considered as simplifications of form Ro-1.

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