Universidad de Huelva

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Problemas de centro e isocronía : linealización t-homogénea de campos vectoriales

Memoria para optar al grado de doctor presentada por:

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Manuel Reyes Columé

Tesis Doctoral

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Memoria presentada por Manuel Reyes Columé para optar al grado de Doctor en Ciencias Matemáticas.

Fdo. Manuel Reyes Columé

Certifico que la presente memoria ha sido realizada por Manuel Reyes Columé bajo mi supervisión y constituye su Tesis para aspirar al grado de Doctor en Matemáticas.

V$^{\underline{o}}$ B$^{\underline{o}}$ del Director:

Fdo. Antonio Algaba Durán

Huelva, 1 de Abril de 2009

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A mis padres, Pablo y Pepa.

A mis hijos, Ale y Marta.

A Bea, mi compañera.

Contents

Introducción[1]

El estudio de los sistemas dinámicos (modelos matemáticos que aproximan la evolución a través del tiempo de un sistema observado) comienza dentro del entorno de las ecuaciones diferenciales ordinarias cuya teoría se inicia en el siglo XVII con Newton y Leibnitz, siendo algunos problemas relacionados con el movimiento de un cuerpo rígido y de Mecánica Celeste los que motivaron su desarrollo.

Hasta finales del siglo XIX, que surge la teoría cualitativa de las ecuaciones diferenciales, desarrollada por Poincaré, fueron muchos los científicos que se dedicaron a intentar resolver por medio de cuadraturas las ecuaciones diferenciales. Destacándose los trabajos de Darboux [64] basados en las soluciones algebraicas de una ecuación diferencial extendida al plano proyectivo complejo.

El método introducido por Poincaré [135] dió un impulso a la formulación y desarrollo de una teoría de sistemas dinámicos. Éste consiste básicamente en estudiar el comportamiento de las soluciones sin el conocimiento explícito de éstas.

La teoría cualitativa donde más ha sido utilizada es en el estudio de los sistemas planos y uno de los más básicos y fundamentales resultados es el conocido Teorema de Poincaré-Bendixon, el cual afirma que cualquier solución acotada que no tienda a un punto singular del sistema debe ser

[1] De conformidad con el artículo 53 del Reglamento de los Estudios Oficiales de Posgrado de la Universidad de Huelva, toda tesis redactada en otro idioma distinto del castellano, deberá incluir un resumen en castellano.

necesariamente una órbita periódica o un grafo (solución formada por la unión de puntos críticos y órbitas).

En relación con el estudio de las órbitas periódicas de un sistema, el problema más famoso, y que en la actualidad sigue siendo un problema abierto, es el problema de determinar el número máximo $H(n)$ de ciclos límites (órbitas periódicas aisladas) que pueden aparecer en un sistema polinomial de grado n. Este problema fue propuesto por Hilbert en el Segundo Congreso Internacional de Matemáticas que tuvo lugar en Paris en 1900 y es uno de los dos únicos problemas que quedan por resolver. En la actualidad, sólo se conocen resultados parciales, siendo el más destacable el resultado dado por Il'yashenko [102] y Écalle [67] en el que afirma que el número de ciclos límites es finito. Atendiendo al valor de n, se sabe, por ejemplo, que $H(2) \geq 4$ y $H(3) \geq 12$ (ver [61, 103]). Las técnicas más usadas para la obtención de estas cotas superiores son fundamentalmente el análisis de bifurcaciones en un entorno de un foco débil y el estudio de las perturbaciones de sistemas integrables.

En esta memoria, estudiamos varios problemas relacionados con el análisis cualitativo de las ecuaciones diferenciales. Concretamente, abordamos el problema de caracterizar cuando un centro o un foco es isócrono, determinar la existencia de integrales primeras de un campo vectorial y caracterizar cuando un campo vectorial es linealizable mediante cambios de coordenadas en los cuales también pueden intervenir parametrizaciones de la variable tiempo.

A continuación, describimos estos problemas y mostramos algunos de los resultados más conocidos relacionados con los problemas planteados.

Problema de centro

Uno de los objetivos en el análisis de sistemas dinámicos es dar una completa caracterización de la geometría de la estructura orbital cerca de un punto fijo.

La caracterización del retrato de fases de un sistema autónomo plano de ecuaciones diferenciales cerca de un punto singular aislado es un problema resuelto casi en su totalidad, ver Arnold & Il'yashenko [26]. Sin embargo, a pesar de los esfuerzos realizados en los últimos años, aún sigue siendo un problema abierto determinar la estructura orbital del sistema cerca de un punto monodrómico, es decir, un punto con la propiedad de que las órbitas próximas al punto rodean a éste. Si el sistema es analítico se sabe, ver Il'yashenko [102], que un punto monodrómico sólo puede ser un centro o un foco.

Recordamos ambos conceptos. Dado un sistema plano de ecuaciones diferenciales

$$\dot{\mathbf{x}} = \mathcal{X}(\mathbf{x}) \tag{0.0.1}$$

de clase $\mathcal{C}^k(U)$, con $k \geq 1$ y U un conjunto abierto de $\mathbb{R}^2$, un *centro* del sistema (0.0.1) es un punto singular aislado $p \in U$ (es decir, hay un entorno $V \subset U$ tal que para cualquier $q \in V \setminus \{p\}$ se verifica $|\mathcal{X}(q)| \neq 0$ con $\mathcal{X}(p) = \mathbf{0}$) y la curva solución que pasa por q es una curva cerrada que rodea a p.
Un *foco* del sistema (0.0.1) es un punto singular aislado $p \in U$ tal que para cualquier $q \in V \setminus \{p\}$, la curva solución que pasa por q es una espiral que rodea a p.

Haciendo una traslación, podemos asumir sin pérdida de generalidad que p es el origen de coordenadas. Si usamos coordenadas polares $x = rcos\,\theta,\ y = rsen\,\theta$ y denominamos $r(t, r_0, \theta_0)$ y $\theta(t, r_0, \theta_0)$ la solución del sistema (0.0.1) que pasa por el punto (r_0, θ_0), decimos que el origen es un foco estable (respectivamente, inestable) si existe $\delta > 0$ tal que para cada $0 < r_0 < \delta$ y $\theta_0 \in \mathbb{R}$ se tiene que $r(t, r_0, \theta_0) \to 0$ y $|\theta(t, r_0, \theta_0)| \to \infty$ cuando $t \to \infty$ (respectivamente, $t \to -\infty$).

El problema de distinguir cuando un punto monodrómico de un sistema es un centro o un foco es conocido como *problema de centro.*

Cuando el punto singular es *hiperbólico* (los autovalores de la matriz de la parte lineal del sistema en el punto singular no están sobre los ejes imaginarios), Hartman [98] y Grobman [96] prueban, de forma independiente, que la estructura orbital de un sistema de clase $\mathcal{C}^1$ cerca del punto singular es cualitativamente la misma que la del sistema lineal asociado, $\dot{\mathbf{x}} = D\mathcal{X}(O)\mathbf{x}$. Además el cambio de variables que transforma el sistema en su parte lineal es un homeomorfismo, es decir, el sistema es $\mathcal{C}^0$ linealizable.

Por otra parte, si el punto singular no es hiperbólico, distinguimos los siguientes casos por separado:

Punto no hiperbólico no degenerado:

Si los autovalores de la matriz de la parte lineal en el punto singular son imaginarios no nulos (punto no hiperbólico no degenerado) el punto es monodrómico. Mediante un cambio lineal de variables y un escalado del tiempo, el sistema puede escribirse en la forma

$$\dot{x} = -y + P(x, y), \qquad \dot{y} = x + Q(x, y), \tag{0.0.2}$$

donde P y Q son funciones con términos de orden mayor o igual que dos. A estos sistemas se les conoce como *sistemas de tipo centro o foco.* Y al problema de centro para estos campos vectoriales se le denomina *problema de centro no degenerado.*

Las técnicas más conocidas para abordar el problema de centro en el caso no degenerado son las siguientes:

• En coordenadas polares, el sistema (0.0.2) es

$$\begin{aligned} \dot{r} &= cos\theta P(r\cos\theta, r\sin\theta) + sin\theta Q(r\cos\theta, r\sin\theta), \\ \dot{\theta} &= 1 + \tfrac{1}{r}(cos\theta Q(r\cos\theta, r\sin\theta) - sin\theta P(r\cos\theta, r\sin\theta)), \end{aligned} \tag{0.0.3}$$

es decir, el sistema tiene la forma

$$\dot{r} = \mathcal{O}(r^2), \qquad \dot{\theta} = 1 + \mathcal{O}(r).$$

Sea $(r(t, r_0, \theta_0),\ \theta(t, r_0, \theta_0))$ la solución de (0.0.3) que pasa por (r_0, θ_0). Para r_0 suficientemente pequeño, $\theta(t, r_0, \theta_0)$ es una función de t estrictamente creciente con lo cual existe $t(\theta, r_0, \theta_0)$ la función inversa de $\theta(t, r_0, \theta_0)$. Para θ_0 fijado, se define la función

$$P(r_0) = r(t(\theta_0 + 2\pi, r_0, \theta_0), r_0, \theta_0)$$

para r_0 suficientemente pequeño.

Si el sistema (0.0.2) es analítico, P es analítica para todo r_0 suficientemente pequeño y se conoce como la *aplicación de Poincaré* de (0.0.2) en el origen (ver Andronov *et al.* [24]).

Consideramos la aplicación de Poincaré asociada al sistema (0.0.2) en los puntos del semieje $OX^+(\theta_0 = 0)$ y $r < r_0$, desarrollamos en serie de potencias de $r = \sqrt{x^2 + y^2}$ y restamos r. De esta forma, obtenemos una serie de potencias en r de la forma

$$d(r) = P(r) - r = d_1 r + d_2 r^2 + d_3 r^3 + \cdots$$

la cual se conoce como *función desplazamiento.*

A partir de esta expresión de la función desplazamiento, deducimos las siguientes propiedades:

1. El origen de (0.0.2) es un centro si y sólo si la aplicación de Poincaré es la identidad, es decir, todas las constantes d_i son nulas.

2. Si la aplicación de Poincaré no es la identidad, se tiene:

   (a) el signo de la primera constante d_i diferente de cero determina la estabilidad del foco. Concretamente, el foco es inestable si es positiva y estable si es negativa,

   (b) cada raíz de la función desplazamiento equivale a una trayectoria cerrada aislada cerca del origen que denominamos *ciclo límite de pequeña amplitud.* Por tanto, si d_k es la primera constante distinta de cero (en este caso se dice que el origen es un *foco débil de orden k*), perturbando los coeficientes del sistema podemos asegurar que a lo más hay k ciclos límites próximos al origen. Además, bajo ciertas condiciones podemos asegurar la existencia de exactamente k ciclos límites.

• La existencia de una integral primera de un sistema de ecuaciones diferenciales está también vinculada con el problema de centro. Una función H es una *integral primera* del sistema (0.0.2) si H es una función constante sobre cada solución del sistema (0.0.2).

Poincaré [135] demuestra que el origen de un sistema polinomial (0.0.2) es un centro si y sólo si existe una integral primera analítica del sistema definida en un entorno del origen. Liapunov [110] extiende el resultado anterior a sistemas (0.0.2) analíticos.

La siguiente técnica para abordar el problema de centro no degenerado fue introducida por Poincaré [135] y desarrollada por Liapunov [110]. Esta técnica consiste en buscar una serie formal de potencias

$$H(x,y) = (x^2+y^2)/2 + H_3(x,y) + H_4(x,y) + \cdots$$

con H_i polinomios homogéneos de grado i verificando

$$\dot{H} = \frac{\partial H}{\partial x}(-y+P) + \frac{\partial H}{\partial x}(x+Q) = \sum_{i=1}^{\infty} v_i (x^2+y^2)^i,$$

donde $\dot{H}$ es la derivada de H con respecto a la variable t. La serie formal H no es única y se le conoce por *función de Liapunov* y a las constantes v_i se les denominan *constantes de Poincaré-Liapunov* de (0.0.2).

Si todas las constantes son nulas, H es una función constante sobre cada solución del sistema (0.0.2) en un entorno del origen y por tanto H es una integral primera del sistema (0.0.2).

Poincaré [135] para el caso polinomial y Liapunov [110] para el caso analítico prueban que cuando todas las constantes son nulas, existe una serie H convergente. Así, como consecuencia se tiene el teorema de centro no degenerado:

> *El origen del sistema analítico (0.0.2) es un centro si y sólo si existe una integral primera analítica de la forma $H(x, y) = (x^2 + y^2)/2 + \cdots$*

Frommer [73] prueba que $d_p = 2\pi v_p$ módulo la anulación de las constantes d_i anteriores. Por tanto, las constantes de Poincaré-Liapunov también pueden ser usadas para determinar el número de ciclos límites de pequeña amplitud que pueden bifurcar desde el origen.

• Otra herramienta usada para resolver el problema de centro de un sistema dinámico se basa en la Teoría de Formas Normales. Está probado que para cualquier orden $l \in \mathbb{N}$, existe un cambio de coordenadas polinomial que transforma el sistema (0.0.2) en

$$\begin{aligned}
\dot{x} &= -y + (\alpha_3 x - \beta_3 y)(x^2 + y^2) + (\alpha_5 x - \beta_5 y)(x^2 + y^2)^2 + \cdots \\
&\qquad + (\alpha_{2l+1} x - \beta_{2l+1} y)(x^2 + y^2)^l + \mathcal{O}(|x, y|^{2l+3}), \\
\dot{y} &= x + (\alpha_3 y + \beta_3 x)(x^2 + y^2) + (\alpha_5 y + \beta_5 x)(x^2 + y^2)^2 + \cdots \\
&\qquad + (\alpha_{2l+1} y + \beta_{2l+1} x)(x^2 + y^2)^l + \mathcal{O}(|x, y|^{2l+3}),
\end{aligned}$$

la cual es conocida como la forma normal de Poincaré-Dulac a orden $2l + 1$ de (0.0.2). O bién, si usamos coordenadas polares

$$\begin{aligned}
\dot{r} &= \alpha_3 r^3 + \alpha_5 r^5 + \cdots + \alpha_{2l+1} r^{2l+1} + \mathcal{O}(r^{2l+2}), \\
\dot{\theta} &= 1 + \beta_3 r^2 + \beta_5 r^4 + \cdots + \beta_{2l+1} r^{2l} + \mathcal{O}(r^{2l+1}).
\end{aligned}$$

De la expresión de la forma normal de (0.0.2) se deduce que $\alpha_{2p+1} = v_p$ módulo la anulación de las constantes α_{2i+1} anteriores. Por tanto, si todas las constantes α_{2i+1} son nulas, el origen es un centro, y en otro caso, si α_{2p+1} es la primera constante distinta de cero, el origen de (0.0.2) es un foco débil de orden p.

Si P y Q son polinomios y si consideramos sus coeficientes como parámetros del sistema, las constantes v_i, d_i y α_{2i+1} son polinomios en los coeficientes del sistema (0.0.2). Como el anillo de los polinomios con variables los coeficientes del sistema es Noetheriano, por el Teorema de la Base de Hilbert, el ideal generado por los coeficientes del sistema es de dimensión finita y por tanto el número de condiciones para probar la existencia de un centro es un número finito, el cual sólo es conocido en algunos casos particulares.

Las expresiones de las primeras constantes han sido obtenidas mediante diferentes métodos: la determinación de una función de Lyapunov (ver Alwash [18], Blows & Lloyd [30], Göbber & Willamowski [93]), estudiando la aplicación de Poincaré (ver Gasull *et al.* [78, 79], Romanovski [137]), usando técnicas de formas normales (ver Algaba *et al.* [1]), entre otros.

• Cherkas [52] da un resultado que caracteriza cuando el origen de una ecuación analítica de Liénard $\ddot{x} + f(x)\dot{x} + g(x) = 0$ es un centro.

Más tarde, Christopher & Lloyd [62] usan este resultado para determinar los centros de un sistema el cual mediante cambio de variables puede ser transformado en una ecuación de Liénard.

La mayoría de los métodos usados para averiguar si un sistema tiene un centro, son combinaciones de condiciones suficientes (la reversibilidad del sistema, existencia de una integral primera, etc) y condiciones necesarias (la anulación de los términos de la aplicación de Poincaré, las constantes de Poincaré-Liapunov, etc) el método desarrollado por Christopher & Lloyd, basado en el resultado de Cherkas, da una condición necesaria y suficiente, este hecho hace que este método sea muy útil para estudiar el problema de centro para ecuaciones diferenciales que pueden transformarse en una ecuación de Liénard.

Se han realizado numerosos trabajos encaminados a resolver el problema de centro. A continuación citamos algunas familias de centros conocidas. Por ejemplo, si el sistema es polinomial, el caso cuadrático

$$\dot{x} = -y + P_2(x, y), \qquad \dot{y} = x + Q_2(x, y)$$

donde P_2 y Q_2 son polinomios homogéneos de grado 2, ha sido resuelto por Dulac [66], Kapteyn [106], Frommer [73], Bautin [28] and Zoladek [163]. De estos trabajos, se deduce que los sistemas (0.0.2) cuadráticos tienen tres constantes d_i Poincaré-Liapunov independientes, es decir, las tres primeras constantes generan el ideal de todas las constantes, siendo Bautin quien las determina por primera vez y caracteriza sus centros. Lunkevich & Sibirskii [121] encuentran las integrales primeras analíticas para estos centros.

El problema de centro de los sistemas cúbicos

$$\dot{x} = -y + P_2(x,y) + P_3(x,y), \qquad \dot{y} = x + Q_2(x,y) + Q_3(x,y)$$

donde P_i y Q_i, $i = 2, 3$ son polinomios homogéneos de grado i, también ha sido estudiado por diversos autores y sin embargo aún sigue siendo un problema sin estar resuelto en su totalidad. Los centros de los sistemas cúbicos con nolinealidades homogéneas ($P_2 = Q_2 \equiv 0$) han sido encontrados por Blows & Lloyd [30], Sibirskii [150] y Zoladek [163], obteniéndose como consecuencia que hay cinco constantes independientes, es decir, que la anulación de las cinco primeras constantes de Poincaré-Liapunov implica la anulación de las restantes. Zoladek [165, 168] caracteriza los sistemas reversibles cúbicos. Los centros de los sistemas de Kukles

$$\dot{x} = y, \quad \dot{y} = -x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$$

parcialmente caracterizados por Kukles [108], por Jin & Wang [105] y por Lloyd & Pearson [116].
Los sistemas cúbicos con degeneración en el infinito (es decir, P_3 y Q_3 son polinomios homogéneos verificando $xQ_3(x,y) - yP_3(x,y) \equiv 0$) estudiados por Lloyd *et al.* [115], Chavarriga & Giné en [43], entre otros. Lloyd *et al.* [117] calculan los centros de una clase de sistemas cúbicos de Kolmogorov. Romanovski & Şubă [139] resuelven el problema de centro de dos familias de sistemas cúbicos con ocho parámetros. Los sistemas

cúbicos, mediante un giro pueden transformarse en un sistema cúbico cuyo coeficiente de y^2 en P_2 es cero, ver Ye [161]. Si además, los coeficientes de los términos xy^2 y y^3 de P_3 son iguales a cero, Amelkin *et al.* [21] prueban que estos sistemas cúbicos pueden transformar en un sistema de Liénard. Cherkas & Romanovski [53] resuelven el problema de centro de una subfamilia amplia de estos sistemas, la cual contiene el sistema cúbico con el centro de ciclicidad 11 encontrado por Zoladek [166].

Chavarriga *et al.* [46, 47] dan resultados parciales a los sistemas con nolinealidades homogéneas de grados cuatro y cinco, respectivamente.

Punto no hiperbólico degenerado.

Un punto es un *hiperbólico degenerado* si los autovalores de la matriz de la parte lineal en el origen son nulos. En tal caso, si la matriz no es idénticamente nula, el punto aislado se conoce como *punto nilpotente* y si es nula se denomina *punto degenerado.*

Un sistema analítico plano de ecuaciones diferenciales cuyo origen es un punto singular nilpotente, en unas coordenadas adecuadas, puede escribirse como

$$\dot{x} = y + P(x,y), \qquad \dot{y} = Q(x,y), \tag{0.0.4}$$

donde P y Q son funciones analíticas definidas en un cierto entorno del origen, sin constantes ni términos lineales en su desarrollo.

Por Andreev [22] podemos conocer el comportamiento de las soluciones en un entorno del punto singular, excepto si se trata de un centro o un foco, (*problema de centro nilpotente*).

La existencia de una integral primera analítica de estos sistemas es una condición suficiente para que un punto singular monodronico sea centro, pero no necesaria. Cuando la parte lineal del sistema es nilpotente, Moussu [130] y Sadovskii [148] caracterizan teóricamente los centros. Giacomini *et al.* [85] analizan los centros nilpotentes como límites de los centros no degenerados con parte lineal $(-y, x\epsilon)^T$ con ϵ suficientemente pequeño. De esta manera, pueden aplicar a este problema las técnicas antes consideradas en el problema de centro no degenerado. Strozyna & Zoladek [153] obtienen los centros con primera integral analítica a través de la forma normal orbital (es decir, formas normales en las que también pueden realizarse, a parte de cambios de coordenadas, reparametrizaciones de la variable tiempo). Concretamente, Strozyna & Zoladek [153] demuestran que los centros nilpotentes con una primera integral analítica local en el origen tiene una forma normal orbital de la forma

$$\dot{x} = y, \qquad \dot{y} = -x^{2m-1},\ m \in \mathbb{N}.$$

Chavarriga *et al.* [42] prueban que si un centro nilpotente de un sistema analítico tiene una integral primera formal (analítica), entonces tiene una integral primera formal (analítica) dada por $H = y^2 + F(x, y)$, donde el desarrollo de F en serie de potencias comienza con términos de orden mayor que dos. Este resultado se utiliza como una herramienta para detectar si un determinado centro nilpotente analítico tiene o no una integral primera analítica. Así, por ejemplo, el sistema

$$\dot{x} = y + x^2,\ \dot{y} = -x^3$$

tiene un centro en el origen, puesto que es monodrómico y reversible al cambio $(x, y, t) \to (-x, y, -t)$, pero no existe una integral primera analítica definida en el origen, ni siquiera una integral primera formal. En este trabajo, los autores también demuestran que los sistemas nilpotentes reversibles bajo el cambio de variables $(x, y, t) \to (x, -y, -t)$ tienen una integral primera analítica en el origen.

Hay sólo unas pocas familias de sistemas (0.0.4) polinomiales cuyos centros se conocen. El problema de centro para los sistemas (0.0.4), donde $P(x,y) = P_{2n+1}(x,y)$ y $Q(x,y) = Q_{2n+1}(x,y)$ son polinomios homogéneos de grado $2n+1$, está resuelto para $n=1$ por Andreev, para $n=2$ por Sadovskii y para $n=3$ por Andreev y otros, ver [23]. En este artículo, los autores también obtienen la multiplicidad de los focos para $n=1,2$ y 3. Sadovskii [148] encuentra los centros de la familia (0.0.4) para $P(x,y) = P_2(x,y)$ y $Q(x,y) = Q_3(x,y)$.

Gasull & Torregrosa [82] extienden el método de Cherkas a ecuaciones Liénard degeneradas y aplican esta extensión para resolver el problema de centro de varias familias de ecuaciones diferenciales polinómicas. Entre otros, caracterizan los centros de los sistemas

$$\dot{x} = y + a_1 xy + a_2 x^{q+1}, \qquad \dot{y} = -x^{2q-1} + b_1 y^2 + b_2 x^q y + b_3 x^{2q},$$

y

$$\dot{x} = y - (a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5), \qquad \dot{y} = -(b_3 x^3 + b_4 x^4 + b_5 x^5),$$

donde $a_i, b_i,\ i = 1, ..., 5$ son parámetros reales y $q \in \mathbb{N}$.
Álvarez & Gasull [16, 17] calculan las dos primeras constantes de Lyapunov generalizadas de (0.0.4), introducidas por Lyapunov [110, 111], y resuelven el problema de la estabilidad de un foco de varias familias polinomiales.

Por último, si la matriz de la parte lineal en el origen es idénticamente nula (punto singular *degenerado*) el problema de la determinación topológica fue resuelto por Brunella & Miari [32], excepto cuando el punto es monodrómico, el cual ha sido caracterizado por Medvedeva [129], si bién, García [75] aporta un algoritmo de monodromía más simple teniendo en cuenta la parte conservativa y disipativa del campo vectorial.

Análogamente al caso nilpotente, en general, no podemos esperar que los centros sean analíticamente integrables. Nemytskii & Stepanov [131] dan un ejemplo de un centro degenerado que no tiene primera integral analítica.

Hay escasos avances en el problema de distinguir si el punto es un centro o un foco (*problema de centro degenerado*). Destacamos el trabajo de Gasull *et al.* [80] que caracteriza los sistemas planos definidos por la suma de dos campos vectoriales homogéneos cuyo origen es un punto singular degenerado monodrómico y estudia su estabilidad. También, Gasull *et al.* [81] analizan la monodromía y la estabilidad de una clase de sistemas con un punto degenerado mediante la determinación de la expresión del término principal del desarrollo asintótico de la aplicación retorno del origen.

Problema de isocronía.

El problema de isocronía consiste en determinar si un punto monodrómico es isócrono. En el caso de centro, el origen es isócrono si el tiempo de vuelo de las órbitas que rodean al centro es el mismo para cada órbita, mientras que en el caso de foco, éste es isócrono si existe una curva que pasa por el origen tal que el tiempo que tarda una órbita cualquiera en volver a cortar la curva es constante.

Aunque ambos problemas, centro isócrono y foco isócrono, son de naturaleza muy distinta, ellos están muy relacionados. A continuación detallamos por separado cada caso y presentamos los resultados más destacables de cada problema.

Consideremos un sistema autónomo plano cuyo origen es un centro. Sea $\mathcal{V}$ el anillo periódico del origen, es decir, el entorno de O rellenado por las órbitas periódicas del sistema que lo rodean. El conjunto de las órbitas periódicas en $\mathcal{V}$ puede ser parametrizado por una curva analítica

$$\Gamma := \{\gamma(\xi) \in \mathcal{V} \,|\, 0 < \xi < \xi_1,\ \gamma(0) = O\},$$

transversal a ellas. Así, la *función periodo* $T(\xi)$ se define como el periodo de la órbita periódica que pasa a través del punto $\gamma(\xi)$. Está probado que esta definición no depende ni de la elección de Γ ni de su parametrización.

Un centro se dice *centro isócrono* si $T(\xi)$ no depende de ξ, es decir, la función periodo es constante. El problema de determinar si un centro es un centro isócrono se denomina *problema de centro isócrono.*

El comportamiento de la función periodo de un centro ha sido estudiado en algunas clases particulares de campos vectoriales.

Hay sólo unas pocas familias de sistemas diferenciales polinomiales en la que es conocida una clasificación de sus centros isócronos. Urabe [158] demuestra que la unica ecuación $\ddot{x}+g(x)=0$ teniendo un centro isócrono, con g una función impar, es $\ddot{x}+x=0$. El resultado de Urabe fue extendido por Chicone & Jacobs [54] en el caso de un polinomio g no impar. Loud [119] clasifica los centros isócronos cuadráticos. Pleshkan [133] ha determinado los centros isócronos cúbicos con parte nolineal homogénea. Los centros isócronos de la familia de Kukles fue estudiada en Christopher, Devlin [59]. Todos estos centros son time-reversible. Mardesic *et al.* [122] encuentran una familia de sistemas cúbicos isócronos los cuales contienen sistemas de Kolmogorov cúbicos isócronos.

Lukasevic [120] muestra que si P y Q son funciones armónicas conjugadas, entonces los centros del sistema (0.0.2) son isócronos. La ecuación de Liénard $\ddot{x}+f(x)\dot{x}+g(x)=0$ y la isocronía de sus centros han motivado a varios investigadores, por ejemplo Christopher & Lloyd [62], Sabatini [144]. Recientemente, Sabatini [146] y Chouikha [56] han estudiado los centros y los centros isócronos de la ecuación $\ddot{x}+f(x)\dot{x}^2+g(x)=0$.

El trabajo de Sabatini & Chavarriga [50] es un resumen de los resultados que se conocen sobre isocronía de centros.

En otro sentido, Chicone & Jacobs [54, 55] muestran propiedades de la función periodo de una determinada clase de sistemas analizando las posibles bifurcaciones de puntos críticos de la función periodo. Recientemente, Gasull & Yu [84] dan condiciones necesarias y suficientes para la existencia de periodos críticos de las órbitas de sistemas isócronos cuadráticos cuando son peturbados dentro de la familia de sistemas cuadráticos reversibles.

Villarini [159] prueba que los centros no degenerados son los únicos que pueden ser isócronos, es decir, sólo existen centros isócronos en los sistemas (0.0.2). Antes, Poincaré [135] muestra que el centro no degenerado de un sistema analítico es isócrono si y sólo si existe un cambio analítico de coordenadas en un entorno del origen que linealiza el sistema, es decir, existe siempre un cambio local analítico que transforma el sistema (0.0.2) en

$$\dot{u} = -v, \quad \dot{v} = u.$$

Chavarriga *et al.* [46, 47, 44] usan esta propiedad para obtener un algoritmo iterativo basado en la construcción del cambio, para encontrar las condiciones que implican la isocronía de un centro.

Esta propiedad también ofrece un método sencillo para generar centros isócronos de campos vectoriales hamiltonianos. Este método, dado por Sabatini [143], se basa en la búsqueda de dos funciones f y g con $f(0,0) = g(0,0) = 0$ tal que la aplicación $(x,y) \to (f(x,y), g(x,y))$ es una aplicación jacobiana, es decir, el determinante de su matriz jacobiana es constante y distinto de cero. En este caso, el sistema hamiltoniano asociado a $H = f^2 + g^2$ es linealizable mediante el cambio canónico de coordenadas $u = f(x,y),\ v = g(x,y)$.

El cálculo de la función periodo es otra técnica utilizada en el problema de centro isócrono. En coordenadas polares, el sistema (0.0.2) se escribe como

$$\dot{r} = R(r,\theta), \quad \dot{\theta} = 1 + \Theta(r,\theta).$$

Si definimos $r(\theta,\rho)$ la solución periódica tal que $r(0,\rho) = \rho$ con ρ suficientemente pequeño, la función periodo viene dada por

$$T(\rho) = \int_0^{2\pi} \frac{d\theta}{1 + \Theta(r(\theta,\rho),\theta)}.$$

El periodo $T(\rho)$ de esta órbita periódica es una función analítica que tiene la forma $T(\rho) = 2\pi + \sum_{n=1}^{\infty} T_n \rho^n$, ver Chicone & Jacobs [54] y Algaba *et al.* [2]. Las constantes T_n se denominan *constantes de periodo* del sistema (0.0.2).
Algaba *et al.* [2] prueban que $T_{2i+1} = 0$ para todo $i \geq 1$ y dan una relación entre las constantes de periodo y los coeficientes azimutales de una forma normal de Poincaré-Dulac del sistema. Concretamente,

$$T_{2n} = 2\pi \sum_{\substack{n_1+\cdots+n_l=2n \\ n_i \text{ even, } l\geq 1}} (-1)^l \beta_{n_1+1} \cdots \beta_{n_l+1}.$$

Por lo tanto, $T_{2n} = -2\pi\beta_{2n+1}$ si las anteriores constantes de periodo, $T_{2i},\ i = 1, ..., n-1$, son cero. En consecuencia, el origen del sistema (0.0.2) es un centro isócrono si y sólo si todas las constantes β_{2i+1} son cero, es decir, el sistema puede ser transformado en un sistema con velocidad angular constante. Estos sistemas se denominan *sistemas rigidos* y juegan un papel fundamental en el problema de centro isócrono. En estos sistemas, el origen es el único equilibrio y si es un centro, será isócrono, de hecho, será uniformemente isócrono.

Resaltamos que la isocronía de un centro no implica necesariamente que su velocidad angular $\dot{\theta}$ sea constante.

El problema de la isocronía de un centro es equivalente a la existencia de un conmutador analítico con parte lineal no degenerada. Recordamos brevemente el concepto de un conmutador de un campo vectorial.
Decimos que los campos vectoriales $\mathcal{X}$ y $\mathcal{U}$ *conmutan* si su corchete de Lie es nulo, es decir, si $[\mathcal{X}, \mathcal{U}] := D\mathcal{X} \cdot \mathcal{U} - D\mathcal{U} \cdot \mathcal{X} = 0$, donde $D\mathcal{X}, D\mathcal{U}$ son las derivadas de $\mathcal{X}$ y $\mathcal{U}$. Además, si $\mathcal{X}$ y U son tranversales, decimos que $\mathcal{U}$ es un *conmutador* de $\mathcal{X}$.

Es conocido, ver Algaba *et al.* [2] y Sabatini [142], que para cualquier sistema analítico cuyo campo vectorial asociado $\mathcal{X}$ tiene parte lineal $(-y, x)^T$, la existencia de un conmutador analítico con parte lineal $(x, y)^T$ es una condición necesaria y suficiente para que el origen sea un centro isócrono.

