The interacting boson model with SU(3) charge symmetry and its application to even–even N≈Z nuclei

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Abstract

The isospin-invariant interacting boson model IBM-3 is analyzed in situations where $SU_T(3)$ charge symmetry [or, equivalently, $U_L(6)$ sd symmetry] is conserved. Analytic expressions for energies, electromagnetic transitions, two-nucleon transfer probabilities, and boson-number expectation values are obtained for the three possible dynamical symmetry limits, $U(5)$, $SU(3)$, and $O(6)$. Results found in IBM-3 are related to corresponding ones in IBM-1 and IBM-2. Numerical calculations are presented for $f_{7/2}$-shell nuclei and some features that distinguish IBM-3 from its predecessors IBM-1 and IBM-2 are pointed out.

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I. INTRODUCTION

The interacting boson model (IBM) [1,2] was originally proposed as a phenomenological model couched in terms of s and d bosons of which the microscopic structure (e.g., in terms of the shell model) was unknown. This state of affairs quickly changed with the realization that the bosons could be interpreted as correlated (Cooper) pairs with angular momentum $J = 0$ and $J = 2$, formed by the nucleons in the valence shell. A connection with an underlying shell-model picture could be established by making a distinction between a proton pair (π boson) and a neutron pair (ν boson), the resulting model being referred to as the proton–neutron interacting boson model or IBM-2 [3–5].

The IBM-2 has been applied extensively and successfully to even–even medium-mass and heavy nuclei [2] where the protons and neutrons occupy different valence shells. In the latter situation it is natural to assume correlated proton–proton and neutron–neutron pairs, and to include (longe-range) proton–neutron correlations through a quadrupole interaction in the Hamiltonian. In lighter $N \approx Z$ nuclei where the protons and the neutrons occupy the same valence shell, this approach no longer is valid since there is no reason not to include proton–neutron $T = 1$ pairs in such nuclei. The inclusion of proton–neutron $T = 1$ pairs (δ bosons) has indeed been proposed by Elliott and White [6] and the resulting model has been named IBM-3. Since the IBM-3 contains a complete isospin triplet of $T = 1$ bosons, it is possible to construct IBM-3 Hamiltonians that conserve isospin symmetry and this feature can be exploited to establish a more direct correspondence between the IBM-3 and the shell model [7]. As a final refinement of the interacting boson model, Elliott and Evans [8] proposed the inclusion of a proton–neutron $T = 0$ pair (σ boson) leading to a version referred to as IBM-4. Both $T = 0$ and $T = 1$ proton–neutron ($T_z = 0$) bosons play an important role in $N = Z$ nuclei but their importance decreases with increasing difference $|N - Z|$. Furthermore, $T = 0$ bosons are crucial in odd–odd nuclei while even–even nuclei seem to be adequately described with $T = 1$ bosons only. These observations then define the scope of the present work: the IBM-3 is geared towards applications to even–even $N \approx Z$

nuclei although its results can be extended to isobaric analog states in neighboring odd–odd nuclei.

The IBM-3 has a rich algebraic structure that starts from the dynamical algebra $U(18)$ and allows several dynamical symmetries containing the $O_T(3)$ subalgebra, necessary to conserve isospin symmetry. In this paper the dynamical symmetries are analyzed that arise after the reduction of $U(18)$ to the direct product $U_L(6) \otimes SU_T(3)$ and thus assume a separation of the orbital (sd) and isospin sectors. The latter approximation plays the same role in IBM-3 as does the assumption of F-spin symmetry in IBM-2 [3]. In realistic IBM-2 calculations the lowest-energy states are (approximately) symmetric under the exchange of proton and neutron indices, that is, they have a large component in the symmetric representation $[N]$ of $U_L(6)$ (or, equivalently, a large component with maximal F spin, $F_{\max} = N/2$). Non-symmetric states $[N-1,1]$ occur at higher energies [9]. Although realistic IBM-2 Hamiltonians may contain important F-spin mixing interactions, F spin is usually an approximately conserved symmetry because of the existence of a relatively large Majorana interaction. The Majorana operator is diagonal in $U_L(6)$ [or $SU_F(2)$] and separates the different $U_L(6)$ representations $[N], [N-1,1], \ldots$ [or $SU_F(2)$ representations with $F = F_{\max}, F_{\max}-1, \ldots$]. As a result, the orbital and charge (or F-spin) spaces are approximately decoupled in IBM-2. The situation is very similar in IBM-3, the $U_L(6) \otimes SU_F(2)$ of IBM-2 being replaced by $U_L(6) \otimes SU_T(3)$, which by analogy can be expected to be an approximate symmetry algebra. This is indeed confirmed in some realistic IBM-3 calculations for pf-shell nuclei [7,10–12] which show that, at low energy, states can be approximately classified by $U_L(6)$ representations $[N]$ or $[N-1,1]$. Representations of lower symmetry such as $[N-2,2]$ are less well realized but these are not considered in this paper.

The structure of this paper is as follows. First, in Section II, a general overview of the algebraic structure of IBM-3 is given; in Section III the IBM-3 Hamiltonian is specified in different representations. In Section IV a brief review is given of the three symmetries that are analyzed in this paper. Sections V and VI are devoted to the analysis of electromagnetic transitions and two-nucleon transfer probabilities, respectively, and in Section VII analytic expressions are given for boson-number expectation values. In Section VIII the most relevant predictions of IBM-3 (as contrasted with IBM-1 and IBM-2) are pointed out and results of numerical calculations in the $f_{7/2}$ shell are shown. Finally, conclusions are presented in Section IX.

II. ALGEBRAIC STRUCTURE

The basic building blocks of the IBM-3 are assigned the orbital angular momenta $l = 0$ and $l = 2$ (s and d bosons) and isospin $T = 1$ with isospin projection $\mu = +1, 0, -1$ for the π, δ, and ν boson (the proton–proton, neutron–proton, and neutron–neutron pairs), respectively. The corresponding creation and annihilation operators can be written as

$$b^{\dagger}_{lm,1\mu}, \qquad b_{lm,1\mu}. \tag{1}$$

These operators are assumed to satisfy the customary boson commutation relations

$$[b_{lm,1\mu}, b^{\dagger}_{l'm',1\mu'}] = \delta_{ll'}\delta_{mm'}\delta_{\mu\mu'}, \qquad [b^{\dagger}_{lm,1\mu}, b^{\dagger}_{l'm',1\mu'}] = [b_{lm,1\mu}, b_{l'm',1\mu'}] = 0. \tag{2}$$

With the operators (1) 324 bilinear combinations,

$$b^{\dagger}_{lm,1\mu} b_{l'm',1\mu'}, \tag{3}$$

can be constructed that generate the algebra $U(18)$, as can be shown explicitly from the commutation relations

$$\begin{aligned} &[b^{\dagger}_{lm,1\mu} b_{l'm',1\mu'}, b^{\dagger}_{l''m'',1\mu''} b_{l'''m''',1\mu'''}] \\ &= b^{\dagger}_{lm,1\mu} b_{l'''m''',1\mu'''} \delta_{l'l''}\delta_{m'm''}\delta_{\mu'\mu''} - b^{\dagger}_{l''m'',1\mu''} b_{l'm',1\mu'} \delta_{ll'''}\delta_{mm'''}\delta_{\mu\mu'''}. \end{aligned} \tag{4}$$

The operators of all physical observables will be expressed in terms of the generators (3) and, consequently, the dynamical algebra of IBM-3 is $U(18)$.

Two important invariances occur in the context of IBM-3: rotational and isospin invariance. The first one is an exact symmetry and leads to conservation of total angular momentum, here denoted as L; the second gives rise to the isospin quantum number T. Isospin is only an approximate symmetry (mainly broken by the Coulomb interaction) but throughout this paper it is assumed to be exact. Given these two invariances it is convenient to take combinations of the operators (3) that have definite transformation properties under rotations in physical and isospin space. The generators of $U(18)$ can thus also be written as

$$(b^\dagger_{l,1} \times \tilde{b}_{l',1})^{(L,T)}_{M_L,M_T} = \sum_{mm'\mu\mu'} \langle lm\; l'm' | LM_L \rangle \langle 1\mu\; 1\mu' | TM_T \rangle b^\dagger_{lm,1\mu} \tilde{b}_{l'm',1\mu'}, \tag{5}$$

where the symbol between angle brackets is a Clebsch–Gordan coefficient and $\tilde{b}_{lm,1\mu} \equiv (-)^{l-m+1-\mu} b_{l-m,1-\mu}$ has the appropriate transformation properties under rotations in physical and isospin space. For completeness we also give the commutation relations among the coupled generators of $U(18)$:

$$\begin{aligned}
&[(b^\dagger_{l,1} \times \tilde{b}_{l',1})^{(L,T)}_{M_L,M_T}, (b^\dagger_{l'',1} \times \tilde{b}_{l''',1})^{(L',T')}_{M'_L,M'_T}] \\
&= \sum_{L''M''_LT''M''_T} \hat{L}\hat{L}'\hat{T}\hat{T}' \langle LM_L\; L'M'_L | L''M''_L \rangle \langle TM_T\; T'M'_T | T''M''_T \rangle \\
&\times \left[(-)^{L''+T''} \begin{Bmatrix} L & L' & L'' \\ l''' & l & l' \end{Bmatrix} \begin{Bmatrix} T & T' & T'' \\ 1 & 1 & 1 \end{Bmatrix} \delta_{l'l''} (b^\dagger_{l,1} \times \tilde{b}_{l''',1})^{(L'',T'')}_{M''_L,M''_T} \right. \\
&\left. \quad -(-)^{L+L'+T+T'} \begin{Bmatrix} L & L' & L'' \\ l'' & l' & l \end{Bmatrix} \begin{Bmatrix} T & T' & T'' \\ 1 & 1 & 1 \end{Bmatrix} \delta_{ll'''} (b^\dagger_{l'',1} \times \tilde{b}_{l',1})^{(L'',T'')}_{M''_L,M''_T} \right],
\end{aligned} \tag{6}$$

where $\hat{L} \equiv \sqrt{2L+1}$ and the symbol between curly brackets is a Racah coefficient.

The dynamical algebra $U(18)$ has a rich substructure. One is, however, not interested in all possible algebraic decompositions of $U(18)$ but only in those that conserve angular momentum and isospin, that is, the ones containing the angular momentum algebra $O_L(3)$ and the isospin algebra $O_T(3)$. A possible way to impose these symmetries is to consider the reduction

$$\begin{array}{ccccc}
U(18) & \supset & U_L(6) & \otimes & SU_T(3) \\
\downarrow & & \downarrow & & \downarrow \\
[N] & & [N_1,N_2,N_3] & & (\lambda_T,\mu_T)
\end{array}, \tag{7}$$

which corresponds to a decomposition of states into an orbital (or *sd*) and an isospin part. Because of the overall symmetry in $U(18)$, the $U_L(6)$ and $SU_T(3)$ representations are the same (i.e., they correspond to the same Young diagram) and this leads to $U_L(6)$ representations $[N_1, N_2, N_3]$ that can have up to three rows with length N_i and with $N_1+N_2+N_3 = N$. This situation should be compared with IBM-2 where at most two-rowed representations can occur in $U(6)$ and IBM-1 which only contains symmetric (one-rowed) $U(6)$ representations. In (7) Elliott's $SU(3)$ labels are used [13] which are related to the usual row labels by $\lambda_T = N_1 - N_2$ and $\mu_T = N_2 - N_3$.

The generators of $U_L(6)$ and $SU_T(3)$ are obtained by contracting in the isospin and orbital indices, respectively. The following coupled form of the generators results:

$$\begin{aligned} U_L(6) &: (b^\dagger_{l,1} \times \tilde{b}_{l',1})^{(L,0)}_{M_L,0}, \\ U_T(3) &: \sum_l \sqrt{2l+1}(b^\dagger_{l,1} \times \tilde{b}_{l,1})^{(0,T)}_{0,M_T}, \end{aligned} \tag{8}$$

with $l, l' = 0, 2$, and L and T running over all values compatible with angular momentum coupling. The $SU_T(3)$ algebra consists of the generators of $U_T(3)$ with $T = 1$ and $T = 2$.

The classification (7) is *sufficient* to ensure invariance under rotations in physical and isospin space. It is, however, not a *necessary* condition and classes of Hamiltonians have been shown to exist [14] that conserve angular momentum and isospin but do not proceed via the reduction (7). In this paper the latter reductions are not considered but only those that conserve $SU_T(3)$ which can be considered as a charge symmetry algebra. Note that $SU_T(3)$ is not a fundamental symmetry such as angular momentum or isospin, and that it may well be broken by specific boson–boson interactions. The requirement of $SU_T(3)$ charge symmetry obviously restricts the applicability of the results derived here. On the other hand, it must be emphasized that $SU_T(3)$ is equivalent to $U_L(6)$ which is a symmetry of basic importance in the IBM.

III. THE IBM-3 HAMILTONIAN

Any Hamiltonian which is invariant under rotations in physical and isospin space can be written as L- and T-scalar combinations of the generators (5). If up to two-body interactions in the bosons are taken, the most general Hamiltonian is

$$\hat{H} = \sum_{l} \epsilon_l \sqrt{3(2l+1)} (b^\dagger_{l,1} \times \tilde{b}_{l,1})^{(0,0)}_{0,0} + \sum_{\substack{l_1 l_2 l'_1 l'_2 \\ LT}} v^{LT}_{l_1 l_2 l'_1 l'_2} \sqrt{\frac{(2L+1)(2T+1)}{(1+\delta_{l_1 l_2})(1+\delta_{l'_1 l'_2})}} \Big((b^\dagger_{l_1,1} \times b^\dagger_{l_2,1})^{(L,T)} \times (\tilde{b}_{l'_1,1} \times \tilde{b}_{l'_2,1})^{(L,T)} \Big)^{(0,0)}_{0,0}. \tag{9}$$

The coefficients ϵ_0 and ϵ_2 are the s- and d-boson energies, which by virtue of isospin invariance are independent of the nature of the bosons (π, δ, or ν). The coefficients $v^{LT}_{l_1 l_2 l'_1 l'_2}$ are the interaction matrix elements between normalized two-boson states,

$$v^{LT}_{l_1 l_2 l'_1 l'_2} \equiv \langle l_1 l_2; LT | \hat{H} | l'_1 l'_2; LT \rangle. \tag{10}$$

The form (9) is referred to as the standard representation of the IBM-3 Hamiltonian.

The IBM-3 Hamiltonian alternatively can be written in a multipole expansion form as

$$\hat{H}_{\rm mul} = \sum_{l} \eta_l \sqrt{3} \hat{l} (b^\dagger_{l,1} \times \tilde{b}_{l,1})^{(0,0)}_{0,0} + \sum_{LT} \kappa_{LT} \hat{L} \hat{T} (\hat{T}^{(L,T)} \times \hat{T}^{(L,T)})^{(0,0)}_{0,0}, \tag{11}$$

where $\hat{T}^{(L,T)}$ are isoscalar ($T = 0$), isovector ($T = 1$), and isotensor ($T = 2$) multipole operators of the form

$$\hat{T}^{(L,T)}_{M_L,M_T} = \sum_{l_1 l_2} \chi^{LT}_{l_1 l_2} (b^\dagger_{l_1,1} \times \tilde{b}_{l_2,1})^{(L,T)}_{M_L,M_T}. \tag{12}$$

The last term in (11) is not a pure two-body term but it contains one-body pieces as well and therefore the parameters η_l in (11) do not coincide with the single-boson energies ϵ_l of (9).

The standard and multipole forms (9) and (11) are two equivalent ways of writing the IBM-3 Hamiltonian, which can be related as follows:

$$\epsilon_l = \eta_l + \sum_{l'LT} (-)^{L+T} \frac{(2L+1)(2T+1)}{3(2l+1)} \kappa_{LT} \chi^{LT}_{ll'} \chi^{LT}_{l'l},$$

$$v^{LT}_{l_1 l_2 l'_1 l'_2} = \sum_{L'T'} (-)^{L+T+L'+T'} \frac{2(2L'+1)(2T'+1)}{(1+\delta_{l_1 l_2})(1+\delta_{l'_1 l'_2})} \kappa_{L'T'} \left\{ \begin{array}{ccc} 1 & 1 & T \\ 1 & 1 & T' \end{array} \right\}$$
$$\times \left[\chi^{L'T'}_{l_1 l'_1} \chi^{L'T'}_{l_2 l'_2} \left\{ \begin{array}{ccc} l_1 & l_2 & L \\ l'_2 & l'_1 & L' \end{array} \right\} + (-)^{L+T} \chi^{L'T'}_{l_1 l'_2} \chi^{L'T'}_{l_2 l'_1} \left\{ \begin{array}{ccc} l_1 & l_2 & L \\ l'_1 & l'_2 & L' \end{array} \right\} \right]. \tag{13}$$

A third way of parametrizing the IBM-3 Hamiltonian exists which relies on the algebraic (sub)structure of $U_L(6) \otimes SU_T(3)$ and the associated Casimir operators. The starting point is the classification (7) and the possible reductions of the orbital algebra $U_L(6)$ as in IBM-1:

$$U(18) \supset \left(U_L(6) \supset \left\{ \begin{array}{c} U_L(5) \supset O_L(5) \\ SU_L(3) \\ O_L(6) \supset O_L(5) \end{array} \right\} \supset O_L(3) \right) \otimes \big(SU_T(3) \supset O_T(3) \big). \tag{14}$$

The most general IBM-3 Hamiltonian *that conserves* $U_L(6)$ *or* $SU_T(3)$ *symmetry* consists of a combination of Casimir operators of the algebras appearing in (14). If up to two-body terms are considered, the following Casimir form results:

$$\begin{aligned} \hat{H}_{\rm cas} &= A_1 \hat{C}_1[U_L(6)] + A_2 \hat{C}_2[U_L(6)] + B_1 \hat{C}_1[U_L(5)] + B_2 \hat{C}_2[U_L(5)] \\ &+ C_2 \hat{C}_2[SU_L(3)] + D_2 \hat{C}_2[O_L(6)] + E_2 \hat{C}_2[O_L(5)] + F_2 \hat{C}_2[O_L(3)] \\ &+ \alpha_2 \hat{C}_2[SU_T(3)] + \beta_2 \hat{C}_2[O_T(3)], \end{aligned} \tag{15}$$

where $\hat{C}_n[G]$ denotes the nth order Casimir operator of the algebra G. No terms in $\hat{C}_1[O_L(2)]$ or $\hat{C}_1[O_T(2)]$ are added since they would break $O_L(3)$ or $O_T(3)$, which are assumed to be true symmetries of the Hamiltonian. It must be emphasized that $\hat{H}_{\rm cas}$ corresponds to a subclass of the general Hamiltonians (9) or (11) and this is so since all Casimir operators in (15) belong to the reduction scheme (14). As already mentioned, alternative classifications exist that conserve L and T, and Casimir operators of those alternative algebras would be required to construct the most general IBM-3 Hamiltonian. The criterion for the choice of (15) is simply that it consists of all linear and quadratic Casimir operators of subalgebras appearing in (14).

The relation between the Hamiltonians in standard and Casimir form cannot be expressed in a compact way and it is more convenient to write the equations that should be satisfied by the standard parameters in the Hamiltonian (9) in order that it reduces to the form (15). These relations read

$$
\begin{aligned}
&v^{02}_{0022} = v^{00}_{0022}\ ,\ v^{22}_{0222} = v^{22}_{0222}, \\
&v^{02}_{0000} - v^{00}_{0000} = v^{22}_{0202} - v^{20}_{0202} = v^{02}_{2222} - v^{00}_{2222} = v^{22}_{2222} - v^{20}_{2222} = v^{42}_{2222} - v^{40}_{2222}, \\
&v^{00}_{0000} = v^{20}_{0202} - \tfrac{1}{\sqrt{5}} v^{00}_{0022}, v^{02}_{0000} = v^{22}_{0202} - \tfrac{1}{\sqrt{5}} v^{02}_{0022}, \\
&v^{11}_{2222} = -\tfrac{8}{7} v^{40}_{2222} + \tfrac{1}{7} v^{20}_{2222} - \tfrac{3}{\sqrt{14}} v^{20}_{0222} + v^{20}_{0202} + v^{21}_{0202}, \\
&v^{31}_{2222} = -\tfrac{3}{7} v^{40}_{2222} - \tfrac{4}{7} v^{20}_{2222} + \sqrt{\tfrac{2}{7}} v^{20}_{0222} + v^{20}_{0202} + v^{21}_{0202}.
\end{aligned}
\tag{16}
$$

A frequently used term in the Hamiltonian is the so-called Majorana interaction which is a pure two-body operator that gives zero acting on states with full symmetry $[N]$ in $U_L(6)$ and non-zero on non-symmetric states. Such an operator has the form

$$
\begin{aligned}
\hat{M} = &-\sqrt{15}\Big((s^\dagger \times d^\dagger)^{(2,1)} \times (\tilde{s} \times \tilde{d})^{(2,1)}\Big)^{(0,0)}_{0,0} \\
&+ \frac{1}{2} \sum_{L=1,3} \sqrt{3(2L+1)}\Big((d^\dagger \times d^\dagger)^{(L,1)} \times (\tilde{d} \times \tilde{d})^{(L,1)}\Big)^{(0,0)}_{0,0},
\end{aligned}
\tag{17}
$$

and can be expressed in terms of the linear and quadratic operators of $U_L(6)$,

$$
\hat{M} = \frac{1}{4}\Big(N(N+5) - \hat{C}_2[U_L(6)]\Big). \tag{18}
$$

As already mentioned, the existence of a sizeable Majorana interaction in the Hamiltonian is essential for an approximate decoupling between orbital and isospin spaces in cases when $U_L(6)$ symmetry is not exactly conserved.

The eigenvalue problem associated with the Hamiltonians (9), (11), or (15) should, in general, be solved numerically. Computer codes exist [15,16] that diagonalize the IBM-3 Hamiltonian given in any of the three forms and subsequently calculate electromagnetic transition probabilities. Some special Hamiltonians can be solved analytically and these are considered in the next section.

IV. DYNAMICAL SYMMETRIES

We start by discussing some symmetry properties of the Hamiltonian (15) that are valid for any combination of parameters. In general, an eigenstate of (15) can be written as

$$|[N_1, N_2, N_3]\phi L M_L; T M_T\rangle. \tag{19}$$

The labels $[N_1, N_2, N_3]$, L, M_L, T, and M_T are always good quantum numbers because the algebras $U_L(6)$, $O_L(3)$, $O_L(2)$, $O_T(3)$, and $O_T(2)$ are common to all classifications in (14). The (λ_T, μ_T) of $SU_T(3)$ are not shown in (19) since they are equivalent to $[N_1, N_2, N_3]$; all additional labels necessary to completely specify the state are denoted by ϕ.

A related consequence of the conservation of quantum numbers is that the associated Casimir operators are diagonal in the basis (19):

$$\begin{aligned}
\langle \hat{C}_1[U_L(6)]\rangle &= N_1 + N_2 + N_3, \\
\langle \hat{C}_2[U_L(6)]\rangle &= N_1(N_1+5) + N_2(N_2+3) + N_3(N_3+1), \\
\langle \hat{C}_2[O_L(3)]\rangle &= L(L+1), \\
\langle \hat{C}_2[SU_T(3)]\rangle &= \lambda_T^2 + \mu_T^2 + 3(\lambda_T + \mu_T) + \lambda_T\mu_T, \\
\langle \hat{C}_2[O_T(3)]\rangle &= T(T+1),
\end{aligned} \tag{20}$$

where $\langle \hat{C}_n[G]\rangle$ denotes $\langle [N_1, N_2, N_3]\phi L M_L; T M_T|\hat{C}_n[G]|[N_1, N_2, N_3]\phi L M_L; T M_T\rangle$. In the symmetric representation $[N]$ of $U(18)$ the Casimir operators $\hat{C}_2[U_L(6)]$ and $\hat{C}_2[SU_T(3)]$ are not independent but they are related through

$$\hat{C}_2[SU_T(3)] = \frac{3}{2}\hat{C}_2[U_L(6)] - \frac{9}{2}N - \frac{1}{2}N^2. \tag{21}$$

The isospin T values contained in a given $U_L(6)$ representation can be obtained from the reduction of the corresponding $SU_T(3)$ representation to $O_T(3)$. The $SU(3) \supset O(3)$ branching rule is known in general from [13]; for the lowest $U_L(6)$ representations one finds

$$\begin{aligned}
[N] &: T = N, N-2, \ldots, 1 \text{ or } 0, \\
[N-1, 1] &: T = N-1, N-2, \ldots, 1.
\end{aligned} \tag{22}$$

A. The $U(5)$ Limit

The orbital reduction in this limit is

$$\begin{array}{ccccccccc} U_L(6) & \supset & U_L(5) & \supset & O_L(5) & \supset & O_L(3) & \supset & O_L(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N_1,N_2,N_3] & & (n_1,n_2,n_3) & & (\upsilon_1,\upsilon_2) & \alpha & L & & M_L \end{array} , \tag{23}$$

where α is a missing label, necessary to completely specify the $O(5) \supset O(3)$ reduction. Wave functions in this limit are thus characterized by

$$|[N_1,N_2,N_3](n_1,n_2,n_3)(\upsilon_1,\upsilon_2)\alpha L M_L; T M_T\rangle. \tag{24}$$

The lowest $U(5)$ eigenstates are listed in Table I together with a short-hand notation for them. Reduction rules for symmetric $[N]$ and non-symmetric $[N-1,1]$ states can be found in [17] and [20], respectively.

The Hamiltonian in this limit is

$$\begin{aligned} \hat{H}_{\rm cas} &= A_1\hat{C}_1[U_L(6)] + A_2\hat{C}_2[U_L(6)] + B_1\hat{C}_1[U_L(5)] + B_2\hat{C}_2[U_L(5)] \\ &\quad + E_2\hat{C}_2[O_L(5)] + F_2\hat{C}_2[O_L(3)] + \alpha_2\hat{C}_2[SU_T(3)] + \beta_2\hat{C}_2[O_T(3)], \end{aligned} \tag{25}$$

with eigenvalues

$$\begin{aligned} E &= A_1(N_1+N_2+N_3) + A_2[N_1(N_1+5) + N_2(N_2+3) + N_3(N_3+1)] \\ &\quad + B_1(n_1+n_2+n_3) + B_2[n_1(n_1+4) + n_2(n_2+2) + n_3^2] \\ &\quad + E_2[\upsilon_1(\upsilon_1+3) + \upsilon_2(\upsilon_2+1)] + F_2 L(L+1) \\ &\quad + \alpha_2[\lambda_T^2 + \mu_T^2 + 3(\lambda_T+\mu_T) + \lambda_T\mu_T] + \beta_2 T(T+1). \end{aligned} \tag{26}$$

A typical energy spectrum is shown in Fig. 1.

B. The $SU(3)$ Limit

The orbital reduction in this limit is

$$\begin{array}{ccccccc} U_L(6) & \supset & SU_L(3) & \supset & O_L(3) & \supset & O_L(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N_1,N_2,N_3] & \beta & (\lambda,\mu) & \kappa & L & & M_L \end{array} , \tag{27}$$

where β and κ are missing labels, necessary to completely specify the $U(6) \supset SU(3)$ and $SU(3) \supset O(3)$ reductions. Wave functions in this limit are thus characterized by

$$|[N_1,N_2,N_3]\beta(\lambda,\mu)\kappa L M_L; T M_T\rangle. \tag{28}$$

The lowest $SU(3)$ eigenstates are listed in Table II together with a short-hand notation for them. Reduction rules for symmetric $[N]$ and non-symmetric $[N-1,1]$ states can be found in [18] and [20], respectively.

The Hamiltonian associated with this group chain is

$$\begin{aligned} \hat{H}_{\text{cas}} = & A_1\hat{C}_1[U_L(6)] + A_2\hat{C}_2[U_L(6)] + C_2\hat{C}_2[SU_L(3)] + F_2\hat{C}_2[O_L(3)] \\ & + \alpha_2\hat{C}_2[SU_T(3)] + \beta_2\hat{C}_2[O_T(3)], \end{aligned} \tag{29}$$

with eigenvalues

$$\begin{aligned} E = & A_1(N_1+N_2+N_3) + A_2[N_1(N_1+5) + N_2(N_2+3) + N_3(N_3+1)] \\ & + C_2[\lambda^2+\mu^2+3(\lambda+\mu)+\lambda\mu] + F_2L(L+1) \\ & + \alpha_2[\lambda_T^2+\mu_T^2+3(\lambda_T+\mu_T)+\lambda_T\mu_T] + \beta_2T(T+1). \end{aligned} \tag{30}$$

A typical energy spectrum is shown in Fig. 2.