En diversas ocasiones, encontrar un conmutador es más sencillo que encontrar un cambio linealizante. Por ejemplo, Sabatini [141] muestra que los sistemas cuadráticos tienen conmutadores polinomiales de grado menor o igual que cuatro, mientras que no se conocen cambios linealizantes polinomiales de sistemas cuadráticos. Las linealizaciones más simples conocidas son racionales y pueden encontrarse en Mardesic, *et al.* [123].

Por otra parte, no es cierto que todos los sistemas polinomiales con un centro isócrono tengan un conmutador polinomial. El primer ejemplo de un centro isócrono polinomial sin un conmutador polinomial se encuentra en Devlin [65]. Se trata de un sistema cuártico donde coexisten un centro isócrono y otros dos no isócronos. Chavarriga *et al.* [44] desarrollan un método el cual usa una solución algebraica del campo vectorial para encontrar un conmutador transversal del campo.

Freire *et al.* [69] prueban que si $\mathcal{V}$ es un *normalizador* transversal a $\mathcal{X}$, es decir, un campo vectorial $\mathcal{V}$ tal que $[\mathcal{X}, \mathcal{V}] = \mu\mathcal{X}$ siendo $[\mathcal{X}, \mathcal{V}]$ el corchete de Lie de $\mathcal{X}$ y $\mathcal{V}$ y μ una función escalar, y si se denota por $\psi(s)$ una parametrización de la trayectoria de $\mathcal{V}$ tal que $\psi(s_0)$ está en el anillo periódico de O, entonces

$$T'(s_0) = \int_0^{T(s_0)} \mu(x(t), y(t))dt,$$

donde $(x(t), y(t))$ es la órbita de $\mathcal{X}$ tal que $(x(0), y(0)) = \psi(s_0)$ y $T(s)$ el periodo de la órbita de $\mathcal{X}$ que pasa por $\psi(s)$. Esto permite obtener información sobre la monotonía de la función periodo.

Freire *et al.* [70] caracterizan un centro isócrono si existe un normalizador tal que la expresión integral anterior sobre cualquier órbita periódica de $\mathcal{X}$ es cero. Finalmente, Sabatini [147] da algunas propiedades de la función periodo de los sistemas planos sin normalizadores conocidos y aporta una nueva caracterización de centro isócrono que generaliza a la dada en [70].

Abordamos a continuación el problema de determinar cuando un foco del sistema (0.0.2) es un foco isócrono, *problema de foco isócrono.*
El sistema analítico (0.0.2) en coordenadas polares viene dado por

$$\dot{r} = rf(r,\theta), \qquad \dot{\theta} = g(r,\theta)$$

donde f y g son funciones analíticas en un entorno de $r = 0$.

Giné & Grau [90] definen el origen como un punto isócrono de (0.0.2) si puede ser transformado mediante un cambio de coordenadas analítico en un sistema con $g(r,\theta) = g(\theta)$.

Un sistema (0.0.2) con un punto isócrono en el origen es más fácilmente escrito usando la variable longitud de arco φ definida por $\varphi = \int \frac{d\theta}{g(\theta)}$ como la nueva variable angular. En esta formulación, el punto O se dice ser isócrono si existe un cambio analítico de variables ϕ tal que es transformado en

$$\dot{r} = rf(r,\varphi), \qquad \dot{\varphi} = k, \qquad k \in \mathbb{R}.$$

Nótese que si el origen es un foco, en general, la transformación ϕ no es convergente (ver Bruno [34] y Tokarev [155]).

Giné [87] muestra varias familias de focos isócronos. En el caso analítico, Giné & Grau [90] caracterizan los focos isócronos mediante la existencia de un campo vectorial analítico $\mathcal{Y} = (x,y)^T + \mathcal{O}(2)$ verificando $[\mathcal{X},\mathcal{Y}] = \mu\mathcal{Y}$ con $\mu(O) = 0$.

Sabatini [145] estudia la existencia de secciones isócronas de un foco, es decir, una curva que corta a todas las órbitas de (0.0.2) contenidas en un entorno del origen, tal que cualquier órbita corta a ésta en un mismo intervalo mínimo de tiempo.
Sabatini prueba que si el origen del sistema (0.0.2) tiene una sección isócrona, entonces existe un campo vectorial $\mathcal{Y}$, al menos de clase $\mathcal{C}^2$, que tiene la propiedad anterior.

Tambien pueden existir secciones isócronas cerca de una órbita periódica aislada. Freire *et al.* [71] prueban la existencia de secciones isócronas cerca de un ciclo límite de un campo vectorial teniendo un campo normalizador y dan una expresión del mutiplicador característico del ciclo límite. Sabatini [147] y Guillamón & Sabatini [99] extienden estos resultados sin necesidad del conocimiento de campos vectoriales normalizadores.

Integrabilidad.

El conocimiento de una integral primera puede usarse para determinar el retrato de fases local cerca de un punto singular de un sistema plano. También, en general, permite un mejor entendimiento de la dinámica del sistema y puede usarse para mostrar algunas propiedades topológicas.

La existencia de integrales primeras en un entorno de un punto regular es una consecuencia de algunos teoremas clásicos del cálculo diferencial, en consecuencia, uno de los problemas relacionados con los sistemas de ecuaciones diferenciales analíticos

$$(\dot{x}, \dot{y})^T = \mathcal{X}(x, y) = (P(x, y), Q(x, y))^T, \tag{0.0.5}$$

es determinar cuando el sistema tiene una integral primera definida en un entorno del punto singular. En general, los sistemas planos integrables que no son hamiltonianos son muy difíciles de detectar.

Si la función H es una integral primera de (0.0.5) de clase $\mathcal{C}^1$ en un conjunto abierto U de $\mathbb{R}^2$ (es decir, H es una función no constante en U la cual es constante sobre cada curva solución de (0.0.5)) es equivalente a $\nabla H \cdot \mathcal{X} \equiv 0$. Así, decimos que una serie formal de potencias

$$H = a_{00} + a_{10}x + a_{01}y + \cdots$$

es una integral primera formal de (0.0.5) si formalmente verifica que $\partial H/\partial x P(x,y) + \partial H/\partial y Q(x,y) \equiv 0$. Si existe una integral primera formal (analítica) de (0.0.5) se dice que (0.0.5) es formalmente (analíticamente) integrable.

Obsérvese que esta definición de integrabilidad sólo es válida para sistemas planos. En dimensiones superiores, si el espacio de fases es de dimensión n, es necesario determinar $n-1$ integrales primeras funcionalmente independientes del sistema para que el sistema sea completamente integrable.

Ya hemos visto que la existencia de una integral primera se utiliza para determinar cuando un sistema tipo centro-foco tiene un centro en el origen. Por lo tanto, será interesante saber cuando un punto singular monodrómico es analíticamente integrable, ya que en tal caso, se trata de un centro.

Se sabe que cualquier sistema diferencial analítico que tenga un centro en el origen tiene una integral primera analítica definida en un entorno del origen excluyendo al origen, ver Li *et al.* [109]. Es también conocido, ver Mazzi & Sabatini [128], que cualquier sistema diferencial $\mathcal{C}^\infty$ con un centro en el origen, tiene una integral primera de clase $\mathcal{C}^\infty$ definida en un entorno del origen. Por lo tanto, otro problema de interés sera reconocer cuando un centro es o no analíticamente integrable.

Por otra parte, también existen sistemas analíticamente integrables con un punto crítico no monodrómico. En este sentido, algunos trabajos relacionados con la integrabilidad de sistemas nilpotentes y degenerados son, por ejemplo, Chavarriga *et al.* [48] donde los autores han obtenido una familia de sistemas cúbicos nilpotentes con una integral primera analítica; Giné [88] determina los sistemas nilpotentes

$$(\dot{x}, \dot{y})^T = (y + P_n, Q_n)^T$$

con una integral primera formal de la forma $y^2+F(x,y)$, donde P_n, Q_n son polinomios homogéneos de grado $n = 2, 3, 4, 5$ y F comienza con términos de grados mayor que dos. Giné [89], caracteriza los sistemas polinomiales degenerados, denominados sistemas nilpotentes generalizados,

$$(\dot{x}, \dot{y})^T = (y^m + P_n, Q_n)^T$$

con una integral primera formal de la forma $y^k + F(x,y)$, donde P_n, Q_n son polinomios homogéneos de grado $n = 2, 3, 4, 5,\ m = 1, 2, 3,\ m < n$ y F comienza con términos de grado mayor que k con $k = 1, 2, 3, 4, 5, 6$ y 7.

Diversos autores han estudiado la integrabilidad de los sistemas casihomogéneos n dimensionales ver Furta [74], Goriely [94], Yoshida [162], Tsygvintsev [157] y Llibre & Zhang [112]. La integrabilidad de los sistemas casihomogéneos planos ha sido caracterizada a través de los exponentes de Kowalevskaya del sistema. Tsygvintsev [157] caracteriza los sistemas homogéneos cuadráticos integrables y Llibre & Zhang [112] y Cairó & Llibre [36] han calculado los sistemas planos $\mathbf{t}$-homogéneos de grado uno y dos.
Recientemente, Algaba *et al.* [6] caracterizan la integrabilidad de los sistemas que son perturbaciones de un sistema plano hamiltoniano cuya función de Hamilton no contiene factores múltiples.

Darboux [64] muestra una técnica para construir integrales primeras de un sistema polinomial con un número suficiente de soluciones algebraicas. Se dice que $f(x,y) = 0$ con $f \in \mathbb{C}[x,y]$ es una curva algebraica invariante de (0.0.5) si para algún polinomio K (*cofactor* de f) se tiene que $\nabla f \cdot \mathcal{X} = Kf$. Y la curva $f(x,y) = 0$ es una solución algebraica de

(0.0.5) si es una curva algebraica invariante polinomial irreducible sobre $\mathbb{C}[x,y]$. La idea de Darboux consiste en encontrar integrales primeras de la forma $\prod_{i=1}^{q} f_i^{\lambda_i}$ donde λ_i son números complejos y f_i son curvas algebraicas invariantes. Darboux prueba que si un sistema polinomial de grado m ($m = \text{máx}(grad(P), grad(Q))$) tiene al menos $\frac{m(m+1)}{2}+1$ soluciones algebraicas, entonces el sistema tiene una integral primera de tipo Darboux. Resaltamos que tal integral puede no estar definida en el origen. Chavarriga *et al.* [49] prueban que en determinadas ocasiones el número de soluciones algebraicas suficiente para poder construir una integral primera tipo Darboux puede ser más reducido. Jouanoulou [104] prueba que si el sistema tiene al menos $\frac{m(m+1)}{2}+2$ soluciones algebraicas, entonces tiene una integral primera racional y todas las soluciones del sistema son algebraicas. Prelle & Singer [134] demuestran que si un sistema polinomial tiene una integral primera elemental, entonces esta integral primera puede ser obtenida a partir de las soluciones algebraicas del sistema.

Otros resultados relacionados con la integrabilidad de Darboux de un sistema pueden verse en Christopher & Kooij [63], Christopher & Llibre [60] y Zoladek [167].

Linealización

Una técnica usada para dar una completa caracterización de la estructura de las órbitas cerca de un punto crítico es expresar el campo vectorial como una perturbación de otro dado, y analizar cuando ambos campos tienen la misma dinámica.

Un método consiste en transformar el sistema en otro cuya estructura orbital es conocida. Así, por ejemplo, el método ha sido usado para caracterizar los centros, centros isócronos y puntos isócronos. Poincaré [136] prueba que, para un centro no degenerado de un sistema analítico (0.0.2), siempre existe un cambio analítico de variables y una función analítica Ψ con $\Psi(0,0)=0$ tal que el sistema es transformado en

$$\dot{u} = -v(1+\Psi(u^2+v^2)), \quad \dot{v} = u(1+\Psi(u^2+v^2)).$$

Además, el origen es un centro isócrono de (0.0.2) si y sólo si el sistema puede transformarse en

$$\dot{u} = -v, \quad \dot{v} = u.$$

Y el origen es un punto isócrono si el sistema (0.0.2) puede ser transformado en un sistema rigido.

Un problema clásico es saber cuando un campo vectorial n dimensional

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^n,$$

puede ser transformado, mediante cambios de variables en un campo vectorial lineal (en ese caso, se dice que el campo vectorial es *linealizable*). Se conoce que la estructura orbital cerca de un punto singular hiperbólico (los autovalores de la aproximación lineal en un punto singular tienen parte real distinta de cero) es cualitativamente la misma que la estructura órbital del correspondiente sistema dinámico lineal. Sin embargo, en general, el difeomorfismo no es analítico en un entorno del punto singular. Para más detalle sobre la naturaleza del cambio de variable linealizante ver Hartman [98] y Grobman [96].
En este sentido, como una consecuencia de los trabajos de Poincaré [135] y Chen [51], si los autovalores de la matriz $D\mathbf{F}(\mathbf{0})$ no son resonantes (es decir, para cualquier $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ con $\sum_{j=1}^n \alpha_j \geq 2$, se tiene que $\lambda_j \neq \sum_{i=1}^n \alpha_i \lambda_i$ para todo autovalor λ_j de $D\mathbf{F}(\mathbf{0})$) el campo vectorial es linealizable bajo $\mathcal{C}^\infty$ conjugación.
Si, además, los autovalores $\lambda_1, \ldots, \lambda_n$ de la aproximación lineal pertenecen al dominio de Poincaré, es decir, si la envolvente convexa de los puntos $\lambda_1, \ldots, \lambda_n$ en el plano complejo no contiene al cero, Poincaré [135] prueba que existe una transformación convergente (el campo vectorial es

analíticamente conjugado a su parte lineal).

Otras condiciones que implican la convergencia de las transformaciones que llevan un campo vectorial con un punto singular hiperbólico a su parte lineal son dadas por Siegel, ver Arnold [25] y Chow *et al.* [58].

También destacamos los trabajos de Bruno sobre convergencia de transformaciones normalizantes (transformaciones que llevan un campo vectorial a su forma normal). Bruno [33] prueba que si una forma normal de Poincaré-Dulac de un campo vectorial verifica la condición "A" y la parte lineal verifica la condición "ω," entonces existe una transformación normalizante convergente. Estas condiciones son las siguientes:

CONDICIÓN A: Una forma normal $\hat{\mathbf{F}}$ de un campo vectorial $\mathbf{F}$ satisface la condición A si $\hat{\mathbf{F}}$ tiene la forma $\hat{\mathbf{F}} = A\mathbf{x} + \alpha(\mathbf{x})A\mathbf{x}$, donde $\alpha(\mathbf{x})$ es una serie de potencias escalar (con $\alpha(\mathbf{0}) = 0$),

CONDICIÓN ω : Sea $\omega_k = min|(Q,\Lambda) - \lambda_j|$ para todo $j = 1, \ldots, n$ y n enteros $q_i \geq 0$ tal que $1 < \sum_{i=1}^{n} q_i < 2^k$ y $(Q,\Lambda) \equiv \sum_{i=1}^{n} q_i\lambda_i \neq \lambda_j$, entonces $\sum_{k=1}^{\infty} 2^{-k} ln(\omega_k^{-1}) < \infty$.

La condición "A" puede ser debilitada para algunas $\mathbf{F}$'s.

Como una consecuencia, Bruno probó que si un punto singular es un centro de un sistema con parte lineal $(-y, x)^T$, entonces existe una transformación normalizante convergente, mientras que si el punto singular es un foco débil, no podemos garantizar la existencia de una transformación convergente.

En los últimos años, ambos problemas, linealización de campos vectoriales y la convergencia de tales transformaciones han sido analizados por medio de la existencia de simetrias en un entorno del punto singular. Recordamos que un campo vectorial de dimensión n, $\mathbf{F}$, admite una simetría no trivial si existe un campo vectorial analítico $\mathbf{G}$, transversal a $\mathbf{F}$ en el origen, tal que el corchete de Lie de ambos campos, $[\mathbf{F}, \mathbf{G}] = (D\mathbf{F})\mathbf{G} - (D\mathbf{G})\mathbf{F}$ es nulo, es decir, existe un campo vectorial $\mathbf{G}$ que conmuta con $\mathbf{F}$.

En relación al problema de convergencia, para los sistemas planos con parte lineal no nula, se sigue a partir de los resultados de Markhashov [125] y Bruno & Walcher [35], que existe una transformación convergente que lleva el sistema a la forma normal de Poincaré-Dulac si y sólo si existe una simetría no trivial. Cicogna [38] extiende parte de estos resultados a campos de dimensiones superiores.

Insistimos que, en general, la existencia de un conmutador del campo vectorial no asegura que el campo sea linealizable. Por ejemplo, los sistemas

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + xP_{2l}(x,y)\sum_{j=0}^{r} a_j(x^2+y^2)^j, \\ \dot{y} & = & x + yP_{2l}(x,y)\sum_{j=0}^{r} a_j(x^2+y^2)^j, \end{array} \right.$$

con $P_{2l}(x,y)$ polinomios homogéneos de grado $2l, l \geq 0$, y a_j números reales arbitrarios, $j = 0, \ldots, r$, tienen una transformación normalizante convergente, pues ellos conmutan con

$$(x\sum_{j=0}^{r} a_j(x^2+y^2)^{j+l}, y\sum_{j=0}^{r} a_j(x^2+y^2)^{j+l})^T.$$

Sin embargo, hay campos vectoriales de esta familia cuyo origen es un foco y por lo tanto no son linealizables.

Por otra parte, en relación con el problema de linealización, ya hemos mencionado la equivalencia entre linealización y conmutación para los campos vectoriales de dimensión dos con un punto singular tipo centro o foco. Concretamente, un sistema de tipo centro o foco es linealizable bajo conjugación analítica si y sólo si existe un conmutador analítico con parte lineal $(x, y)^T$. Bambusi *et al.* [27] extienden este resultado y caracterizan, al menos teóricamente, los campos vectoriales linealizables. Ellos probaron que un campo vectorial de dimensión n, $\mathbf{F}$ formal (analítico) es linealizable por un cambio de variable formal (analítico) si y sólo si la ecuación $[\mathbf{F}, \mathbf{G}] = \mathbf{0}$ admite una solución analítica con $(D\mathbf{G})(\mathbf{0}) = I$. Además, en tal caso, existe un cambio de variables formal (analítico) que linealiza tanto a $\mathbf{F}$ como a $\mathbf{G}$.

Evidentemente, la diferencia entre los resultados de Bruno & Walcher [35] y Bambusi *et al.* [27] es que el conmutador tenga o no tenga parte lineal nula.

Para los centros, en Algaba *et al.* [2] está también probada la relación entre linealización bajo equivalencia orbital analítica y la existencia de un normalizador analítico del campo vectorial con parte lineal $(x, y)^T$. Así, un sistema (0.0.2) puede transformarse bajo equivalencia orbital en $\dot{x} = -y$, $\dot{y} = x$ si y sólo si existe un campo vectorial analítico $\mathcal{Y} = (x, y)^T + \mathcal{O}(2)$ verificando $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{X}$ y $\mu(O) = 0$. Giné & Grau [91] prueban un resultado similar para campos vectoriales planos cuyo determinante de la parte lineal en el origen es distinto de cero.
García & Maza [76] obtienen la expresión del cambio de variables que linealiza un centro isócrono a partir del conmutador. Giné & Maza [92] extienden esta técnica para sistemas con puntos singulares elementales (no todos los autovalores de la parte lineal nulos).

Después de esta introducción de los diferentes problemas de sistemas dinámicos que hemos presentado y que abordaremos, pasamos a describir la organización y contenido de la presente memoria.

En el capítulo 1, abordamos el problema de isocronía, tanto en el caso de un centro como de un foco. Comenzamos el capítulo dando una nueva definición de punto isócrono, más general que la existente. En la sección 1.2, caracterizamos un foco isócrono débil de orden finito a partir de la forma normal de Poincaré-Dulac del sistema, es decir, usamos su forma normal para resolver el problema de isocronía de un foco, (Theorem 1.2.5). Damos un algoritmo que permite calcular la expresión del primer coeficiente de la forma normal de Poincaré-Dulac diferente de cero en términos de los coeficientes del sistema. En la sección 1.3, probamos, en el caso de un foco isócrono, la existencia de un campo vectorial normalizado y de secciones isócronas que parten del origen (Theorem 1.3.10). También damos una condición más débil en el caso de que el orden del foco es conocido (Theorem 1.3.12). Aportamos algoritmos que calculan los términos de un campo vectorial normalizado y las secciones isócronas de un sistema con un foco isócrono. En la sección 1.4, caracterizamos un centro isócrono (Theorem 1.4.18), un foco isócrono (Theorem 1.4.19) y un punto isócrono (Theorem 1.4.22) mediante la existencia de un conmutador del campo vectorial cuya parte lineal puede ser nula. En particular, Theorem 1.4.18 generaliza los resultados obtenidos por Sabatini [142] y Algaba *et al.* [2].

Como aplicaciones de los resultados obtenidos, en la sección 1.5 extendemos los trabajos de Loud [119] y Pleshkan [133] clasificando los focos isócronos de los sistemas cuadráticos y de los sistemas con nolinealidades cúbicas y probamos que el orden máximo de un foco débil isócrono de estos sistemas es dos y tres, respectivamente. Damos una familia triparamétrica de ecuaciones de Rayleigh cuya función periodo llega a tener cuatro puntos críticos locales.

En 1.5.5, aplicamos el algoritmo desarrollado en la sección 1.6 para resolver el problema de centro y centro isócrono de dos familias polinomiales de sistemas de ecuaciones diferenciales. La primera es una familia cuártica con ocho parámetros de la forma $(\dot{x}, \dot{y})^t = (yP(x), xQ(y))^T$, con P, Q polinomios de grado menor o igual que tres. Esta familia incluye a una subfamilia de sistemas de Kolmogorov con un centro isócrono (Theorem 1.5.27). Y la segunda familia es una clase de sistemas polinomiales degenerados de Kolmogorov. Encontramos sus centros y damos una familia con tres ciclos límites de pequeña amplitud (Theorem 1.5.28).

También hallamos los sistemas de la familia con un centro isócrono (Theorem 1.5.29). En 1.5.6, aportamos varios ejemplos de familias de sistemas con conmutadores con parte lineal nula.
Terminamos el capítulo, desarrollando un método basado en el procedimiento de linealización de Carleman para calcular los coeficientes de la forma normal de Poincaré-Dulac de un sistema plano. Éste tiene algunas ventajas sobre otros métodos conocidos pues los coeficientes se obtienen resolviendo un único sistema de ecuaciones lineales cuya matriz de los coeficientes es triangular y en consecuencia, el cómputo se reduce considerablemente. También, de forma genérica podemos deducir algunas propiedades de los coeficientes de la forma normal. En la sección 1.6, damos un algoritmo para calcular los coeficientes radiales y azimutales de la forma normal (Theorem 1.6.36) y Corollary 1.6.37 es un algoritmo que permite calcular condiciones necesarias para que un sistema tenga un centro isócrono.

En los capítulos 2 y 3 estudiamos la existencia de conmutadores de dos clases de sistemas planos polinomiales. La mayoría de los conmutadores que se conocen de los sistemas con un centro isócrono son polinomiales, es por ello, que nos planteamos averiguar que sistemas tienen conmutador polinomial y, en caso afirmativo, calcular su expresión. El capítulo 2 trata este problema para los sistemas polinomiales que se conocen como sistemas con infinito degenerado. En la sección 2.2, mostramos varias propiedades que deben verificar un campo con un conmutador polinomial y damos condiciones para la existencia de tales conmutadores (Theorem 2.2.45). En la sección 2.3, estudiamos la expresión de los conmutadores polinomiales de los sistemas

$$\begin{aligned}\dot{x} &= -y + P_s + x(H_k + H_3 + \cdots + H_{n-1}),\\ \dot{y} &= x + Q_s + y(H_k + H_3 + \cdots + H_{n-1}),\end{aligned}$$

con $s \leq k$, P_s, Q_s polinomios homogéneos de grado s y $H_i = H_i(x, y)$ polinomios homogéneos de grado i, y caracterizamos los sistemas con conmutador polinomial para $s = 2$ (Theorem 2.3.47). En sección 2.4

estudiamos los sistemas polinomiales con velocidad angular constante y determinamos los que tienen conmutador polinomial con parte lineal nula (Corollary 2.4.49) y los que tienen parte lineal de la forma $(x, y)^T$ (Theorem 2.4.51). Por último damos el retrato de fases de los sistemas cuárticos y quínticos con un centro uniformemente isócrono con conmutador polinomial.

En el capítulo 3, abordamos el problema de determinar los sistemas con velocidad angular constante (sistemas rígidos) con conmutador analítico y en definitiva, los sistemas que tienen un centro. Está dividido en cuatro secciones. La sección 3.2 contiene el principal resultado del capítulo, Theorem 3.2.54, el cual da unas condiciones que caracterizan los centros de los sistemas rígidos, y mostramos la equivalencia entre estas condiciones y la anulación de los coeficientes de la componente radial de la forma normal de Poincaré-Dulac (Theorem 3.2.55). En la siguiente sección desarrollamos un algoritmo recursivo que nos permite obtener condiciones sobre los coeficientes del sistema para que el origen sea centro que, por su naturaleza, será centro isócrono. En la sección 3.4, mostramos varias subfamilias que tienen un centro. Obtenemos los centros de las familias $(-y + x(H_1 + H_m), x + y(H_1 + H_m))^T$ y $(-y + x(H_2 + H_{2n}), x + y(H_2 + H_{2n}))^T$, con H_i polinomios homogéneos en x, y de grado i y también determinamos el número máximo de ciclos límites de pequeña amplitud que pueden bifurcar a partir de un foco débil. Por último, calculamos los centros uniformemente isócronos cuárticos y quínticos.

En el capítulo 4, fijado $\mathbf{t} = (p, q) \in \mathbb{N}^2$ y $n \in \mathbb{N}$, trabajamos con los sistemas planos nilpotentes cuya primera componente $\mathbf{t}$-homogénea, según su grado, es $(y, 0)^T$, la segunda está formada por términos $\mathbf{t}$-homogéneos de grado $(2n + 1)p - q$ y contiene al término $(0, -x^{2n+1})^T$, y las aristas de su diagrama de Newton de las restantes componentes son paralelas a la arista asociada al término $\mathbf{t}$-homogéneo de grado $(2n+1)p-q$. Esta clase contiene, entre otros, a los sistemas nilpotentes invariantes al cambio de variables $(x,y) \to (-x,-y)$. En la sección 4.2, definimos los conceptos de función y campo vectorial $\mathbf{t}$-homogéneos y mostramos varias propiedades de ellos, las cuales usaremos en éste capítulo y en los posteriores. En la tercera sección probamos que esta clase particular de sistemas nilpotentes tiene una función de Lyapunov generalizada de clase $\mathcal{C}^\infty$ la cual tiene un desarrollo de la forma $W = \frac{1}{2}y^2 + \sum_{l=1}^{\infty} W_{2q+2sl}$ donde W_{2q+2sl} son funciones $\mathbf{t}$-homogéneas de grado $2q+2sl$, $l \geq 1$ siendo $s=(n+1)p-q>0$ (Theorem 4.3.75). En la sección 4.4, damos el desarrollo de Taylor en el origen de la aplicación de Poincaré de estos sistemas (Theorem 4.4.78). Ambos resultados nos permiten resolver teóricamente el problema de centro, determinar la ciclicidad de un foco y obtener sistemas con ciclos límites que bifurcan del origen. Finalmente, en la sección 4.5, mostramos varias familias contenidas en esta clase de sistemas nilpotentes, resolviendo el problema de centro y obtenemos la ciclicidad del foco débil, encontrando una familia con ocho ciclos límites. También resaltamos que todos los centros obtenidos son analíticamente integrables y ellos pueden escribirse en la forma

$$\dot{x} = y + v_y K(v, y^2) + y\Psi(v, y^2), \qquad \dot{y} = -v_x K(v, y^2)$$

donde v, K, Ψ son funciones analíticas definidas en un entorno del origen con $\Psi(O) = 0$.

En el capítulo 5 se estudia la integrabilidad de los sistemas planos casi homogéneos y su relación con los exponentes de Kowalevskaya de estos sistemas. Los resultados obtenidos en el capítulo están estrechamente relacionados con la descomposición conservativa-disipativa de un campo vectorial $\mathbf{t}$-homogéneo, Lemma 5.2.92. En la sección 5.2, damos una sencilla caracterización de los sistemas polinomiales $\mathbf{t}$-homogéneos con una integral primera racional (Theorems 5.2.96 y 5.2.98). En la sección 5.3, calculamos los exponentes de Kowalevskaya de los sistemas $\mathbf{t}$-homogéneos y mostramos como obtenerlos en el caso de que el sistema sea racionalmente (polinomialmente) integrable (Theorem 5.3.101). Concluimos este capítulo encontrando los sistemas $(1,2)$-polinomiales homogéneos de grado dos con una integral primera racional (Theorems 5.4.102 y 5.4.103).

En el capítulo 6, damos condiciones necesarias y suficientes para que un campo vectorial n dimensional tenga la misma estructura orbital que un campo vectorial casihomogéneo, el cual es no lineal, en general. Concretamente, fijado un tipo $\mathbf{t} \in \mathbb{N}^n$, caracterizamos los campos que son formalmente (analíticamente) conjugados (Theorem 6.2.109) y orbitalmente equivalentes (Theorem 6.2.111) a su parte $\mathbf{t}$-homogénea de menor grado, mediante la existencia de un campo vectorial normalizado $\mathcal{C}^\infty$ (analítico). Finalizamos, dando varias aplicaciones de los resultados obtenidos.

Chapter 1

Isochronous points.

1.1 Introduction

Let us consider the autonomous differential system

$$\dot{x} = -y + \lambda x + P(x, y), \qquad \dot{y} = x + \lambda y + Q(x, y), \tag{1.1.1}$$

where $\lambda \in \mathbb{R}$ and P and Q are smooth functions of order greater than or equal to two. This system is associated to the vector field $(-y + \lambda x + P(x, y), x + \lambda y + Q(x, y))^T$ which we will denote as

$$\mathcal{X} = (-y + \lambda x + P)\partial_x + (x + \lambda y + Q)\partial_y. \tag{1.1.2}$$

The origin O is an isolated singular point of (1.1.2) and is also a monodromic point.
In the analyticity case, if $\lambda \neq 0$, O is a strong focus of (1.1.2). Otherwise, O can be either a centre or a weak focus.

We now take the following into consideration in order to understand the concept of isochronous focus. The smooth vector field (1.1.2) for $\lambda = 0$ in polar coordinates has the form $\mathcal{X} = f(r, \theta)\partial_r + g(r, \theta)\partial_\theta$ with $g(r, \theta) = 1 + \frac{1}{r}(cos\theta Q(r\cos\theta, r\sin\theta) - sin\theta P(r\cos\theta, r\sin\theta))$, that is,

$$g(r, \theta) = 1 + \sum_{i \geq 1} r^i g_i(\theta) + G(r, \theta),$$

where G is a smooth function in a neighborhood of $r = 0$ and flat in $r = 0$. Giné and Grau [90] define O as an isochronous point of $\mathcal{X}$ if it can be transformed by means of an analytic near-identity change of variables into a vector field with $g(r, \theta) = g(\theta)$. This implies that $g_i(\theta) = 0$, for any $i \geq 1$ and for every $\theta \in [0, 2\pi)$. In such a case, the return time of the orbits of $\mathcal{X}$ is constant on every ray of the origin; concretely, it is 2π. Thus, in cartesian coordinates, the vector field will have the form $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a smooth scalar function. This leads us to the following definition, which is less restrictive than that given in [90], since we do not require the analyticity of the change of variables.

Definition 1.1.1. *The origin of (1.1.2) is said to be an isochronous point if there exists a smooth near-identity change of variables which transforms $\mathcal{X}$ into $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a smooth scalar function with $H(O) = 0$, i.e. $\mathcal{X}$ is smoothly conjugated to $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$.*

On the one hand, if O is an isochronous centre of an analytic system, the diffeomorphism can be chosen analytic since the normal form of (1.1.2) verifies the conditions "A" and "ω" (see [33]) and also $H(x, y) = 0$. The vector field (1.1.2) is analytically linearizable.
On the other hand, if O is a focus of an analytic system, in general, we cannot guarantee the existence of a convergent transformation (see [33, 155]), i.e. it can exist an analytic vector fields whose O is an isochronous focus, according to Definition 1.1.1, but do not verify the definition provided in [90].

Now, we list the results provided in this chapter. The Poincaré-Dulac normal form of a plane vector field at a critical point has been used to study the centre problem. In next section, we characterize an isochronous focus from its normal form, i.e. we use this for solving the problem of isochronicity of a focus. (Theorem 1.2.5). Also, we give an algorithm which allows to obtain the expression of the first coefficient non-zero of the Poincaré-Dulac normal form.

In the section 1.3, we prove, in the case of isochronous focus, the existence of a normalized vector field and an isochronous section which arrives at the origin with defined direction. We also provide algorithms that compute the terms of normalized vector field and of isochronous section of a system.

We characterize the systems which have either an isochronous centre or an isochronous focus at the origin by means of the existence of a commutator of the field.

As applications, we analyze the weak isochronous foci for quadratic systems and for systems with cubic non-linearities. Moreover, we prove that the maximum order of a weak isochronous focus for quadratic systems is two and for systems with cubic non linearities is three. We give a three-parameter family of Rayleigh equations with four local critical periods. Finally, we shown several examples of families of systems with commutators whose linear part is null.

1.2 Isochronicity and normal forms

Let $\mathcal{H}_i$ be the linear vector space of the homogeneous polynomial vector field of degree i in x and y. Every vector field $\mathbf{F}$ with $\mathbf{F}(0,0) = (0,0)^T$ can be expanded by $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots$ where $\mathbf{F}_i$ are homogeneous polynomial vector fields which belong to $\mathcal{H}_i$. The homological operator which determines the normal form of (1.1.2) is

$$L_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i, \qquad L_i(\mathbf{F}_i) = [\mathbf{F}_i, (-y + \lambda x)\partial_x + (x + \lambda y)\partial_y],$$

being $[\mathbf{F}, \mathbf{G}]$ the Lie product of the vector fields F and G, see [2].

If $\lambda \neq 0$, $L_i(\mathcal{H}_i) = \mathcal{H}_i$. Thus, the system (1.1.2) is smoothly linearizable (see [152]); therefore, O is an isochronous focus.

If $\lambda = 0$, it is easy to prove that $\mathcal{C}_i = Ker\, L_i$ is a subspace in $\mathcal{H}_i$ complementary to $\mathcal{R}_i$, the range of the linear operator L_i. Additionally, $\mathcal{C}_{2i} = \{0\}$

and $(x^2+y^2)^i(x\partial_x+y\partial_y)$, $(x^2+y^2)^i(-y\partial_x+x\partial_y)$ is a basis for $\mathcal{C}_{2i+1}$, which we will denote by $(1,0)_{\mathcal{C}_{2i+1}}$, $(0,1)_{\mathcal{C}_{2i+1}}$, respectively. Thus, by the classical normal form Theorem (see [58]), (1.1.2) can be transformed by means of a smooth near-identity change of variables into

$$(0,1)_{\mathcal{C}_1}+\sum_{i\geq 1}(\alpha_{2i+1},\beta_{2i+1})_{\mathcal{C}_{2i+1}}+\mathbf{F}, \tag{1.2.3}$$

where $\mathbf{F}$ is a smooth function in a neighborhood of O and flat at O. The constants α_{2i+1} and β_{2i+1} are called the i-th radial and azimuthal coefficient of (1.1.2).

It is known that if $\mathcal{X}$ is analytic and their radial coefficients are zero, then O is a centre (in fact, there is a convergent normalizing transformation, see [34]). If α_{2r+1} is the first one non-zero, that is $\alpha_3=\cdots=\alpha_{2r-1}=0, \alpha_{2r+1}\neq 0$, O is a weak focus of order r (but, in this case, the existence of a convergent normalizing transformation is not guaranteed).

In what follows, we denote by ϕ_* and ϕ^* the push-forward and pull-back defined by the smooth diffeomorphism ϕ, respectively (see [124]). Given U a smooth vector field with $U(O)=O$, we denote by ϕ_U to the change of variables $\phi_U(x,y)=u(x,y,1)$ where u is the unique solution of the initial-value problem

$$\frac{\partial}{\partial\epsilon}u(x,y,\varepsilon)=U(u(x,y,\varepsilon)),\quad u(x,y,0)=(x,y).$$

By the transformation theory based on Lie transforms, the diffeomorphism ϕ_U transforms $\mathcal{X}$ into

$$\mathcal{Y}=\mathcal{X}+[\mathcal{X},U]+\frac{1}{2!}[[\mathcal{X},U],U]+\frac{1}{3!}[[[\mathcal{X},U],U],U]+\cdots, \tag{1.2.4}$$

see Algaba et al. [1].

We cite two properties of the Lie bracket of two vector field, which are easily obtained.

Lemma 1.2.2. *Let* $\mathcal{H}_i^R = \{P_{i-1}(x\partial_x + y\partial_y) \in \mathcal{H}_i,\ deg(P_{i-1}) = i-1,\ i \geq 1\}$. *The following properties hold:*
(a) if $[-y\partial_x + x\partial_y, U_{i+2}] \in \mathcal{H}_{i+1}^R$ *then* $U_{i+2} \in \mathcal{H}_{i+2}^R$,
(b) if $U_i \in \mathcal{H}_i^R,\ V_j \in \mathcal{H}_j^R$ *then* $[U_i, V_j] \in \mathcal{H}_{i+j-1}^R$.

We now provide some properties of the Poincaré-Dulac normal form of (1.1.2) which we will use in order to prove the main results of this chapter.

Lemma 1.2.3. *Let* $\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + (\alpha_{2r+1}, 0)_{\mathcal{C}_{2r+1}} + F$ *with* $\alpha_{2r+1} \neq 0$ *and* F *a smooth function of order greater than* $2r+1$. *Then,* $\hat{\mathcal{X}}$ *is smoothly conjugated to a smooth vector field of the form* $(0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}}$, *with* $\alpha_{2i+1} \in \mathbb{R}$.

Proof.
From (1.2.4), the Lie transform whose generator is $U = U_{2k+1} = (0, B)_{\mathcal{C}_{2k+1}}$, with $k \geq 1$, transforms $\hat{\mathcal{X}}$ into $\phi_{U^*}\hat{\mathcal{X}}$, which is also in normal form, since $[\hat{\mathcal{X}}_{2i+1}, U_{2k+1}] \in \mathcal{C}_{2k+2i+1}$.
As $U_{2k+1} \in \mathcal{C}_{2k+1}$ and $\mathcal{C}_{2k+1} = Ker\, L_{2k+1}$,we have that the degree of $[\hat{\mathcal{X}}, U]$ is greater than or equal to $2k + 2r + 1$. Therefore the transformed vector field remains unaltered up to order $2k + 2r - 1$, and the term of order $(2k + 2r + 1)$ is $\hat{\mathcal{X}}_{2k+2r+1} + [\hat{\mathcal{X}}_{2r+1}, U_{2k+1}]$, that is $(\alpha_{2k+2r+1}, \beta_{2k+2r+1} - 2kB\alpha_{2r+1})_{\mathcal{C}_{2k+2r+1}}$. Thus, taking $B = \frac{\beta_{2k+2r+1}}{2k\alpha_{2r+1}}$, the $(k+r)$-th azimuthal constant of $\phi_{U^*}\hat{\mathcal{X}}$ is annihilated.
Making successive changes of variables over $\hat{\mathcal{X}}$ and by Borel's Theorem (see [98]), we can assert that there exists a smooth diffeomorphism ϕ such that

$$\phi_*\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}} + \hat{F},$$

where $\hat{F}$ is a smooth function in a neighbourhood of the origin and flat in O.

By Tokarev [155], there exists a smooth diffeomorphism ψ such that

$$\psi_*\phi_*\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1},0)_{\mathcal{C}_{2i+1}} + \bar{f}(x^2+y^2)(1,0)_{\mathcal{C}_1} + \bar{g}(x^2+y^2)(0,1)_{\mathcal{C}_1}$$

where $\bar{f}, \bar{g}$ are smooth functions in a neighbourhood of 0 and flat in 0. By Takens [154], there is a smooth change of variables φ of the form $\varphi(x,y) = (x + \varphi_1(x^2+y^2)x)\partial_x + (y + \varphi_2(x^2+y^2)y)\partial_y$ such that

$$\varphi_*\psi_*\phi_*\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1},0)_{\mathcal{C}_{2i+1}} + h(x^2+y^2)(0,1)_{\mathcal{C}_1}$$

where h is a smooth function in a neighbourhood of 0 and flat in 0. Finally, we complete the proof, by performing the smooth change

$$(x - G(x^2+y^2)y)\partial_x + (y + G(x^2+y^2)x))\partial_y$$

where G is the function $G(z) = -\int_0^z \frac{h(\bar{z})}{\sum_{i\geq r}\alpha_{2i+1}\bar{z}^{2i}}d\bar{z}$ flat in O. ■

The following lemma provides a normal form of the vector fields with constant angular speed.

Lemma 1.2.4. *Let $\mathcal{X} = (0,1)_{\mathcal{C}_1} + H(1,0)_{\mathcal{C}_1}$ with H a smooth scalar function in a neighbourhood of the origin. $\mathcal{X}$ is smoothly conjugated either to $(0,1)_{\mathcal{C}_1}$ or to a smooth vector field of the form $(0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1},0)_{\mathcal{C}_{2i+1}}$ with $\alpha_{2i+1} \in \mathbb{R}$ and $\alpha_{2r+1} \neq 0$.*

Proof.
From Lemma 1.2.2, the generator U which transforms $\mathcal{X}$ into $\hat{\mathcal{X}} = \phi_{U^*}\mathcal{X}$ can be chosen such that $U_i \in \mathcal{H}_i^R$ and $\hat{\mathcal{X}}_i \in \mathcal{H}_i^R$ too.
So, if all $\hat{\mathcal{X}}_i$ are zero, $\mathcal{X}$ is smoothly conjugated to $(0,1)_{\mathcal{C}_1}$. Otherwise, $\mathcal{X}$ is smoothly conjugated to $(0,1)_{\mathcal{C}_1} + \sum_{i\geq r}(\alpha_{2i+1},0)_{\mathcal{C}_{2i+1}}$ with $\alpha_{2r+1} \neq 0$. ■

The first related to the problem of the isochronicity of a weak focus of finite order

Theorem 1.2.5. *The following statements are equivalent:*
(i) origin of (1.1.2) is a weak isochronous focus of order $r \geq 0$,

(ii) $\lambda = 0, \alpha_3 = ... = \alpha_{2r-1} = 0, \alpha_{2r+1} \neq 0, \beta_3 = \beta_5 = ... = \beta_{2r+1} = 0,$ *where* $\alpha_{2i+1}, \beta_{2i+1}$ *are the radial and azimuthal coefficients of order* i *of a Poincaré-Dulac normal form of (1.1.2), respectively.*

Proof.

(i)⇒(ii) If O is a weak isochronous focus of order r, there exists a change of variables ϕ such that $\phi_*\mathcal{X}$ takes the form $-y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a scalar smooth function. From Lemma 1.2.4, we have $\lambda = 0, \alpha_3 = ... = \alpha_{2r-1} = 0, \alpha_{2r+1} \neq 0$ and $\beta_{2i+1} = 0$, for all i. In particular, the first r azimuthal coefficients are zero.

(ii)⇒(i) From Lemma 1.2.3, we have that there exists a normal form of $\mathcal{X}$ whose azimuthal coefficients are zero. That is, O is an isochronous focus of $\mathcal{X}$. ∎

We emphasize that origin is a weak isochronous focus of finite order if a finite number of azimuthal coefficients of (1.1.2) are zero. Nevertheless, if O is a centre with all its azimuthal coefficients equal to zero, does not imply that the origin is an isochronous centre, for example

$$\dot{x} = -y - ye^{-1/(x^2+y^2)}, \qquad \dot{y} = x + xe^{-1/(x^2+y^2)}.$$

Analogously, the origin of

$$\dot{x} = -y + (x-y)e^{-1/(x^2+y^2)}, \qquad \dot{y} = x + (x+y)e^{-1/(x^2+y^2)}$$

is a weak focus of infinity order with all its azimuthal coefficients equal to zero and it is a non-isochronous focus.

Now, we provide an algorithm that allows us to obtain the radial and azimuthal coefficients. Thereby, we define the recursive sequence $T_{i_1,i_2,...,i_k} \in \mathcal{H}_{i_1+i_2+\cdots+i_k-k+1}$ given by

$$T_{i_1} = \mathcal{X}_{i_1},\ i_1 \geq 2,$$
$$T_{i_1,i_2,...,i_k} = [\mathcal{X}_{i_1}, L^{-1}(T_{i_2,...,i_k})],\quad k \geq 2,\ i_1, ...,\ i_k \geq 2.$$

Theorem 1.2.6. *Let vector field (1.1.2) be with* $\lambda = 0$, $\alpha_{2i+1} = \beta_{2i+1} = 0$, $i = 1, ..., r-1$. *The pair* $(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}}$ *is given by the projection into* $\mathcal{C}_{2r+1}$ *of*

$$\sum_{k=0}^{2r-1} \Big(\sum_{j_1=1}^{2r-1} \sum_{\substack{j_1+j_2=1 \\ j_2 \geq 1}}^{2r-1} \cdots \sum_{\substack{j_1+\cdots+j_k=1 \\ j_k \geq 1}}^{2r-1} (2r - j_1 - \cdots - j_k) T_{j_1+1, j_2+1, ..., j_k+1, 2r+1-j_1-\cdots-j_k} \Big), \tag{1.2.5}$$

where k *is the number of* j_i *greater than or equal to one.*

Proof.

To prove this theorem, we will use the transformation theory based on Lie transforms, which provides an efficient procedure to obtain the expression of the transformed vector field by means a near-identity change of variables. Basically, the key in this approach is that if $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \cdots$, $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2 + \cdots$ and $U = U_2 + \cdots$ are their decompositions into homogeneous polynomial in the variables x, y, respectively, for each $k \geq 1$, we can determine $\mathcal{Y}_k$ satisfying (1.2.4) from $\mathcal{X}_1, ..., \mathcal{X}_k; U_2, ..., U_{k-1}$, by defining the sequence $\{V_{k,l}\}$ by

$$\begin{aligned} V_{k,0} &= k! \mathcal{X}_{k+1}, \; k \geq 0, \\ V_{k,l} &= V_{k,l-1} + \sum_{j=0}^{k-l} \begin{pmatrix} k-l \\ j \end{pmatrix} [V_{k-j-1,l-1}, U_{j+2}], \; 1 \leq l \leq k. \end{aligned}$$

The above recurrent succession of vector fields satisfies $k! \mathcal{Y}_{k+1} = V_{k,k}$, for all $k \geq 0$. The computation of the above succession can be accomplished by means of the Lie triangle:

$$\begin{array}{lllll} V_{0,0} & & & & \\ V_{1,0} & V_{1,1} & & & \\ V_{2,0} & V_{2,1} & V_{2,2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ V_{k,0} & V_{k,1} & V_{k,2} & \ldots & V_{k,k} \end{array}$$

where each row can be computed from the previous rows. It is easy to see that the elements of the $(k+1)$-th row can be written as $V_{k,l} = W_{k,l} - L_{k+1}(U_{k+1})$, where $W_{k,l}$ only depends on $U_2, \ ..., \ U_k$. In particular, $k!\mathcal{Y}_{k+1} = V_{k,k} = W_{k,k} - L_{k+1}(U_{k+1})$. So, we can choose U_{k+1} to obtain a normal form up to order $k+1$, see Algaba et al. [3].

We are interested in computing α_{2r+1} and β_{2r+1} of vector field (1.1.2), by assuming that $\lambda = 0; \ \alpha_{2i+1} = \beta_{2i+1} = 0, \ i = 1, ..., r-1$. Thereby, we make the following considerations:
The Lie transform which brings $\mathcal{X}$ to $(0,1)_{\mathcal{C}_1} + \sum_{i \geq r}(\alpha_{2i+1}, \beta_{2i+1})_{\mathcal{C}_{2i+1}}$ verifies:

- the first column of Lie triangle, up to order $2r$, is

$$V_{0,0} = 0!\mathcal{X}_1, \ ..., \ V_{2r-1,0} = (2r-1)!\mathcal{X}_{2r},$$

- from $V_{i,i} = 0, \ 1 \leq i \leq 2r-1$, it has that the i-column, $2 \leq i \leq 2r-1$, up to order $2r$ are zero. Therefore, $V_{i,1} = V_{i,2} = ... = V_{i,i} = 0, \ 1 \leq i \leq 2r-1$.
  In particular, the second column, up to order $2r+1$, is given by

$$\begin{aligned} V_{1,1} &= \mathcal{X}_2 + [\mathcal{X}_1, U_2], \\ V_{2,1} &= 2!\mathcal{X}_3 + [\mathcal{X}_2, U_2] + [\mathcal{X}_1, U_3], \\ V_{3,1} &= 3!\mathcal{X}_4 + 2[2!\mathcal{X}_3, U_2] + 2[\mathcal{X}_2, U_3] + [\mathcal{X}_1, U_4]. \end{aligned}$$

  So, $V_{n,1}, \ 1 \leq n \leq 2r$ is

$$V_{n,1} = (n-1)! \left(n\mathcal{X}_{n+1} + \sum_{i=1}^{n} \frac{1}{(i-1)!}[\mathcal{X}_{n+1-i}, U_{i+1}] \right).$$

  From Lie triangle, it follows easily that $V_{2r,2r} = V_{2r,1}$. So, if we split $V_{2r,1} = V_{2r,1}^r + V_{2r,1}^c$, where $V_{2r,1}^r \in \mathcal{R}_{2r+1}$ and $V_{2r,1}^c \in \mathcal{C}_{2r+1}$, we have that

$$(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}} = V_{2r,1}^c.$$

Basically, to obtain the expression (1.2.5), it is enough to take into account, on the one hand, that for each j, with $2 \leq j \leq 2r$, by means of a recursive procedure we obtain U_j verifying

$$L_j(U_j) = (j-2)! \left((j-1)\mathcal{X}_j + \sum_{i=1}^{j-2} \frac{1}{(i-1)!}[\mathcal{X}_{j-i}, U_{i+1}] \right).$$

on the other hand, that $(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}}$ is

$$Proy_{\mathcal{C}_{2r+1}} \left((2r)\mathcal{X}_{2r+1} + \sum_{i=1}^{2r-1} \frac{1}{(i-1)!}[\mathcal{X}_{2r+1-i}, U_{i+1}] \right). \blacksquare$$

Let us note that Theorem 1.2.6 provides closed conditions so that the origin of (1.1.2) be isochronous centre.

So, for instance, the expression of (1.2.5) for $r = 1$ is $2T_3 + T_{2,2}$, and for $r = 2$ is

$$4T_5 + 3T_{2,4} + 2T_{3,3} + T_{4,2} + 2T_{2,2,3} + T_{2,3,2} + T_{3,2,2} + T_{2,2,2,2}.$$

If $\mathcal{X}_i = \sum_{j=0}^{i} (a_{j,i-j}, b_{j,i-j})^T x^j y^{i-j}$, applying Theorem 1.2.6, we have

$$\begin{aligned} \alpha_3 = \;& \tfrac{1}{8}(3a_{30} + a_{12} + b_{21} + 3b_{03} + 2a_{02}b_{02} + a_{11}a_{02} + a_{11}a_{20} - 2b_{20}a_{20} \\ & -b_{20}b_{11} - b_{02}b_{11}), \\ \beta_3 = \;& \tfrac{1}{24}(-10b_{20}b_{02} - 4a_{20}^2 - 3a_{21} + 9b_{30} + 3b_{12} - 9a_{03} - b_{11}^2 + a_{02}b_{11} \\ & +a_{11}b_{20} - 10a_{02}a_{20} + 5a_{11}b_{02} + 5a_{20}b_{11} - a_{11}^2 - 4b_{02}^2 - 10b_{20}^2 \\ & -10a_{02}^2). \end{aligned}$$

In the particular case of $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_m$, these conditions are very simple, since

$$T_{j_1+1, j_2+1, \ldots, j_k+1, n-j_1-j_2-\cdots-j_k+1} \equiv 0,$$

except for $j_1 + 1 = j_2 + 1 = \cdots = j_k + 1 = n - j_1 - \cdots - j_k + 1 = m$, that is, $n = (k+1)(m-1)$. The following corollary provides conditions of isochronicity for these vector fields.

Corollary 1.2.7. *The origin of (1.1.2) with* $\mathcal{X} = -y\partial_x + x\partial_y + \mathcal{X}_m$ *is an isochronous centre if and only if the projection into* $\mathcal{C}_{(j+1)(m-1)+1}$ *of* D_j *is zero for every* j*, where* $D_j \in \mathcal{H}_{(j+1)(m-1)+1}$ *is defined by*

$$D_1 = \mathcal{X}_m,$$
$$D_j = [\mathcal{X}_m, L^{-1}(D_{j-1})], \quad j \geq 2.$$

Finally, if moreover the vector field is uniformly isochronous (i.e. it has constant angular velocity), $\mathcal{X}_m = H_{m-1}(x\partial_x + y\partial_y)$ with H_{m-1} homogeneous polynomial of degree $m-1$ in the variables x and y, it has that $L^{-1}(D_1) = K_{m-1}(x\partial_x + y\partial_y)$, with K_{m-1} homogeneous polynomial of degree $m-1$. In this case

$$D_2 = [H_{m-1}(x\partial_x + y\partial_y), K_{m-1}(x\partial_x + y\partial_y)] \equiv 0,$$

and so $D_j \equiv 0$, for all $j \geq 2$. Therefore, it has the following result.

Corollary 1.2.8. *Let (1.1.2) be with* $\mathcal{X} = -y\partial_x + x\partial_y + H_{m-1}(x\partial_x + y\partial_y)$*. It holds:*
i) *if* m *is even, the origin is an isochronous centre,*
ii) *if* m *is odd, the origin is an isochronous centre if and only if* $H_{m-1}(x\partial_x + y\partial_y) \in \mathcal{R}_m$*.*

A similar result is given in Conti [39].

1.3 Isochronous sections and normalized vector fields.

Next, we introduce the concept of an isochronous section of a monodromic point, which helps us to analyze several geometric aspects of vector field (1.1.2) whose origin is an isochronous point. For every $(x, y) \in \mathbb{R}^2$, the flow of vector field (1.1.2) is denoted by $\Phi_{\mathcal{X}}(t; x, y)$.

Definition 1.3.9. *An isochronous section of (1.1.2) at the origin is a smooth curve* η*, transversal to* $\mathcal{X}$*, defined in* $[0,1)$*, verifying* $\eta(0) = O$*,* $\eta'(0) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *and such that:*
(i) given $s \in (0,1)$*, there exists* $\bar{s} \in (0,1)$ *with* $\Phi_{\mathcal{X}}(2\pi;\eta(s)) = \eta(\bar{s})$*,*
(ii) for every $t \in (0,2\pi)$*,* $s \in (0,1)$*, it has that* $\Phi_{\mathcal{X}}(t;\eta(s)) \notin \{\eta(s),\ s \in (0,1)\}$.

This definition is more demanding than the one given by Sabatini [145] and used in Giné and Grau [90], since we also impose that the curve arrives at origin with defined direction and the return time is 2π.

The following result relates the concepts of isochronous point, normalized vector field and isochronous sections at the origin.

Theorem 1.3.10. *The origin is an isochronous point of (1.1.2) if and only if there exists a smooth vector field* $\mathcal{Y} = x\partial_x + y\partial_y + \mathcal{O}(2)$ *such that* $[\mathcal{X},\mathcal{Y}] = \mu\mathcal{Y}$*, where* μ *is a smooth scalar function with* $\mu(O) = 0$*, i.e.* $\mathcal{Y}$ *is a normalized vector field by* $\mathcal{X}$*.*
Moreover, every orbit of $\mathcal{Y}$ *contained in a neighbourhood of* O *is an isochronous section of (1.1.2) at the origin.*

Proof.
We prove the first part. We assume that O is an isochronous point of (1.1.2). Then there exists a smooth change of variables ϕ with $D\phi(O) = I$ such that $\phi_*\mathcal{X} = -y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$ where H is a smooth scalar function.
The vector field $x\partial_x + y\partial_y$ verifies $[\phi_*\mathcal{X}, x\partial_x + y\partial_y] = \mu(x\partial_x + y\partial_y)$ with $\mu(x,y) = (xH_x + yH_y)H(x,y)$, where H_x, H_y stand the partial derivatives of H. Therefore, the smooth vector field $\phi^*(x\partial_x + y\partial_y)$ verifies

$$[\mathcal{X}, \phi^*(x\partial_x + y\partial_y)] = \nu\phi^*(x\partial_x + y\partial_y)$$

with $\phi^*(x\partial_x + y\partial_y) = x\partial_x + y\partial_y + \mathcal{O}(2)$ and $\nu(O) = 0$.

Conversely, from the Sternberg's Theorem [152], the vector field $\mathcal{Y}$ is linearizable, then there exists a change of variables $\psi = x\partial_x + y\partial_y + \mathcal{O}(2)$ such that $\psi_*\mathcal{Y} = x\partial_x + y\partial_y$. Thus, $[\psi_*\mathcal{X}, x\partial_x + y\partial_y] = \sigma(x\partial_x + y\partial_y)$. Therefore, $\psi_*\mathcal{X}$ transforms every ray of the origin $R_\xi = \{(r,\xi),\ \theta = \xi\}$ in other ray of the origin $R_{\bar{\xi}}$, that is $\psi_*\mathcal{X}$ has constant angular speed.
Now we undertake the second part. The change of variables ϕ which brings $\mathcal{X}$ into $\phi_*\mathcal{X} = -y\partial_x + x\partial_y + H(x\partial_x + y\partial_y)$, brings $\mathcal{Y}$ into $\phi_*\mathcal{Y} = x\partial_x + y\partial_y + \mathcal{O}(2)$. From Sternberg [152], $\phi_*\mathcal{Y}$ is linearizable. Also it is easy to prove that the change of variables φ which transforms $\phi_*\mathcal{Y}$ into $\varphi_*\phi_*\mathcal{Y} = (1+\alpha)(x\partial_x + y\partial_y)$, where α is a smooth scalar function with $\alpha(0,0) = 0$, can be chosen radial. Thus, $\varphi_*\phi_*\mathcal{X} = -y\partial_x + x\partial_y + K(x\partial_x + y\partial_y)$. The trajectories of $\varphi_*\phi_*\mathcal{Y}$ (straight-lines passing by O) are isochronous sections of $\varphi_*\phi_*\mathcal{X}$. So, the orbits of $\mathcal{Y}$ are isochronous sections of vector field (1.1.2). ■

We emphasize that the vector field $\mathcal{Y}$ is not unique. We give the following relation between two normalized vector fields by $\mathcal{X}$.

Corollary 1.3.11. *The following statements hold:*
(i) If $\mathcal{Y}_1 = x\partial_x + y\partial_y + \mathcal{O}(2)$ and $\mathcal{Y}_2 = x\partial_x + y\partial_y + \mathcal{O}(2)$, are normalized vector fields by $\mathcal{X}$, then $\mathcal{Y}_1 = \alpha\mathcal{Y}_2$, where α is a smooth scalar function with $\alpha(O) = 1$.
(ii) If $\mathcal{Y}_1 = \alpha\mathcal{Y}_2$ where α is a smooth scalar function with $\alpha(O) = 1$ and $\mathcal{Y}_1$ is a normalized vector field by $\mathcal{X}$, then $\mathcal{Y}_2$ is also a normalized vector field by $\mathcal{X}$.

Proof.
(i) From Theorem 1.3.10, the orbits of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are isochronous sections of $\mathcal{X}$ which fill a neighbourhood of the origin. Therefore $\mathcal{Y}_1$ and $\mathcal{Y}_2$ have the same orbital structure at O, thus $\mathcal{Y}_1 = \alpha\mathcal{Y}_2$.
(ii) From Theorem 1.3.10, the statement easily follows since under such conditions the orbits of $\mathcal{Y}_2$ also are isochronous sections of $\mathcal{X}$. ■

The following result shows that if O is a focus, the study of its isochronicity is reduced to prove the existence of a normalized vector field up to certain order. So, in such a case, the problem is easier than if O were a centre.

Theorem 1.3.12. *Let O be a weak focus of order r of (1.1.2). O is an isochronous focus if and only if there exists a polynomial vector field $\mathcal{Y}_p$ of degree $2r+1$ of the form $\mathcal{Y}_p = x\partial_x + y\partial_y + \mathcal{O}(2)$ such that $[\mathcal{X}, \mathcal{Y}_p] = \mu_p \mathcal{Y}_p + \mathcal{O}(2r+2)$, where μ_p is a polynomial of degree $2r$ with $\mu_p(O) = 0$, i.e. $\mathcal{Y}_p$ is a normalized vector field by $\mathcal{X}$ up to order $2r+1$.*

Proof.
The necessity is deduced from Theorem 1.3.10.
To prove the sufficiency, we impose the existence of $\mathcal{Y}_p$ a smooth normalized vector field by $\mathcal{X}$ up to order $2r+1$, i.e. $\mathcal{J}^{2r+1}[\mathcal{X}, \mathcal{Y}_p] = \mu_p \mathcal{Y}_p$, where $\mathcal{J}^k f$ denotes the k-jet of f at origin.

As origin of vector field (1.1.2) is a weak focus of order r, there exists a smooth near-identity change ϕ such that $\phi_* \mathcal{X}$ takes the form $\phi_* \mathcal{X} = \hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + \sum_{i \geq 1} (\alpha_{2i+1}, \beta_{2i+1})_{\mathcal{C}_{2i+1}}$ with $\alpha_3 = \alpha_5 = ... = \alpha_{2r-1} = 0$, $\alpha_{2r+1} \neq 0$, and $\phi_* \mathcal{Y}_p = \hat{\mathcal{W}} = (1,0)_{\mathcal{C}_1} + \cdots$. Moreover, it has that $\mathcal{J}^{2r+1}[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = \hat{\mu} \hat{\mathcal{W}}$, with $\hat{\mu}(O) = 0$. Writing $\hat{\mathcal{X}} = \hat{\mathcal{X}}_1 + \hat{\mathcal{X}}_2 + ...$ and $\hat{\mathcal{W}} = \hat{\mathcal{W}}_1 + \hat{\mathcal{W}}_2 + ...$ with $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{W}}_i$ homogeneous polynomial vector fields of order $i \geq 1$ and $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2 + ...$, with $\hat{\mu}_i$ homogeneous polynomial of order $i \geq 1$, we have
1. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_1] \equiv 0$,
2. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_2] = \hat{\mu}_1 \hat{\mathcal{W}}_1$. From Lemma 1.2.2, this leads to $\hat{\mathcal{W}}_2 \in \mathcal{H}_2^R$.
In a similar way, for order $2i$ we have $\hat{\mathcal{W}}_{2i} \in \mathcal{H}_{2i}^R$, for $i \leq r$.
3. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_3] + [\hat{\mathcal{X}}_3, \hat{\mathcal{W}}_1] = \hat{\mu}_1 \hat{\mathcal{W}}_2 + \hat{\mu}_2 \hat{\mathcal{W}}_1$. Then

$$L_3(\hat{\mathcal{W}}_3) = 2\beta_3 (x^2 + y^2)(0,1)_{\mathcal{C}_1} - \hat{\mu}_1 \hat{\mathcal{W}}_2 - \hat{\mu}_2 \hat{\mathcal{W}}_1.$$

Projecting the above equality onto the range of L_3 and onto $\mathcal{C}_3$, we deduce that $\beta_3 = 0$, and hence $\hat{\mathcal{W}}_3 \in \mathcal{H}_3^R$.

4. Analogously, taking into account the $(2i+1)$-th order terms of $[\hat{\mathcal{X}}, \hat{\mathcal{W}}]$, we have that $\beta_{2i+1} = 0$, $\hat{\mathcal{W}}_{2i+1} \in \mathcal{H}^R_{2i+1}$, for $i \leq r-1$.
5. For order $2r+1$, we get

$$[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2r+1}] + [\hat{\mathcal{X}}_{2r+1}, \hat{\mathcal{W}}_1] = \sum_{j=1}^{2r} \hat{\mu}_j \hat{\mathcal{W}}_{2r+1-j},$$

that is, $L_{2r+1}(\hat{\mathcal{W}}_{2r+1}) = 2r(\alpha_{2r+1}, \beta_{2r+1})_{\mathcal{C}_{2r+1}} + \sum_{j=1}^{2r} \hat{\mu}_j \hat{\mathcal{W}}_{2r+1-j}$. Then $\beta_{2r+1} = 0$.
Thus, from Theorem 1.2.5, origin is a weak isochronous focus. ∎

We offer a similar result to the one obtained in [90].

Theorem 1.3.13. *The origin is an isochronous point of (1.1.2) if and only if it has an infinite number of isochronous sections.*
Moreover, if the origin is a focus, it has that:
(a) the isochronous sections are disjoint two to two and they fill a neighbourhood of the origin,
(b) given a non-zero vector, there is an unique isochronous section which arrives at the origin with that direction.