C. The $O(6)$ Limit

The orbital reduction in this limit is

$$\begin{array}{ccccccccc} U_L(6) & \supset & O_L(6) & \supset & O_L(5) & \supset & O_L(3) & \supset & O_L(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N_1,N_2,N_3] & & (\sigma_1,\sigma_2,\sigma_3) & & (\upsilon_1,\upsilon_2) & \alpha & L & & M_L \end{array} , \tag{31}$$

where α is a missing label, necessary to completely specify the $O(5) \supset O(3)$ reduction. Wave functions in this limit are thus characterized by

$$|[N_1, N_2, N_3](\sigma_1, \sigma_2, \sigma_3)(\upsilon_1, \upsilon_2)\alpha L M_L; T M_T\rangle. \tag{32}$$

The lowest $O(6)$ eigenstates are listed in Table III together with a short-hand notation for them. Reduction rules for symmetric $[N]$ and non-symmetric $[N-1,1]$ states can be found in [19] and [20], respectively.

The Hamiltonian associated with this group chain is

$$\begin{aligned} \hat{H}_{\rm cas} &= A_1\hat{C}_1[U_L(6)] + A_2\hat{C}_2[U_L(6)] + D_2\hat{C}_2[O_L(6)] + E_2\hat{C}_2[O_L(5)] \\ &\quad + F_2\hat{C}_2[O_L(3)] + \alpha_2\hat{C}_2[SU_T(3)] + \beta_2\hat{C}_2[O_T(3)], \end{aligned} \tag{33}$$

with eigenvalues

$$\begin{aligned} E &= A_1(N_1+N_2+N_3) + A_2[N_1(N_1+5) + N_2(N_2+3) + N_3(N_3+1)] \\ &\quad + D_2[\sigma_1(\sigma_1+4) + \sigma_2(\sigma_2+2) + \sigma_3^2] \\ &\quad + E_2[\upsilon_1(\upsilon_1+3) + \upsilon_2(\upsilon_2+1)] + F_2L(L+1) \\ &\quad + \alpha_2[\lambda_T^2 + \mu_T^2 + 3(\lambda_T+\mu_T) + \lambda_T\mu_T] + \beta_2T(T+1). \end{aligned} \tag{34}$$

A typical energy spectrum is shown in Fig. 3.

V. ELECTROMAGNETIC TRANSITIONS

A general one-body electromagnetic operator in IBM-3 consists of isoscalar, isovector, and isotensor parts,

$$\hat{T}^{(l)}_{m_l}(l_1,l_2) = a_0\hat{T}^{(l,0)}_{m_l,0}(l_1,l_2) + a_1\hat{T}^{(l,1)}_{m_l,0}(l_1,l_2) + a_2\hat{T}^{(l,2)}_{m_l,0}(l_1,l_2), \tag{35}$$

where, as in (12), the superscripts in the operators on the rhs refer to the angular momentum and the isospin, respectively, whereas on the lhs only the angular momentum is given since the operator corresponds to an admixture of isospins. In previous IBM-3 studies only isoscalar and isovector electromagnetic operators are considered [21]; isotensor contributions are included in this paper for completeness. The relevance of this contribution is discussed in Section VIII when comparing with experimental results. The parameters a_t are boson g factors, boson electric charges, etc. depending on the multipolarity of the operator, and the operators $\hat{T}^{(l,t)}_{m_l,0}(l_1,l_2)$ are defined as

$$\begin{aligned}
\hat{T}^{(l,0)}_{m_l,0}(l_1,l_2) &= \sqrt{3}(b^\dagger_{l_1,1}\times\tilde{b}_{l_2,1})^{(l,0)}_{m_l,0},\\
\hat{T}^{(l,1)}_{m_l,0}(l_1,l_2) &= \sqrt{2}(b^\dagger_{l_1,1}\times\tilde{b}_{l_2,1})^{(l,1)}_{m_l,0},\\
\hat{T}^{(l,2)}_{m_l,m_t}(l_1,l_2) &= -\sqrt{6}(b^\dagger_{l_1,1}\times\tilde{b}_{l_2,1})^{(l,2)}_{m_l,0}.
\end{aligned} \tag{36}$$

The factors are taken for later convenience and lead to the explicit forms

$$\begin{aligned}
\hat{T}^{(l,0)}_{m_l,0}(l_1,l_2) &= (b^\dagger_{l_1,\pi}\times\tilde{b}_{l_2,\pi})^{(l)}_{m_l} + (b^\dagger_{l_1,\delta}\times\tilde{b}_{l_2,\delta})^{(l)}_{m_l} + (b^\dagger_{l_1,\nu}\times\tilde{b}_{l_2,\nu})^{(l)}_{m_l},\\
\hat{T}^{(l,1)}_{m_l,0}(l_1,l_2) &= (b^\dagger_{l_1,\pi}\times\tilde{b}_{l_2,\pi})^{(l)}_{m_l} - (b^\dagger_{l_1,\nu}\times\tilde{b}_{l_2,\nu})^{(l)}_{m_l},\\
\hat{T}^{(l,2)}_{m_l,0}(l_1,l_2) &= (b^\dagger_{l_1,\pi}\times\tilde{b}_{l_2,\pi})^{(l)}_{m_l} + 2(b^\dagger_{l_1,\delta}\times\tilde{b}_{l_2,\delta})^{(l)}_{m_l} - (b^\dagger_{l_1,\nu}\times\tilde{b}_{l_2,\nu})^{(l)}_{m_l}
\end{aligned} \tag{37}$$

We may thus write the operators (35) alternatively as

$$\hat{T}^{(l)}_{m_l}(l_1,l_2) = a_\pi(b^\dagger_{l_1,\pi}\times\tilde{b}_{l_2,\pi})^{(l)}_{m_l} + a_\delta(b^\dagger_{l_1,\delta}\times\tilde{b}_{l_2,\delta})^{(l)}_{m_l} + a_\nu(b^\dagger_{l_1,\nu}\times\tilde{b}_{l_2,\nu})^{(l)}_{m_l}, \tag{38}$$

where the a_ρ with $\rho=\pi,\delta,\nu$ are related to the a_t through

$$a_0 = a_\pi + a_\delta + a_\nu, \qquad a_1 = a_\pi - a_\nu, \qquad a_2 = -a_\pi + 2a_\delta - a_\nu. \tag{39}$$

For the calculation of matrix elements it is also of interest to know the tensor properties of the transition operators under $SU_T(3)\supset O_T(3)\supset O_T(2)$. Given that $s^\dagger$ or $d^\dagger_\mu$ transforms as a $\hat{T}^{(1,0)1}$ tensor under $SU_T(3)\supset O_T(3)$ and that s or $\tilde{d}_\mu$ transforms as $\hat{T}^{(0,1)1}$, one finds that the full tensor character of the coupled operator is uniquely determined by its isospin coupling, that is,

$$\begin{aligned}
\hat{T}^{(l,0)}_{m_l,0}(l_1,l_2) &\to \hat{T}^{(0,0)00},\\
\hat{T}^{(l,1)}_{m_l,m_t}(l_1,l_2) &\to \hat{T}^{(1,1)1m_t},\\
\hat{T}^{(l,2)}_{m_l,m_t}(l_1,l_2) &\to \hat{T}^{(1,1)2m_t}.
\end{aligned} \tag{40}$$

Since the isovector and isotensor operators belong to the same $SU(3)$ representation (1,1), their $O_T(3) \subset SU_T(3)$ reduced matrix elements are related,

$$\frac{\langle [N_1,N_2,N_3]\alpha \| \hat{T}^{(l,2)}_{m_l,*}(l_1,l_2) \| [N'_1,N'_2,N'_3]\alpha' \rangle}{\langle [N_1,N_2,N_3]\alpha \| \hat{T}^{(l,1)}_{m_l,*}(l_1,l_2) \| [N'_1,N'_2,N'_3]\alpha' \rangle} = -\sqrt{3}. \tag{41}$$

This ratio is different from one due to the specific normalization of the tensor operators (36).

The orbital structure of the electromagnetic transition operators is the usual one as it occurs, for example, in IBM-1 or IBM-2. In particular, for M1, E2, and M3 transitions the following form is taken:

$$\begin{aligned}
\hat{T}(\mathrm{M1}) &= \sqrt{\frac{3}{4\pi}} g \sqrt{10} (d^\dagger \times \tilde{d})^{(1)}, \\
\hat{T}(\mathrm{E2}) &= e\Big((s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)} + \chi (d^\dagger \times \tilde{d})^{(2)} \Big), \\
\hat{T}(\mathrm{M3}) &= \sqrt{\frac{35}{8\pi}} \Omega (d^\dagger \times \tilde{d})^{(3)}.
\end{aligned} \tag{42}$$

A. Symmetric-to-Symmetric Transitions

Symmetric states of the IBM-3 are defined as having $U_L(6)$ quantum numbers $[N_1,N_2,N_3] = [N,0,0] \equiv [N]$ or, equivalently, $SU_T(3)$ quantum numbers $(\lambda_T,\mu_T) = (N,0)$. There is a one-to-one correspondence between the symmetric states of IBM-3 and all states of IBM-1 and, because of this, matrix elements between symmetric IBM-3 states can be related to the corresponding ones in IBM-1, as shown in this subsection.

The starting point is to consider a matrix element between IBM-3 states with $M_T = -N$ (all-neutron states), for which, by virtue of (37), the following relations are satisfied:

$$\begin{aligned}
\langle [N]\phi' L'; N,-N \| \hat{T}^{(l,0)}_{*,0}(l_1,l_2) \| [N]\phi L; N-N \rangle_{\mathrm{IBM3}} &= \langle [N]\phi' L' \| \hat{T}^{(l)}(l_1,l_2) \| [N]\phi L \rangle_{\mathrm{IBM1}}, \\
\langle [N]\phi' L'; N,-N \| \hat{T}^{(l,1)}_{*,0}(l_1,l_2) \| [N]\phi L; N,-N \rangle_{\mathrm{IBM3}} &= -\langle [N]\phi' L' \| \hat{T}^{(l)}(l_1,l_2) \| [N]\phi L \rangle_{\mathrm{IBM1}}, \\
\langle [N]\phi' L'; N,-N \| \hat{T}^{(l,2)}_{*,0}(l_1,l_2) \| [N]\phi L; N,-N \rangle_{\mathrm{IBM3}} &= -\langle [N]\phi' L' \| \hat{T}^{(l)}(l_1,l_2) \| [N]\phi L \rangle_{\mathrm{IBM1}},
\end{aligned} \tag{43}$$

where ϕ represents any label in between $U_L(6)$ and $O_L(3)$ and all states have $T = -M_T = N$. The symbols $\|$ indicate a matrix element reduced from $O_L(2)$ to $O_L(3)$; no $O_L(2)$ indices are required in bra, ket, or operator, and in the latter this is indicated by an asterisk. To find the general relationship between corresponding IBM-1 and IBM-3 matrix elements, one may use the Wigner–Eckart theorem for $O_T(3) \subset SU_T(3)$. Expressions can be derived for any T and M_T, but only the case $T = -M_T$ is listed here since it corresponds to states lowest in energy. (An exception to this rule occurs in self-conjugate odd–odd nuclei where $T = 0$ and $T = 1$ states are close in energy; these nuclei, however, are not amenable to the IBM-3 description considered in this paper.) For an isoscalar operator one finds the relation

$$\frac{\langle [N]\phi' L'; T, -T \| \hat{T}^{(l,0)}_{*,0}(l_1, l_2) \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}}}{\langle [N]\phi' L'; N, -N \| \hat{T}^{(l,0)}_{*,0}(l_1, l_2) \| [N]\phi L; N, -N \rangle_{\mathrm{IBM3}}} = \frac{\left\langle \begin{array}{cc|c} (N,0) & (0,0) & (N,0) \\ T & 0 & T \end{array} \right\rangle \langle T - T\; 00 | T - T \rangle}{\left\langle \begin{array}{cc|c} (N,0) & (0,0) & (N,0) \\ N & 0 & N \end{array} \right\rangle \langle N - N\; 00 | N - N \rangle}, \tag{44}$$

where the symbols between big angle brackets are $SU(3) \supset O(3)$ isoscalar factors [22], which in this case trivially are equal to one. The following relation is thus found for an isoscalar operator:

$$\langle [N]\phi' L'; T, -T \| \hat{T}^{(l,0)}_{*,0}(l_1, l_2) \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}} = \langle [N]\phi' L' \| \hat{T}^{(l)}(l_1, l_2) \| [N]\phi L \rangle_{\mathrm{IBM1}}. \tag{45}$$

The analogous relation for isovector ($t = 1$) and isotensor ($t = 2$) operators is

$$\frac{\langle [N]\phi' L'; T, -T \| \hat{T}^{(l,t)}_{*,0}(l_1, l_2) \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}}}{\langle [N]\phi' L'; N, -N \| \hat{T}^{(l,t)}_{*,0}(l_1, l_2) \| [N]\phi L; N, -N \rangle_{\mathrm{IBM3}}} = \frac{\left\langle \begin{array}{cc|c} (N,0) & (1,1) & (N,0) \\ T & t & T \end{array} \right\rangle \langle T - T\; t0 | T - T \rangle}{\left\langle \begin{array}{cc|c} (N,0) & (1,1) & (N,0) \\ N & t & N \end{array} \right\rangle \langle N - N\; t0 | N - N \rangle}, \tag{46}$$

such that

$$\langle [N]\phi' L'; T, -T \| \hat{T}^{(l,1)}_{*,0}(l_1, l_2) \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}}$$

$$= -\frac{T}{N}\langle [N]\phi' L' \| \hat{T}^{(l)}(l_1,l_2) \| [N]\phi L\rangle_{\rm IBM1},$$

$$\langle [N]\phi' L'; T, -T \| \hat{T}^{(l,2)}_{*,0}(l_1,l_2) \| [N]\phi L; T, -T\rangle_{\rm IBM3}$$

$$= -\frac{T(2N+3)}{(2T+3)N}\langle [N]\phi' L' \| \hat{T}^{(l)}(l_1,l_2) \| [N]\phi L\rangle_{\rm IBM1},$$

$$\langle [N]\phi' L'; T+2, -T \| \hat{T}^{(l,2)}_{*,0}(l_1,l_2) \| [N]\phi L; T, -T\rangle_{\rm IBM3}$$

$$= 6\sqrt{\frac{(T+1)(N-T)(N+T+3)}{(2T+3)^2(2T+5)N^2}}\langle [N]\phi' L' \| \hat{T}^{(l)}(l_1,l_2) \| [N]\phi L\rangle_{\rm IBM1}. \qquad (47)$$

Note that all results are symmetric under interchange of the orbital parts ϕL and $\phi' L'$. An alternative derivation of these results can be found in [10].