Proof.
By definition, there is a diffeomorphism ϕ such that $\phi_*\mathcal{X}$ has constant angular velocity. The ray $R_\xi = \{(r, \xi),\ \theta = \xi\}$ is an isochronous section of $\phi_*\mathcal{X}$. Therefore, the curve counter-images of the ray, $\phi^{-1}(R_\xi)$ is an isochronous section of (1.1.2), for every $\xi \in [0, 2\pi]$.
These curves are transversal to the vector field and their derivatives are equal to the derivatives of the rays. Moreover, in the case of a focus, these curves are disjoint, since, by Theorem 1.3.10, they are different trajectories of a smooth vector field.
Now, we prove the sufficient condition. Let η be a isochronous section, verifying $\eta'(0) = (1, 0)$.
The curve $\eta_\xi :\ [0, 1) \to \mathbf{R}^2$ defined by $\eta_\xi(s) = \Phi_{\mathcal{X}}(\xi; \eta(s))$ is a transversal

isochronous section of (1.1.2) at O, for every $\xi \in [0, 2\pi)$.
We define the function $\Psi(\xi, s) = \Phi_{\mathcal{X}}(\xi; \eta(s)),\ \xi \in [0, 2\pi),\ s \in (0, 1)$. We have

$$\frac{\partial}{\partial \xi}\Psi(\xi, s) = \frac{\partial}{\partial \xi}\Phi_{\mathcal{X}}(\xi; \eta(s)) = \mathcal{X}(\Phi_{\mathcal{X}}(\xi; \eta(s)) = \mathcal{X}(\eta_\xi(s)),$$
$$\frac{\partial}{\partial s}\Psi(\xi, s) = \frac{\partial}{\partial s}\Phi_{\mathcal{X}}(\xi; \eta(s)) = \eta'_\xi(s).\eta'(s).$$

Thus, $|D\Psi(t, s)| = \mathcal{X}(\eta_\xi(s)) \wedge \eta'_\xi(s) \neq 0$, by transversality. Therefore Ψ is a diffeomorphism. Also, $\Psi_* \mathcal{X}$ verifies

$$\Phi_{\Psi_* \mathcal{X}}(\xi; R_t) = R_{\xi + t}.$$

So, $\Psi_* \mathcal{X}$ is an uniformly isochronous focus, therefore $\mathcal{X}$ has an isochronous focus at O. ∎

1.3.1 Computing isochronous sections at the origin from a normalized vector field.

We assume that O is an isochronous focus of (1.1.2) and we know the expression of a normalized vector field up to finite order, $(x + R)\partial_x + (y + S)\partial_y$ with R and S of order greater than or equal to two.

Our objective is to provide a recursive algorithm to allow us to compute the coefficients of Taylor expansion at the origin of the isochronous section η whose derivative at the origin is known.
We will denote by $(t, \eta(t)),\ t \in [0, 1)$, a parametrization of this isochronous section, i.e. $y = \eta(x)$.

Lemma 1.3.14. *Let η be isochronous section at the origin with $\eta'(0) = A_1$. If $R(x, \eta(x)) = \sum_{j \geq 2} r_j x^j$ and $S(x, \eta(x)) = \sum_{j \geq 2} s_j x^j$, it has that $\eta(x) = A_1 x + A_2 x^2 + A_3 x^3 + \cdots$ where*

$$\begin{aligned} A_2 &= s_2, \\ A_j &= \tfrac{1}{j-1}\big(s_j - \textstyle\sum_{k=1}^{j-2}(k+1)A_{k+1} r_{j-k}\big),\ j \geq 3. \end{aligned} \tag{1.3.6}$$

Proof.
The curve $y = \eta(x) = A_1x + A_2x^2 + A_3x^3 + \cdots$ is the trajectory of $(x+R)\partial_x + (y+S)\partial_y$ with $\eta'(0) = A_1$. So, $\dot{y} = \eta'(x)\dot{x}$, that is

$$\eta(x) + S(x, \eta(x)) = \eta'(x)(x + R(x, \eta(x))).$$

Therefore, it has that

$$\begin{aligned}\textstyle\sum_{j\geq 1} A_j x^j + \sum_{j\geq 2} s_j x^j &= \textstyle\sum_{j\geq 1} jA_j x^j \\ &\quad \textstyle+ \big(\sum_{j\geq 0}(j+1)A_{j+1}x^j\big)\big(\sum_{j\geq 2} r_j x^j\big) \\ &= \textstyle\sum_{j\geq 1} jA_j x^j \\ &\quad \textstyle+ \sum_{j\geq 2}\big(\sum_{k=1}^{j-2}(k+1)A_{k+1}r_{j-k}\big)x^j. \blacksquare\end{aligned}$$

We now offer a formula to obtain the coefficients r_m and s_m from the expressions of R and S and $A_1, A_2, ..., A_{m-1}$.

Lemma 1.3.15. *Let η isochronous section at the origin be with $\eta'(0) = A_1 \neq 0$. If $R(x,y) = c_{20}x^2 + c_{11}xy + c_{02}y^2 + \cdots$, $S(x,y) = d_{20}x^2 + d_{11}xy + d_{02}y^2 + \cdots$ and $\eta(x) = A_1x + A_2x^2 + A_3x^3 + \cdots$, it has that $R(x,\eta(x)) = \sum_{j\geq 2} r_j x^j$ and $S(x, \eta(x)) = \sum_{j\geq 2} s_j x^j$ with*

$$r_m = c_{m0} + \sum_{k=1}^{m-1} c_{k1}A_{m-k} + \sum_{j=2}^{m}\sum_{k=0}^{m-j} c_{kj}C^{(j)}_{m-k-j+1}, \quad m \geq 2. \tag{1.3.7}$$

$$s_m = d_{m0} + \sum_{k=1}^{m-1} d_{k1}A_{m-k} + \sum_{j=2}^{m}\sum_{k=0}^{m-j} d_{kj}C^{(j)}_{m-k-j+1}, \quad m \geq 2. \tag{1.3.8}$$

where the constants $C^{(j)}_m$ are given by

$$\begin{aligned} C^{(j)}_1 &= A_1^j, \\ C^{(j)}_m &= \tfrac{1}{(m-1)A_1}\textstyle\sum_{k=1}^{m-1}(kj - m + k + 1)A_{k+1}C^{(j)}_{m-k}, \quad m \geq 2. \end{aligned} \tag{1.3.9}$$

To prove Lemma 1.3.15, we use the following result, which can be seen in [95].

Lemma 1.3.16. *If* $(\sum_{m=1} A_m x^m)^j = x^j(\sum_{m=1} C_m^{(j)} x^{m-1}),\ A_1 \neq 0,\ j \geq 2$, *it has that* $C_m^{(j)}$ *satisfy (1.3.9).*

Proof.
Rewriting $R(x, \eta(x))$ of the form

$$\begin{aligned} R(x,\eta(x)) &= c_{20}x^2 + c_{11}x\eta(x) + c_{02}\eta(x)^2 + c_{30}x^3 + c_{21}x^2\eta(x) + \cdots \\ &= c_{20}x^2 + c_{30}x^3 + c_{40}x^4 + \cdots \\ &\quad + c_{11}x\eta(x) + c_{21}x^2\eta(x) + c_{31}x^3\eta(x) + \cdots \\ &\quad + c_{02}\eta(x)^2 + c_{12}x\eta(x)^2 + c_{22}x^2\eta(x)^2 + \cdots \\ &= c_{20}x^2 + c_{30}x^3 + c_{40}x^4 + \cdots \\ &\quad + (c_{11}x + c_{21}x^2 + c_{31}x^3 + \cdots)\eta(x) \\ &\quad + (c_{02} + c_{12}x + c_{22}x^2 + \cdots)\eta(x)^2 + \cdots \\ &\quad + (c_{0n} + c_{1n}x + c_{2n}x^2 + \cdots)\eta(x)^n + \cdots , \end{aligned}$$

and applying the above lemma, we arrive at (1.3.7). Analogously it has (1.3.8). ■

To compute the isochronous section with derivative A_1 at the origin, we use the following recursive procedure:
Step 1. Given A_1, from (1.3.9), we calculate $C_1^{(2)}$ and from (1.3.7) and (1.3.8), we compute r_2, s_2; and from (1.3.6) we obtain A_2.
Step 2. In general, given $A_1, A_2, ..., A_k, C_m^{(n)}$ with $m \leq k-1,\ 2 \leq n \leq k$, and $(r_2, s_2), ..., (r_k, s_k)$, from (1.3.9) we compute $C_m^{(k+1)},\ m = 1, \cdots, k$ and $C_k^{(n)},\ n = 2, \cdots, k$. From (1.3.7) and (1.3.8), we calculate r_{k+1}, s_{k+1}; and from (1.3.6) we obtain A_{k+1}.

The coefficients A_2 and A_3 are:

$$\begin{aligned} A_2 = & \; d_{20} + d_{11}A_1 + d_{02}A_1^2, \\ A_3 = & \; \tfrac{1}{2}c_{30} + \tfrac{1}{2}d_{11}d_{20} - d_{20}c_{20} \\ & +(\tfrac{1}{2}d_{21} + \tfrac{1}{2}d_{11}^2 + d_{02}d_{20} - d_{20}c_{11} - d_{11}c_{20})A_1 \\ & +(\tfrac{3}{2}d_{11}d_{02} + \tfrac{1}{2}d_{12} - d_{20}c_{02} - d_{11}c_{11} - d_{02}c_{20})A_1^2 \\ & +(d_{02}^2 + \tfrac{1}{2}d_{03} - d_{11}c_{02} - d_{02}c_{11})A_1^3 \\ & -d_{02}c_{02}A_1^4. \end{aligned} \tag{1.3.10}$$

Recently, Guillamón and Huguet in [97] compute the isochronous sections near a limit cycle.

1.4 Isochronicity and commutators.

We now provide the following lemma that we will use to prove the main result of this section. From now on, $\mathcal{J}^k f$ denotes the k-jet of f at the origin.

Lemma 1.4.17. *Let $j \in \mathbb{N}$ and $\hat{\mathcal{X}} = (0,1)_{\mathcal{C}_1} + \sum_{i=1}^{j}(\alpha_{2i+1}, \beta_{2i+1})_{\mathcal{C}_{2i+1}} + \mathcal{O}(2j+2)$ a $\mathcal{C}^\infty$-vector field in a neighborhood of the origin. There exists $\hat{\mathcal{W}} = (1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$ a $\mathcal{C}^\infty$-vector field such that $\mathcal{J}^{4j+1}[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = 0$ if and only if $\alpha_3 = ... = \alpha_{2j-1} = 0, \beta_3 = ... = \beta_{2j+1} = 0$.*

Proof.
We prove the necessity. We impose the existence of $\hat{\mathcal{W}}$ a $\mathcal{C}^\infty$-vector field verifying $\mathcal{J}^{4j+1}[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = 0$. Writing $\hat{\mathcal{X}} = \hat{\mathcal{X}}_1 + \hat{\mathcal{X}}_2 + ...$ and $\hat{\mathcal{W}} = \hat{\mathcal{W}}_{2j+1} + \hat{\mathcal{W}}_{2j+2} + ...$ with $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{W}}_i$ homogeneous polynomials of order $i \geq 1$, we have
1. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2j+1}] \equiv 0$,
2. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2j+2}] = 0$, which leads to $\hat{\mathcal{W}}_{2j+2} = 0$. In a similar way, for order $2i$ it has $\hat{\mathcal{W}}_{2i} = 0$, for all i.

3. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2j+3}] + [\hat{\mathcal{X}}_3, \hat{\mathcal{W}}_{2j+1}] = 0$, then

$$-L_{2j+3}(\hat{\mathcal{W}}_{2j+3}) + (2(1-j)\alpha_3, 2\beta_3)_{\mathcal{C}_{2j+3}} = 0.$$

Projecting the above equality onto the range of L_{2j+3} and onto the complement $\mathcal{C}_{2j+3}$, we deduce: $\beta_3 = 0$, $\alpha_3 = 0$ if $j \neq 1$ and $\hat{\mathcal{W}}_{2j+3} \in \mathcal{C}_{2j+3}$, that is, $\hat{\mathcal{W}}_{2j+3} = (a_{2j+3}, b_{2j+3})_{\mathcal{C}_{2j+3}}$ with a_{2j+3} and b_{2j+3} arbitrary constants.
4. Analogously, taking into account the $(2i+1)$-th order terms of $[\hat{\mathcal{X}}, \hat{\mathcal{W}}]$ with $i = j, ..., 2j-1$, we have $2(i-j)\alpha_{2i+1} = 0$, $2i\beta_{2i+1} = 0$, for $i = 1, ..., j-1$; thus, $\alpha_{2i+1} = \beta_{2i+1} = 0$ and $\hat{\mathcal{W}}_{2i+1} \in \mathcal{C}_{2i+1}$.
5. For order $4j+1$, we get

$$[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{4j+1}] + [\hat{\mathcal{X}}_{2j+1}, \hat{\mathcal{W}}_{2j+1}] = -L_{4j+1}(\hat{\mathcal{W}}_{4j+1}) + (0, 2j\beta_{2j+1})_{\mathcal{C}_{4j+1}} = 0$$

then $\beta_{2j+1} = 0$ and $\hat{\mathcal{W}}_{4j+1} \in \mathcal{C}_{4j+1}$.
Thus, $\hat{\mathcal{W}} = (1,0)_{\mathcal{C}_{2j+1}} + \sum_{i=j+1}^{2j} \hat{\mathcal{W}}_{2i+1}$ verifies $\mathcal{J}^{4j+1}[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = 0$.
The sufficient condition is follows easily. ∎

The following theorem is a generalization of the result of Sabatini [142] and Algaba et al. [2] (case $j = 0$). The examples 1.5.30, 1.5.31 and 1.5.32, in the section 1.5.6, show the usefulness of this result.

Theorem 1.4.18. *Let system (1.1.2) be analytic. The following properties holds:*
(i) If O is an isochronous centre of (1.1.2) then there exists $\mathcal{W}$ commuting with $\mathcal{X}$ of the form $\mathcal{W} = (1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$ for any integer $j \geq 0$.
(ii) If there exists an integer $j \geq 0$ such that $\mathcal{W} = (1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$ commutes with $\mathcal{X}$ and $\alpha_{2j+1} = 0$, where α_{2j+1} is the radial coefficient of order j of a normal form of (1.1.2), then O is an isochronous centre of (1.1.2).

Proof.
(i) By definition, if (1.1.2) has an isochronous centre at the origin, there

exists a change ϕ bringing $\mathcal{X}$ to the form $\hat{\mathcal{X}} = (-y, x)^T$, it which commutes with $\hat{\mathcal{W}} = (1,0)_{\mathcal{C}_{2j+1}}$, for every $j \geq 0$. The vector field $\phi^*\hat{\mathcal{W}}$ has the form $(1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$ and commutes with $\mathcal{X}$ since $[\phi^*\hat{\mathcal{X}}, \phi^*\hat{\mathcal{W}}] = \phi^*[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = 0$.
(ii) Let a change ϕ be with $\phi(O) = O$ and $D\phi(O) = I$ bringing $\mathcal{X}$ to its normal form (1.2.3) where $\lambda = \alpha_3 = \alpha_5 = ... = \alpha_{2j+1} = 0$ and let $\hat{\mathcal{X}}$ be its vector field associated. The vector field $\hat{\mathcal{W}} = \phi_*\mathcal{W}$ also has the form $(1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$. By Lemma 1.4.17, $\beta_{2j+1} = 0$, that is $\hat{\mathcal{X}} = (-y, x)^T + \mathcal{O}(2j+3)$ and $\hat{\mathcal{W}} = (1,0)_{\mathcal{C}_{2j+1}} + \sum_{i=j+1}^{2j} \hat{\mathcal{W}}_{2i+1} + \mathcal{O}(4j+3)$ with $\hat{\mathcal{W}}_{2i+1} = (a_{2i+1}, b_{2i+1})_{\mathcal{C}_{2i+1}}$ where $a_{2i+1}, b_{2i+1}, i = j+1, ..., 2j$ are arbitrary constants.
For order $4j+3$, $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{4j+3}] + [\hat{\mathcal{X}}_{2j+3}, \hat{\mathcal{W}}_{2j+1}] = 0$, that is, $\hat{\mathcal{W}}_{4j+3} \in \mathcal{C}_{4j+3}$ and $2\alpha_{2j+3} = 0, \quad 2(j+1)\beta_{2j+3} = 0$. Thus, we have $\alpha_{2j+3} = \beta_{2j+3} = 0$. And so on it arrives to $\alpha_{2i+1} = \beta_{2i+1} = 0$, for any $i \geq 1$. Therefore, $\hat{\mathcal{X}} = (-y, x)^T$, i.e. $\mathcal{X}$ has an isochronous centre at O. ■

The following result characterizes the weak isochronous focus of order j. Let us note that only is necessary the existence of a commutator only up to order $4j+1$.

Theorem 1.4.19. *Let $j \geq 0$ integer and we assume that O is a weak focus of order j for $\mathcal{X}$. The following statements are equivalent:*
(i) The origin is a weak isochronous focus of order j for $\mathcal{X}$,
(ii) there exists $\mathcal{W} = (1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$ such that $\mathcal{J}^{4j+1}[\mathcal{X}, \mathcal{W}] = 0$.

Proof.
(i)⇒(ii) If O is a weak isochronous focus of order j for $\mathcal{X}$, there exists a change of variables ϕ such that $\phi_*\mathcal{X}$ takes the form $(-y, x)^T + (x, y)^T H(x, y)$ where H is a scalar $\mathcal{C}^\infty$-function. From Lemma 1.2.4 it has $\lambda = 0, \alpha_3 = ... = \alpha_{2j-1} = 0, \alpha_{2j+1} \neq 0$ and $\beta_{2i+1} = 0$, for all i. From Lemma 1.4.17, there exists $\hat{\mathcal{W}} = (1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$ such that $\mathcal{J}^{4j+1}[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = 0$. Thus, $\mathcal{W} = \phi^*\hat{\mathcal{W}}$ verifies $\mathcal{J}^{4j+1}[\mathcal{X}, \mathcal{W}] = \mathcal{J}^{4j+1}\phi^*[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = 0$.

(ii)$\Rightarrow$(i) From Lemma 1.4.17, it has that $\beta_3 = ... = \beta_{2j+1} = 0$, and by Lemma 1.2.3, $\beta_{2i+1} = 0$, for all $i \geq 1$. That is, O is an isochronous focus for $\mathcal{X}$. ∎

A strong focus is a focus of order zero. Taking $\mathcal{W} = (x, y)^T + \mathcal{O}(2)$, as $\mathcal{J}^1[\mathcal{X}, \mathcal{W}] = 0$, from Theorem 1.4.19, we have the following result:

Corollary 1.4.20. *Every strong focus is an isochronous focus.*

Also, from Theorems 1.4.18 and 1.4.19, it easily has

Corollary 1.4.21. *$\mathcal{X}$ has a commutator of the form $(x, y)^T + \mathcal{O}(2)$ if and only if either $\lambda \neq 0$ or the system (1.1.2) has an isochronous centre at origin.*

The following theorem is a new characterization of isochronous point. It completes the results of Giné and Grau [90] and Sabatini [145].

Theorem 1.4.22. *The origin is an isochronous point of (1.1.2) if and only if there exists $j \geq 0$ integer and $\mathcal{W} = (1, 0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j + 2)$ such that $[\mathcal{X}, \mathcal{W}] = 0$.*
Moreover, if the origin is a focus for $\mathcal{X}$, the normal form of $\mathcal{X}$ determines univocally to $\mathcal{W}$.

Proof.
If O is either an isochronous centre or an isochronous strong focus ($\lambda \neq 0$), by Corollary 1.4.21, the vector field $\mathcal{X}$ has a commutator with linear part $(x, y)^T$.
If (1.1.2) has a weak isochronous focus of order j at the origin, then there exists a change ϕ with $\phi(O) = O$ and $D\phi(O) = I$ transforming $\mathcal{X}$ into $\hat{\mathcal{X}} = (-y, x)^T + \sum_{i\geq j}(\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}}$ with $\alpha_{2j+1} \neq 0$. It is easy to check that this field commutes with $\hat{\mathcal{W}} = (1, 0)_{\mathcal{C}_{2j+1}} + \frac{1}{\alpha_{2j+1}} \sum_{i\geq j+1}(\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}}$. Thus, the vector field $\mathcal{W} = \phi^*\hat{\mathcal{W}}$ has the form $(1, 0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j + 2)$ and also $[\mathcal{X}, \mathcal{W}] = 0$.

Conversely, we assume that there exists a change of variables ϕ which transforms $\mathcal{X}$ into $\hat{\mathcal{X}}$, the vector field associated to (1.2.3). In such case, $\phi_*\mathcal{W} = \hat{\mathcal{W}}$ has the form $(1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+2)$. By Lemma 1.4.17, $\hat{\mathcal{X}} = (-y,x)^T + \alpha_{2j+1}(1,0)_{\mathcal{C}_{2j+1}} + \mathcal{O}(2j+3)$ and by Lemma 1.2.3, O is an isochronous focus of (1.1.2).
We see that the commutator is unique. We can assume that $\hat{\mathcal{X}} = (-y,x)^T + \alpha_{2j+1}(1,0)_{\mathcal{C}_{2j+1}} + \sum_{i\geq j+1}(\alpha_{2i+1},0)_{\mathcal{C}_{2i+1}}$. From Lemma 1.4.17 we have $\hat{\mathcal{W}} = (1,0)_{\mathcal{C}_{2j+1}} + \sum_{i=j+1}^{2j} \hat{\mathcal{W}}_{2i+1} + \mathcal{O}(4j+2)$ with $\hat{\mathcal{W}}_{2i} = 0$ and $\hat{\mathcal{W}}_{2i+1} = (a_{2i+1}, b_{2i+1})_{\mathcal{C}_{2i+1}}$, where a_{2i+1} and b_{2i+1} are arbitrary constants. For order $4j+3$, $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{4j+3}] + [\hat{\mathcal{X}}_{2j+1}, \hat{\mathcal{W}}_{2j+3}] + [\hat{\mathcal{X}}_{2j+3}, \hat{\mathcal{W}}_{2j+1}] = 0$, that is, $\hat{\mathcal{W}}_{4j+3} \in \mathcal{C}_{4j+3}$ and $-2\alpha_{2j+1}a_{2j+3} + 2\alpha_{2j+3} = 0$, $-2(j+1)\alpha_{2j+1}b_{2j+3} = 0$. Thus, $a_{2j+3} = \frac{\alpha_{2j+3}}{\alpha_{2j+1}}$ and $b_{2j+3} = 0$.
In a similar way, for order $4j+2i+1$ we have $\hat{\mathcal{W}}_{4j+2i+1} \in \mathcal{C}_{4j+2i+1}$, and

$$-2i\alpha_{2j+1}a_{2j+2i+1} + 2i\alpha_{2j+2i+1} = 0, \quad -2(j+i)\alpha_{2j+1}b_{2j+2i+1} = 0.$$

Thus, $a_{2j+2i+1} = \frac{\alpha_{2j+2i+1}}{\alpha_{2j+1}}$, $b_{2j+2i+1} = 0$, for $i \geq 1$. Therefore,

$$\hat{\mathcal{W}} = (1,0)_{\mathcal{C}_{2j+1}} + \frac{1}{\alpha_{2j+1}} \sum_{i\geq j+1} (\alpha_{2i+1}, 0)_{\mathcal{C}_{2i+1}}$$

commutes with $\hat{\mathcal{X}}$. ∎

1.5 Several applications and examples.

1.5.1 Quadratic isochronous points.

The quadratic systems can be transformed by means of a rotation of axis into the form provided by Bautin [28],

$$\begin{aligned} \dot{x} &= -y + \lambda x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= \phantom{-y+{}} x + \lambda y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{aligned} \tag{1.5.11}$$

He found the weak foci and their orders, and their centres.
Later, Loud [119] obtained their isochronous centres. The following result completes the work of Loud, characterizing the isochronous foci of (1.5.11) and their orders.

Theorem 1.5.23. *The origin of system (1.5.11) is an isochronous focus if and only if one of the following five series of conditions holds:*
i) $\lambda \neq 0$ *(Strong focus).*
ii) $\lambda = 0,\ \lambda_5 \neq 0,\ \lambda_3 \neq \lambda_6,\ (\lambda_5 + 4\lambda_2)^2 = -\lambda_4^2 - 18\lambda_3^2 - 10\lambda_6^2 - 9\lambda_3\lambda_4 + 12\lambda_3\lambda_6 + \lambda_4\lambda_6 \geq 0.$ *(Weak focus of order 1)*
iii) $\lambda = 0,\ \lambda_5 = 0,\ \lambda_3 \neq \lambda_6,\ \lambda_6 \neq 0,\ \lambda_4 = 6(\lambda_6 - \lambda_3),\ \lambda_2 \neq 0,\ \lambda_6(3\lambda_3 - 5\lambda_6) = 2\lambda_2^2$ *(Weak focus of order 2)*
iv) $\lambda = 0,\ \lambda_5 = 0,\ \lambda_3 \neq \lambda_6,\ \lambda_6 \neq 0,\ \lambda_4 \neq 0,\ \lambda_2 \neq 0,\ 14\lambda_4 = 55\lambda_6 - 71\lambda_3 + sgn(\lambda_6)\sqrt{1065\lambda_6^2 - 1650\lambda_3\lambda_6 + 841\lambda_3^2}$, *with* $\lambda_3 > 4\lambda_6$ *if* $\lambda_6 > 0$ *or* $\lambda_3 < 4\lambda_6$ *if* $\lambda_6 < 0$ *(Weak focus of order 2)*
v) $\lambda = 0,\ \lambda_5 = 0,\ \lambda_3 \neq \lambda_6,\ \lambda_6 \neq 0,\ \lambda_4 \neq 0,\ \lambda_2 \neq 0,\ 14\lambda_4 = 55\lambda_6 - 71\lambda_3 - sgn(\lambda_6)\sqrt{1065\lambda_6^2 - 1650\lambda_3\lambda_6 + 841\lambda_3^2}$, *with* $(15 - \sqrt{105})\lambda_6 < 4\lambda_3 < (15 + \sqrt{105})\lambda_6$ *if* $\lambda_6 > 0$, *or* $(15 + \sqrt{105})\lambda_6 < 4\lambda_3 < (15 - \sqrt{105})\lambda_6$ *if* $\lambda_6 < 0$, *(Weak focus of order 2)*
Moreover, there isn't an isochronous focus of order greater than or equal to three.

Proof.
If $\lambda \neq 0$, then (1.5.11) is a strong focus which is smoothly linearizable, see Sternberg [152]; therefore, O is an isochronous focus, (case **i**)).
The first radial coefficient is $\alpha_3 = -\lambda_5(\lambda_3 - \lambda_6)$. So, the systems (1.5.11) whose origin is an isochronous focus of order 1, must hold $\lambda = 0,\ \lambda_3 \neq \lambda_6, \lambda_5 \neq 0$. Under these conditions, it has $\beta_3 = \frac{1}{24}(r - (\lambda_5 + 4\lambda_2)^2)$, with

$$r := r(\lambda_3, \lambda_4, \lambda_6) = -\lambda_4^2 - 18\lambda_3^2 - 10\lambda_6^2 - 9\lambda_3\lambda_4 + 12\lambda_3\lambda_6 + \lambda_4\lambda_6.$$

Therefore $\beta_3 = 0$ if and only if $r = (\lambda_5 + 4\lambda_2)^2$ (case **ii**)).
We now obtain systems (1.5.11) whose origin is a weak isochronous focus

of order 2. By imposing $\alpha_3 = 0$, it has that

$$\alpha_5 = \lambda_2\lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5\lambda_3 - 5\lambda_6).$$

That is, from Theorem 1.2.5, such systems must verify

$$\lambda = \lambda_5 = 0,\ \lambda_2 \neq 0, 0 \neq \lambda_4 \neq 5(\lambda_6 - \lambda_3) \neq 0,$$

and $\beta_3 = \beta_5 = 0$. In this case $r = 16\lambda_2^2 > 0$ and

$$\beta_5 = \frac{1}{768}(\lambda_3 - \lambda_6)(\lambda_4 + 6\lambda_3 - 6\lambda_6)\Sigma(\lambda_3, \lambda_4, \lambda_6)$$

where $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 70\lambda_6^2 - 220\lambda_6\lambda_3 - 55\lambda_6\lambda_4 + 150\lambda_3^2 + 71\lambda_3\lambda_4 + 7\lambda_4^2$. If λ_6 were zero, we would have

$$r = -(3\lambda_3 + \lambda_4)(6\lambda_3 + \lambda_4), \quad \beta_5 = -\frac{1}{768}\lambda_3(50\lambda_3 + 7\lambda_4)r.$$

As $\lambda_3 \neq 0$ and $r > 0$, $\beta_5 = 0$ only if $\lambda_4 = -50/7\lambda_3$; in this case, r is negative. Therefore, without loss of generality, we can assume that $\lambda_6 \neq 0$.

The azimuthal coefficient of order 2, β_5, is zero if and only if either $\lambda_4 = 6(\lambda_6 - \lambda_3)$ or $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 0$.

In the first case, that is $\lambda_4 = 6(\lambda_6 - \lambda_3)$, it has $r = 8\lambda_6(3\lambda_3 - 5\lambda_6)$, therefore if also $\lambda_6(3\lambda_3 - 5\lambda_6) = 2\lambda_2^2$, the origin is a weak isochronous focus of order 2 (case **iii**)).

In the second case, $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 0$, O is a weak isochronous focus of order 2 if and only if λ_3, λ_4 and λ_6 also verify $r(\lambda_3, \lambda_4, \lambda_6) > 0$. Making $\bar{\lambda}_3 = \frac{\lambda_3}{\lambda_6}$, $\bar{\lambda}_4 = \frac{\lambda_4}{\lambda_6}$ we have

$$\Sigma(\lambda_3, \lambda_4, \lambda_6) = \lambda_6^2\bar{\Sigma}(\bar{\lambda}_3, \bar{\lambda}_4), \qquad r(\lambda_3, \lambda_4, \lambda_6) = \lambda_6^2\bar{r}(\bar{\lambda}_3, \bar{\lambda}_4),$$

therefore O will be isochronous focus on the region where $\bar{\lambda}_3$ and $\bar{\lambda}_4$ verify $\bar{\Sigma}(\bar{\lambda}_3, \bar{\lambda}_4) = 0$ and $\bar{r}(\bar{\lambda}_3, \bar{\lambda}_4) > 0$.

It is easy to prove that the hyperbolas $\bar{\Sigma}(\bar{\lambda}_3, \bar{\lambda}_4) = 0$ and $\bar{r}(\bar{\lambda}_3, \bar{\lambda}_4) = 0$

meet at the points $P_0(4,-10)$, $P_1(\frac{1}{4}(15+\sqrt{105}),-\frac{1}{4}(85+7\sqrt{105}))$ and $P_2(\frac{1}{4}(15-\sqrt{105}),-\frac{1}{4}(85-7\sqrt{105}))$. The point P_0 is on the branch of $\bar{\Sigma}$

$$14\bar{\lambda}_4 = 55 - 71\bar{\lambda}_3 + sgn(\lambda_6)\sqrt{1065 - 1650\bar{\lambda}_3 + 841\bar{\lambda}_3^2}$$

and P_1, P_2 are on the branch

$$14\bar{\lambda}_4 = 55 - 71\bar{\lambda}_3 - sgn(\lambda_6)\sqrt{1065 - 1650\bar{\lambda}_3 + 841\bar{\lambda}_3^2}.$$

Studying the relative position of both conics, it leads to the cases **iv**) and **v**).
Finally, we compute the systems (1.5.11) whose origin is a weak isochronous focus of order 3. It has that

$$v_7 = -\lambda_2\lambda_4(\lambda_3 - \lambda_6)^2(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2).$$

So, it arrives at

$$\lambda = \lambda_5 = 0, \lambda_4 = 5(\lambda_6 - \lambda_3) \neq 0, \lambda_2^2 \neq \lambda_6(\lambda_3 - 2\lambda_6).$$

Under these conditions, also it must hold $\beta_3 = \beta_5 = \beta_7 = 0$. In this case, β_3 is zero if and only if $8\lambda_2^2 = \lambda_3^2 + 6\lambda_6\lambda_3 - 15\lambda_6^2$. Substituting λ_2^2 in β_5 it has $\beta_5 = -\frac{5}{128}(\lambda_3 - \lambda_6)^4$, which is nonzero. Therefore, O cannot be a weak isochronous focus of order 3. ■

As a consequence, it has that an isochronous focus of system (1.5.11) with order greater than two is an isochronous centre.

1.5.2 Isochronous points of systems with non linearities cubic.

The cubic systems centre-focus type without quadratic terms can be written by means of a rotation of axes in the following form given by

Sibirskii [150],

$$\begin{aligned}\dot{x} = & -y + \lambda x + (\mu_3 - \mu_1 - \mu_2)x^3 + (3\mu_5 - \mu_4)x^2y \\ & \qquad +(3\mu_2 - 3\mu_1 - 2\mu_3 + \mu_6)xy^2 + (\mu_7 - \mu_5)y^3, \\ \dot{y} = & \quad x + \lambda y + (\mu_5 + \mu_7)x^3 + (3\mu_1 + 3\mu_2 + 2\mu_3)x^2y \\ & \qquad +(\mu_4 - 3\mu_5)xy^2 + (\mu_1 - \mu_2 - \mu_3)y^3.\end{aligned} \tag{1.5.12}$$

He gave the weak foci and their orders, and their centres.
Later, Pleshkan [133] found the systems (1.5.12) with an isochronous centre at the origin. We now give the systems whose origin is a weak isochronous focus and their orders.

Theorem 1.5.24. *The origin of system (1.5.12) is an isochronous focus if and only if*
i) $\lambda \neq 0$ *(Strong focus).*
ii) $\lambda = \mu_4 = 0,\ \mu_6 \neq 0$ *(Weak focus of order 1)*
iii) $\lambda = \mu_4 = \mu_6 = 0,\ (4\mu_1 + \mu_3)(6\mu_1 - \mu_3) = -6(\mu_2^2 + \mu_5^2 + \mu_7^2),\ \mu_3 \neq 0,\ \mu_7 \neq 0,$ *(Weak focus of order 2).*
iv) $\lambda = \mu_4 = \mu_6 = 0,\ \mu_7 = 0,\ \mu_5 = 0,\ (4\mu_1+\mu_3)(6\mu_1-\mu_3) = -6\mu_2^2,\ \mu_3 \neq 0,\ \mu_1 \neq 0,\ \mu_2 \neq 0,$ *(Weak focus of order 3).*
v) $\lambda = \mu_4 = \mu_6 = 0,\ \mu_7 = 0,\ \mu_3 = -24\mu_1,\ \mu_2^2 + \mu_5^2 = 100\mu_1^2,\ \mu_3 \neq 0,\ \mu_1 \neq 0,\ \mu_2 \neq 0,$ *(Weak focus of order 3).*
Moreover, there isn't an isochronous focus of order greater than or equal to four.

Proof.
If $\lambda \neq 0$ the system (1.5.12) is a strong focus which is smoothly linearizable (see [152]); therefore, O is an isochronous focus, (case **i**)).
Now, we assume that $\lambda = 0$. The first radial constants, up to a positive

factor, are

$$
\begin{aligned}
&\alpha_3 = \mu_6,\\
&\alpha_5 = -\mu_3\mu_7,\ \text{if } \alpha_3 = 0,\\
&\alpha_7 = \mu_3\mu_2\mu_1,\ \text{if } \alpha_3 = \alpha_5 = 0,\\
&\alpha_9 = \mu_3^2\mu_2\mu_4,\ \text{if } \alpha_3 = \alpha_5 = \alpha_7 = 0,\\
&\alpha_{11} = -\mu_3^2\mu_2[4(\mu_2^2+\mu_5^2)-\mu_3^2],\ \text{if } \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = 0,\\
&\alpha_{2k+1} = 0,\ k \geq 6,\ \text{if } \alpha_{2i+1} = 0,\ i = 1,2,3,4,5.
\end{aligned}
$$

The first azimuthal constant is, up to a positive factor, $\beta_3 = \mu_4$. Therefore, from Theorem 1.2.5, O is a weak isochronous focus of order 1 if and only if $\mu_4 = 0,\ \mu_6 \neq 0$ (case **ii**)).
The system (1.5.12) has a weak focus of order 2 at the origin if $\alpha_3 = 0$ and $\alpha_5 \neq 0$, i.e. $\mu_6 = 0, \mu_7 \neq 0,\ \mu_3 \neq 0$, and it will be isochronous if also $\beta_3 = \beta_5 = 0$, that is $\mu_4 = 0$ and in this case,

$$\beta_5 = -\frac{1}{8}[6(\mu_2^2+\mu_5^2+\mu_7^2)+(4\mu_1+\mu_3)(6\mu_1-\mu_3)]$$

is zero (case **iii**)).
From the expression for α_5 and α_7 we have that O is focus of order 3 if and only if $\mu_6 = \mu_7 = 0,\ \mu_1 \neq 0,\ \mu_2 \neq 0$ and $\mu_3 \neq 0$. In this case, O is an isochronous focus if $\beta_3 = 0$ (i.e. $\mu_4 = 0$) and

$$
\begin{aligned}
&\beta_5 = -\tfrac{1}{8}[6(\mu_2^2+\mu_5^2)+(4\mu_1+\mu_3)(6\mu_1-\mu_3)],\\
&\beta_7 = -\tfrac{5}{16}\mu_1\mu_5(\mu_3+24\mu_1)
\end{aligned}
$$

are zero.
If $\mu_5 = 0$, then $(4\mu_1+\mu_3)(6\mu_1-\mu_3) = -6\mu_2^2$ holds (case **iv**)), and if $\mu_3 = -24\mu_1$ we have that $\mu_2^2+\mu_5^2 = 100\mu_1^2$ (case **v**)).
The origin is a weak focus of order 4 if it verifies $\alpha_3 = \alpha_5 = \alpha_7 = 0$ and $\alpha_9 \neq 0$, that is $\mu_6 = \mu_7 = \mu_1 = 0,\ \mu_4 \neq 0,\ \mu_2 \neq 0$ and $\mu_3 \neq 0$. So, O cannot be isochronous since $\beta_3 = 0$ only if $\mu_4 = 0$.
Finally, we compute the systems whose origin is a weak isochronous focus of order five. They must verify

$$\mu_6 = \mu_7 = \mu_1 = \mu_4 = 0,\ \mu_2 \neq 0,\ \mu_3 \neq 0,\ \mu_3^2 \neq 4(\mu_2^2+\mu_5^2),$$

and $\mu_4 = 0$ ($\beta_3 = 0$); $\mu_3^2 = 6(\mu_2^2 + \mu_5^2)$ ($\beta_5 = 0$). This contradicts that $\mu_3 \neq 0$. Therefore, there is not a weak isochronous focus of order 5. ∎

As a consequence, an isochronous focus of system (1.5.12) with order greater than three is an isochronous centre.

1.5.3 A three-parameters family of Rayleigh equations with four local critical periods.

Aside from its interest in physical applications, the study of the period function is essential for approaching some problems of differential equations. So, for instance, the monotonicity of the period function is strictly related to the existence and uniqueness of solutions of some boundary values, bifurcation or perturbation problems.
Our following application shows how the calculation of the azimuthal coefficients of a normal form of the system allows us to solve the problem of determining the number of local critical points of the period function (*local critical periods*) which can appear by perturbation of a system in the neighbourhood of a centre.

Let us consider the family of second order differential equations, so-called Rayleigh equations, $\ddot{x} + h(\dot{x}) + x = 0$, where $h(x) = a_2x^2 + a_4x^4 + a_6x^6$. Each differential equation of the family will be denoted by $\mathrm{R}(a_2, a_4, a_6)$. It is easy to prove that O is a centre of $\mathrm{R}(a_2, a_4, a_6)$, for all a_2, a_4, a_6 real numbers.
We have the following result.

Theorem 1.5.25. *The following properties hold:*
i) *There are, at the most, four critical periods of the family* $\mathrm{R}(a_2, a_4, a_6)$ *in a neighbourhood of the origin, for* $(a_2, a_4, a_6) \neq (0, 0, 0)$.
ii) *Moreover, given a neighbourhood of the origin, there are* $a_2, a_4, a_6 \in \mathbb{R}$ *such that* $\mathrm{R}(a_2, a_4, a_6)$ *has exactly four critical periods in a neighbourhood of the origin.*

Proof.
A plane differential system associated to the Rayleigh equations $\mathrm{R}(a_2, a_4, a_6)$ is the Liénard system

$$\dot{x} = -y, \qquad \dot{y} = x + a_2 y^2 + a_4 y^4 + a_6 y^6. \tag{1.5.13}$$

An analysis on the monotonicity of the period function of a Liénard system can be seen in [144].
This system is time-reversible, therefore O is a centre.
If we denote by $r(\theta, \rho)$ the periodic solution, expressed in polar coordinates, such that $r(0, \rho) = \rho$, where ρ is small enough, the period $T(\rho)$ of this periodic orbit is an analytic function of the form $T(\rho) = 2\pi + \sum_{n=1}^{\infty} T_n \rho^{2n}$, see [54, 2]. The constants T_n are called period constants of system (1.5.13).
Algaba et al. [2], provide a relation between the period constants and the azimuthal coefficients of a normal form of the system. Concretely,

$$T_n = 2\pi \sum_{\substack{n_1+\cdots+n_l=2n \\ n_i \text{ even, } l\geq 1}} (-1)^l \beta_{n_1+1} \cdots \beta_{n_l+1}.$$

Thus, $T_n = -2\pi\beta_{2n+1}$ if the above period constants, $T_i,\ i = 1...n-1$, are zero.
The first azimuthal coefficients of (1.5.13) are

$$\begin{aligned}
\beta_3 = \quad & -\tfrac{1}{6}a_2^2 \\
\beta_5 = \quad & -\tfrac{13}{144}a_2^4 - \tfrac{1}{3}a_2 a_4 \\
\beta_7 = \quad & -\tfrac{6937}{77760}a_2^6 - \tfrac{2711}{4320}a_4 a_2^3 - \tfrac{5}{16}a_6 a_2 - \tfrac{7}{40}a_4^2 \\
\beta_9 = \quad & -\tfrac{71053}{746496}a_2^8 - \tfrac{154129}{155520}a_4 a_2^5 - \tfrac{239}{270}a_6 a_2^3 - \tfrac{1399}{960}a_4^2 a_2^2 - \tfrac{27}{80}a_4 a_6 \\
\beta_{11} = \quad & -\tfrac{106311847}{895795200}a_2^{10} - \tfrac{60925411}{38707200}a_4 a_2^7 - \tfrac{212609}{161280}a_2^5 a_6 - \tfrac{25518229}{5806080}a_4^2 a_2^4 \\
& -\tfrac{3825733}{967680}a_6 a_4 a_2^2 - \tfrac{297}{1792}a_6^2 - \tfrac{111769}{86400}a_4^3 a_2.
\end{aligned}$$

If $(a_2, a_4, a_6) = (0, 0, 0)$, is an isochronous centre. Otherwise, by applying the Malgrange Theorem, to the equation $T'(\rho) = 0$, for $0 < \rho << 1$, we

arrive at that 4 local critical periods, at the most, can bifurcate from the origin in the family $\mathrm{R}(a_2, a_4, a_6)$.
We prove the second part. The system $\mathrm{R}(0,0,1)$ verifies $T_1 = T_2 = T_3 = T_4 = 0$ and $T_5 \neq 0$. If we take the following perturbation of $\mathrm{R}(0,0,1)$,

$$a_2 = \epsilon^{2n},\ a_4 = -\epsilon^n,\ a_6 = 1,\ n \text{ sufficiently big,}$$

it has that

$$\begin{aligned}
\beta_3 &= -\tfrac{1}{6}\epsilon^{4n}, \\
\beta_5 &= \tfrac{48}{144}\epsilon^{3n} + o(\epsilon^{8n}), \\
\beta_7 &= -(\tfrac{7}{40} + \tfrac{5}{16})\epsilon^{2n} + o(\epsilon^{7n}), \\
\beta_9 &= \tfrac{27}{80}\epsilon^{n} + o(\epsilon^{6n}), \\
\beta_{11} &= -\tfrac{297}{1792} + o(\epsilon^{5n}),
\end{aligned}$$

Taking n sufficiently big, it has that $0 < |T_1| << |T_2| << |T_3| << |T_4| << |T_5|$, and the period constants alternate sign, then they form a Sturm sequence. Therefore, we can assert that 4 critical points of the period function bifurcate from the origin. ■

The azimuthal constants have been obtained by using Theorem 1.2.6.

1.5.4 An isochronous focus of a cubic Kukles system.

We consider the following family of cubic Kukles systems depending on the parameters a_1 and b_2,

$$\begin{aligned}
\dot{x} &= -y, \\
\dot{y} &= \quad x + a_1(x^2+y^2) + a_1^2(x^2+5y^2)x + b_2(x^2+y^2)y,
\end{aligned} \tag{1.5.14}$$

with $a_1 b_2 \neq 0$. The first radial coefficient for this family is $\alpha_3 = b_2$. Hence, when $b_2 > 0$ the origin is an unstable weak focus and when $b_2 < 0$ the origin is a stable weak focus. We have the following result.

Theorem 1.5.26. *The family (1.5.14) has a weak isochronous focus at the origin. An isochronous section of derivative 1 is*

$$\eta(x) = x - (\frac{3}{8}a_1^2 + \frac{1}{4}b_2)x^3 + \mathcal{O}(4).$$

Proof.
The origin of (1.5.14) is a weak focus of order 1. The field $Y_p = (x + R)\partial_x + (y + S)\partial_y$ where

$$\begin{aligned} R = & \ -a_1(x^2+y^2) - \tfrac{1}{4}a_1^2 x(5x^2+y^2) - \tfrac{1}{2}b_2 y(y^2+x^2) \\ S = & \ -\tfrac{1}{4}a_1^2 y(5x^2-7y^2) - \tfrac{1}{2}b_2 x(x^2+y^2), \end{aligned}$$

is a normalized vector field up to order 3, i.e. $[\mathcal{X}, \mathcal{Y}_p] = \mu\mathcal{Y}_p + \mathcal{O}(4)$ with $\mu = b_2(x^2+y^2)$. From Theorem 1.3.12, the origin of (1.5.14) is a weak isochronous focus of order 1.
The expression of the isochronous section is obtained from (1.3.10). ∎

1.5.5 Centres and isochronous centres of two families of polynomial systems.

We apply our approach to the centre and isochronous centre problems of two families of polynomial differential equation systems. The first one is the cuartic family with eight-parameters $(\dot{x}, \dot{y})^t = (yP(x), xQ(y))^t$, with P, Q polynomials of degree less or equal to three. This family includes a class with an isochronous centre at the origin, which contains isochronous cuartic Kolmogorov systems.
The Kolmogorov system have the form $(\dot{x}, \dot{y})^t = (xF(x,y), yG(x,y))^t$. These systems are widely used in ecology to describe the interaction between two populations, see Bazykin [29] and May [126]. In that case, attention is restricted to the behavior of orbits in the realistic quadrant, $x > 0$ and $y > 0$. It is well known that, in the quadratic case, these systems have not limit cycle, and there is at most one critical point in

the realistic quadrant and that, if it were centre, it would be non isochronous. Sáez and Szántó [149] find a one-parameter family of isochronous cubic Kolmogorov systems. Also, Lloyd et. al. [117] obtain a cubic Kolmogorov family with six limit cycles.
The second family studied is a class of polynomial degenerate Kolmogorov system. We have found a three-parameter subclass of cuartic Kolmogorov systems with an isochronous centre in the realistic quadrant and give a polynomial family with three limit cycles.

First, we consider the systems

$$\begin{cases} \dot{x} &= yP(x), \\ \dot{y} &= xQ(y), \end{cases} \tag{1.5.15}$$

where P and Q are polynomials with real coefficients. We note that the origin is a centre if and only if $P(0)Q(0) < 0$, because the matrix of the linear part of the system has imaginary eigenvalues and these systems have the analytic first integral $\int_0^x \frac{sds}{P(s)} - \int_0^y \frac{sds}{Q(s)}$.
Now, we characterize the quartic systems (1.5.15) which have an isochronous centre at the origin.

Theorem 1.5.27. *The system (1.5.15) with P, Q polynomials of degree at most three and with $P(0)Q(0) < 0$, has an isochronous centre at the origin if and only if, up to linear change of the state variables and a constant scaling of the time variable, has the form*

$$\begin{cases} \dot{x} &= -\frac{1}{27}y(-3+a_1x)(-9+6a_1x+(c-a_1^2)x^2), \\ \dot{y} &= \frac{1}{27}x(3+b_1y)(9+6b_1y+(c+b_1^2)y^2), \end{cases} \tag{1.5.16}$$

with a_1, b_1, c arbitrary real constants.

Proof.
By means of the change $x \to \frac{y}{\sqrt{-P(0)Q(0)}}$, $y \to \frac{x}{P(0)}$, and multiplying the system by the constant $1/\sqrt{-P(0)Q(0)}$, we can suppose (1.5.15) with $P(x) = -1 + a_1x + a_2x^2 + a_3x^3$ and $Q(y) = 1 + b_1y + b_2y^2 + b_3y^3$. Next

we compute necessary conditions so that the origin be an isochronous centre. In complex coordinates, this system takes the form

$$\begin{aligned}\dot{z} = iz \quad &+\tfrac{1}{4}(b_1 - ia_1)z^2 + \tfrac{1}{4}(-b_1 + ia_1)\bar{z}^2 \\ &-\tfrac{i}{8}(a_2 + b_2)z^3 + \tfrac{i}{8}(b_2 - a_2)z^2\bar{z} + \tfrac{i}{8}(a_2 + b_2)z\bar{z}^2 + \tfrac{i}{8}(a_2 - b_2)\bar{z}^3 \\ &-\tfrac{1}{16}(b_3 + ia_3)z^4 + \tfrac{1}{8}(b_3 - ia_3)z^3\bar{z} + \tfrac{1}{8}(-b_3 + ia_3)z\bar{z}^3 \\ &+\tfrac{1}{16}(b_3 + ia_3)\bar{z}^4.\end{aligned}$$

Using the algorithm included at the end of the chapter, we have

$$\begin{aligned}\gamma_1 = &\ -\tfrac{i}{24}(3a_2 + a_1^2 + b_1^2 - 3b_2) \\ \gamma_2 = &\ -\tfrac{i}{256}(3a_2 + a_1^2 + b_1^2 - 3b_2)(a_2 - a_1^2 - b_1^2 - b_2)\end{aligned}$$

From the vanishing of γ_1 it has $b_2 = a_2 + \frac{1}{3}(a_1^2 + b_1^2)$. In this case, γ_2 is zero and the constant γ_3 becomes

$$\gamma_3 = \frac{i}{155520}\left[(2a_1^3 + 9a_1a_2 + 27a_3)^2 + (b_1^3 + 3a_1^2b_1 + 9a_2b_1 - 27b_3)^2\right]$$

and this will be zero if and only if $a_3 = -\frac{1}{27}a_1(2a_1^2 + 9a_2)$ and $b_3 = \frac{1}{27}b_1(b_1^2 + 3a_1^2 + 9a_2)$.
In cartesian coordinates, the system comes given by (1.5.16) with $c = 3(a_1^2 + 3a_2)$.

Conversely, we prove that the origin is an isochronous centre.
By means of the analytic change of variable

$$\begin{aligned}u(x,y) &= \frac{(3+b_1y)x}{(-3+a_1x)\sqrt{9+6b_1y+(c+b_1^2)y^2}}, \\ v(y) &= \frac{y}{\sqrt{9+6b_1y+(c+b_1^2)y^2}},\end{aligned}$$

the system (1.5.16) is transformed into

$$\left\{\begin{array}{lcl}\dot{u} & = & -vH(u,v), \\ \dot{v} & = & uH(u,v),\end{array}\right. \tag{1.5.17}$$

with $H(u,v)=3(3+b_1y(v))(-3+a_1x(u,v))$, being $(x(u,v),\,y(v))$ the inverse function of $(u(x,y),\,v(y))$.
The period of the solution that passes by $(r,0)$ is

$$T(r)=\int_0^{2\pi}\frac{d\,\theta}{H(r sin\,\theta, r cos\,\theta)}.$$

Eliminating x from the expression of u and substituting it in the integral, we obtain

$$\begin{aligned} T(r)= & \int_0^{2\pi}\tfrac{a_1 r sin\,\theta\sqrt{9+6b_1y(r cos\,\theta)+(c+b_1^2)y(r cos\,\theta)^2}}{9(3+b_1y(r cos\,\theta))^2}d\,\theta \\ & -\int_0^{2\pi}\tfrac{d\,\theta}{9(3+b_1y(r cos\,\theta))}. \end{aligned}$$

The first integral is zero, because it is the integral of an odd function on the interval $[0,2\pi]$, eliminating y of the expression of v and substituting it in the second integral, we have

$$\begin{aligned} T(r) &= -\int_0^{2\pi}\left(\tfrac{1}{9}+\tfrac{b_1 r cos\,\theta(r cos\,\theta-\sqrt{1-c(r cos\,\theta)^2})}{9(-1+c(r cos\,\theta)^2+b_1 r cos\,\theta\sqrt{1-c(r cos\,\theta)^2})}\right)d\,\theta \\ &= -\tfrac{2\pi}{9}, \end{aligned}$$

thus, it follows the result.
Remark. Let us note that if $a_1=b_1=0$, (1.5.16) is a cubic reversible system; if $a_1=c=0$, (1.5.16) is an isochronous quartic Newton's system, studied by Volokitin and Ivanov [160].
We also observe that the systems with an isochronous centre given by Theorem 1.5.27, with a_1 and b_1 different from zero, by means of an appropriate change of variables are quartic Kolmogorov systems. Concretely, making $x\to-\frac{3}{a_1}(x-1),\ y\to\frac{3}{b_1}(y-1)$, the system (1.5.16) becomes

$$\left\{\begin{array}{lcl} \dot{x} &=& -\frac{1}{a_1b_1}x(y-1)(c(x-1)^2-a_1^2x^2), \\ \dot{y} &=& -\frac{1}{a_1b_1}y(x-1)(c(y-1)^2+b_1^2y^2). \end{array}\right. \tag{1.5.18}$$

The Kolmogorov systems are widely used in ecology to describe the interaction between two populations. In this case, the attention is restricted to the behavior of orbits in the realistic quadrant, $x > 0$ and $y > 0$. From Theorem 1.5.27, case $c = 0$, the degenerated system

$$\begin{cases} \dot{x} &= ax^3(y-1), \\ \dot{y} &= by^3(x-1), \end{cases} \tag{1.5.19}$$

with $ab < 0$, has an isochronous centre at $(1, 1)$.

Finally, we characterize the centres and isochronous centres of the systems

$$\begin{cases} \dot{x} &= x^p(a_1x + b_1y + c_1), \\ \dot{y} &= y^q(a_2x + b_2y + c_2), \end{cases} \tag{1.5.20}$$

with $c_1c_2 \neq 0$ and $p, q \geq 1$.
If $p = q = 1$, it is well known that there is at most one critical point in the realistic quadrant and that, if it were centre, it would be non isochronous, see Bazykin [29].
The following results characterize the centres and isochronous centres of the polynomial family (1.5.20) for any $p, q \geq 1$.

Theorem 1.5.28. *The polynomial system (1.5.20) has a critical point that is a centre if and only if, by means of a change of variables, it can be transformed either into*

$$\begin{cases} \dot{x} &= ax^p(y-1), \\ \dot{y} &= by^q(x-1), \end{cases} \tag{1.5.21}$$

with $ab < 0$, *or into*

$$\begin{cases} \dot{x} &= x^p(a_1x - a_2y + a_2 - a_1), \\ \dot{y} &= y^p(a_2x - a_1y + a_1 - a_2), \end{cases} \tag{1.5.22}$$

with $p \geq 2$, $a_1a_2 \neq 0$ *and* $a_1 \neq a_2$.
Furthermore, three small amplitude limit cycles can bifurcate from the point $(1, 1)$.

Proof.
Without loss of generality, we can suppose $p \leq q$, since otherwise, we interchange x by y.
We look for systems of the family (1.5.20) with a critical point which is a centre. Obviously this point cannot be on the axes, since they are invariant. This leads to suppose $a_1b_2 - a_2b_1 \neq 0$ and $c_1c_2 \neq 0$.
By rescaling the state variables we can suppose that the straight lines $r \equiv a_1x + b_1y + c_1 = 0$ and $s \equiv a_2x + b_2y + c_2 = 0$, cut at the point $(1,1)$, that is, we have $r \equiv a_1x + b_1y - a_1 - b_1 = 0$ and $s \equiv a_2x + b_2y - a_2 - b_2 = 0$. Making the change $x \to x-1,\ y \to y-1$, the system (1.5.20) takes the form

$$\left\{\begin{array}{lcl} \dot{x} & = & (x+1)^p(a_1x + b_1y), \\ \dot{y} & = & (y+1)^q(a_2x + b_2y), \end{array}\right. \tag{1.5.23}$$

whose trace and determinant of the matrix of linear part are $a_1 + b_2$ and $-a_1^2 - b_1a_2$, respectively. Thus, the origin is non-degenerated centre if it holds $b_2 = -a_1,\ w_0^2 = -a_1^2 - a_2b_1 > 0$, that is, $a_2 \neq 0$, and $b_1 = -\frac{1}{a_2}(w_0^2 + a_1^2)$. The change of variables $x \to w_0x + a_1y,\ y \to a_2y$, transforms the system (1.5.23) into

$$\left\{\begin{array}{lcl} \dot{x} & = & (w_0x + a_1y + 1)^p(a_1x - w_0y) - (a_2y+1)^qa_1x, \\ \dot{y} & = & w_0(a_2y+1)^qx,\ a_2 \neq 0, \end{array}\right. \tag{1.5.24}$$

with $w_0^2 > 0,\ p \leq q$ and the linear part $(-w_0y,\ w_0x)^T$.
Now, we compute the first coefficients of the Poincaré-Dulac normal form of the system (1.5.24), for any pair p, q with $1 \leq p \leq q$.
We have

$$Re(\gamma_1) = -\frac{1}{16}a_1\left(q(q-1)a_2^2 - p(p-1)(w_0^2 + a_1^2)\right),$$

it is zero if and only if either $a_1 = 0$, (in this case, the system (1.5.21), by means of the change $x \to x-1,\ y \to 1-y$, is transformed into a system of the family (1.5.15), which is centre), or if $q(q-1)a_2^2 = p(p-1)(w_0^2 + a_1^2)$ with $a_1 \neq 0$; in this case, if $p = q = 1$, the result is known, see Bazykin

[29]; if $p=1,\ q>1$, the first radial coefficient of the normal form is non zero; and for $p>1$ and $q>1$, it has

$$Re(\gamma_2)=\frac{1}{576}(p-q)(a_1^2+w_0^2)a_1a_2\left((10pq-9q-9p+9)a_2+8(p^2-p)a_1\right).$$

which is zero if $p=q$ or $(10pq-9q-9p+9)a_2+8(p^2-p)a_1=0$. In the first case, we have the system (1.5.22); in this system, the point $(1,1)$ is a centre since, the change $x\to 1/2(x+y),\ y\to 1/2(y-x)$, transforms it in a reversible system. In the other case, from the vanishing of the first two radial coefficients, we obtain a_1 and a_2 and replacing them on $Re(\gamma_3)$, we have a expression of $Re(\gamma_3)$ with denominator different from zero and whose numerator comes given by $(p-1)^4(q-1)(p-q)qp^4w_0^6a_1\Phi(p,q)$, with $\Phi(p,q)$ a polynomial in p,q of degree eight in p and q, which is different from zero for any p,q. Therefore, the third radial coefficient is different to zero, that is, the system (1.5.24) does not have a centre at the origin.
Last on, from the proof, it follows that the maximal order of a fine focus of the family (1.5.20) is three. In those cases, i.e. $p\neq q,\ a_1\neq 0$ and $q(q-1)a_2^2=p(p-1)(w_0^2+a_1^2)$, it is easy to see that we can introduce a sequence of perturbations, each of which reduces the order of the fine focus by one and reverses the stability of the critical point. At each reversal of stability, a limit cycle bifurcates; therefore, three small amplitude limit cycles can bifurcate when the origin is of maximal order.
■
We now characterize the isochronous centres of the family (1.5.20).

Theorem 1.5.29. *The polynomial system (1.5.20) has a critical point which is an isochronous centre if and only if, by means of a change of variables, the system becomes*

$$\begin{cases}\dot{x} &= ax^3(y-1),\\ \dot{y} &= by^3(x-1).\end{cases}\tag{1.5.25}$$

Proof.

From Theorem 1.5.28, the centres of the family (1.5.20) are (1.5.21) and (1.5.22). In the first case, following the proof of the above theorem, these can transform in the systems (1.5.24) with $a_1 = 0$, $p \leq q$, whose first constant of period is $Im(\gamma_1) = \frac{1}{48} w_0 \left(q(q-3)a_2^2 + p(p-3)w_0^2 \right)$. Therefore, for that this constant can be zero, is necessary that $p \leq 3, q \geq 3$. If $p = q = 3$, we have the system (1.5.25), that is the isochronous centre given by Theorem 1.5.27, for $c = 0$. And if $p < 3 < q$ from the vanishing of $Im(\gamma_1)$ it follows that $q(q-3)a_2^2 = -p(p-3)w_0^2$. Eliminating a_2^2 and substituting on $Im(\gamma_2)$, we obtain

$$Im(\gamma_2) = \frac{1}{192} \frac{(p-3)p}{(q-3)q} \left((q-3)q(p(p-3)+1) + p(p-3) \right) w_0^5,$$

which is different from zero.

In the second case, the system (1.5.22), by means of a change of variables, becomes (1.5.24) with $q(q-1)a_2^2 = p(p-1)(w_0^2 + a_1^2)$, $a_1 \neq 0$ and $p = q$. It has $Im(\gamma_1) = \frac{1}{24w_0}(-a_1 + a_2)(-3a_2 + a_2 q + 3qa_1 - 3a_1)qa_2^2$, that is, $a_1 = \frac{(q-3)a_2}{3(q-1)}$ and we have

$$Im(\gamma_2) = -\frac{1}{4374 w_0^3 (q-1)^4}(2q-1)(-3+2q)^3 a_2^8 (q-3) q^3,$$

which is zero only if $q = 3$, in whose case $a_1 = 0$. ∎

1.5.6 Several families with a commutator with null linear part.

As application of Theorem 1.4.18, we show several families with a commutator with null linear part.

Example 1.5.30. *We consider the system*

$$\begin{aligned} \dot{x} &= -y + x(ax+by)(bx-ay)(1-(x^2+y^2)^2), \\ \dot{y} &= x + y(ax+by)(bx-ay)(1-(x^2+y^2)^2), \ a, b \in \mathbb{R}, \end{aligned} \tag{1.5.26}$$

which is included in the family $(\dot{x},\dot{y})^T = (-y,x)^T + (x,y)^T H(x,y)$. *The system (1.5.26) holds* $\alpha_3 = 0$ *and it commutes with* $\mathcal{W} = (x^2+y^2)(1-(x^2+y^2)^2)(x,y)^T$. *From Theorem 1.4.18, the origin of (1.5.26) is an isochronous centre.*

Let us note that this system has no polynomial commutator with non null linear part (see [10]). Therefore, the problem of finding or to be able to guarantee the existence of an analytic or $\mathcal{C}^\infty$-commutator with non null linear part, without applying Theorem 1.4.18, is a problem really difficult.

Example 1.5.31. *The system*

$$\dot{x} = -y - \frac{4}{9}x^4 - \frac{40}{9}x^2y^2 + \frac{4}{3}y^4, \qquad \dot{y} = x + \frac{20}{9}x^3y - \frac{28}{9}xy^3, \quad (1.5.27)$$

has been studied by Chavarriga et al. [46]. The authors proved that (1.5.27) commutes with

$$\mathcal{W} = (x^2+y^2)U(x,y)^{-\frac{1}{6}}(x(3-16y^3), y(3+12x^2y-4y^3))^T,$$

where $U(x,y) = 9 + 24x^2y + 16x^4y^2 - 24y^3 + 32x^2y^4 + 16x^6$. *As* $\alpha_3 = 0$, *from Theorem 1.4.18, the origin of (1.5.27) is an isochronous centre.*

Example 1.5.32. *We consider the quintic polynomial system*

$$\dot{x} = -y + y(-x^4 - 4x^2y^2 + y^4), \qquad \dot{y} = x + 2xy^2(x^2-y^2). \quad (1.5.28)$$

In [47], the authors prove that

$$\mathcal{W} = (x^2+y^2)(x(1-x^2y^2-5y^4), y(1+3x^2y^2-y^4))^T$$

commutes with (1.5.28). As $\alpha_3 = 0$, *from Theorem 1.4.18 it follows that the origin of (1.5.28) is an isochronous centre.*

Theorem 1.4.19 provides a effective method for obtaining the isochronous focus of a family of systems.

Example 1.5.33. *Let us consider the cubic Lienard system*

$$\dot{x} = -y + a_2 x^2 + a_2 b_2 x^3, \qquad \dot{y} = x + b_2 x^2 + \frac{2}{9}(a_2^2 + 5b_2^2)x^3, \quad (1.5.29)$$

with a_2 *y* b_2 *different from zero. The origin of (1.5.29) is a weak focus of order one since* $c = 0$ *and* $\alpha_3 = a_2 b_2 \neq 0$*. The field commutes up to order 5 with* $(R, S)^T$ *where*

$$\begin{aligned} R = {} & (x^2+y^2)x + \tfrac{1}{3}b_2x^4 - \tfrac{4}{3}a_2x^3y - b_2x^2y^2 - \tfrac{2}{3}a_2xy^3 - \tfrac{2}{3}b_2y^4 \\ & + \tfrac{1}{9}(b_2^2 - a_2^2)x^5 - \tfrac{11}{18}a_2b_2x^4y + \tfrac{2}{9}(a_2^2 - 3b_2^2)x^3y^2 + \tfrac{2}{3}a_2b_2x^2y^3, \\ S = {} & (x^2+y^2)y + \tfrac{1}{3}a_2x^4 + \tfrac{4}{3}b_2x^3y - a_2x^2y^2 + \tfrac{2}{3}b_2xy^3 - \tfrac{2}{3}a_2y^4 \\ & + \tfrac{23}{18}a_2b_2x^5 + \tfrac{5}{9}(3b_2^2 - a_2^2)x^4y - \tfrac{4}{9}a_2b_2x^3y^2 + \tfrac{2}{3}b_2^2x^2y^3. \end{aligned}$$

From Theorem 1.4.19, the origin of (1.5.29) is a weak isochronous focus of order one.