Expressions for symmetric-to-symmetric transitions in IBM-3 can now be derived from the corresponding ones in IBM-1 [17–19]. In Tables IV, V, VI, and VII all non-zero M1, E2, and M3 transitions out of the ground state are listed for the U(5), SU(3), and O(6) limits. Isoscalar, isovector, and isotensor parts are given separately; the E2 operator is defined with $\chi = -\frac{\sqrt{7}}{2}$ and $\chi = 0$ in the $SU(3)$ and $O(6)$ limits, respectively.

B. Symmetric-to-Non-Symmetric Transitions

Generally, non-symmetric states in IBM-3 have $U_L(6)$ quantum numbers different from $[N]$. Usually, however, only $[N-1,1]$ states are considered in the analysis and this is what will be done here. Just as there exists a one-to-one correspondence between IBM-1 and symmetric IBM-3 states, a one-to-one correspondence can be established between IBM-2 and non-symmetric, two-rowed IBM-3 states. This fact can be exploited to derive relations between matrix elements involving non-symmetric $[N-1,1]$ states in IBM-3 and corresponding ones in IBM-2 known from [23].

The IBM-2 states that can be related to the IBM-3 classification (14) are classified themselves according to

$$U(12) \supset \left(U_L(6) \supset \left\{ \begin{array}{c} U_L(5) \supset O_L(5) \\ SU_L(3) \\ O_L(6) \supset O_L(5) \end{array} \right\} \supset O_L(3) \right) \otimes SU_F(2). \qquad (48)$$

Comparison of (14) and (48) shows that the correspondence between IBM-2 and IBM-3 can be established via the algebra $U_L(6)$. All IBM-2 states belong to $U_L(6)$ representations of the type $[N-f,f]$ where $f=0,1,\ldots,\min(N_\pi,N_\nu)$ and these form a subset of the possible $U_L(6)$ representations in IBM-3. The F spin is defined as $F=N/2-f$. An F-scalar operator cannot contribute to symmetric-to-non-symmetric transitions. Since a one-body operator is either F scalar or F vector, only the latter can connect symmetric with a non-symmetric state. Furthermore, an F-vector operator in IBM-2 coincides with its isovector counterpart in IBM-3 [see (39)],

$$\hat{T}^{(L,F=1)}_{M_L,M_F}(l_1,l_2)=\hat{T}^{(L,T=1)}_{M_L,M_T}(l_1,l_2). \tag{49}$$

To obtain the relation between IBM-2 and IBM-3 transitions, one must first establish the tensor character of the transition operator under $U_L(6)$. The boson operators $b^\dagger_{lm,1\mu}$ transform as a $\hat{T}^{[1]}$ tensor operator under $U_L(6)$ while $\tilde{b}_{lm,1\mu}$ transforms as $\hat{T}^{[1^5]}$. Consequently, the $U_L(6)$ tensor character of an F-vector or isovector one-body operator is $\hat{T}^{[2,1^4]}$. Because of this property the ratio between corresponding $[N]\rightarrow[N-1,1]$ matrix elements of an F-vector operator in IBM-2 and an isovector operator in IBM-3, is a unique function of N_π, N_ν, N, and T, independent of the particular states or of the operator. For simplicity the $SU_L(3)$ limit is analyzed but the proof can be extended to any limit, the only requirement being that it has $U_L(6)$ symmetry. First the ratio of arbitrary matrix elements can be shown to be related to a ratio of specific ones,

$$\begin{aligned}&\frac{\langle[N-1,1]\beta'(\lambda',\mu')\kappa'L';T,-T\|\hat{T}^{[2,1^4](\bar\lambda,\bar\mu)(l,1)}_{*,0}\|[N]\beta(\lambda,\mu)\kappa L;T,-T\rangle_{\rm IBM3}}{\langle[N-1,1]\beta'(\lambda',\mu')\kappa'L'\|\hat{T}^{[2,1^4](\bar\lambda,\bar\mu)(l,1)}_{*,0}\|[N]\beta(\lambda,\mu)\kappa L\rangle_{\rm IBM2}}\\&=\frac{\langle[N-1,1]1^+_M;T,-T\|\hat{T}(M1)\|[N]0^+_1;T,-T\rangle_{\rm IBM3}}{\langle[N-1,1]1^+_M\|\hat{T}(M1)\|[N]0^+_1\rangle_{\rm IBM2}},\end{aligned} \tag{50}$$

where $\|$ indicates a $O_L(2)\subset O_L(3)$ reduced matrix element. The identity (50) follows from the Wigner–Eckart theorem in $O_L(3)\subset SU_L(3)\subset U_L(6)$, which leads to the same $U_L(6)$-reduced matrix element in both the general and the specific case. Hence

$$\langle[N-1,1]\beta'(\lambda',\mu')\kappa'L'\|\hat{T}^{[2,1^4](\bar\lambda,\bar\mu)(l,1)}_{*,0}\|[N]\beta(\lambda,\mu)\kappa L\rangle$$

$$
= \langle [N-1,1]1^+_M \| \hat{T}(M1) \| [N]0^+_1 \rangle
$$

$$
\times \frac{\left\langle \begin{array}{cc|c} [N] & [2,1^4] & [N-1,1] \\ \beta(\lambda,\mu) & (\bar{\lambda},\bar{\mu}) & \beta'(\lambda',\mu') \end{array} \right\rangle \left\langle \begin{array}{cc|c} (\lambda,\mu) & (\bar{\lambda},\bar{\mu}) & (\lambda',\mu') \\ \kappa L & l & \kappa' L' \end{array} \right\rangle}{\left\langle \begin{array}{cc|c} [N] & [2,1^4] & [N-1,1] \\ (2N,0) & (1,1) & (2N-2,1) \end{array} \right\rangle \left\langle \begin{array}{cc|c} (2N,0) & (1,1) & (2N-2,1) \\ 0 & 1 & 1 \end{array} \right\rangle}, \tag{51}
$$

where the symbols between big angle brackets are $U(6) \supset SU(3)$ or $SU(3) \supset O(3)$ isoscalar factors. The F-spin and isospin labels can be omitted from the matrix elements because the result (51) is identical in IBM-2 and IBM-3. Taking the ratio of an IBM-2 and an IBM-3 matrix element, the isoscalar factors cancel out and the result (50) is obtained. Since both matrix elements on the rhs of (50) are known from [23] and [24], and since the derivation can be generalized to any state with good $U_L(6)$ symmetry, the following relation results:

$$
\begin{aligned}
&\langle [N-1,1]\phi' L'; T, -T \| \hat{T}^{[2,1^4](l,1)}_{*,0} \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}} \\
&= \sqrt{\frac{1}{4N_\pi N_\nu} \frac{T(N-T)(N+T+1)}{T+1}} \langle [N-1,1]\phi' L' \| \hat{T}^{[2,1^4](l,1)}_{*,0} \| [N]\phi L \rangle_{\mathrm{IBM2}}.
\end{aligned} \tag{52}
$$

It is clear that this ratio is independent of the orbital structure of the states and/or the electromagnetic operator. Analogous relations for isotensor operators and $T \neq |M_T|$ states can be obtained using (52) together with the Wigner–Eckart theorem,

$$
\begin{aligned}
&\langle [N-1,1]\phi' L'; T+1, -T \| \hat{T}^{[2,1^4](l,1)}_{*,0} \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}} \\
&= \sqrt{\frac{1}{4N_\pi N_\nu} \frac{(T+2)N(N-T)}{(T+1)(2T+3)}} \langle [N-1,1]\phi' L' \| \hat{T}^{[2,1^4](l,1)}_{*,0} \| [N]\phi L \rangle_{\mathrm{IBM2}}, \\
&\langle [N-1,1]\phi' L'; T, -T \| \hat{T}^{[2,1^4](l,2)}_{*,0} \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}} \\
&= -\sqrt{\frac{3}{4N_\pi N_\nu} \frac{3T(N-T)(N+T+1)}{(T+1)(2T+3)^2}} \langle [N-1,1]\phi' L' \| \hat{T}^{[2,1^4](l,1)}_{*,0} \| [N]\phi L \rangle_{\mathrm{IBM2}}, \\
&\langle [N-1,1]\phi' L'; T+1, -T \| \hat{T}^{[2,1^4](l,2)}_{*,0} \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}} \\
&= \sqrt{\frac{3}{4N_\pi N_\nu} \frac{3T^2 N(N-T)}{(T+1)(T+2)(2T+3)}} \langle [N-1,1]\phi' L' \| \hat{T}^{[2,1^4](l,1)}_{*,0} \| [N]\phi L \rangle_{\mathrm{IBM2}}, \\
&\langle [N-1,1]\phi' L'; T+2, -T \| \hat{T}^{[2,1^4](l,2)}_{*,0} \| [N]\phi L; T, -T \rangle_{\mathrm{IBM3}} \\
&= \sqrt{\frac{3}{4N_\pi N_\nu} \frac{12(T+1)(T+3)(N-T)(N-T-2)}{(T+2)(2T+3)^2(2T+5)}}
\end{aligned}
$$

$$\times \langle [N-1,1]\phi' L' \| \hat{T}^{[2,1^4](l,1)}_{*,0} \| [N]\phi L\rangle_{\rm IBM2}. \tag{53}$$

Note an additional factor $-\sqrt{3}$ for the isotensor matrix elements due to the specific normalization of the isovector and isotensor operators. An alternative derivation of these results can be found in [10].

Expressions for symmetric-to-non-symmetric transitions in IBM-3 can now be derived from the corresponding ones in IBM-2 [23]. In Tables IV, V, VI, and VII all non-zero M1, E2, and M3 transitions out of the ground state are listed for the U(5), SU(3), and O(6) limits.

VI. TWO-NUCLEON TRANSFER PROBABILITIES

Two-nucleon transfer reactions have been studied in IBM-1 [2], generally showing a good agreement with experimental observations [27]. To see whether any specific effects are found for two-nucleon transfer properties if a proton–neutron pair is included in the boson basis, it is of interest to compare the IBM-1 analysis with the corresponding one in IBM-3, which is the object the present section.

A general one-boson transfer operator has the form

$$P^{(l)}_{+,\rho,m} = p_{\rho,l} b^\dagger_{lm,1\rho}, \qquad P^{(l)}_{-,\rho,m} = p_{\rho,l} b_{lm,1\rho}, \tag{54}$$

for the addition or the removal of two nucleons, respectively, and where $\rho = \pi, \delta, \nu$. A measurable quantity is the transfer intensity I which is defined as

$$\begin{aligned} I(N\phi L;\xi \to N+1\ \phi' L';\xi') &= \frac{p^2_{\rho,l}}{2L+1} |\langle N+1\ \phi' L';\xi' \| P^{(l)}_{+,\rho} \| N\phi L;\xi\rangle|^2, \\ I(N\phi L;\xi \to N-1\ \phi' L';\xi') &= \frac{p^2_{\rho,l}}{2L+1} |\langle N-1\ \phi' L';\xi' \| P^{(l)}_{-,\rho} \| N\phi L;\xi\rangle|^2, \end{aligned} \tag{55}$$

where ϕ and ξ denote all additional orbital and isospin labels, respectively, to completely specify the state and the symbol $\|$ indicates the reduction of the matrix element from $O_L(2)$ to $O_L(3)$.

The analysis in this section will be confined to transitions between symmetric states with $T = -M_T$. Analytical expressions for transfer intensities in IBM-3 can be obtained in much the same way as in Section V A by relating them to IBM-1 using the property that $b^\dagger$ and $\tilde{b}$ transform as $\hat{T}^{(1,0)1}$ and $\hat{T}^{(0,1)1}$ tensors under $SU_T(3) \supset O_T(3)$. The starting point is again the matrix element between IBM-3 states with $M_T = -N$,

$$\langle [N+1]\phi' L'; N+1, -N-1 \| b^\dagger_\nu \| [N]\phi L; N, -N\rangle_{\rm IBM3} = \langle [N+1]\phi' L' \| b^\dagger \| [N]\phi L\rangle_{\rm IBM1}, \quad (56)$$

from where the following relation results:

$$\langle [N+1]\phi' L'; T-\mu, -T+\mu \| b^\dagger_{l,1\mu} \| [N]\phi L; T, -T\rangle_{\rm IBM3}$$
$$= \frac{\left\langle \begin{matrix} (N,0) & (1,0) \\ T & \mu \end{matrix} \right| \left. \begin{matrix} (N+1,0) \\ T-\mu \end{matrix} \right\rangle \langle T -T\ 1\mu | T-\mu\ \ -T+\mu\rangle}{\left\langle \begin{matrix} (N,0) & (1,0) \\ N & 1 \end{matrix} \right| \left. \begin{matrix} (N+1,0) \\ N+1 \end{matrix} \right\rangle \langle N -N\ 1-1 | N+1\ \ -N-1\rangle} \langle [N]\phi' L' \| b^\dagger_l \| [N]\phi L\rangle_{\rm IBM1}, \quad (57)$$

for the π ($\mu = +1$), δ ($\mu = 0$), or ν ($\mu = -1$) boson. Insertion of the appropriate isoscalar factors and Clebsch–Gordan coefficients leads to

$$\begin{aligned} & I([N]\phi L; T, -T \to [N+1]\phi' L'; T-1, -T+1) \\ &= \frac{p^2_{\pi,l}}{2L+1}\frac{T(N-T+2)}{(2T+1)(N+1)}\left|\langle [N+1]\phi' L' \| b^\dagger_l \| [N]\phi L\rangle_{\rm IBM1}\right|^2, \\ & I([N]\phi L; T, -T \to [N+1]\phi' L'; T, -T) = 0, \\ & I([N]\phi L; T, -T \to [N+1]\phi' L'; T+1, -T-1) \\ &= \frac{p^2_{\nu,l}}{2L+1}\frac{(T+1)(N+T+3)}{(2T+3)(N+1)}\left|\langle [N+1]\phi' L' \| b^\dagger_l \| [N]\phi L\rangle_{\rm IBM1}\right|^2. \end{aligned} \quad (58)$$

From the relation

$$\left|\langle [N-1]\phi' L' \| \tilde{b}_l \| [N]\phi L\rangle_{\rm IBM1}\right|^2 = \left|\langle [N]\phi L \| b^\dagger_l \| [N-1]\phi' L'\rangle_{\rm IBM1}\right|^2, \quad (59)$$

expressions for the inverse reaction are obtained,

$$I([N]\phi L; T, -T \to [N-1]\phi' L'; T+1, -T-1)$$

$$= \frac{p_{\pi,l}^2}{2L+1}\frac{(T+1)(N-T)}{(2T+3)N}\left|\langle[N]\phi L\|b_l^\dagger\|[N-1]\phi' L'\rangle_{\text{IBM1}}\right|^2,$$

$$I([N]\phi L; T,-T \to [N-1]\phi' L'; T,-T) = 0,$$

$$I([N]\phi L; T,-T \to [N-1]\phi' L'; T-1,-T+1)$$
$$= \frac{p_{\nu,l}^2}{2L+1}\frac{T(N+T+1)}{(2T+1)N}\left|\langle[N]\phi L\|b_l^\dagger\|[N-1]\phi' L'\rangle_{\text{IBM1}}\right|^2. \quad (60)$$

As can be seen from (58) and (60) the probability for the transfer of a δ boson is zero. This is due to a vanishing isoscalar factor,

$$\left\langle \begin{matrix} (N,0) & (1,0) \\ T & 1 \end{matrix} \middle| \begin{matrix} (N+1,0) \\ T \end{matrix} \right\rangle = 0. \quad (61)$$

To illustrate these formulas, some particular cases in the $SU(3)$ are given in Table VIII.