Example 1.5.34. *Let us consider the system*

$$\dot{x} = -y + 2xy - 2y^3 + (x+y^2)H(x,y), \qquad \dot{y} = x - y^2 + yH(x,y), \quad (1.5.30)$$

with $H(x,y) = x^2 + y^2 - 2xy^2 + y^4$*, (example 5, [87]). The field commutes with* $H(x,y)(x+y^2, y)^T$*, it which has the expression*

$$(x^2+y^2)(x,y)^T + (y^2(y^2 - x^2 - y^2x + y^4), (y^2 - 2x)y^3)^T.$$

From Theorem 1.4.22, the origin of (1.5.30) is an isochronous point.

1.6 Procedure for computing the coefficients of the normal form of Poincaré-Dulac using Carleman linearization.

The system (1.1.1) with $\lambda = 0$, in complex coordinates, $z = x + iy$, has the form

$$\dot{z} \;=\; iz + \textstyle\sum_{m,n\geq 0} a_{mn} z^m \bar{z}^n, \qquad (1.6.31)$$

with $a_{mn} \in \mathbb{C}$, $\bar{a}_{mn} = a_{nm}, 2 \leq m+n$, and its Poincaré-Dulac normal form has the expression

$$\dot{Z} \;=\; iZ + \textstyle\sum_{l\geq 1} \gamma_l Z^{l+1}\bar{Z}^l, \tag{1.6.32}$$

with $\gamma_l = \alpha_{2l+1} + i\beta_{2l+1} \in \mathbb{C}$. The expression of the first coefficients of the monomials $Z^{l+1}\bar{Z}^l$, so-called resonant monomials, can be seen in Hassard and Wan [101], Algaba et al. [2], and Gasull et al. [78, 79].

In this section, we develop a method based on the relation of equivalence between the matrices of Carleman of the systems (1.6.31) and (1.6.32), ver Carleman [37], which allows us to calculate any coefficient γ_l. This method has some advantages respect to the ones mentioned above. All the coefficients are obtained by solving an unique system of linear equations whose matrix is triangular, thus, the computation is reduced considerably; in the general case, we can deduce some algebraic properties of the constants. Moreover, for the isochronicity problem, we obtain the value of each one without knowing the previous ones. We give a computationally efficient algorithm that allows us to obtain necessary conditions so that the system has an isochronous centre.

Next, we recall the Carleman linearization procedure for the system (1.6.31), see [151]. Let us consider the system (1.6.31) truncated to order $l \geq 2$,

$$\dot{z} \;=\; iz + \sum_{\substack{m,n\geq 0\\ 2\leq m+n\leq l}} a_{mn} z^m \bar{z}^n. \tag{1.6.33}$$

The Carleman linearization procedure consists in studying properties of (1.6.33) through the analysis of the linear system of dimension $\frac{l(l+3)}{2}$ which we obtain introducing the powers

$$z,\ \bar{z},\ z^2,\ z\bar{z},\ \bar{z}^2, \cdots, z\bar{z}^{l-1},\ \bar{z}^l,$$

as variables of the new system. That is, if we denote

$$z^{[h]} = (z^h,\ z^{h-1}\bar{z},\ z^{h-2}\bar{z}^2, \cdots,\ z\bar{z}^{h-1},\ \bar{z}^h)^t, \quad 1 \leq h \leq l,$$

we have a linear system of the form

$$\begin{pmatrix} \dot{z}^{[1]} \\ \dot{z}^{[2]} \\ \vdots \\ \dot{z}^{[l]} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1\,l} \\ 0 & A_{22} & \cdots & A_{2\,l} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{l\,l} \end{pmatrix} \begin{pmatrix} z^{[1]} \\ z^{[2]} \\ \vdots \\ z^{[l]} \end{pmatrix}.$$

The matrix of the coefficients is called *Carleman matrix of order l of the system (1.6.33)*, and denoted by $C_l = ((A_{mn}))$. It is a square matrix of order $\frac{l(l+3)}{2}$, upper triangular, whose blocks A_{mn} have $m+1$ rows and $n+1$ columns and are the complex matrices of the coefficients of the linear system

$$\dot{z}^{[m]} = \sum_{n=m}^{l} A_{mn} z^{[n]}$$

which is obtained by expressing the derivative of the monomials of degree m with respect to the monomials of degree greater or equal to m.

The elements of the matrices A_{mn} can be determined as follows.

Lemma 1.6.35. *The expression of the derivative of $z^{m_1}\bar{z}^{n_1}$ with $1 \leq m_1 + n_1 \leq l$, in the system (1.6.33) is*

$$\frac{d}{dt}(z^{m_1}\bar{z}^{n_1}) = \sum_{1 \leq m_2+n_2 \leq l} a_{m_1,n_1}^{m_2,n_2} z^{m_2}\bar{z}^{n_2},$$

with

1. *if* $m_1 + n_1 > m_2 + n_2$, $a_{m_1,n_1}^{m_2,n_2} = 0$,

2. *if* $m_1 + n_1 = m_2 + n_2$,

$$a_{m_1,n_1}^{m_2,n_2} = \begin{cases} 0, & \text{if } m_1 \neq m_2, \\ (m_1 - n_1)i, & \text{if } m_1 = m_2, \end{cases} \tag{1.6.34}$$

3. *if* $m_1 + n_1 < m_2 + n_2$, $a_{m_1,n_1}^{m_2,n_2} = 0$, *if* $n_1 > n_2 + 1$ *or* $m_1 > m_2 + 1$,

and in another case

$$a^{m_2,n_2}_{m_1,n_1} = \begin{cases} n_1 \bar{a}_{0,r+1}, & \textit{if } n_1 = n_2 + 1, \\ m_1 a_{0,r+1}, & \textit{if } m_1 = m_2 + 1, \\ m_1 a_{m_2-m_1+1,n_2-n_1} & \\ +n_1 \bar{a}_{n_2-n_1+1,m_2-m_1}, & \textit{in the remaining cases,} \end{cases} \tag{1.6.35}$$

where $r = (m_2 + n_2) - (m_1 + n_1)$.

In general, the matrices C_h with $1 < h \leq l$, have the following structure per blocks:

$$C_h = \left(\begin{array}{c|c} & A_{1,h} \\ C_{h-1} & \vdots \\ & A_{h-1,h} \\ \hline 0 & A_{h,h} \end{array} \right), \tag{1.6.36}$$

being C_{h-1} the Carleman matrix of order $h-1$. By (1.6.34), the matrices $A_{m,m}$, $1 \leq m \leq l$, are diagonal matrices whose element of the diagonal of the row j comes given by $(m - 2j + 2)i$, $1 \leq j \leq m$ (that is, the eigenvalues of the Carleman matrix).
By (1.6.35), the elements of the matrices $A_{m,n}$ with $m < n$, are linear combinations of the elements of the matrix $A_{1,n-m+1}$, that is, depend exclusively on the coefficients $a_{i,j}$, $\bar{a}_{ij}$ with $i + j = n - m + 1$, $i, j \geq 0$.
We now develop a method for computing the coefficients of the normal form of Poincaré-Dulac of the system (1.6.31). The method is based on the similarity relationship between Carleman matrices of order $2l + 1$ of the system (1.6.31), $C_{2l+1} \in \mathbb{C}^{c \times c}$, with $c := (2l + 1)(l + 2)$, and of the system (1.6.32) which we denote by $J_{2l+1} = ((B_{m,n}))$, where the $B_{m,n}$ are $(m + 1) \times (n + 1)$ complex matrices defined as:

$$B_{m,m} = A_{m,m}, \ m \geq 1,$$

$$B_{m,m+2n} = \left(\begin{array}{c|c|c} 0_{(m+1)\times n} & D_{m+1,m} & 0_{(m+1)\times(n+1)} \end{array} \right)$$

with $m, n \geq 1$, being $D_{m+1,m}$ the $(m+1)\times m$ complex matrix given by

$$D_{m+1,m} = \begin{pmatrix} m\gamma_n & 0 & 0 & \cdots & 0 \\ \bar{\gamma}_n & (m-1)\gamma_n & 0 & \cdots & 0 \\ 0 & 2\bar{\gamma}_n & (m-2)\gamma_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & (m-1)\bar{\gamma}_n & \gamma_n \\ 0 & 0 & 0 & 0 & m\bar{\gamma}_n \end{pmatrix}$$

and the remaining are null matrices.
From Poincaré-Dulac normal form Theorem, exists a nonsingular matrix of order c,

$$P = \begin{pmatrix} \vdots & \vdots & & \vdots \\ p_1 & p_2 & \cdots & p_c \\ \vdots & \vdots & & \vdots \end{pmatrix},$$

such that $P^{-1}C_{2l+1}P = J_{2l+1}$. Furthermore, without loss of generality, we can choose P unit upper triangular.
From now on, we will denote by $p_{j,h}$ to the j-th component of the vector p_h and $i_r(j) := 2(j+1)^2 - 1,\ j \geq 1$.
By definition, it has that

$$(C_{2l+1} - iI_c)P = P(J_{2l+1} - iI_c), \tag{1.6.37}$$

and if we denote $w_h := p_{i_r(h)}$, with $0 \leq h \leq l$, it has easily the following result.

Theorem 1.6.36. *The coefficient γ_l with $l \geq 1$, of the system (1.6.32), comes determined by the $l+1$ systems of linear equations*

$$(C_{2l+1} - iI_c)w_0 = 0, \tag{1.6.38}$$

$$(C_{2l+1} - iI_c)w_j = \sum_{h=0}^{j-1}((h+1)\gamma_{j-h} + h\bar{\gamma}_{j-h})w_h,\ j \leq l. \tag{1.6.39}$$

where C_{2l+1} *is Carleman matrix of order* $2l+1$ *and* w_j *with* $0 \le j \le l$, *vectors of* c *components, such that* $w_{i_r(l),l} = 1$ *and* $w_{j,l} = 0$, *for* $j > i_r(l)$.

Note that i appears in the diagonal of C_{2l+1} in the positions that correspond to the equations of the derivatives of the resonant monomials, i.e. $z,\ z^2\bar{z},\ z^3\bar{z}^2,\ \ldots,\ z^{l+1}\bar{z}^l$ (the first, seventh, ..., $i_r(l)-$th position) respectively, that is, the systems (1.6.39) are upper triangular with zeros in the diagonal that correspond to such positions, therefore, they are systems with parameters which must verify certain compatibility conditions.

Now, let us see how to compute, using Theorem 1.6.36, the first two coefficients of the resonant monomials of the Poincaré-Dulac normal form of (1.6.31). The coefficient γ_1 comes determined by

$$(C_3 - iI_9)w_1 = \gamma_1 w_0, \tag{1.6.40}$$

where w_0 and w_1 are vectors of nine components with $w_0 = [1, 0, , \ldots, 0]^T$ and $w_1 = [w_{1,1}, \ldots, w_{6,1}, 1, 0, 0]^T$.

The matrix $C_3 - iI_9$ is

$$\left(\begin{array}{cc|ccc|cccc}
0 & 0 & a_{20} & a_{11} & a_{02} & a_{30} & a_{21} & a_{12} & a_{03} \\
0 & -2i & \bar{a}_{02} & \bar{a}_{11} & \bar{a}_{20} & \bar{a}_{03} & \bar{a}_{12} & \bar{a}_{21} & \bar{a}_{30} \\
\hline
 & & i & 0 & 0 & 2a_{20} & 2a_{11} & 2a_{02} & 0 \\
 & 0 & 0 & -i & 0 & \bar{a}_{02} & \bar{a}_{11} + a_{20} & a_{11} + \bar{a}_{20} & a_{02} \\
 & & 0 & 0 & -3i & 0 & 2\bar{a}_{02} & 2\bar{a}_{11} & 2\bar{a}_{20} \\
\hline
 & & & & & 2i & 0 & 0 & 0 \\
 & 0 & & 0 & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & -2i & 0 \\
 & & & & & 0 & 0 & 0 & -4i
\end{array}\right).$$

Given the structure of the equation (1.6.40), and concretely of the matrix $C_3 - iI_9$, in order to determine γ_1 we only need to calculate the value of $w_{5,1}$, $w_{4,1}$ and $w_{3,1}$ given by the diagonal system,

$$(A_{2,2} - iI_3)\begin{pmatrix} w_{3,1} \\ w_{4,1} \\ w_{5,1} \end{pmatrix} = \begin{pmatrix} 2a_{11} \\ \bar{a}_{11} + a_{20} \\ 2\bar{a}_{02} \end{pmatrix}$$

and substituting these values in the first equation of (1.6.40) we have

$$\gamma_1 = a_{21} + i(a_{11}a_{20} - a_{11}\bar{a}_{11} - \frac{2}{3}a_{02}\bar{a}_{02}).$$

Tsiligiannis and Lyberatos obtained this result, see [156].
In order to determine the coefficient γ_2, is enough to solve the linear system

$$(C_5 - iI_{20})w_2 = \gamma_2 w_0 + (2\gamma_1 + \bar{\gamma}_1)w_1,$$

where w_0, w_1 and w_2 are vectors of twenty components of the form

$$\begin{aligned} w_0 &= [1, 0, ..., 0]^T, \\ w_1 &= [w_{1,1}, ..., w_{6,1}, 1, 0, ..., 0]^T, \\ w_2 &= [w_{1,2}, ..., w_{16,2}, 1, 0, ..., 0]^T. \end{aligned}$$

This system has a very similar structure to (1.6.40). Now, the components corresponding to the block $A_{5,5}$ but for the 17 ($w_{17,2} = 1$) are zero. The coordinates $w_{10,2}, ..., w_{14,2}$ come given by the diagonal system

$$(A_{4,4} - iI_5)\begin{pmatrix} w_{10,2} \\ w_{11,2} \\ w_{12,2} \\ w_{13,2} \\ w_{14,2} \end{pmatrix} = \begin{pmatrix} 4a_{02} \\ 3a_{11} + \bar{a}_{20} \\ 2a_{20} + 2\bar{a}_{11} \\ 3\bar{a}_{02} \\ 0 \end{pmatrix}.$$

Recursively, we calculate, from down upwards, the components of the vector w_2, until we reach at the seventh equation, whose coefficient of the diagonal is zero. It has, in this case, the equality $2\gamma_2 + \bar{\gamma}_2 = 2\gamma_2 + \bar{\gamma}_2$. Continuing with the procedure, we now have an arbitrary parameter, $w_{7,2}$. Calculating the value of $w_{6,2}$, $w_{5,2}$, $w_{4,2}$ and $w_{3,2}$ and replacing it in the first equation. we have the value of γ_2:

$$
\begin{aligned}
& a_{32}-\tfrac{1}{12}(24a_{11}^2a_{30}+24a_{12}\bar{a}_{11}^2-24a_{11}^2\bar{a}_{12}+8a_{20}a_{02}a_{30}-8a_{20}a_{02}\bar{a}_{12}\\
& +96a_{20}a_{21}a_{11}+24a_{20}a_{21}\bar{a}_{20}-12a_{20}a_{12}\bar{a}_{11}-6a_{20}\bar{a}_{02}a_{03}+27a_{03}\bar{a}_{02}\bar{a}_{11}\\
& -32a_{02}\bar{a}_{11}a_{30}+24a_{02}\bar{a}_{11}\bar{a}_{12}-2a_{02}\bar{a}_{20}\bar{a}_{03}+8a_{02}\bar{a}_{02}\bar{a}_{21}-31a_{11}a_{02}\bar{a}_{03}\\
& -36a_{11}\bar{a}_{02}a_{12}+12a_{11}\bar{a}_{02}\bar{a}_{30}+24a_{11}a_{20}\bar{a}_{21}+24a_{11}\bar{a}_{11}a_{21}\\
& +24a_{11}\bar{a}_{11}\bar{a}_{21}+12a_{11}a_{30}\bar{a}_{20})\\
& +\tfrac{1}{12}i(16a_{02}\bar{a}_{11}^3-96a_{20}^2a_{11}^2+24a_{11}^2\bar{a}_{11}^2+24a_{11}^3\bar{a}_{02}-24a_{20}a_{02}\bar{a}_{11}^2\\
& -12a_{12}\bar{a}_{12}+24a_{11}\bar{a}_{11}^2\bar{a}_{20}+12a_{31}\bar{a}_{20}-24a_{22}\bar{a}_{11}-9a_{03}\bar{a}_{03}\\
& +12a_{30}a_{12}-12a_{11}\bar{a}_{22}+12a_{02}\bar{a}_{20}\bar{a}_{02}a_{20}-24a_{20}^2a_{11}\bar{a}_{20}\\
& +66a_{20}a_{02}\bar{a}_{02}a_{11}+18a_{02}\bar{a}_{20}\bar{a}_{02}\bar{a}_{11}-29a_{11}\bar{a}_{02}a_{02}\bar{a}_{11}+36a_{11}a_{20}\bar{a}_{11}\bar{a}_{20}\\
& -12a_{11}^2\bar{a}_{02}\bar{a}_{20}-12\bar{a}_{02}a_{13}+16a_{02}a_{40}+24a_{11}a_{31}-8a_{02}\bar{a}_{13}\\
& +8a_{02}\bar{a}_{11}a_{20}^2+36a_{11}^2a_{20}\bar{a}_{11})\\
& +\tfrac{1}{9}(9a_{11}\bar{a}_{11}+2a_{02}\bar{a}_{02}+27a_{11}a_{20})(2\gamma_1+\bar{\gamma}_1)+\gamma_1 w_{7,2}.
\end{aligned}
$$

Remark. Let us note that if γ_1 is different from zero, then we can choose $w_{7,2}$ such that γ_2 is zero. Therefore, γ_2 is unessential if γ_1 is non-zero.

If we want to calculate the expression of γ_l, under the vanishing of the previous constants, the problem is reduced considerably, since is enough to solve only a system of linear equations. For this reason, we think that this method is more efficient, so theoretically as computationally, that the quoted ones in the introduction.

Corollary 1.6.37. *If $\gamma_j = 0,\ 1 \le j \le l-1$, with $l > 1$, the coefficient γ_l of the system (1.6.32) comes determined by the system of linear equations*

$$(C_{2l+1} - iI_c)w_l = \gamma_l w_0,$$

where C_{2l+1} is Carleman matrix of order $2l+1$, and w_0, w_l vectors of c *components such that*

$$\begin{aligned}
&w_{1,0} = 1, \quad w_{j,0} = 0, \ \textit{for}\ 2 \leq j \leq l,\\
&w_{i_r(l),l} = 1,\\
&w_{j,l} = 0, \ \textit{for}\ j > i_r(l), \ 0 \leq j \leq l.
\end{aligned}$$

Concretely, $\gamma_l = \sum_{i=1}^{c} (C_{2l+1} - iI_c)_{1,i} w_{i,l}$.

The following result shows that to obtain γ_l is not necessary to know $\gamma_j, \ 1 \leq j \leq l-1$, previously.

Theorem 1.6.38. *It holds that, for* $l \geq 2$, $\gamma_l = \gamma_l^* + \sum_{h=1}^{l-1} \gamma_h w_{i_r(h),l}$, *with* γ_l^* *unique and where* w_l *is given by Corollary 1.6.37.*

Proof.
For $l = 2$ by solving $(C_5 - iI_{20})w_2 = \gamma_2 w_0$, it has $\gamma_2 = \gamma_2^+ + \gamma_1 w_{7,2}$ with γ_2^* unique, see the expression of γ_2 in page 84.
Let us suppose that for $l-1$ the equality holds and we prove it for l.
The solution w_l of the form $w_l = [w_{1,l}, w_{2,l}, ..., w_{i_r(l),l}, 0, ..., 0]^T$, with $w_{i_r(l),l} = 1$ of the homogeneous system $(\tilde{C}_{2l+1} - iw\tilde{I}_c)w_l = 0$, with $\tilde{C}_{2l+1}$ and $\tilde{I}_c$ obtained by suppressing the first row of the matrices C_{2l+1} and I_c, respectively, depends on the arbitrary constants $w_{i_r(0),l}, ..., w_{i_r(l-1),l}$, i.e. we write w_l as $w_l = u_l + v_l$ with u_l containing the parameters $w_{i_r(h),l}, \ 0 \leq h \leq l-1$ and v_l solution of $(\tilde{C}_{2l+1} - i\tilde{I}_c)w_l = 0$, with $v_{i_r(h),l} = 0, h \leq l-1, \ v_{i_r(l),l} = 1$.
The vector u_l is solution of $(\tilde{C}_{2l+1} - i\tilde{I}_c)w_l = 0$, and it is of the form $u_l = [u_{1,l}, u_{2,l}, ..., u_{i_r(l-1),l}, 0, ..., 0]^T$. By (1.6.36) we have that u_l is solution of $(\tilde{C}_{2l-1} - iI_{(2l-1)(l+1)})w_{l-1} = 0$, that is,

$$\sum_{i=1}^{c} (C_{2l+1} - iI_c)_{1,i} u_{i,l}$$

is

$$\sum_{i=1}^{(2l-1)(l+1)} (C_{2l-1} - iI_{(2l-1)(l+1)})_{1,i} w_{i,l-1} = \sum_{h=1}^{l-1} \gamma_h w_{i_r(h),l},$$

since the property is certain for $l-1$.
Finally, as v_l is solution of $(\tilde{C}_{2l+1} - i\tilde{I}_c)v_l = 0$, with $v_{i_r(h),l} = 0, h \leq l-1$, $v_{i_r(l),l} = 1$, the expression $\sum_{i=1}^{c}(C_{2l+1} - iI_c)_{1,i}v_{i,l}$ is uniquely determined, thus it has that $\gamma_l = \gamma_l^* + \sum_{h=1}^{l-1}\gamma_h w_{i_r(h),l}$. ∎

Next, we present the *MapleV* code for computing the first constants of the resonant monomials of the system (1.6.32), under the vanishing of the previous constant, based on the algorithm of theorem 1.6.37.

```
#==Program that calculates the resonant coefficients of the Hopf

# ---- We must introduce the number of coefficients that are
needed , ncoef, and the coefficients of Taylor of the derivative
of z=x+I*y, in complex coordinates
\dot{z}=Iz+a20z^2+a11zw+a02w^2+...

#===== Program that calculates the first constants.

for nn from 1 to ncoef do
 nl:=2*nn+1:
 sol:=array(1..nl*(nl+3)/2):
 #-----
 i1j1:=array(1..2):
 i1:=nl+1:j1:=0:
 for k from nl*(nl+3)/2 by -1 to 3 do
  ecuacion:=0:
  if j1=0 then j1:=i1-1:i1:=0: else i1:=i1+1:j1:=j1-1 fi:
  i2:=i1:j2:=j1:
  coefcoci:=(i1-j1-1)*atv:

  #---- We calculate sol[k]
  if coefcoci=0 then
  # ---- We null all the parameters except the last one.
  if k=(2*nn+1)*(nn+2)-(nn+1) then sol[k]:=1: else sol[k]:=0: fi:
  else
   #---- coefcoci<>0
   for h from k+1 to nl*(nl+3)/2 do
    if i2=0 then i2:=j2+1: j2:=0 else i2:=i2-1:j2:=j2+1 fi:
```

```
     #--- we calculate coefficient associated to the row k(i1,j1)
     #--- and column h (i2,j2).
     if (i1+j1)=(i2+j2) then coeficiente:=0
     else
      #--- In this case, the terms that appear are of order  r+1
      r:=(i2+j2)-(i1+j1):
      if j2+1<j1 or j2+1>j1+r+2 then coeficiente:=0:
      else
       if j2+1=j1 then coeficiente:=j1*b.(r+1).`.`.0: fi;
       if j2+1=j1+r+2 then coeficiente:=i1*a.0.`.`.(r+1): fi;
       if j2+1>j1 and j2+1<j1+r+2 then
        coea:=a.(r+j1-j2+1).`.`.(j2-j1);coeb:=b.(r+j1-j2).`.`.(j2-j1+1):
        coeficiente:=i1*coea+j1*coeb:
       fi;
      fi;
     fi;
     ecuacion:=ecuacion+sol[h]*coeficiente:
    od;
    lprint(time(),`Generating sol[`,k,`]...`):
    sol[k]:=simplify(-ecuacion/coefcoci):
  fi:
 od:
 g.(nn):=0:
 ii2:=0:jj2:=1:
 for h1 from 3 to nl*(nl+3)/2 do
  if ii2=0 then ii2:=jj2+1: jj2:=0: else ii2:=ii2-1:jj2:=jj2+1: fi:
  g.(nn):=simplify(g.(nn)+sol[h1]*a.(ii2).`.`.(jj2)):
 od:
 lprint(time(),`It follows the value of.. g`.nn, `g`.nn);
od:
```

Chapter 2

Isochronous centres of systems with degeneracy at infinity with polynomial commutator.

2.1 Introduction

The problem of characterizing isochronous centres have attracted the attention of several authors. However, it is far from being completely solved, even for specific families of vector fields. Consequently, setting up weaker problems whose solutions enable us to get information about the general problem that is equivalent to the existence of an analytic commutator of such vector field, ver Theorem 1.4.18 in section 1.4. In particular, for $j = 0$ in Theorem 1.4.18 we have that for any analytic system with linear part $(-y, x)^T$, the existence of an analytic commutator with linear part $(x, y)^T$ is a necessary and sufficient condition for the origin to be an isochronous centre (see Algaba *et al.* [2] also). In this context, we are interested in to find the vector fields of a specific family that have a polynomial commutator with or without its linear part null.

There are only a few families of polynomial differential systems in which a complete classification of the isochronous centres is known, and almost all of them have polynomial commutator. From a chronological point of view, we should begin mentioning the quadratic isochronous centres, characterized by Loud [119]. In Pleshkan [133], cubic isochronous centres with homogeneous nonlinear part are settled. In Christopher, Devlin [59], the isochronous centres of the Kukles family are obtained. All these centres are time-reversible and moreover, all of them have polynomial commutator. Commutators of quadratic centres are computed in Sabatini [141] (they have degrees three, four and five); commutators of cubic systems with homogeneous nonlinear part can be found in Gasull *et al.* [77] (they have degrees two, three and five); commutators for the Kukles system can be seen in Volokitin, Ivanov [160] (they have degree four). Mardesic *et al.* [122] find a family of isochronous cubic systems which contain Kolmogorov's isochronous cubic systems. The first example of a polynomial isochronous centre without polynomial commutator is found in Devlin [65]. It is a quartic system, with homogeneous nonlinear part, where an isochronous centre at the origin and others two non isochronous centres coexist.

Because of their relevance in Mechanic, we mention the so called Newton equations. These equations are given by

$$\ddot{x} + \sum_{i=0}^{n} q_i(x)\dot{x}^i = 0,$$

where the q_i are polynomials of arbitrary degree with $q_0(0) = 0$, $q_0^{'}(0) = 1$ and, if $n \geq 1$, $q_1(0) = 0$. For $n = 0$ (potential equations), Urabe [158] proved that the origin is an isochronous centre if and only if $q_0(x) = x$. For $n = 1$ (Liénard equations) all isochronous centres are conjectured to be known (see Algaba *et al.* [2], Christopher, Lloyd [62] and Sabatini [144]), without distinguishing between those with and without polynomial commutator. If $n = 2$ or $n \geq 4$, Volokitin and Ivanov ([160]) prove that an isochronous centre can not have a polynomial commutator. For

$n = 3$ (Abel's equations), the same work characterizes the polynomial centres which have a polynomial commutator.

We study the existence of polynomial commutators for the plane systems with degeneracy at infinity and a centre-focus equilibrium at the origin. Such systems, up to a linear change of variable, are given by

$$\left\{\begin{array}{rcccl} \dot{x} & = & P(x,y) & = & -y + P_2(x,y) + P_3(x,y) + \cdots + P_n(x,y), \\ \dot{y} & = & Q(x,y) & = & x + Q_2(x,y) + Q_3(x,y) + \cdots + Q_n(x,y), \end{array}\right. \tag{2.1.1}$$

with P_i, Q_i homogeneous polynomials and $xQ_n(x,y) = yP_n(x,y)$. The name of these systems comes from this last condition, since it implies that all points at infinity are critical.

In the cubic case, all centres have been characterized, and some systems with an isochronous centre different from those given by Collins are found. All of them are reversible and a few have polynomial commutator. We should mention, about these systems, the papers of Chavarriga *et al.* [43, 45] and Lloyd *et al.* [115].

Next, we detail the structure of this chapter and the results obtained. It is divided into five sections. In section 2.2, we study properties, basically in relation to the polynomial cofactors of an invariant curve, polynomial inverse integrating factors and polynomial commutators, if they exist, of the plane systems (2.1.1) with degeneracy at infinity and a centre-focus equilibrium at the origin. We end the section with Theorem 2.2.45, that gives conditions which the homogeneous parts of a polynomial commutator must satisfy. In section 2.3, we characterize the systems

$$(-y + P_s + \sum_{j=k}^{n-1} xH_j, x + Q_s + \sum_{j=k}^{n-1} yH_j)^T$$

with polynomial commutator, being $P_j,\ Q_j, H_j, K_j$ homogeneous polynomials. In section 2.4, we characterize the uniformly isochronous centres with polynomial commutator and show that the commutator of the

centres of the analytic systems whose angular speed is constant can be chosen of radial form.
We finish this chapter by giving the geometry of the orbits of the quartic and quintic rigid systems with a polynomial commutator.

2.2 Several properties of the systems with polynomial commutators.

We first recall briefly the notions of invariant algebraic curves, inverse integrating factor and commutator of a vector field. Let us start with a polynomial system (2.1.1).

Definition 2.2.39. *An invariant algebraic curve of system (2.1.1) is an algebraic polynomial curve $f(x, y) = 0$ satisfying $P\partial_x f + Q\partial_y f = Kf$, where $\partial_x f$ and $\partial_y f$ are the partial derivatives of f respect to x and y. The polynomial K is called the cofactor associated to f. Notice that the cofactor's degree is less than or equal to $n - 1$.*

Definition 2.2.40. *An inverse integrating factor of system (2.1.1) is any function $V(x, y)$ satisfying $P\partial_x V + Q\partial_y V = (\partial_x P + \partial_y Q)V$.*

This name comes from the fact that function $1/V$ is an integrating factor of the system; that is, $\partial_x(P/V) + \partial_y(Q/V) = 0$. Notice that an algebraic inverse integrating factor is an invariant algebraic curve whose cofactor is given by the divergence of the system.

Lemma 2.2.41. *Let $K = \sum_{j=1}^{n-1} K_j$ be a cofactor of an invariant algebraic curve of degree m of system (2.1.1), with $P_n = xH_{n-1}$ and $Q_n = yH_{n-1}$. Then $K_{n-1} = mH_{n-1}$.*

Proof.
We consider $f = \sum_{j=0}^{m} f_j$, an invariant algebraic curve of the system

(2.1.1), and its associated cofactor $K = \sum_{j=0}^{n-1} K_j$. By definition, we have

$$(-y + P_2 + P_3 + \cdots + P_n)\partial_x f + (x + Q_2 + Q_3 + \cdots + Q_n)\partial_y f = Kf.$$

This equality is a polynomial equation of degree $m + n - 1$ over the variables x and y. From the higher degree terms, we deduce that

$$H_{n-1}(x\partial_x f_m + y\partial_y f_m) = K_{n-1} f_m.$$

By Euler's Theorem for homogeneous functions we have that $mH_{n-1}f_m = K_{n-1}f_m$, what leads us to $K_{n-1} = mH_{n-1}$. ■

Lemma 2.2.42. *Any polynomial inverse integrating factor of the system (2.1.1), with $P_n = xH_{n-1}$ and $Q_n = yH_{n-1}$ has degree $n + 1$.*

Proof.
Let f be a polynomial inverse integrating factor of degree m. By definition, f is an algebraic invariant curve of (2.1.1) whose associated cofactor is the divergence of the system. From Lemma 2.2.41, the higher order terms satisfy

$$mH_{n-1} = \partial_x(xH_{n-1}) + \partial_y(yH_{n-1}),$$

We conclude from Euler's Theorem that $mH_{n-1} = (n+1)H_{n-1}$, and this clearly forces $m = n + 1$. ■

Lemma 2.2.43. *Any polynomial commutator of the system (2.1.1) with $P_n = xH_{n-1}$ and $Q_n = yH_{n-1}$ has the same degree that the system.*

Proof.
Let $(U, V)^T = (\sum_{j=1}^{l} U_j, \sum_{j=1}^{l} V_j)^T$ be a polynomial commutator of (2.1.1). The function $V^* = PV - QU$ is an inverse integrating factor of system (2.1.1), of degree $n + l$ at most. But from Lemma 2.2.42 we know it has degree $n + 1$; therefore, terms of degree higher than $n + 1$ are zero. In

particular, the term of degree $n+l$, $xH_{n-1}V_l - yH_{n-1}U_l$, is zero, what implies that $xV_l = yU_l$.
On the other hand, terms of higher degree in the Lie bracket between the system and the commutator $\left[(xH_{n-1},\ yH_{n-1})^T, (U_l,\ V_l)^T\right]$ are null:

$$\begin{pmatrix} (H_{n-1} + x\partial_x H_{n-1})U_l + x\partial_y H_{n-1} V_l - xH_{n-1}\partial_x U_l - yH_{n-1}\partial_y U_l \\ y\partial_x H_{n-1} U_l + (H_{n-1} + y\partial_y H_{n-1})V_l - xH_{n-1}\partial_x V_l - yH_{n-1}\partial_y V_l \end{pmatrix}.$$

As $xV_l = yU_l$ we have

$$\left[(xH_{n-1},\ yH_{n-1})^T, (U_l,\ V_l)^T\right] = (n-l)H_{n-1}\,(U_l,\ V_l)^T = (0,\ 0)^T.$$

From this, we deduce that $U_l = V_l = 0$, and finally that $U_j = V_j = 0$, for $j = n+1, \ldots, l$. ∎

Lemma 2.2.44. *If $(U,V)^T = (\sum_{j=1}^n U_j, \sum_{j=1}^n V_j)^T$ is a polynomial commutator of the system (2.1.1) with $P_j = xH_{j-1}$, $Q_j = yH_{j-1}$, $j = k, \ldots, n$, then $xV_j = yU_j$, $j = k, \ldots, n$.*

Proof.
From Lemma 2.2.42, the inverse integrating factor $V^* = \sum_{j=1}^{2n} V_j^* = PV - QU$ has degree $n+1$, what implies that terms of degrees $n+k, n+k+1, \ldots, 2n$ are null. As the term of degree $2n$ of the inverse integrating factor is $V_{2n}^* = xH_{n-1}V_n - yH_{n-1}U_n$, we have $xV_n = yU_n$. The one of degree $2n-1$ is given by

$$V_{2n-1}^* = H_{n-1}(xV_{n-1} - yU_{n-1}) + H_{n-2}(xV_n - yU_n) = 0,$$

and since $xV_n = yU_n$ we have $xV_{n-1} = yU_{n-1}$. In general, V_{n+i}^* with $k \le i \le n$ is null and is given by

$$V_{n+i}^* = H_{n-1}(xV_i - yU_i) + H_{n-2}(xV_{i+1} - yU_{i+1}) + \cdots + H_{i-1}(xV_n - yU_n).$$

Consequently it follows that $xV_j = yU_j$, $j = k, \ldots, n$. ∎

Theorem 2.2.45. *If* $(U,V)^T = (\sum_{j=1}^{n} U_j, \sum_{j=1}^{n} V_j)^T$ *is a polynomial commutator of system (2.1.1) with* $P_j = xH_{j-1}$, $Q_j = yH_{j-1}$, $U_j = xK_{j-1}$, $V_j = yK_{j-1}$, $j = k, \ldots, n$, *then:*

1. $H_pK_q = H_qK_p$ *for every pair* p, q *with* $k-1 \le p \le n-1$, $k-1 \le q \le n-1$.

2. $H_{n-1}(xV_j - yU_j) = K_{n-1}(xQ_j - yP_j)$, *for every* j *with* $2 \le j \le k-1$.

3. *Let* $s = max\{j,\ yP_j - xQ_j \ne 0\} < k$. *If* $P_j = Q_j = 0$, $j = s+1, \ldots, k-1$, *then* $U_j = V_j = 0$, $j = s+1, \ldots, k-1$.

Proof.
We prove the first part. If we compute and develop the terms of degree greater or equal than $n+k-1$ of the Lie bracket between the field and the polynomial commutator we easily get the following relations:

$$\sum_{j=k'}^{n} (n+k'-2j)H_{n+k'-1-j}K_{j-1} = 0, \quad \text{with } k \le k' \le n-1.$$

From the higher degree term ($k' = n-1$), we have $H_{n-1}K_{n-2} = H_{n-2}K_{n-1}$; for $k' = n-2$ we deduce that $H_{n-1}K_{n-3} = H_{n-3}K_{n-1}$; for $k' = n-3$ we have the expression $3H_{n-1}K_{n-4} + H_{n-2}K_{n-3} - H_{n-3}K_{n-2} - 3H_{n-4}K_{n-1} = 0$, which can be simplified, using the previous expressions, to $H_{n-1}K_{n-4} = H_{n-4}K_{n-1}$. Following in this fashion, the first part follows.
In order to prove the second part, we use the fact that terms of orders $n+2, \ldots, n+k-1$ of the inverse integrating factor V^* are zero.
As $V^*_{n+k-1} = H_{n-1}(xV_{k-1} - yU_{k-1}) + K_{n-1}(yP_{k-1} - xQ_{k-1})$, we have $H_{n-1}(xV_{k-1} - yU_{k-1}) = K_{n-1}(xQ_{k-1} - yP_{k-1})$.
As V^*_{n+k-2} is zero, it follows that

$$H_{n-1}(xV_{k-2} - yU_{k-2}) + H_{n-2}(xV_{k-1} - yU_{k-1}) \\ + K_{n-2}(yP_{k-1} - xQ_{k-1}) + K_{n-1}(yP_{k-2} - xQ_{k-2}) = 0,$$

and since $H_{n-1}K_{n-2} = K_{n-1}H_{n-2}$, we have $H_{n-1}(xV_{k-1} - yU_{k-1}) = K_{n-1}(xQ_{k-1} - yP_{k-1})$.

Finally, since $V_{n+2}^*=0$ we have $H_{n-1}(xV_2-yU_2)=K_{n-1}(xQ_2-yP_2)$. We see the third part. If $P_j = Q_j = 0$, $j = s+1$, ..., $k-1$, being $s = max\{j,\ yP_j - xQ_j \neq 0\} < k$, then $P_j = xH_{j-1}$, $Q_j = yH_{j-1}$, $j=s+1,\dots,n$, therefore, from Lemma 2.2.44, $U_j = xK_{j-1}$, $V_j = yK_{j-1}$, $j=s+1,\dots,n$, and applying the first part with $p=n-1,\ q=j$, it has that $K_{j-1}=0$, that is, $U_j=V_j=0, j=s+1,\ ...,\ k-1$. ∎

2.3 Vector fields $(-y+P_s+\sum_{j=k}^{n-1} xH_j, x+Q_s+\sum_{j=k}^{n-1} yK_j)^T$ with polynomial commutator.

Let us consider the system

$$\begin{cases} \dot{x} &=& -y+P_s+x(H_k+H_{k+1}+\cdots+H_{n-1}), \\ \dot{y} &=& x+Q_s+y(H_k+H_{k+1}+\cdots+H_{n-1}), \end{cases} \tag{2.3.2}$$

with $2\le s\le k$, P_s, Q_s homogeneous polynomial of degree s and $H_i = H_i(x,y)$ homogeneous polynomial of degree i. From Theorem 2.2.45, if the system (2.3.2) have polynomial commutator, then either it has the form

$$(U,V)^T = (x+U_s+x\sum_{j=k}^{n-1} K_j, y+V_s+y\sum_{j=k}^{n-1} K_j)^T, \tag{2.3.3}$$

or

$$(U,V)^T = (U_s+x\sum_{j=k}^{n-1} K_j, V_s+y\sum_{j=k}^{n-1} K_j)^T, \tag{2.3.4}$$

with $(yP_s-xQ_s)K_j = (yU_s-xV_s)H_j,\ j=k,\ ...,\ n-1$.

The next lemma characterizes the terms $U_s,\ V_s$ of a polynomial commutator of (2.3.2).

Lemma 2.3.46. *If the vector field (2.3.3) is a polynomial commutator of the system (2.3.2) then U_s, V_s verify*

$$((y\partial_x - x\partial_y)^2 + id)(U_s) = -(s-1)(Q_s + y\partial_x P_s - x\partial_y P_s),$$
$$((y\partial_x - x\partial_y)^2 + id)(V_s) = (s-1)(P_s - y\partial_x Q_s + x\partial_y Q_s),$$

with $y\partial_x U_s - x\partial_y U_s = V_s - (s-1)P_s,\ y\partial_x V_s - x\partial_y V_s = -U_s - (s-1)Q_s$. If the vector field (2.3.4) is a polynomial commutator of the system (2.3.2) then U_s, V_s verify

$$((y\partial_x - x\partial_y)^2 + id)(U_s) = 0, \quad ((y\partial_x - x\partial_y)^2 + id)(V_s) = 0,$$

with $y\partial_x U_s - x\partial_y U_s = V_s,\ y\partial_x V_s - x\partial_y V_s = -U_s$.

Proof.
The term of degree s of the Lies bracket between the field and the polynomial commutator (2.3.3) is given by $[(-y,x)^T,(U_s,V_s)^T]+[(P_s,Q_s)^T,(x,y)^T]$. Expanding and applying Euler's Theorem, it becomes

$$(-V_s + y\partial_x U_s - x\partial_y U_s + (s-1)P_s, U_s + y\partial_x V_s - x\partial_y V_s + (s-1)Q_s)^T.$$

As it must be null, we have that U_s and V_s must verify $y\partial_x U_s - x\partial_y U_s = V_s - (s-1)P_s$ and $y\partial_x V_s - x\partial_y V_s = -U_s - (s-1)Q_s$. Applying to both expressions the operator $y\partial_x - x\partial_y$ we arrive to

$$((y\partial_x - x\partial_y)^2 + id)(U_s) = -(s-1)(Q_s + y\partial_x P_s - x\partial_y P_s),$$
$$((y\partial_x - x\partial_y)^2 + id)(V_s) = (s-1)(P_s - y\partial_x Q_s + x\partial_y Q_s),$$

that is, we can calculate in a explicit way the polynomials U_s V_s. Analogously, it proves the case when the system (2.3.2) commutes with (2.3.4). ∎

The next result characterizes the systems (2.3.2) with $s=2$ that have polynomial commutator.

Theorem 2.3.47. *The system (2.3.2) with $s=2$, $P_2^2+Q_2^2 \neq 0$, has polynomial commutator if and only if either it has constant angular speed* *(see Corollary 2.4.49 and Theorem 2.4.51), or, up to rotation, it is of the form*

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + p_{20}x^2 - 2q_{20}xy + q_{20}p_{20}x^3 + ax^2y, \\ \dot{y} & = & x + q_{20}x^2 + p_{20}xy - q_{20}y^2 + q_{20}p_{20}x^2y + axy^2, \end{array} \right. \tag{2.3.5}$$

and its commutator is

$$\begin{array}{rcl} U & = & x + q_{20}x^2 + p_{20}xy - q_{20}y^2 + q_{20}p_{20}x^2y + axy^2, \\ V & = & y + 2q_{20}xy + p_{20}y^2 + q_{20}p_{20}xy^2 + ay^3, \end{array} \tag{2.3.6}$$

with p_{20}, q_{20}, a real numbers and $q_{20} \neq 0$.

Proof.
We consider the system (2.3.2) with $P_s = P_2 = p_{20}x^2 + p_{11}xy + p_{02}y^2$, $Q_s = Q_2 = q_{20}x^2 + q_{11}xy + q_{02}y^2$. Making a rotation, we can suppose $p_{02} = 0$. The operator $(y\partial_x - x\partial_y)^2 + id$ on the set of the homogeneous polynomials of degree s in the variables x, y is invertible, and $((y\partial_x - x\partial_y)^2 + id)^{-1}$ comes given by

$$\begin{array}{l} ((y\partial_x - x\partial_y)^2 + id)^{-1}(m_{20}x^2 + m_{11}xy + m_{02}y^2) \\ \qquad = \frac{1}{3}(m_{20} + 2m_{02})x^2 - \frac{1}{3}m_{11}xy + \frac{1}{3}(2m_{20} + m_{02})y^2. \end{array}$$

We first suppose that the system commutes with (2.3.3), applying Lemma 2.3.46, the quadratic terms of the polynomial commutator U_2, V_2 are unique and take the expression

$$\begin{array}{l} U_2 = \frac{1}{3}(-p_{11} - q_{20} - 2q_{02})x^2 + \frac{1}{3}(2p_{20} + q_{11})xy + \frac{1}{3}(p_{11} - 2q_{20} - q_{02})y^2, \\ V_2 = \frac{1}{3}(p_{20} - q_{11})x^2 + \frac{1}{3}(-p_{11} + 2q_{20} - 2q_{02})xy + \frac{1}{3}(2p_{20} + q_{11})y^2, \end{array}$$

furthermore, it is easy to show that they satisfy the conditions

$$y\partial_x U_2 - x\partial_y U_2 = V_2 - P_2, \quad y\partial_x V_2 - x\partial_y V_2 = -U_2 - Q_2.$$

The cubic term of the Lie bracket is

$$\begin{pmatrix} x \\ y \end{pmatrix} (y\partial_x K_2 - x\partial_y K_2 + 2H_2) + \left[\begin{pmatrix} P_2 \\ Q_2 \end{pmatrix}, \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} \right]. \tag{2.3.7}$$

If we multiply the first component by y, the second one by x, and subtracting both expressions, we obtain a quartic term that does not depend on the coefficients of H_2, K_2 and that must be identically null, i.e. we get necessary conditions that the coefficients of P_2, Q_2 must verify for the existence of the polynomial commutator. The simultaneous annulment of these coefficients leads us to

$$P_2 = p_{20}x^2 - 2q_{20}xy, \quad Q_2 = q_{20}x^2 + p_{20}xy - q_{20}y^2.$$

As $yP_2 - xQ_2 = -q_{20}x(x^2+y^2)$ and $yU_2 - xV_2 = -q_{20}y(x^2+y^2)$, from Theorem 2.2.45, the polynomials H_j, K_j verify $xK_j = yH_j$, for all j, i.e. $H_j = xM_{j-1}, K_j = yM_{j-1},\ j = k,\ \ldots,\ n-1$. In this case, $xK_2 = yH_2$ we arrive to $H_2 = q_{20}p_{20}x^2 + axy$, $K_2 = q_{20}p_{20}xy + ay^2$.

It is easy to check that the Lie bracket of the vector fields associated to the systems (2.3.5) and (2.3.6) is null, therefore both fields commute. Next, we prove that if $(U,V)^T = (x+U_2+x\sum_{j=k}^{n-1} K_j, y+V_2+y\sum_{j=k}^{n-1} K_j)^T$ is a polynomial commutator of (2.3.2) then

$$\begin{aligned}\left[\begin{pmatrix} P_2 \\ Q_2 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} K_j\right] + \left[\begin{pmatrix} x \\ y \end{pmatrix} H_j, \begin{pmatrix} U_2 \\ V_2 \end{pmatrix}\right] \\ = (j-2)q_{20}\begin{pmatrix} x \\ y \end{pmatrix}(x^2+y^2)M_{j-1},\end{aligned} \tag{2.3.8}$$

for all $j = k, \ldots, n-1$.
Expanding the sum of both brackets we have

$$\begin{pmatrix} x \\ y \end{pmatrix}(U_2\partial_x H_j + V_2\partial_y H_j - P_2\partial_x K_j - Q_2\partial_y K_j) + \begin{pmatrix} P_2K_j - U_2H_j \\ Q_2K_j - V_2H_j \end{pmatrix}. \tag{2.3.9}$$

The second summand of the right-hand sides of the expression also has radial form since

$$y(P_2K_j - U_2H_j) - x(Q_2K_j - V_2H_j) = (yP_2 - xQ_2)K_j - (yU_2 - xV_2)H_j = 0,$$

from Theorem 2.2.45. In this case,

$$U_2\partial_x H_j + V_2\partial_y H_j - P_2\partial_x K_j - Q_2\partial_y K_j = (j-1)q_{20}(x^2+y^2)M_{j-1},$$
$$P_2K_j - U_2H_j = -q_{20}(x^2+y^2)M_{j-1},$$

therefore, it verifies (2.3.8).

For any j with $2 \le j < n-1$, using Theorem 2.2.45 and by (2.3.8), the term of degree $j+2$ of the Lie bracket of the field with the commutator comes given by

$$\begin{pmatrix} x \\ y \end{pmatrix} (y\partial_x K_{j+1} - x\partial_y K_{j+1} + (j+1)H_{j+1} + (j-2)q_{20}(x^2+y^2)M_{j-1}), \tag{2.3.10}$$

with $2 \le j < n-1$. For $j = n-1$, the annulment of the term of degree $n+1$ of the Lie bracket leads us $(n-3)q_{20}(x^2+y^2)M_{n-2}$, being $H_{n-1} = xM_{n-2}$, $K_{n-1} = yM_{n-2}$. That is, either M_{n-2}, M_{n-3}, ..., M_{k-1} are nulls, therefore the system is (2.3.5) with $q_{20} \neq 0$, or $q_{20} = 0$, i.e. $P_2 = p_{20}x^2$, $Q_2 = p_{20}xy$, therefore the field has constant angular speed. Finally, if the system commutes with (2.3.4), it has $U_2 = V_2 = 0$, since the operator $(y\partial_x - x\partial_y)^2 + \mathrm{id}$ is invertible, and from Theorem 2.2.45 it has $yP_2 = xQ_2$, i. e. the system has constant angular speed. ∎

2.4 Isochronous uniformly centres with a polynomial commutator.

Let us consider the system

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + x(H_1 + H_2 + \cdots + H_{n-1}), \\ \dot{y} & = & x + y(H_1 + H_2 + \cdots + H_{n-1}), \end{array} \right. \tag{2.4.11}$$

with $H_i = H_i(x,y)$ homogeneous polynomial of degree i. Using the previous results we know that if a polynomial commutator exists, it will take the form

$$(U,V)^T = (xK, yK)^T = (x\sum_{j=0}^{n-1} K_j, y\sum_{j=0}^{n-1} K_j)^T \tag{2.4.12}$$

with $K_i = K_i(x,y)$ homogeneous polynomial of degree i.

Lemma 2.4.48. *The vector field (2.4.12) is a polynomial commutator of the system (2.4.11) if and only if the polynomials K_j verify:*

1. *$H_pK_q = H_qK_p$ for any pair p, q with $1 \le p \le n-1$, $1 \le q \le n-1$.*
2. *(a) If $K_0 \neq 0$, then $-jH_j + x\partial_y K_j - y\partial_x K_j = 0$ with $1 \le j \le n-1$,*
   (b) If $K_0 = 0$, then $x\partial_y K_j - y\partial_x K_j = 0$ with $1 \le j \le n-1$.

Proof.
Applying Theorem 2.2.45 we get the first part.
In the case $K_0 \neq 0$, expanding and simplifying the Lie bracket of the field with the commutator we have

$$(x,\ y)^T(x\partial_x H + y\partial_y H - (x\partial_y K - y\partial_x K) + K(x\partial_x H + y\partial_y H) \\ -H(x\partial_x K + y\partial_y K)).$$

If we denote the homogeneous part of degree j of the polynomial P by $(\,P\,)_j$, we will have

$$\textstyle\sum_{j=1}^{n-1}(x\partial_x H + y\partial_y H)_j - \sum_{j=1}^{n-1}(x\partial_y K - y\partial_x K)_j \\ +\textstyle\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(K_j(x\partial_x H + y\partial_y H)_i - H_i(x\partial_x K + y\partial_y K)_j).$$

That is,

$$\textstyle\sum_{j=1}^{n-1} jH_j - x\sum_{j=1}^{n-1}\partial_y K_j + y\sum_{j=1}^{n-1}\partial_x K_j \\ +\textstyle\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(i-j)H_iK_j = 0.$$

Taking into account the terms degree by degree we get 2.(a).
If $K_0 = 0$, the Lie bracket is

$$\textstyle\sum_{j=1}^{n-1} jH_j - x\sum_{j=1}^{n-1}\partial_y K_j + y\sum_{j=1}^{n-1}\partial_x K_j \\ +\textstyle\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(i-j)H_iK_j = 0.$$

Therefore $x\partial_y K_j - y\partial_x K_j = 0,\ 1 \leq j \leq n-1$.
Reciprocal implication is obvious, since under conditions *1* and *2* the field (2.4.12) is a commutator of the system (2.4.11). ■

These relations allow us to compute polynomials K_j using polynomials H_j. Notice that if j is odd the polynomial K_j is unique, but if j is even, K_j is settled module the vector space generated by $(x^2+y^2)^{j/2}$.
From Lemma 2.4.48, it has the following result.

Corollary 2.4.49. *The system (2.4.11) has polynomial commutator with null linear part if and only if it has the form*

$$\begin{cases} \dot{x} &= -y + xP_{2l}(x,y)\sum_{j=0}^{r} a_j(x^2+y^2)^j, \\ \dot{y} &= \quad x + yP_{2l}(x,y)\sum_{j=0}^{r} a_j(x^2+y^2)^j, \end{cases} \tag{2.4.13}$$

with $P_{2l}(x,y)$ homogeneous polynomial of degree $2l, l \geq 0$, and $a_j,\ j = 0,\ \cdots,\ r$, arbitrary real numbers. In this case, the commutator (2.4.12) is given by

$$(x\sum_{j=0}^{r} a_j(x^2+y^2)^{j+l}, y\sum_{j=0}^{r} a_j(x^2+y^2)^{j+l})^T. \tag{2.4.14}$$

We now study the case when the commutator has linear part $(x,y)^T$. Next, we prove a result that will be used in the proof of the main theorem.

Lemma 2.4.50. *Let M, N be homogeneous polynomials of orders m and n respectively, with $m \leq n$, verifying*

$$mM(x\partial_y N - y\partial_x N) = nN(x\partial_y M - y\partial_x M). \tag{2.4.15}$$

Then M divides to N.

Proof.
The relation (2.4.15) can be expressed as

$$\frac{\frac{1}{n}(x\partial_y N - y\partial_x N)}{N} = \frac{\frac{1}{m}(x\partial_y M - y\partial_x M)}{M},$$

that is, $\frac{1}{n}(x\partial_y - y\partial_x) ln\, N = \frac{1}{m}(x\partial_y - y\partial_x) ln\, M$, so, $(x\partial_y - y\partial_x) ln\, \frac{N^m}{M^n} = 0$. Consequently $ln\, \frac{N^m}{M^n} = f(x^2+y^2)$ and therefore $N^m = M^n e^{f(x^2+y^2)}$ being f an analytic real function.
Since N^m and M^n are polynomials, we have that $e^{f(x^2+y^2)}$ is constant. So, $N^m = \gamma M^n$ with γ a constant. Since M and N are polynomials, that can only take place if M divides to N. ∎

Theorem 2.4.51. *The system (2.4.11) with $H_1 = H_2 = \cdots = H_{j-1} = 0$, $H_j \neq 0$ $(j \geq 1)$ has polynomial commutator with linear part $(x, y)^T$ if and only if there are polynomials α_l, β_l of order l $(l \leq j$, l divides to $j)$ verifying $x\partial_y \beta_l - y\partial_x \beta_l = l\alpha_l$, such that the system (2.4.11) is given by*

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + x\alpha_l \sum_{k=j/l-1}^{r-1} a_k \beta_l^k, \\ \dot{y} & = & x + y\alpha_l \sum_{k=j/l-1}^{r-1} a_k \beta_l^k, \end{array} \right. \tag{2.4.16}$$

with a_k, $k = j/l - 1, \cdots, r-1$, arbitrary real numbers and $r = [\frac{n-1}{l}]$. In this case, the commutator (2.4.12) is given by

$$\left\{ \begin{array}{rcl} U & = & x + x \sum_{k=j/l-1}^{r-1} a_k \beta_l^{k+1}, \\ V & = & y + y \sum_{k=j/l-1}^{r-1} a_k \beta_l^{k+1}. \end{array} \right. \tag{2.4.17}$$

Proof.
If the field (2.4.11) is homogeneous, it is enough to take $l = j$.
If it is not homogeneous, there is a polynomial H_i non null with $i > j$. Let m be the natural number such that $mj < i \leq (m+1)j$.
The proof consists, basically, of studying what has to satisfy the degree of H_i and stating the expression of H_i in terms of K_i.
From Lemma 2.4.48, the factors K_i, K_j corresponding to the terms of degree i, j of the commutator are also non null and have to verify

$$\frac{\frac{1}{i}(x\partial_y K_i - y\partial_x K_i)}{K_i} = \frac{\frac{1}{j}(x\partial_y K_j - y\partial_x K_j)}{K_j},$$

that is, $jK_j(x\partial_y K_i - y\partial_x K_i) = iK_i(x\partial_y K_j - y\partial_x K_j)$. Therefore, from Lemma 2.4.50, we have $K_i = K_j M_{i-j}$, with M_{i-j} homogeneous polynomial over x, y, of order $i-j$. If we substitute K_i, in the previous expression we will get $jK_j(x\partial_y M_{i-j} - y\partial_x M_{i-j}) = (i-j)M_{i-j}(x\partial_y K_j - y\partial_x K_j)$, that is, now M_{i-j}, K_j verify the hypothesis of Lemma 2.4.50. At which there are M_{i-2j}, $M_{i-3j}, \ldots$, M_{i-mj}, homogeneous polynomials of the degree given by the subscript, such that

$$M_{i-2j} = K_j M_{i-3j},\ M_{i-3j} = K_j M_{i-4j}, \ldots,\ M_{i-(m-1)j} = K_j M_{i-mj},$$

and so $K_i = M_{i-mj}K_j^m$.
We distinguish two situations:

- If $i=(m+1)j$, $K_j = M_{i-mj}$ up to a constant, at which $K_{(m+1)j}$ and $H_{(m+1)j}$ are K_j^{m+1} and $H_jK_j^m$ up to a constant, respectively. Notice therefore, that if the non null factors H_i take the form H_{mj} we have that $l=j$ and $\alpha_j = H_j$, $\beta_j = K_j$.

- The other way, $mj < i < (m+1)j$, applying Lemma 2.4.50 on polynomials K_j, M_{i-mj} the existence of a polynomial of degree $(m+1)j-i$, $B_{(m+1)j-i}$, such that $K_j = M_{i-mj}B_{(m+1)j-i}$. Substituting we have that $B_{(m+1)j-i}$ and M_{i-mj} satisfy the Lemma again.
  If the obtained polynomials have the same degree, that is, $2i=(2m+1)j$, they will be proportional, at which we will have $M_{j/2} = \gamma B_{j/2}$ and therefore

$$\begin{aligned} K_j &= \gamma B_{j/2}^2, \\ H_j &= \tfrac{1}{j}(x\partial_y \gamma B_{j/2}^2 - y\partial_x \gamma B_{j/2}^2) = \tfrac{1}{j/2}\gamma B_{j/2}(x\partial_y B_{j/2} - y\partial_x B_{j/2}) \\ &= A_{j/2}B_{j/2}. \end{aligned}$$

  That is, if there was not any other H_i non null, $l=j/2$, $\alpha_{j/2} = A_{j/2}$ and $\beta_{j/2} = B_{j/2}$.
  If they have different degrees, the one with the lowest degree will divide to the other one. That is, either $K_j = B_{(m+1)j-i}^2 M_{2i-(2m+1)j}$ or $K_j = B_{(2m+1)j-2i}M_{i-mj}^2$.
  If these factors do not have the same degree, it will lead us to the

existence of polynomials B and M such that K_j is given by B^3M, B^2M^2 or BM^3.
Anyway, we have that K_j is the product of two factors, that is, $K_j = B^pM^q$ with B, M homogeneous polynomials verifying the hypothesis of Lemma 2.4.50 and p, q natural numbers and moreover both polynomials have the same degree, l. So, $M = \delta B$, what implies $K_j = \delta^q B^{p+q}$. That is, K_j has the form $K_j = \widetilde{\delta} B^{j/l}$ with B homogeneous polynomial of degree l, $\widetilde{\delta}$ constant and

$$H_j = \frac{1}{j}(x\partial_y \widetilde{\delta} B^{j/l} - y\partial_x \widetilde{\delta} B^{j/l}) = \widetilde{\delta}\frac{1}{l}(x\partial_y B - y\partial_x B)B^{j/l-1} = \widetilde{\delta}\alpha\beta^{j/l-1}.$$

If there only is a H_i non null, the results follows. Otherwise, if there were more than one H_i non null, for instance, H_{i_1}, H_{i_2} (so K_{i_1}, K_{i_2} are not null), applying the previous reasoning we deduce the existence of two natural numbers l_1, l_2 and polynomials α_{l_1}, β_{l_1} and α_{l_2}, β_{l_2} of degrees l_1, l_2 respectively, such that, up to constants,

$$K_{i_1} = (\beta_{l_1})^{i_1/l_1}, \quad K_{i_2} = (\beta_{l_2})^{i_2/l_2},$$

at which we will have two expressions for K_j and comparing both of them we get $\beta_{l_1}^{l_2} = \beta_{l_2}^{l_1}$.
If $l_1 < l_2$, since β_{l_1}, β_{l_2} are polynomials, we have that $\beta_{l_2} = \beta_{l_1}^{l_2/l_1}$, at which $K_{i_2} = (\beta_{l_1})^{\frac{l_2 i_2}{l_1 l_2}} = (\beta_{l_1})^{i_2/l_1}$. In this case $l = l_1$.
If there were r homogeneous no null polynomials $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$ of degrees $i_1, i_2, \ldots, i_r$, respectively (polynomials $K_{i_1}, K_{i_2}, \ldots, K_{i_r}$ are also non null) we would have at most r expressions of K_i as a power of homogeneous polynomials of degrees $l_1, l_2, \ldots, l_r$, and in this case, $l = min(l_1, l_2, \ldots, l_r)$.

In order to prove the other implication, it is enough to check that the systems (2.4.16) and (2.4.17) verify the Lemma 2.4.48. ∎

2.5 Geometry of quartic and quintic rigid systems with a polynomial commutator.

As we know, the isochronous quadratic and cubic systems with a constant angular speed have a polynomial commutator. The phase portraits on the Poincaré disc are represented in Mardesic *et al.* [123] and Collins [40], respectively. In this section, we classify, applying the Theorem 2.4.51, the quartic and quintic systems with this property, and we display on the Poincaré disc the phase portraits about these systems with a non-homogeneous H.
If H is a homogeneous polynomial of degree three or four, we will just give a first integral of the system, remaining for another work the study of their phase portrait. That integral has been picked up from Mazzi, Sabatini [127]. Concretely, from the result which states that if H is an homogeneous polynomial of degree n satisfying the Conti's condition, then a first integral of the system is

$$I(x,y) = \frac{(x^2+y^2)^{n-1}}{(1+K(x,y))^2}, \tag{2.5.18}$$

where K is a commutator polynomial whose expression can be obtained by Theorem 2.4.51.

We classify the systems (2.4.16) with non-homogeneous H upon the values of l. If $l=1$, doing a rotation and a rescaling, we get

$$\left\{\begin{array}{rcl} \dot{x} &=& -y + x^2(ba_1 + b^2a_2y + b^3a_3y^2 + b^4a_4y^3), \\ \dot{y} &=& x + xy(ba_1 + b^2a_2y + b^3a_3y^2 + b^4a_4y^3), \end{array}\right.$$

being $b,\ a_1,\ a_2,\ a_3$ and a_4 real numbers, with $b \neq 0,\ a_3^2 + a_4^2 \neq 0$.
If $a_3 = a_4 = 0$, we can find the phase portraits for $a_2 = 0$ in [123], and for $a_2 \neq 0$ in [40].

If $a_3 \neq 0$ and $a_4 = 0$, scaling the variables, we have

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + x^2(B_1 + B_2 y - y^2), \\ \dot{y} & = & x + xy(B_1 + B_2 y - y^2). \end{array} \right. \tag{2.5.19}$$

If $B_1 = B_2 = 0$, a first integral of the system is given by (2.5.18) with $n = 3,\ K(x, y) = -y^3$.

Next we study the dynamic system (2.5.19) with $B_1^2 + B_2^2 \neq 0$. Its polynomial commutator is $(x, y)^T p(y)$ with $p(y) = 1 + B_1 y + B_2 y^2 - y^3$. The function $\frac{1}{(x^2+y^2)p(y)}$ is a integrating factor and, therefore,

$$F(x, y) = \frac{1}{2} ln(x^2 + y^2) - \int_0^y \frac{B_1 + B_2 s - s^2}{p(s)} ds$$

is a first integral of (2.5.19) The polynomial $p(y)$ plays a essential role, since every real root give us an invariant straight line. Studying the multiple roots of the polynomial, we have drawn the bifurcation diagrams shown in the Figure 1. The phase portraits on the Poincaré disc has been drawn in it.

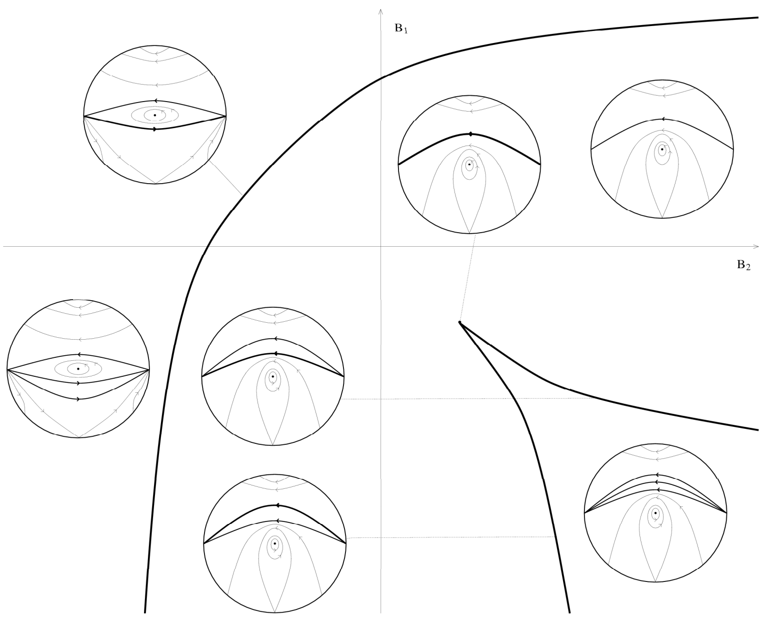

Figure 1. Phase portraits of the family (2.5.19) with $B_1^2 + B_2^2 \neq 0$.