VII. BOSON-NUMBER EXPECTATION VALUES

The average number of bosons is a useful quantity for comparing with other models, for example, Shell Model Monte Carlo calculations (SMMC) [25] or the isovector-pairing $SO(5)$ seniority model [26]. In this section analytic expressions are derived for the average number of the different kinds of bosons as obtained for symmetric states with $T = |M_T|$, though expressions for higher isospin can be obtained through the Wigner–Eckart theorem.

Consider the following operators:

$$\hat{N}_0^s \equiv (s^\dagger \times \tilde{s})^{(L=0,T=0)}_{M_L=0,M_T=0} = \frac{1}{\sqrt{3}}\Big((s_\pi^\dagger s_\pi) + (s_\delta^\dagger s_\delta) + (s_\nu^\dagger s_\nu)\Big),$$
$$\hat{N}_1^s \equiv (s^\dagger \times \tilde{s})^{(L=0,T=1)}_{M_L=0,M_T=0} = \frac{1}{\sqrt{2}}\Big((s_\pi^\dagger s_\pi) - (s_\nu^\dagger s_\nu)\Big),$$
$$\hat{N}_2^s \equiv (s^\dagger \times \tilde{s})^{(L=0,T=2)}_{M_L=0,M_T=0} = \frac{1}{\sqrt{6}}\Big((s_\pi^\dagger s_\pi) - 2(s_\delta^\dagger s_\delta) + (s_\nu^\dagger s_\nu)\Big), \quad (62)$$

and

$$\hat{N}_0^d \equiv (d^\dagger \times \tilde{d})^{(L=0,T=0)}_{M_L=0,M_T=0} = \frac{1}{\sqrt{3}}\Big((d_\pi^\dagger \times \tilde{d}_\pi)^{(L=0)}_{M_L=0} + (d_\delta^\dagger \times \tilde{d}_\delta)^{(L=0)}_{M_L=0} + (d_\nu^\dagger \times \tilde{d}_\nu)^{(L=0)}_{M_L=0}\Big),$$
$$\hat{N}_1^d \equiv (d^\dagger \times \tilde{d})^{(L=0,T=1)}_{M_L=0,M_T=0} = \frac{1}{\sqrt{2}}\Big((d_\pi^\dagger \times \tilde{d}_\pi)^{(L=0)}_{M_L=0} - (d_\nu^\dagger \times \tilde{d}_\nu)^{(L=0)}_{M_L=0}\Big),$$
$$\hat{N}_2^d \equiv (d^\dagger \times \tilde{d})^{(L=0,T=2)}_{M_L=0,M_T=0} = \frac{1}{\sqrt{6}}\Big((d_\pi^\dagger \times \tilde{d}_\pi)^{(L=0)}_{M_L=0} - 2(d_\delta^\dagger \times \tilde{d}_\delta)^{(L=0)}_{M_L=0} + (d_\nu^\dagger \times \tilde{d}_\nu)^{(L=0)}_{M_L=0}\Big). \quad (63)$$

Boson-number operators can be defined in terms of (62) and (63) by introducing

$$\hat{N}_T = \hat{N}^s_T + \sqrt{5}\hat{N}^d_T, \tag{64}$$

for $T = 0, 1, 2$. The completely scalar operator with $L = 0$ and $T = 0$, $\hat{N}_0$, is proportional to the number of bosons N. The number operators for the π, δ, and ν bosons separately can now be defined in terms of the $\hat{N}_T$,

$$\begin{aligned} \hat{N}_\pi &= \frac{1}{\sqrt{3}}\hat{N}_0 - \frac{1}{\sqrt{2}}\hat{N}_1 - \frac{1}{\sqrt{6}}\hat{N}_2, \\ \hat{N}_\delta &= \frac{1}{\sqrt{3}}\hat{N}_0 + \sqrt{\frac{2}{3}}\hat{N}_2, \\ \hat{N}_\nu &= \frac{1}{\sqrt{3}}\hat{N}_0 - \frac{1}{\sqrt{2}}\hat{N}_1 + \frac{1}{\sqrt{6}}\hat{N}_2. \end{aligned} \tag{65}$$

For states with $M_T = -N$, $|[N]\phi L; N, -N\rangle$, that is, for all-neutron states, the matrix elements of $\hat{N}_T$ are easily evaluated,

$$\begin{aligned} \langle [N]\phi L; N, -N|\hat{N}_0|[N]\phi L; N, -N\rangle &= -\frac{N}{\sqrt{3}}, \\ \langle [N]\phi L; N, -N|\hat{N}_1|[N]\phi L; N, -N\rangle &= \frac{N}{\sqrt{2}}, \\ \langle [N]\phi L; N, -N|\hat{N}_2|[N]\phi L; N, -N\rangle &= -\frac{N}{\sqrt{6}}. \end{aligned} \tag{66}$$

To obtain expressions for the more general states $|[N]\phi L; T, -T\rangle$ use can be made of the Wigner–Eckart theorem in $O_T(3) \subset SU_T(3)$. The results are

$$\begin{aligned} \langle [N]\phi L; T, -T|\hat{N}_0|[N]\phi L; T, -T\rangle &= -\frac{N}{\sqrt{3}}, \\ \langle [N]\phi L; T, -T|\hat{N}_1|[N]\phi L; T, -T\rangle &= \frac{T}{\sqrt{2}}, \\ \langle [N]\phi L; T, -T|\hat{N}_2|[N]\phi L; T, -T\rangle &= -\frac{T(2N+3)}{\sqrt{6}(2T+3)}. \end{aligned} \tag{67}$$

Finally, the average boson numbers are obtained through combination of the previous equations,

$$\begin{aligned} \langle \hat{N}_\pi\rangle \equiv \langle [N]\phi L; T, -T|\hat{N}_\pi|[N]\phi L; T, -T\rangle &= \frac{(T+1)(N-T)}{2T+3}, \\ \langle \hat{N}_\delta\rangle \equiv \langle [N]\phi L; T, -T|\hat{N}_\delta|[N]\phi L; T, -T\rangle &= \frac{N-T}{2T+3}, \\ \langle \hat{N}_\nu\rangle \equiv \langle [N]\phi L; T, -T|\hat{N}_\nu|[N]\phi L; T, -T\rangle &= \frac{T(N+T)+(N+2T)}{2T+3}. \end{aligned} \tag{68}$$

These expectation values satisfy the usual relations

$$\langle \hat{N}_\pi \rangle + \langle \hat{N}_\delta \rangle + \langle \hat{N}_\nu \rangle = N, \qquad \langle \hat{N}_\nu \rangle - \langle \hat{N}_\pi \rangle = T. \tag{69}$$

The expressions (68) do not depend on the orbital part of the state, but only on its $SU_T(3)$ symmetry character. Being valid for all symmetric states, they are quite insensitive to the detailed structure of a specific IBM-3 Hamiltonian.

VIII. PREDICTIONS AND COMPARISON WITH EXPERIMENT

Within the context of IBM-1 and IBM-2 extensive calculations have been performed for energy spectra and electromagnetic properties of medium-mass and heavy nuclei, generally yielding a satisfactory agreement with the data [2]. The IBM-3 has been applied less extensively and correspondingly less is known about its viability. In addition, IBM-3 applications differ somewhat from previous IBM-1 and IBM-2 studies, and this in two respects. Firstly, its region of applicability is essentially confined to $N \approx Z$ nuclei (the only nuclei where proton–neutron pairs might play a role at low energies) and hence to lighter nuclei which generally exhibit less collectivity than those with higher mass number. The IBM-3 can only provide a partial description of such nuclei and inevitably misses out a number of levels (of single-particle or intruder character) at low energies. Secondly, because it conserves the isospin quantum number, the IBM-3 can be linked (more naturally than either IBM-1 or IBM-2) to the shell-model and previous IBM-3 studies have been concerned primarily with the connection between IBM-3 and the shell model, rather than with phenomenological applications. In particular, IBM-3 calculations have been performed in the $f_{7/2}$ and pf shells [10–12,28] with Hamiltonians and electromagnetic operators derived from a shell-model mapping. The agreement with experimental observations is good enough for low-lying symmetric states, but for higher-lying states discrepancies occur. An additional problem is that the number of data points for non-symmetric states is quite low.

As far as phenomenological applications of IBM-3 to $N \approx Z \approx 40$ nuclei are concerned, an example can be found in Ref. [29] where a schematic Hamiltonian is taken with parameters obtained through a fit to energy spectra of several nuclei. As an illustration of this type of approach we use here a schematic Hamiltonian with a (d-boson) pairing term and a quadrupole–quadrupole interaction, and apply it to the $f_{7/2}$ nuclei ^{44}Ti, ^{46}Ti, ^{48}Ti, and ^{48}Cr. For simplicity no dependence on N and T is considered in the Hamiltonian parameters and the quadrupole force is assumed to act exclusively between protons and neutrons, and not between identical particles [29]. The Hamiltonian then reads

$$\hat{H} = \epsilon_d \hat{n}_d + \kappa_0 \mathcal{N}[\hat{T}^{(2,0)} : \hat{T}^{(2,0)} + \frac{2}{3}\hat{T}^{(2,1)} : \hat{T}^{(2,1)}] + t\hat{T}^2, \tag{70}$$

where the symbol : denotes a scalar product in isospin and orbital space and $\mathcal{N}[\ldots]$ stands for a normal-ordered product. The structure parameters of the quadrupole operators are assumed to be independent of T: $\chi^{2T}_{02} = \chi^{2T}_{20} = 1$ and $\chi^{2T}_{22} = \chi$. The parameter ϵ_d is obtained from the binding energies and single-particle levels of $^{40-42}$Ca, $^{41-42}$Sc, and ^{42}Ti. The parameters κ_0 and χ are determined through a best fit to low-energy states in ^{44,46,48}Ti and ^{48}Cr. Finally, t is obtained through a comparison with a shell-model calculation [28] since reliable experimental information on states with $T > |M_T|$ is scarce. The parameters thus determined are: $\epsilon_d = 1.5$ MeV (the same for π, δ, and ν bosons), $\kappa_0 = -0.2$ MeV, $\chi = -2.4$, and $t = 1.2$ MeV. It is worth noting that the quadrupole–quadrupole interaction contains a sizeable contribution to the Majorana term [29] and this guarantees an approximate $U_L(6)$ symmetry for the low-lying states. The experimental and theoretical spectra are shown in Fig. 4. An overall agreement is observed for the low-lying states of all nuclei; the degree of agreement is similar to the one obtained in Ref. [12].

Regarding the electromagnetic transitions, some general conclusions independent of the $U_L(6)$ symmetry can be obtained. Unfortunately there are only few measured B(M1) transition rates in this mass region and it is difficult to determine the parameters in the electromagnetic operators. In Table IX the observed energies of non-symmetric states are compared to the ones calculated with IBM-3, the non-symmetric character of the states being proposed on the basis of their decay properties. In Table X the B(M1) values in IBM-3 are compared to the observed ones when known or to the shell-model values otherwise. Two parameters enter the IBM-3 calculation: the isovector boson g factor which is fixed to reproduce the $B(\mathrm{M1}; 2^+_M \to 2^+_1)$ value in ^{44}Ti ($g_1 = 1.20\mu_N$). Although not suggested by microscopy, one may study the influence of an isotensor boson g factor which is also illustrated in Table X with ($g_2 = 0.58\mu_N$), derived from the $B(\mathrm{M1}; 2^+_M \to 2^+_1)$ in ^{46}Ti. Rather unexpectedly, it is seen that the isotensor contribution improves the fit significantly. Note that the isotensor contribution for ^{44}Ti and ^{48}Cr vanishes because $T_z = 0$ for both nuclei.

Although boson-number expectation values are not directly measurable quantities, a comparison with more elaborate calculations can be carried out. In Fig. 5 the expectation value of boson numbers in IBM-3 in the ground state of even–even Fe and Cr isotopes is compared with SMMC calculations [26]. It is seen that the simple formulas (68) qualitatively reproduce the features of full microscopic calculations. The SMMC results are scaled such that the pair-number expectation values are normalized to the total number of bosons, which is only approximately valid for shells with finite size.

The contributions of the Majorana and $\hat{T}^2$ terms in the Hamiltonian are crucial to the relative position of states with different $U_L(6)$ symmetry and isospin. We illustrate this with a schematic calculation that is relevant for $N = Z$ nuclei. It is generally assumed that states with $[N-2, 2]$ $U_L(6)$ symmetry occur at higher energy than those with $[N-1, 1]$. This, however, is not necessarily so in $N = Z$ even–even nuclei. This peculiarity arises because the lowest allowed isospin value in the $[N-1, 1]$ representation is $T = 1$, while it can be $T = 0$ in the $[N-2, 2]$ representation. Thus, although the Majorana term always favors $[N-1, 1]$ over $[N-2, 2]$ states, in $N = Z$ nuclei this effect is counteracted by the $\hat{T}^2$ term. To make the argument somewhat more quantitative, assume a simple Hamiltonian with a $\hat{T}^2$ and a Majorana term besides a general orbital dependence $\hat{H}_L$,

$$\hat{H} = t\hat{T}^2 + m\hat{M} + \hat{H}_L. \tag{71}$$

Appropriate values of the parameters in the $f_{7/2}$ shell are $t = 1.2$ MeV and $m = 3.3$, 2.5, and 0.7 MeV for $N = 2$, 3, and 4, respectively, where the values of m are estimated from the energies of non-symmetric states in ^{44,46,48}Ti [10,11]. Application of the schematic

Hamiltonian (71) to an $N = Z$ nucleus with four bosons (e.g., ^{48}Cr or ^{64}Ge) gives the result shown in Fig. 6. It is seen in particular that the $[N-2,2]T=0$ states occur at a lower energy than those with $[N-1,1]T=1$. An intruiging feature is that the $[N-2,2]$ states cannot decay to the symmetric ground-state configuration if one assumes the electromagnetic operators of one-body type in the bosons. However, this effect is not very likely to persist for realistic Hamiltonians because of $U_L(6)$ mixing between $[N-2,2]$ and $[N-1,1]$ states.

IX. CONCLUSIONS

The main objective of this paper was to present a comprehensive analysis of the symmetry limits of the IBM-3 that conserve $SU_T(3)$ charge or $U_L(6)$ *sd* symmetry. Although particular results were obtained for the three limits $U_L(5)$, $SU_L(3)$, and $O_L(6)$, special emphasis was given to a general analysis independent of the latter limits but only requiring $SU_T(3)$ or $U_L(6)$ symmetry. The origin of this symmetry was shown to be related to the Majorana interaction in the IBM-3 Hamiltonian which leads to a decoupling of the orbital and isospin spaces. In previous, microscopic studies of IBM-3 the $SU_T(3)$ charge symmetry had been shown to be approximately valid and this paper has taken this result as a starting point to derive the properties of all limits with that symmetry.

A numerical application of the IBM-3 was presented involving a simple, phenomenological Hamiltonian with a few parameters either derived from shell-model considerations or fitted to the data. This Hamiltonian was applied to even–even $f_{7/2}$ nuclei where the $SU_T(3)$ charge symmetry is thought to be approximately valid. Reasonable results were obtained but any analysis of this kind will ultimately be hampered by the limited amount of collectivity exhibited by these nuclei. Applications to regions of more collective nuclei should thus be considered (e.g., $28 \leq N, Z \leq 50$).

A valuable aspect of algebraic models is that they usually are simple enough as to give clues concerning key observable quantities. The IBM-3 is no exception. Examples are the expressions derived for the expectation values of the various boson numbers; although these cannot be measured, they can be compared to similar expectation values calculated in more elaborate models. In particular, the IBM-3 expectation values were shown to be in good agreement with Shell Model Monte Carlo calculations. The present analysis has also revealed two intriguing predictions of the IBM-3. The first concerns the transfer probility of a δ boson (i.e., a proton–neutron $T = 1$ pair) which turns out to be zero in IBM-3. As such, this result is too schematic to assign it too much weight: the transfer necessarily takes place from or into an odd–odd nucleus for which an IBM-3 description is incomplete. However, it indicates that the same problem should be revisited in IBM-4 where it possibly can teach us something about the proton–neutron pair structure of $N \sim Z$ nuclei. The second intriguing prediction of the IBM-3 concerns the energy of non-symmetric states. In $N = Z$ nuclei, and in $N = Z$ nuclei *only*, it is conceivable that the $[N - 2, 2]$ states occur at a lower energy than those with $[N - 1, 1]$ symmetry. Assuming exact $SU_T(3)$ charge symmetry, the decay from $[N - 2, 2]$ to the symmetric ground-state configuration $[N]$ is forbidden. The breaking of $SU_T(3)$ charge symmetry will, however, destroy this selection rule; it would nevertheless be of interest to verify whether any remnant of it is still observable.

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TABLES

TABLE I. Lowest Eigenstates of a $U(5)$ Hamiltonian

$\vert 0_1^+;T\rangle$	$=$	$\vert[N](0,0,0)(0,0)0;T\rangle$
$\vert 0_2^+(d^2);T\rangle$	$=$	$\vert[N](2,0,0)(0,0)0;T\rangle$
$\vert 0_3^+(d^3);T\rangle$	$=$	$\vert[N](3,0,0)(3,0)0;T\rangle$
$\vert 1_M^+(d^2);T\rangle$	$=$	$\vert[N-1,1](1,1,0)(1,1)1;T\rangle$
$\vert 2_1^+(d);T\rangle$	$=$	$\vert[N](1,0,0)(1,0)2;T\rangle$
$\vert 2_2^+(d^2);T\rangle$	$=$	$\vert[N](2,0,0)(2,0)2;T\rangle$
$\vert 2_3^+(d^3);T\rangle$	$=$	$\vert[N](3,0,0)(1,0)2;T\rangle$
$\vert 2_M^+(d);T\rangle$	$=$	$\vert[N-1,1](1,0,0)(1,0)2;T\rangle$
$\vert 3_1^+(d^3);T\rangle$	$=$	$\vert[N](3,0,0)(3,0)3;T\rangle$
$\vert 3_M^+(d^2);T\rangle$	$=$	$\vert[N-1,1](1,1,0)(1,1)3;T\rangle$
$\vert 4_1^+(d^2);T\rangle$	$=$	$\vert[N](2,0,0)(2,0)4;T\rangle$
$\vert 4_2^+(d^3);T\rangle$	$=$	$\vert[N](3,0,0)(3,0)4;T\rangle$

TABLE II. Lowest Eigenstates of an $SU(3)$ Hamiltonian

$$
\begin{aligned}
|0_1^+;T\rangle &= |[N](2N,0)0;T\rangle \\
|0_\beta^+;T\rangle &= |[N](2N-4,2)2;T\rangle \\
|1_M^+;T\rangle &= |[N-1,1](2N-2,1)1;T\rangle \\
|2_1^+;T\rangle &= |[N](2N,0)2;T\rangle \\
|2_\beta^+;T\rangle &= |[N](2N-4,2)02;T\rangle \\
|2_\gamma^+;T\rangle &= |[N](2N-4,2)22;T\rangle \\
|2_M^+;T\rangle &= |[N-1,1](2N-2,1)2;T\rangle \\
|3_\gamma^+;T\rangle &= |[N](2N-4,2)3;T\rangle \\
|3_M^+;T\rangle &= |[N-1,1](2N-2,1)3;T\rangle \\
|3_{M'}^+;T\rangle &= |[N-1,1](2N-4,2)3;T\rangle \\
|4_1^+;T\rangle &= |[N](2N,0)4;T\rangle \\
|4_\beta^+;T\rangle &= |[N](2N-4,2)04;T\rangle \\
|4_\gamma^+;T\rangle &= |[N](2N-4,2)24;T\rangle
\end{aligned}
$$

TABLE III. Lowest Eigenstates of an $O(6)$ Hamiltonian

$$
\begin{aligned}
|0_1^+;T\rangle &= |[N](N,0,0\rangle(0,0)0;T\rangle \\
|0_2^+;T\rangle &= |[N](N,0,0)(3,0)0;T\rangle \\
|0_3^+;T\rangle &= |[N](N-2,0,0)(0,0)0;T\rangle \\
|1_M^+;T\rangle &= |[N-1,1](N-1,1,0)(1,1)1;T\rangle \\
|2_1^+;T\rangle &= |[N](N,0,0)(1,0)2;T\rangle \\
|2_2^+;T\rangle &= |[N](N,0,0)(2,0)2;T\rangle \\
|2_3^+;T\rangle &= |[N](N-2,0,0)(1,0)2;T\rangle \\
|2_4^+;T\rangle &= |[N](N-2,0,0)(2,0)2;T\rangle \\
|2_M^+;T\rangle &= |[N-1,1](N-1,1,0)(1,0)2;T\rangle \\
|3_1^+;T\rangle &= |[N](N,0,0)(3,0)3;T\rangle \\
|3_M^+;T\rangle &= |[N-1,1](N-1,1,0)(1,1)3;T\rangle \\
|4_1^+;T\rangle &= |[N](N,0,0)(2,0)4;T\rangle \\
|4_2^+;T\rangle &= |[N](N-2,0,0)(2,0)4;T\rangle
\end{aligned}
$$

TABLE IV. E2 Excitation out of the Ground State for the $U(5)$ Limit

J_f^π	$\mathcal{T}\lambda$	T	$B(\mathcal{T}\lambda, T; 0_1^+ \rightarrow J_f^\pi)$
2_1^+	$E2$	0	$(e_\pi + e_\delta + e_\nu)^2\, 5N$
2_1^+	$E2$	1	$(e_\pi - e_\nu)^2\, \frac{5T^2}{N}$
2_1^+	$E2$	2	$(-e_\pi + 2e_\delta - e_\nu)^2\, \frac{5T^2(2N+3)^2}{N(2T+3)^2}$
2_M^+	$E2$	1	$(e_\pi - e_\nu)^2\, \frac{5T(N-T)(N+T+1)}{N(T+1)}$
2_M^+	$E2$	2	$(-e_\pi + 2e_\delta - e_\nu)^2\, \frac{45T(N-T)(N+T+1)}{N(T+1)(2T+3)^2}$

TABLE V. M1 and E2 Excitation out of the Ground State for the $SU(3)$ Limit

$J_{\rm f}^{\pi}$	$\mathcal{T}\lambda$	T	$B(\mathcal{T}\lambda, T; 0_1^+ \rightarrow J_{\rm f}^{\pi})$
1_M^+	$M1$	1	$\frac{3}{4\pi}(g_\pi - g_\nu)^2 \frac{8T(N-T)(N+T+1)}{(2N-1)(T+1)}$
1_M^+	$M1$	2	$\frac{3}{4\pi}(-g_\pi + 2g_\delta - g_\nu)^2 \frac{72T(N-T)(N+T+1)}{(T+1)(2T+3)^2(2N-1)}$
2_1^+	$E2$	0	$(e_\pi + e_\delta + e_\nu)^2 N(2N+3)$
2_1^+	$E2$	1	$(e_\pi - e_\nu)^2 \frac{T^2(2N+3)}{N}$
2_1^+	$E2$	2	$(-e_\pi + 2e_\delta - e_\nu)^2 \frac{T^2(2N+3)^3}{N(2T+3)^2}$
2_M^+	$E2$	1	$(e_\pi - e_\nu)^2 \frac{3T(N-1)(N-T)(N+T+1)}{N(2N-1)(T+1)}$
2_M^+	$E2$	2	$(-e_\pi + 2e_\delta - e_\nu)^2 \frac{27T(N-T)(N-1)(N+T+1)}{N(2N-1)(T+1)(2T+3)^2}$

TABLE VI. M3 Excitation out of the Ground State for the $SU(3)$ Limit

J^{π}_{f}	$\mathcal{T}\lambda$	T	$B(\mathcal{T}\lambda, T; 0^+_1 \rightarrow J^{\pi}_{\mathrm{f}})$
3^+_{γ}	$M3$	0	$\frac{35}{8\pi}(\Omega_\pi + \Omega_\delta + \Omega_\nu)^2 \frac{8N(N-2)(N-1)}{3(2N-3)(2N-1)}$
3^+_{γ}	$M3$	1	$\frac{35}{8\pi}(\Omega_\pi - \Omega_\nu)^2 \frac{8(N-2)(N-1)T^2}{3N(2N-1)(2N-3)}$
3^+_{γ}	$M3$	2	$\frac{35}{8\pi}(-\Omega_\pi + 2\Omega_\delta - \Omega_\nu)^2 \frac{8T^2(N-1)(N-2)(2N+3)^2}{3N(2N-1)(2N-3)(2T+3)^2}$
3^+_{M}	$M3$	1	$\frac{35}{8\pi}(\Omega_\pi - \Omega_\nu)^2 \frac{4T(N-T)(2N+3)(N+T+1)}{15(N-1)(2N-1)(T+1)}$
3^+_{M}	$M3$	2	$\frac{35}{8\pi}(-\Omega_\pi + 2\Omega_\delta - \Omega_\nu)^2 \frac{12T(N-T)(2N+3)(N+T+1)}{(T+1)(2T+3)^2(N-1)(2N-1)}$
$3^+_{M'}$	$M3$	1	$\frac{35}{8\pi}(\Omega_\pi - \Omega_\nu)^2 \frac{4T(N-T)(N-2)^2(N+T+1)}{3N(N-1)(2N-3)(T+1)}$
$3^+_{M'}$	$M3$	2	$\frac{35}{8\pi}(-\Omega_\pi + 2\Omega_\delta - \Omega_\nu)^2 \frac{12T(N-T)(N-2)^2(N+T+1)}{N(N-1)(2N-3)(T+1)(2T+3)^2}$

TABLE VII. M1, E2, and M3 Excitation out of the Ground State for the $O(6)$ Limit

$J_{\rm f}^{\pi}$	$\mathcal{T}\lambda$	T	$B(\mathcal{T}\lambda, T; 0_1^+ \to J_{\rm f}^{\pi})$
1_M^+	$M1$	1	$\frac{3}{4\pi}(g_\pi - g_\nu)^2 \frac{3T(N-T)(N+T+1)}{(N+1)(T+1)}$
1_M^+	$M1$	2	$\frac{3}{4\pi}(-g_\pi + 2g_\delta - g_\nu)^2 \frac{27T(N-T)(N+T+1)}{(T+1)(2T+3)^2(N+1)}$
2_1^+	$E2$	0	$(e_\pi + e_\delta + e_\nu)^2 N(N+4)$
2_1^+	$E2$	1	$(e_\pi - e_\nu)^2 \frac{T^2(N+4)}{N}$
2_1^+	$E2$	2	$(-e_\pi + 2e_\delta - e_\nu)^2 \frac{T^2(N+4)(2N+3)^2}{N(2T+3)^2}$
2_M^+	$E2$	1	$(e_\pi - e_\nu)^2 \frac{2T(N+2)(N-T)(N+T+1)}{N(N+1)(T+1)}$
2_M^+	$E2$	2	$(-e_\pi + 2e_\delta - e_\nu)^2 \frac{18T(N-T)(N+2)(N+T+1)}{N(N+1)(T+1)(2T+3)^2}$
3_M^+	$M3$	1	$\frac{35}{8\pi}(\Omega_\pi - \Omega_\nu)^2 \frac{7T(N-T)(N+T+1)}{10(N+1)(T+1)}$
3_M^+	$M3$	2	$\frac{35}{8\pi}(-\Omega_\pi + 2\Omega_\delta - \Omega_\nu)^2 \frac{63T(N-T)(N+T+1)}{10(T+1)(2T+3)^2(N+1)}$

TABLE VIII. Two-Neutron Transfer Intensities in the $SU(3)$ Limit

$I([N]0_1^+;T,-T \rightarrow [N+1]0_1^+;T-1,-T+1)$	$= p_{\pi,0}^2 \dfrac{T(N-T+2)(2N+3)}{3(2T+1)(2N+1)}$
$I([N](L-2)_1^+;T,-T \rightarrow [N+1]L_1^+;T-1,-T+1)$	$= p_{\pi,2}^2 \dfrac{L(L-1)(2N+L+1)}{(2L-3)(2L-1)(2T+1)}$
	$\times \dfrac{(2N+L+3)T(N-T+2)}{(2N+1)(2N+2)}$
$I([N]0_1^+;T,-T \rightarrow [N+1]0_1^+;T,-T)$	$= 0$
$I([N](L-2)_1^+;T,-T \rightarrow [N+1]L_1^+;T,-T)$	$= 0$
$I([N]0_1^+;T,-T \rightarrow [N+1]0_1^+;T+1,-T-1)$	$= p_{\nu,0}^2 \dfrac{(T+1)(2N+3)(N+T+3)}{3(2T+3)(2N+1)}$
$I([N](L-2)_1^+;T,-T \rightarrow [N+1]L_1^+;T+1,-T-1)$	$= p_{\nu,2}^2 \dfrac{L(L+1)(2N+L+1)}{(2L-3)(2L-1)}$
	$\times \dfrac{(2N+L+3)(T+1)(N+T+3)}{(2T+3)(2N+1)(2N+2)}$

TABLE IX. Energies of Non-Symmetric States in ^{44,46,48}Ti and ^{48}Cr

Nucleus	State	Energy (MeV)	
		Observed	IBM-3
^{44}Ti	1_1^+	5.7[a]	5.2
	2_M^+	6.6	4.8
^{46}Ti	1_1^+	4.3	2.8
	2_M^+	2.5[a]	2.1
	3_M^+	3.6[a]	3.8
^{48}Ti	1_1^+	3.7	2.9
	2_M^+	2.4	2.2
	3_M^+	3.2	4.3
^{48}Cr	1_1^+	5.5[a]	5.4

[a] Calculated value from Refs. [10,11].

TABLE X. M1 Transition Rates between Symmetric and Non-Symmetric States in ^{44,46,48}Ti and ^{48}Cr

Nucleus	Transition	$B(\mathrm{M1})$ (μ_N^2)		
		Observed	Shell model[a]	IBM-3[b]
^{44}Ti	$2^+_M \to 2^+_1$		1.14	1.14 (1.14)
	$0^+_1 \to 1^+_1$		2.40	1.75 (1.75)
^{46}Ti	$2^+_M \to 2^+_1$		0.73	0.73 (1.13)
	$3^+_M \to 2^+_1$		0.07	0.20 (0.25)
	$3^+_M \to 4^+_1$		0.20	0.41 (0.69)
	$0^+_1 \to 1^+_1$	1.01		1.15 (1.34)
^{48}Ti	$2^+_M \to 2^+_1$	0.50(10)	0.58	0.90 (1.24)
	$3^+_M \to 2^+_1$	0.08(3)	0.003	0.30 (0.34)
	$3^+_M \to 4^+_1$	0.42(16)	0.32	0.49 (0.64)
	$4^+_M \to 4^+_1$	1.4(5)	1.50	
	$0^+_1 \to 1^+_1$	0.50(8)	0.54	1.82 (2.10)
^{48}Cr	$0^+_1 \to 1^+_1$		3.05	4.82 (4.82)

[a] From Refs. [10,11,30].

[b] This work. The numbers in parentheses are without isotensor contribution.

FIGURES

FIG. 1. Energy spectrum of $T = 0$ and $T = 1$ states for a $U(5)$ Hamiltonian (25) with parameters (in MeV) $A_2 = -0.175$, $B_1 = 0.400$, $F_2 = 0.010$, and $\beta_2 = 1.2$. The boson number is $N = 4$.

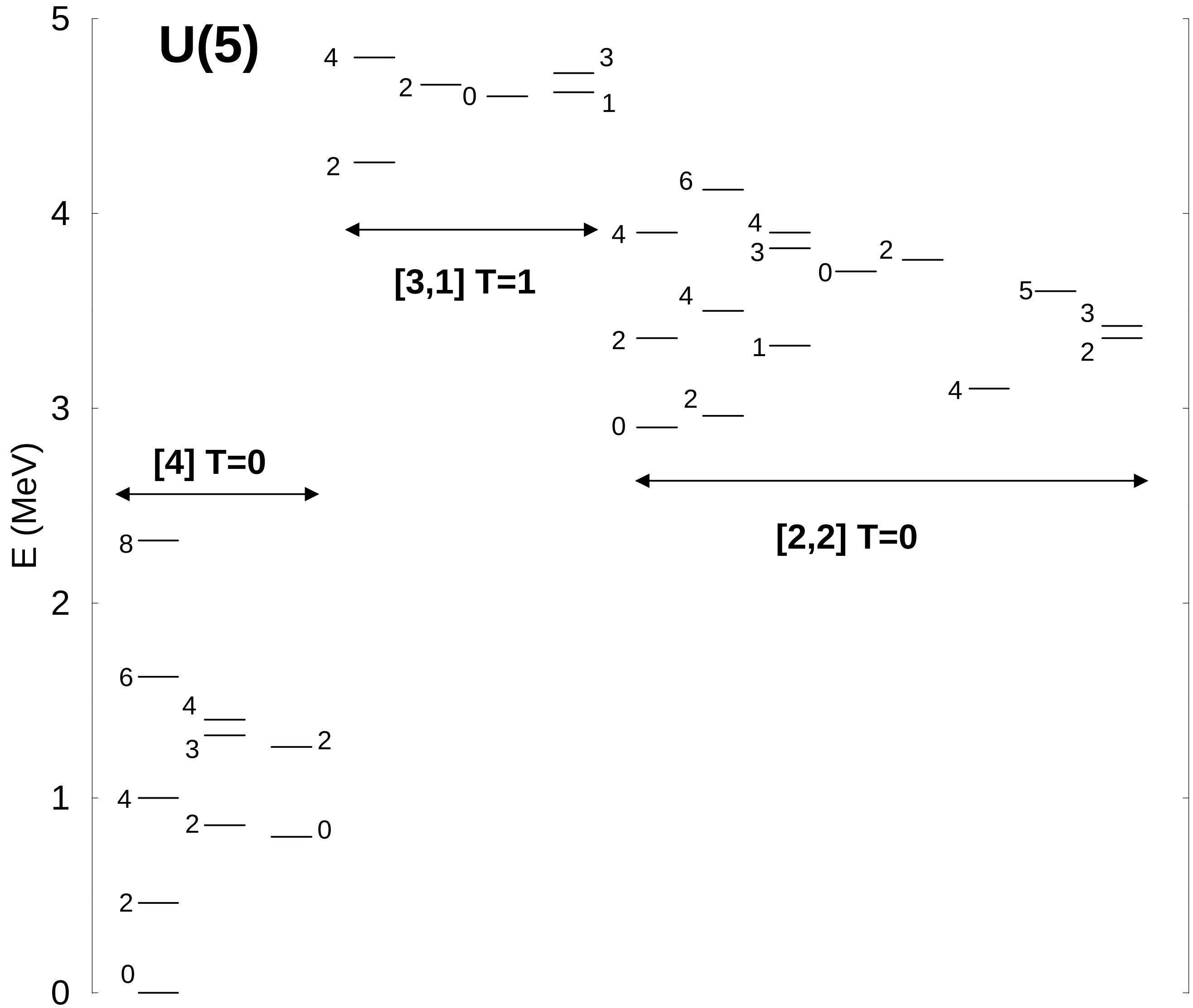
FIG. 2. Energy spectrum of $T = 0$ and $T = 1$ states for an $SU(3)$ Hamiltonian (29) with parameters (in MeV) $A_2 = -0.175$, $C_2 = -0.006$, $F_2 = 0.010$, and $\beta_2 = 1.2$. The boson number is $N = 4$.

FIG. 3. Energy spectrum of $T = 0$ and $T = 1$ states for an $O(6)$ Hamiltonian (33) with parameters (in MeV) $A_2 = -0.175$, $D_2 = -0.035$, $E_2 = 0.035$, $F_2 = 0.010$, and $\beta_2 = 1.2$. The boson number is $N = 4$.

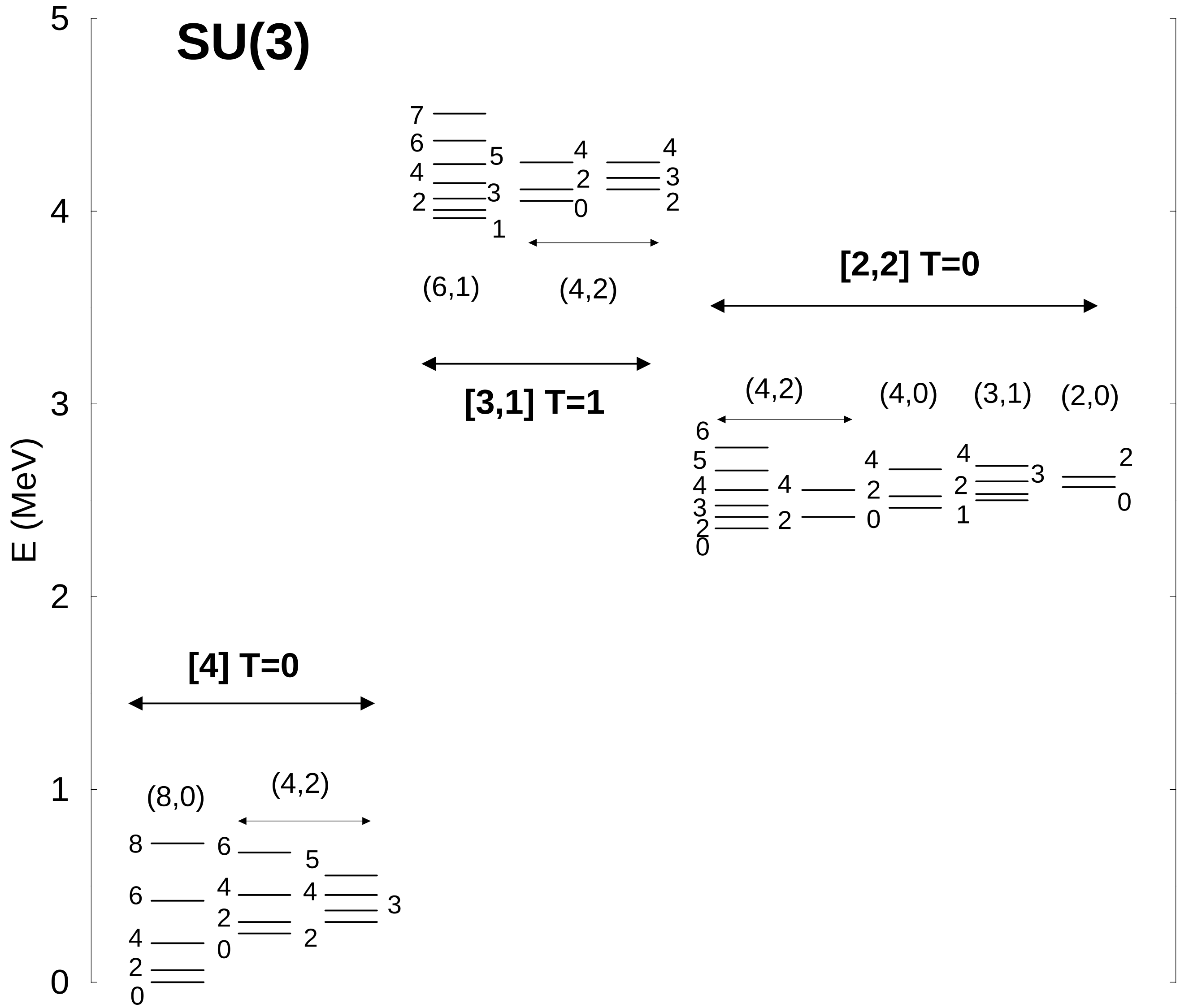
FIG. 4. Comparison of experimental and IBM-3 excitation energies of ^{44,46,48}Ti and ^{48}Cr. The parameters are constant for all isotopes and are given in the text.

FIG. 5. Ground-state boson-number expectation values in Cr and Fe isotopes as a function of $|T_z| = (N - Z)/2$ calculated with SMMC (left) and IBM-3 (right).

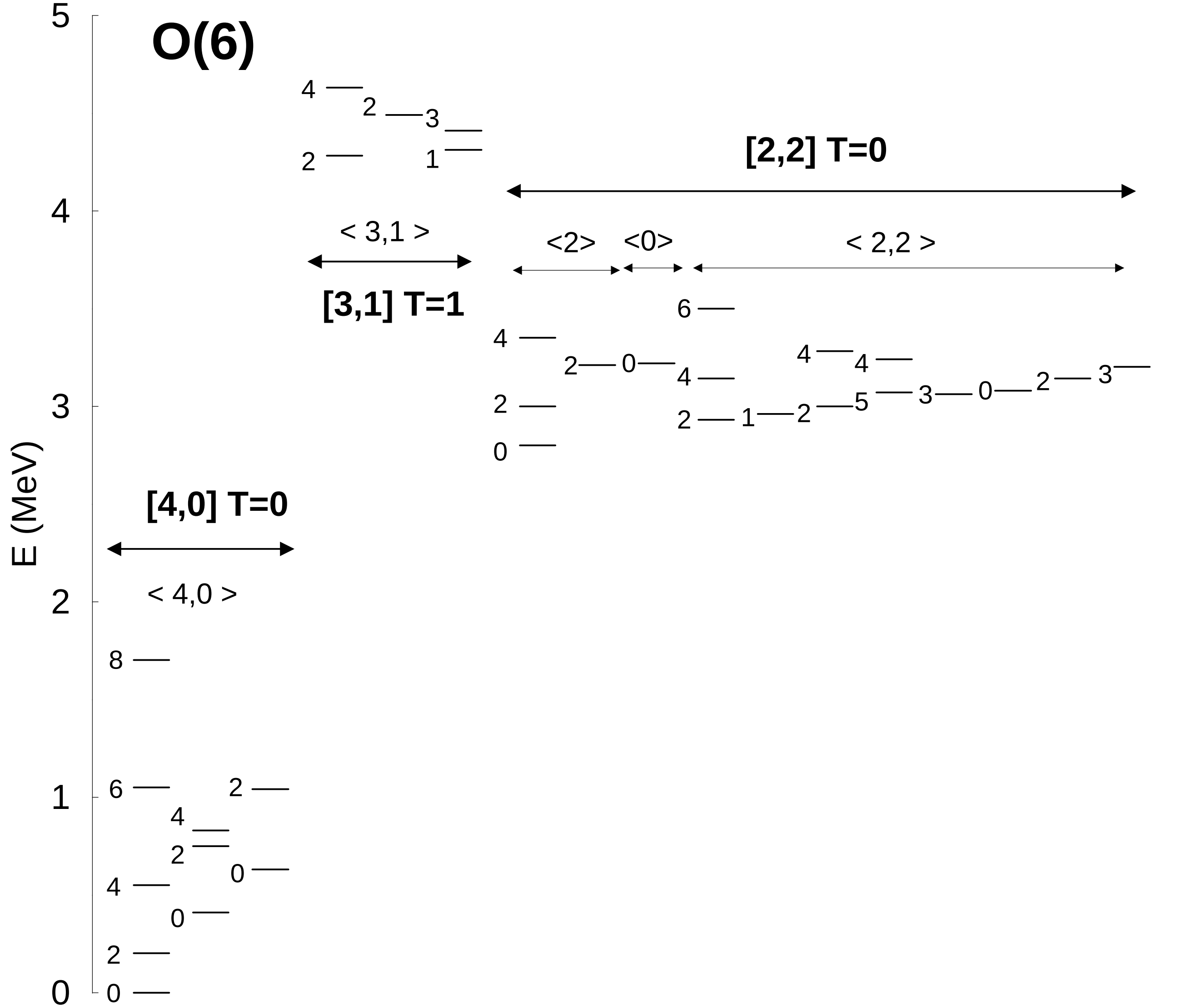
FIG. 6. Energies of [4], [3,1], and [2,2] states in an $N = Z$ nucleus with four bosons. The schematic Hamiltonian (71) is used with parameters (in MeV) $t = 1.2$ and $m = 0.7$. Transitions between [4] and [2,2] (dashed line) are forbidden.



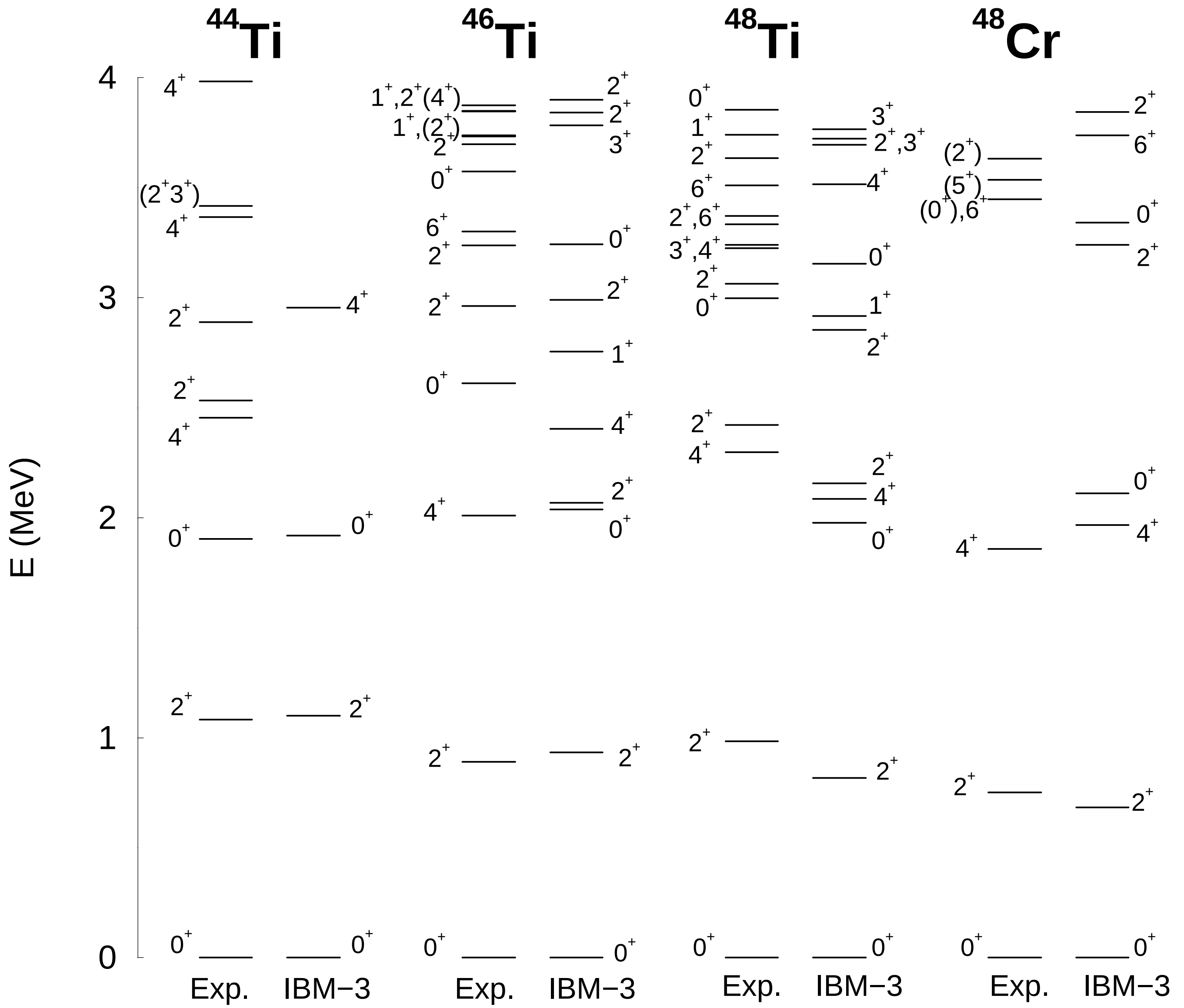
U(5)
E (MeV)
0
1
2
3
4
5
[4] T=0
[3,1] T=1
[2,2] T=0



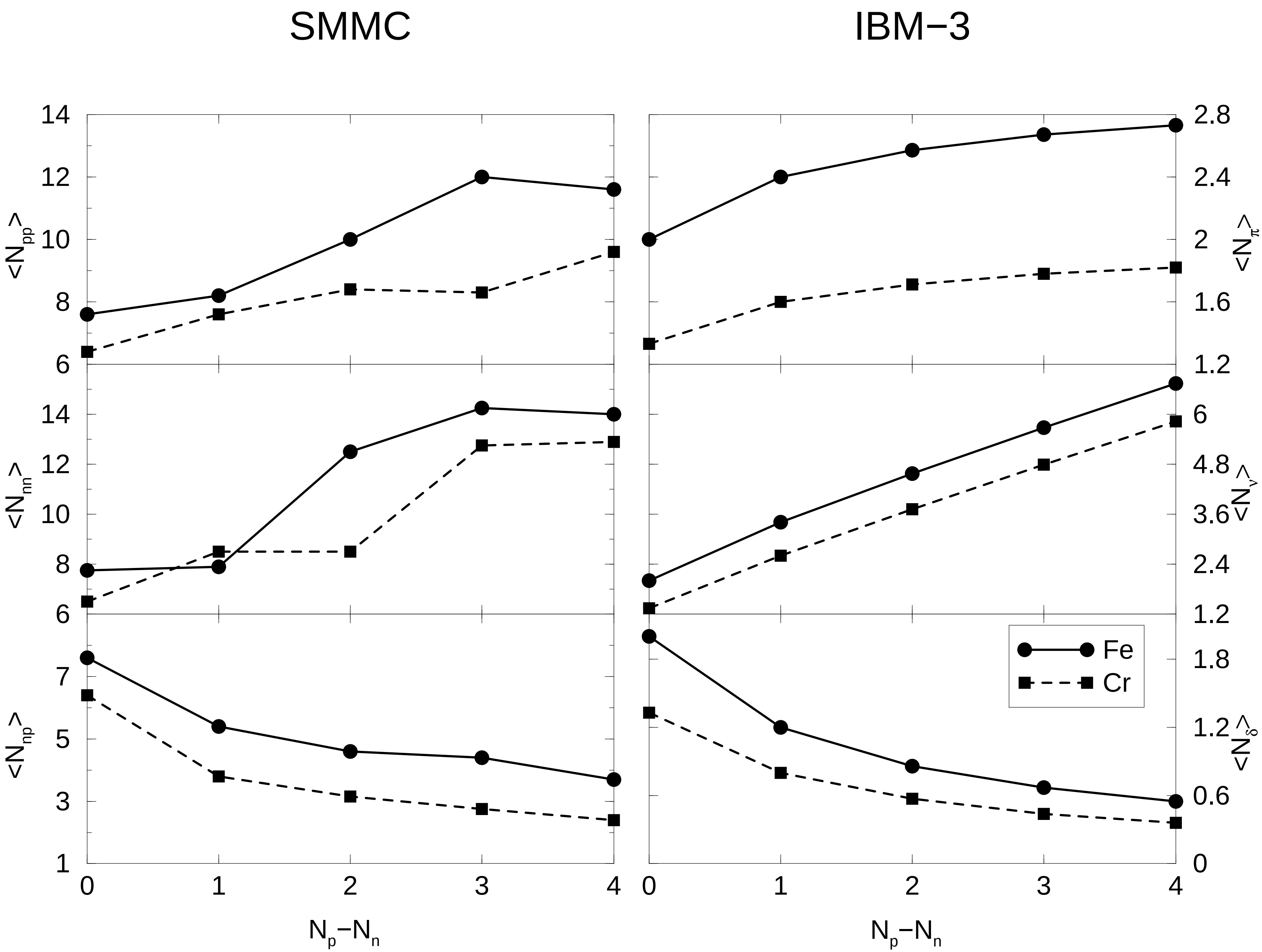
SU(3)
E (MeV)
0
1
2
3
4
5
[4] T=0
(8,0)
(4,2)
8
6
4
2
0
6
4
2
0
5
4
2
3
[3,1] T=1
(6,1)
(4,2)
7
6
4
2
5
3
1
4
2
0
4
3
2
[2,2] T=0
(4,2)
(4,0)
(3,1)
(2,0)
6
5
4
3
2
0
4
2
4
2
0
4
2
1
3
2
0



O(6)
E (MeV)
0
1
2
3
4
5
[4,0] T=0
< 4,0 >
[3,1] T=1
< 3,1 >
[2,2] T=0
<2>
<0>
< 2,2 >



44Ti
46Ti
48Ti
48Cr
E (MeV)
0
1
2
3
4
Exp.
IBM−3
Exp.
IBM−3
Exp.
IBM−3
Exp.
IBM−3


SMMC
IBM−3
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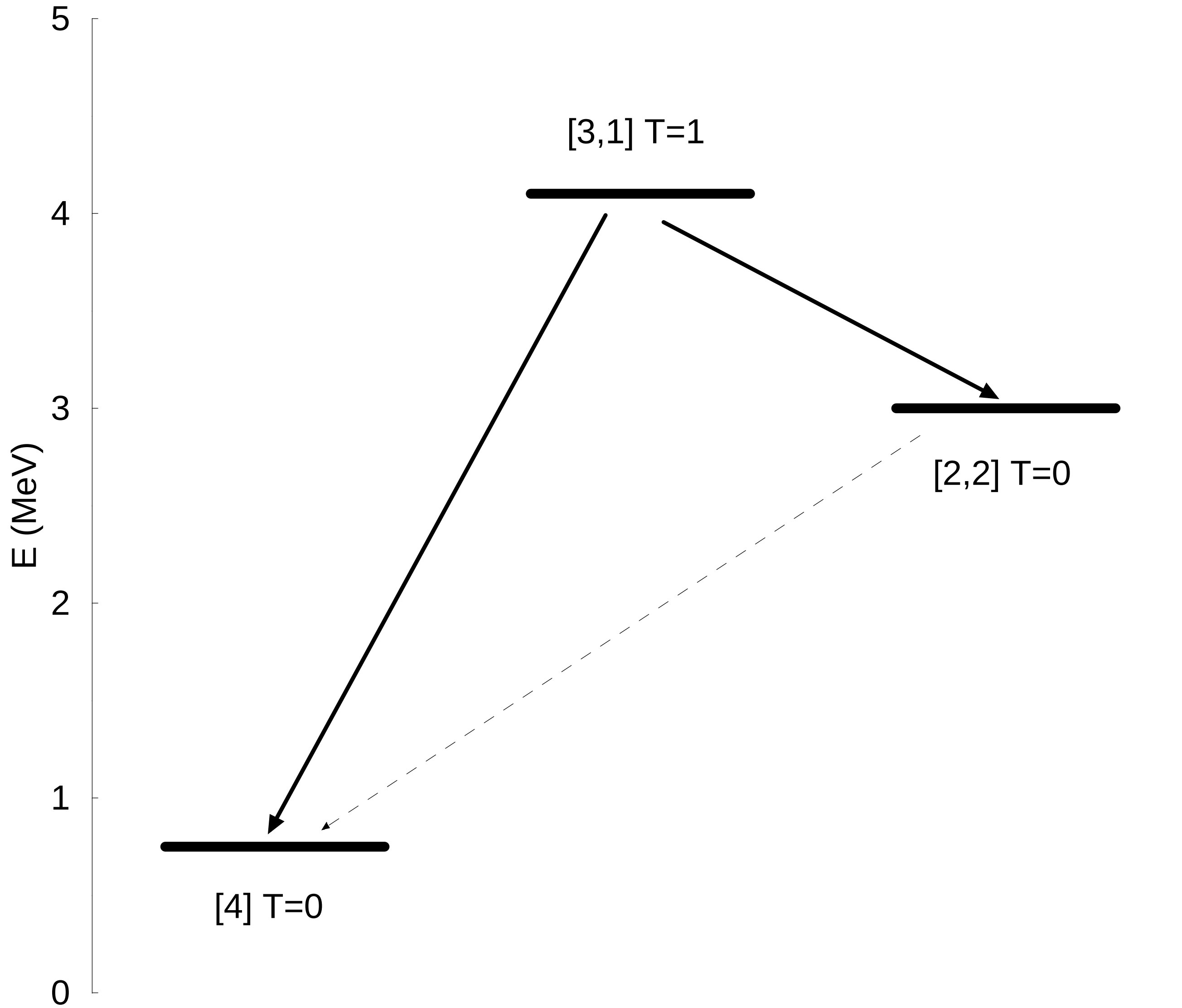
<N_{nn}>
<N_{np}>
<N_π>
<N_ν>
<N_δ>
Fe
Cr
$N_p - N_n$
$N_p - N_n$



5
4
3
2
1
0
E (MeV)
[3,1] T=1
[2,2] T=0
[4] T=0