The dynamic in the finite part of the disc is determined due to the presence of the centre in the origin and the/or invariant line/s. These systems have a common feature, as corresponds to every system with degeneration at the infinity, every point in the infinity is an equilibrium. However, among them, there are four relevant points which are the poles of the disc. In this study, we have used methods of trigonometric blow up, giving us the following results:

- The upper pole is a centre.
- If $p(y)$ has not negative roots, from the bottom pole appears a homoclinic which encloses the periodic ring of the origin. If $p(y)$ has negative roots, the periodic ring is enclosed between two lines and the homoclinic is replaced by two separatrices, one of them comes from the left pole and the other one goes to the right.
- The dynamics in the right and left poles are symmetric due to the reversibility. When $p(y)$ has only a positive root, both points are, respectively, the end and the origin of the invariant respective line. The existence of two or three positive roots is translated in the appearance of a repulsive parabolic sector in the right pole and attractive in the left one. When, at least, there is a real negative root, among the lines corresponding to the positive root and the less negative one appears, a hyperbolic sector, associated to the original periodic ring. In that case, everything under such hyperbolic sector is a repulsive parabolic one for the left pole and attractive for the right.

We are going to expound the study of a specific case. For instance, $B_1 = -1/2$ and $B_2 = -5/2$. The invariant lines are $y = -2$, $y = -1$ and $y = 1/2$. For the study of the left and right poles, we make the following change $x = 1/v$, $y = u/v$ and after multiplying the resultant system by v^2, we obtain $\dot{u} = (1 + u^2)\, v^2$, $\dot{v} = uv^3 + \frac{v^2}{2} + \frac{5\,uv}{2} + u^2$.

The unique equilibrium in $v = 0$ is $(0, 0)$, which is degenerated (null lineal part). Making the change $u = r\cos\theta$, $v = r\sin\theta$ (blow up homogeneous trigonometric) and dividing the field among r we obtain the system

$$\begin{cases} \dot{r} = r^3\cos\theta - r^3\cos^3\theta + \dfrac{r\sin\theta}{2} + \dfrac{r\sin\theta\cos^2\theta}{2} + \dfrac{7\,r\cos\theta}{2} - \dfrac{7\,r\cos^3\theta}{2}, \\ \dot{\theta} = -\sin\theta + \dfrac{7\,\sin\theta\cos^2\theta}{2} + \dfrac{\cos\theta}{2} + \dfrac{\cos^3\theta}{2}, \end{cases} \tag{2.5.20}$$

which in $r = 0$, with $\theta \in [0, 2\pi)$, has six equilibria: one in the first quadrant, two in the second, one in the third and two in the fourth. In short, we detect six input (output) directions in the equilibrium, which correspond to the three invariant lines of the system, what joined to the analysis of the sign of the second component, drives us to the dynamic around the equilibrium, as it is drawn in Figure 2.

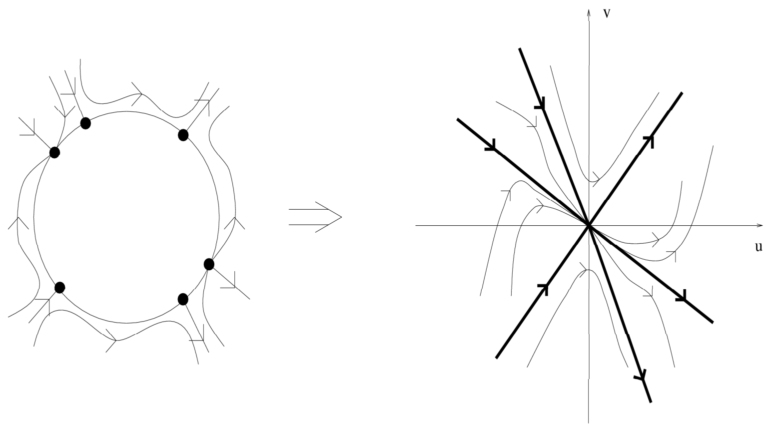

Figure 2. Dynamic close to the right and left poles.

In the study of the others poles we proceed in a analogue way. Making the change $x = u/v$, $y = 1/v$ and multiplying by v^2 we obtain $\dot{u} = (-u^2 - 1)\,v^2$, $\dot{v} = -u\,(2\,v^3 - v^2 - 5\,v - 2)/2$. The $(0, 0)$ is equilibrium of the system with lineal part type Takens-Bogdanov, that is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, tell us how the dynamic close to that point is going to be. However, it is enough to make a quasi-homogeneous trigonometric blow-up of type

$(3,2)$, that is, $u = r^3 cos\theta,\ v = r^2 sin\theta$, and after multiplying the resultant system by $(2 + cos^2\theta)/r$ and studying the equilibria in $r = 0,\ \theta \in [0, 2\pi]$, we obtain the results.

Finally, if $a_4 \neq 0$, scaling the variables, the system adopt the form

$$\begin{cases} \dot{x} &= -y + x^2(C_1 + C_2 y + C_3 y^2 + C_4 y^3), \\ \dot{y} &= x + xy(C_1 + C_2 y + C_3 y^2 + C_4 y^3), \end{cases} \tag{2.5.21}$$

with $C_4 = \pm 1$.
If $C_1 = C_2 = C_3 = 0$, a first integral of the system is given by (2.5.18), with $n = 4$ and $K(x, y) = C_4 y^4$.
If $C_1^2 + C_2^2 + C_3^2 \neq 0$, the polynomial $q(y) = 1 + C_1 y + C_2 y^2 + C_3 y^3 + C_4 y^4$ plays a key role on the first integral of this field and, therefore, over its dynamic. In fact, each root of q means an invariant straight line an so, the study of this family is analogous to the system (2.5.19). Fixed C_2, we obtain the following figures for case $C_4 = 1$ and $C_4 = -1$, respectively, (where n+s (n-s), denote n positive (negative) roots of $q(y)$; n+d (n-d), denote n double positive (negative) roots of $q(y)$, ...)

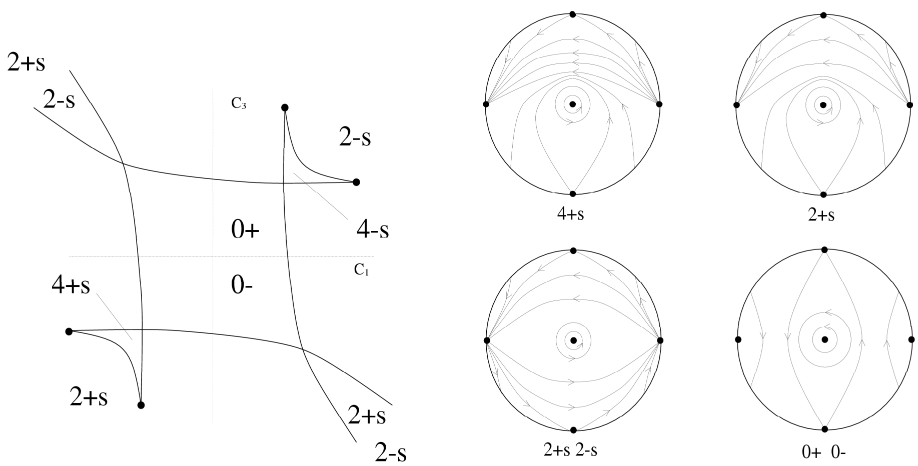

Figure 3. Phase portraits of the family (2.5.21) for $C_1^2 + C_2^2 + C_3^2 \neq 0$ and $C_4 = 1$.

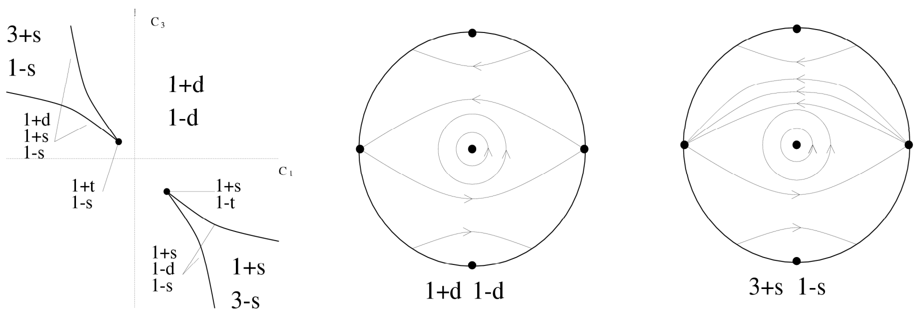

Figure 4. Phase portraits of the family (2.5.21) for $C_1^2 + C_2^2 + C_3^2 \neq 0$ and $C_4 = -1$.

If $l = 2$, up to rotation and scaling, we have

$$\begin{cases} \dot{x} &= -y + x^2y(a_1 + a_2(ax^2 + (a+1)y^2)), \\ \dot{y} &= x + xy^2(a_1 + a_2(ax^2 + (a+1)y^2)), \end{cases}$$

being $a,\ a_1,\ a_2$ real numbers, with $a.a_2 \neq 0$.
Both cases $a = 0$ and $a_2 = 0$ have already been watched. Let us suppose then $aa_2 \neq 0$. Making $(x \to \sqrt[4]{|aa_2|}x,\ y \to \sqrt[4]{|aa_2|}y)$ we transform the system in

$$\begin{cases} \dot{x} &= -y + x^2y(D_1 + D_2x^2 + D_4y^2)), \\ \dot{y} &= x + xy^2(D_1 + D_2x^2 + D_4y^2)). \end{cases} \tag{2.5.22}$$

with $D_2 = \pm 1$.
For $D_1 = D_4 = 0$, a first integral is given by (2.5.18), with $n = 4$ and $K(x, y) = -D_2x^4$.
If $D_1^2 + D_4^2 \neq 0$, studying relevant equilibrium points at infinity, we have obtained the bifurcation diagrams given in Figure 5. For this study we have used trigonometric *blow-up* techniques.
In case $D_2 = 1$, the right/left pole is always a centre, but the character of the upper/lower one depends on D_4. If $D_4 \geq 0$, it is connected by two heteroclinic orbits which bound the periodic ring of the origin. If $D_4 < 0$, it is a centre and two new relevant equilibria appear at infinity.

These are joined by four heteroclinic orbits that bound the periodic ring. When $D_2 = -1$ we have a wider range of behaviours.
If $D_4 < 0$ the situation is the same as for $D_2 = 1$, $D_4 > 0$, up to a $\frac{\pi}{2}$ rotation.
If $D_4 = 0$ we have to distinguish among $D_1 < 2$, $D_1 = 2$ and $D_1 > 2$. Dynamic for $D_1 < 2$ is analogous to the one for $D_4 < 0$. At $D_1 = 2$, the upper/lower pole becomes an equilibrium with two elliptic sectors and two hyperbolic sectors. For $D_1 > 2$, four parabolic sectors add to the others.
If $D_4 > 0$ other two significant equilibria show up at infinity again. The curve represented on Figure 5 is a saddle-node bifurcation curve for a trigonometric polynomial in $[0, 2\pi)$. We obtain this polynomial studying angular speed on curve $r = 0$ after a quasi-homogeneous trigonometric *blow-up* on the equilibria which are not the poles. When this polynomial has four roots in $[0, 2\pi)$, there are four sectors, one elliptic, two parabolic and one hyperbolic, respectively, surrounding each equilibrium. When the roots are only two, the parabolic sectors disappear. In both cases, the periodic ring is bounded by two heteroclinic orbits connecting the upper/lower pole if $D_1 > 0$ or the right/left one if $D_1 < 0$. Finally, when there are no roots, these equilibria are centres and the periodic ring is surrounded by four heteroclinic orbits which join the poles.

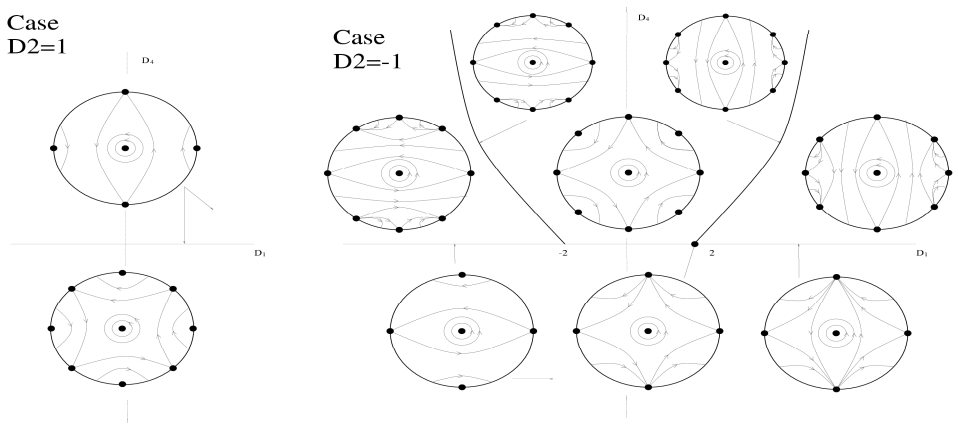

Figure 5. Phase portraits of the family (2.5.22) for $D_1^2 + D_4^2 \neq 0$.

Finally, for $l=3$ and $l=4$ we obtain

$$\left\{\begin{array}{rcl}\dot{x} & = & -y+x\alpha_3,\\ \dot{y} & = & x+y\alpha_3,\end{array}\right. \qquad \left\{\begin{array}{rcl}\dot{x} & = & -y+x\alpha_4,\\ \dot{y} & = & x+y\alpha_4,\end{array}\right.$$

respectively, being

$$I(x,y)=\frac{(x^2+y^2)^2}{(1+\beta_3)^2} \qquad I(x,y)=\frac{(x^2+y^2)^3}{(1+\beta_4)^2},$$

first integrals of both systems, respectively, where

$$\alpha_3=\frac{1}{3}\left(x\frac{\partial}{\partial y}\beta_3-y\frac{\partial}{\partial x}\beta_3\right), \qquad \alpha_4=\frac{1}{4}\left(x\frac{\partial}{\partial y}\beta_4-y\frac{\partial}{\partial x}\beta_4\right)$$

for certain homogeneous polynomial β_3 and β_4 of degree 3 and 4, respectively.

Notice that the fields studied are of the types B^1, B^2 y B^4 following the notation of Collins [40], depending on the heteroclinics number that bound the periodic ring of the origin.

We remark that in a generic way, the dynamic of the family with $H=x\sigma(y)$, depends on the nature of the roots of polynomial $1+y\sigma(y)$, has as many invariant straight lines as roots of the polynomial.

Chapter 3

Isochronous uniformly centres.

3.1 Introduction

In this chapter, we study the problem of centre for the planar analytic systems which have a centre-focus equilibrium at the origin and whose angular speed is constant. In these systems, the origin is the only finite equilibrium and if it is a centre, it will be isochronous. These systems, up to a linear change, take the following expression:

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + xH(x,y), \\ \dot{y} & = & x + yH(x,y), \end{array} \right. \tag{3.1.1}$$

where H is an analytic function which vanishes at the origin.
The interest in studying this family is due, on the one hand, to the importance of these systems in the general problem of the isochronicity, see Definition 1.1.1 in section 1.1. Also Rudenok [140] proves that any analytic system with linear part $(-y, x)^T$ has an isochronous centre at the origin if and only if it is possible to transform it by means of specific analytic change $(x \to x + P(y^2), y \to y + Q(x,y))$ into a system (3.1.1).

On the other hand, these systems in polar coordinates take the form

$$\begin{cases} \dot{r} &= \sum_{k\geq 1} H_k(cos\,\theta, sin\,\theta) r^{k+1}, \\ \dot{\theta} &= 1, \end{cases}$$

where H_k are the homogeneous parts of degree k of H. These systems can be written as a single equation in form Abel generalized

$$\partial_\theta r = \sum_{k\geq 1} H_k(cos\,\theta, sin\,\theta) r^{k+1}, \tag{3.1.2}$$

The study of this equation gives us information about the systems, and vice versa, since the constant solution $r = 0$ of (3.1.2) corresponds to the critical point $x = y = 0$ of (3.1.1), and the periodic solutions of (3.1.2) correspond to closed orbits of (3.1.1), see Alwash & Lloyd [19, 20], Lloyd [114], Lloyd & Pearson [116] and Lloyd *et al.* [118].

Let us do now a more detailed review of the works related to the family (3.1.1). The first place where a subfamily of centres of (3.1.1) is characterized, is Conti [39]. Particularly, it is characterized the case when H is a homogeneous polynomial of arbitrary degree. It is proved that if the degree of H is odd then the origin is a centre, but if it is even must satisfy one condition, which is equivalent to vanish the first coefficient of the normal form in the radial component. In Algaba *et al.* [3], we find the centre condition in terms of the coefficients of the system. In Mardesic *et al.* [123] we have linearizating changes for time-reversible systems of type (3.1.1) with $H = H_1 + H_2$. In fact, as it is proved later in Collins [40], all the centres of this subfamily are time-reversible. Up to rotation, its expression is $(\dot{x}, \dot{y})^T = (-y + x^2\sigma(y),\ x + xy\sigma(y))^T$, with $\sigma(y) = a + by$. Later, Mazzi & Sabatini [127] study the system (3.1.1) when it commutes with a radial field, and they find a first integral and a linearization for (3.1.1). Algaba & Reyes [10] characterize the systems (3.1.1) which have polynomial commutator, appearing not time-reversible centres with H nonhomogeneous.

This chapter is divided into four sections. In section 3.2, we present the main result of this work, derive a few conditions that characterize the centres with angular constant speed. We show the equivalence between these conditions and the vanishing of the coefficients of the radial component of the normal form. In section 3.3, we develop an recursive algorithm that allows us to obtain conditions on the coefficients of the system which they must hold in order that the origin is a centre. In last section, we cite several subfamilies that have a centre. we obtain the centres of the families $(-y + x(H_1 + H_m), x + y(H_1 + H_m))^T$ and $(-y + x(H_2 + H_{2n}), x + y(H_2 + H_{2n}))^T$, being H_i homogeneous polynomial in x, y of degree i and also determine the maximum number of limit cycles which can bifurcate from a fine focus. Finally, we compute the centres of the cuartic polynomial rigid systems, and the quintic with a nonquintic nonlinearity.

3.2 Isochronous uniformly vector fields and their commutators.

We consider the following couple of differential systems

$$(\dot{x}, \dot{y})^T = \mathcal{X}(x, y), \qquad (\dot{x}, \dot{y})^T = \mathcal{U}(x, y),$$

with $\mathcal{X}$ and $\mathcal{U}$ analytic functions in a neighborhood of the origin.

If we denote by $\Phi_{\mathcal{X}}(t,(x, y))$, $\Phi_{\mathcal{U}}(t,(x, y))$ with $\Phi_{\mathcal{X}}(0,(x, y)) = (x, y)$ and $\Phi_{\mathcal{U}}(0,(x, y)) = (x, y)$, the flows of the previous systems, respectively, it is known that $\mathcal{X}$ and $\mathcal{U}$ commute if and only if the local flows $\Phi_{\mathcal{X}}$ and $\Phi_{\mathcal{U}}$ verify

$$\Phi_{\mathcal{X}}(t, \Phi_{\mathcal{U}}(s,(x, y))) = \Phi_{\mathcal{U}}(s, \Phi_{\mathcal{X}}(t,(x, y))),$$

for every t and s such that $\Phi_{\mathcal{X}}(t, \Phi_{\mathcal{U}}(s,(x, y)))$ and $\Phi_{\mathcal{U}}(s, \Phi_{\mathcal{X}}(t,(x, y)))$ exist, see Olver [132].

From Theorem 1.4.18, the problem of the isochronicity of a centre of a vector field is equivalent to the existence of an analytic commutator of such vector field; more precisely, for any analytic system with linear part $(-y, x)^T$, the existence of an analytic commutator with linear part $(x, y)^T$ is a necessary and sufficient condition so that the origin be an isochronous centre (also see Algaba *et al.* [2] and Sabatini [142]).

We are interested in the centres of the analytic systems with constant angular speed,

$$(\dot{x}, \dot{y})^T = \mathcal{X}(x, y) = (-y + xH(x, y), x + yH(x, y))^T, \tag{3.2.3}$$

where H is an analytic function in a neighborhood of the origin and $H(0, 0) = 0$. From Definition 1.1.1 and Theorem 1.4.22, the systems (3.2.3) always have a commutator. We have also proved in the chapter 2 that if H is a polynomial, the polynomial commutators, in case of existing, are of the form

$$(\dot{x}, \dot{y})^T = \mathcal{U}(x, y) = (xK(x, y), yK(x, y))^T, \tag{3.2.4}$$

where K is a polynomial of the same degree that H. We extend this property for the analytic case.

Proposition 3.2.52. *If the analytic system (3.2.3) is centre, then there exists an analytic commutator (3.2.4) around the origin.*

Proof.
We assume that $\mathcal{X}$ is a centre. Let Ψ be, the local flow which defines the solutions of the differential equation $\dot{x} = \alpha(x)$, with α analytic, $\alpha(0) = 0$ and $\alpha(x) < 0$ in $(0, \epsilon)$. Fixed (x, y) near the origin sufficiently, let s_0 be the minimum value $s \geq 0$ such that $\Phi_{\mathcal{X}}(-s, (x, y))$ is on the x-axis. We already define the analytic flow $\Phi_{\mathcal{U}}$ as $\Phi_{\mathcal{U}}(t, (x, y)) = \Phi_{\mathcal{X}}(s_0, (\Psi(t, \Phi_{\mathcal{X}}(-s_0, (x, y)), 0))$. Since $\mathcal{X}$ is uniformly isochronous, $\Phi_{\mathcal{U}}$ does not depend of s_0. Moreover, it commutes with $\Phi_{\mathcal{X}}$ and all the straight lines which through the origin are invariants to the flow. That

is, the associated field $\mathcal{U}$ is analytic and of the form (3.2.4). ∎

Therefore, the problem of looking for those H for that (3.2.3) has a centre in the origin is equivalent to the problem of looking for those H which exists $\mathcal{U}$ of the form (3.2.4) with $[\mathcal{X}, \mathcal{U}] = 0$.

If we use complex coordinates, $z = x + iy$, (3.2.3) turns out

$$(\dot{z}, \dot{\bar{z}})^T = (iz + zH(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}), -i\bar{z} + \bar{z}H(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}))^T$$

and the commutator (3.2.4) is

$$(\dot{z}, \dot{\bar{z}})^T = (zK(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}), \bar{z}K(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}))^T.$$

It will cause no confusion if we write the systems (3.2.3) in complex coordinates as

$$\dot{z} = X(z, \bar{z}) = iz + zH(z, \bar{z}) \tag{3.2.5}$$

and the commutator vector field as

$$\dot{z} = U(z, \bar{z}) = zK(z, \bar{z}) \tag{3.2.6}$$

with $H(z, \bar{z}) = \bar{H}(z, \bar{z})$ and $K(z, \bar{z}) = \bar{K}(z, \bar{z})$.

By definition, the Lie's bracket in complex coordinates is $[X, U] = X_z U + X_{\bar{z}} \bar{U} - U_z X - U_{\bar{z}} \bar{X}$, being X_z, U_z and $X_{\bar{z}}, U_{\bar{z}}$ the first partial derivatives of X and U respect to z and $\bar{z}$, respectively.

If for every m we denote by H_m and K_m the homogeneous parts of degree m of H and K, respectively, the equation $[X, U] = 0$ becomes

$$[iz, z\sum_{m\geq 0} K_m] + [z\sum_{l\geq 1} H_l, z\sum_{j\geq 0} K_j] = 0.$$

The term of degree $m+1$ of the above expression gets

$$[iz, zK_m] = \sum_{\substack{j+l=m \\ j\geq 0,\, l\geq 1}} [zK_j, zH_l]\,. \tag{3.2.7}$$

As $H = \bar{H}$ y $K = \bar{K}$, we obtain

$$\begin{aligned}
&[iz, zK] = iz(\bar{z}K_{\bar{z}} - zK_z), \\
&[zK, zH] = z(H(zK_z + \bar{z}K_{\bar{z}}) - K(zH_z + \bar{z}H_{\bar{z}})).
\end{aligned}$$

From Euler's Theorem for homogeneous functions, we have that

$$[zK_j, zH_l] = z(jH_lK_j - lK_jH_l) = (j-l)zH_lK_j.$$

Let $m \geq 1$, we consider the linear vector space $\mathcal{H}_m$, consisting of homogeneous polynomials, H_m, of degree m in the variables z and $\bar{z}$. And we denote by $\tilde{\mathcal{H}}_m$, the linear vector subspace of $\mathcal{H}_m$ on $\mathbb{R}$ defined by

$$\tilde{\mathcal{H}}_m = \{P \in \mathcal{H}_m / P(z,\bar{z}) = \bar{P}(z,\bar{z})\} = \{\sum_{k=0}^{[\frac{m}{2}]}(a_k z^k \bar{z}^{m-k} + \bar{a}_k z^{m-k}\bar{z}^k),\ a_k \in \mathbb{C}\}.$$

A basis of $\tilde{\mathcal{H}}_m$ is

$$\begin{aligned}\tilde{\mathcal{B}}_m = \ & \{u_k = z^k\bar{z}^{m-k} + z^{m-k}\bar{z}^k,\ 0 \leq k \leq [\tfrac{m}{2}]\} \\ & \cup\{v_k = i(z^k\bar{z}^{m-k} - z^{m-k}\bar{z}^k),\ 0 \leq k \leq [\tfrac{m-1}{2}]\}.\end{aligned}$$

Lemma 3.2.53. *The map*

$$L_m(\sum_{k=0}^{[\frac{m}{2}]}(a_k z^k\bar{z}^{m-k} + \bar{a}_k z^{m-k}\bar{z}^k) = \sum_{k=0}^{[\frac{m}{2}]} i(2k-m)(a_k z^k\bar{z}^{m-k} - \bar{a}_k z^{m-k}\bar{z}^k),$$

is a linear map of $\tilde{\mathcal{H}}_m$ into itself.
Moreover, if m is odd, $L_m(\tilde{\mathcal{H}}_m) = \tilde{\mathcal{H}}_m$ and $Ker(L_m) = \{\mathbf{0}\}$. And if m is even, $dim(L_m(\tilde{\mathcal{H}}_m)) = m$ and $Ker(L_m) = < z^{\frac{m}{2}}\bar{z}^{\frac{m}{2}} >$.

Proof.
The first part can be checked easily and the second one it follows from the action of the map on each element of the basis $\tilde{\mathcal{B}}_m$,

$$\begin{aligned}L_m(u_k) &= (m-2k)v_k,\ 0 \leq k \leq [\tfrac{m}{2}], \\ L_m(v_k) &= (2k-m)u_k,\ 0 \leq k \leq [\tfrac{m-1}{2}].\blacksquare\end{aligned}$$

The equations (3.2.7) can be rewritten as

$$L_m(K_m) = \sum_{j=0}^{m-1}(2j-m)K_jH_{m-j}, \qquad m \geq 1. \tag{3.2.8}$$

And from Lemma 3.2.53, we have the following result which characterizes the systems (3.2.3) with a centre at the origin.

Theorem 3.2.54. *The analytic system (3.2.5) is a centre if and only if there exists a* $\mathcal{C}^\infty$*−function defined in a neighborhood of the origin,* $K = \sum_{m\geq 0} K_m$*, with* $K_0 = 1$*, such that, for any* $m \geq 1$*,*

$$Proy_{Cor\,L_{2m}}\left(\sum_{j=0}^{2m-1} 2(j-m)K_j H_{2m-j}\right) = \{\mathbf{0}\},$$

being $L_{2m}(\tilde{\mathcal{H}}_{2m}) \oplus Cor\, L_{2m} = \tilde{\mathcal{H}}_{2m}$*, and* $L_m(K_m) = \sum_{j=0}^{m-1}(2j-m)K_j H_{m-j}$*.*

For each m, we denote by $Cond_m$ the above proyection on $Cor\ L_{2m}$.

The expression of $Cond_m$ is a polynomial in the coefficients of the polynomials $H_1, ..., H_{2m}$ and $Cond_m = 0$ is the condition that must satisfy so that the homogeneous part of degree $2m$ of the commutator exists, that is, $Cond_m = 0$ is the compatibility's condition of the system of linear equations whose unknown quantities are the coefficients of K_{2m}, which turn out of the equation (3.2.8).
The following result proves that the conditions of compatibility for the existence of the terms of degree even of a commutator, up to a certain degree, is equivalent to the vanishing of the coefficients of radial component of the normal form of the system, up to that degree.

Theorem 3.2.55. *Let fields (3.2.5) and (3.2.6) be such that* $[X, U] = \mathcal{O}(|z, \bar{z}|^{2m+1})$ *(i.e.,* $Cond_1 = \cdots = Cond_{m-1} = 0$*). Then,* $Cond_m \neq 0$ *if and only if the normal form of the system (3.2.5) up to order* $2m$ *is* $\dot{z} = iz + \sum_{j\geq m} a_j z(z\bar{z})^j$*, with* $a_m \neq 0$*.*

Proof.
It is known that if $[X, U] = 0$, then $[\Phi_* X, \Phi_* U] = 0$, where Φ is a dipheomorfism and $\Phi_* X$, $\Phi_* U$ are the transformed vector fields of X, U by Φ, respectively. Besides, if $\mathcal{J}^N[X, U] = 0$, then $\mathcal{J}^N[\Phi_* X, \Phi_* U] = 0$, where $\mathcal{J}^N X$ represents the Taylor's N-jet of the vector field X.

1) Let us $\tilde{X}(z,\bar{z}) = \Phi_* X(z,\bar{z}) = iz + a_m z^{m+1}\bar{z}^m + \mathcal{O}(|z,\bar{z}|^{2m+3})$ a normal form of vector field X. From the structure of the homological operator L_m, $m \geq 2$, the change of variables can be chosen of the form $\Phi(z) = z(1+\Psi(z,\bar{z}))$, i.e., it is radial change.
Since $[\tilde{X}, z] = \mathcal{O}(|z,\bar{z}|^{2m+1})$, then $[\Phi_*^{-1}X, \Phi_*^{-1}z] = \mathcal{O}(|z,\bar{z}|^{2m+1})$ and $U(z,\bar{z}) = \Phi_*^{-1}z$ is a radial type, i.e. $U(z,\bar{z}) = zK(z,\bar{z})$ with $K = \bar{K}$.

2) Let us $\tilde{X}(z,\bar{z}) = \Phi_* X(z,\bar{z}) = iz + \sum_{j=1}^{m} a_j z^{j+1}\bar{z}^j + \mathcal{O}(|z,\bar{z}|^{2m+3})$ a normal form of the vector field X. By hypothesis, we have $[X,U] = \mathcal{O}(|z,\bar{z}|^{2m+1})$, that is, $[\Phi_* X, \Phi_* U] = \mathcal{O}(|z,\bar{z}|^{2m+1})$ with $\tilde{U} = \Phi_* U = zK(z,\bar{z})$, $K = \bar{K}$ and $K(0,0) = 1$. Therefore, $a_1, a_2, \cdots, a_{m-1} = 0$, since $\mathcal{J}^1[\tilde{X},\tilde{U}] = 0$, $\mathcal{J}^2[\tilde{X},\tilde{U}] = 0, \cdots, \mathcal{J}^{2m}[\tilde{X},\tilde{U}] = 0$. ∎

From Lemma 3.2.53, we know that the terms of even degree of the commutator are not uniquely determined, that is, if K_{2m}^* is a particular solution of $L_{2m}(K_{2m}) = P_{2m}$, then $K_{2m}^* + \gamma z^m \bar{z}^m$ also it is, being γ any real parameter.
With the following result, we can consider the above mentioned coefficients null.

Lemma 3.2.56. *Let the fields (3.2.5) and (3.2.6) be, such that* $[X,U] = \mathcal{O}(|z,\bar{z}|^{2m+1})$, *where* $K_{2j} = K_{2j}^* + \gamma_j z^j \bar{z}^j$, *and let* $Cond_m^*$ *be the value which comes from to substitute* $\gamma_1 = \gamma_2 = \cdots = \gamma_{m-1} = 0$ *in* $Cond_m$. *Then,* $Cond_m = Cond_m^*$.

Proof.
By hypothesis, $Cond_1 = Cond_2 = \ldots = Cond_{m-1} = 0$, and from Theorem 3.2.55, we know that the coefficients of radial component of the normal form of the system (3.2.5), $a_1, \ldots, a_{m-1}$, are nulls. $Cond_m$ comes given by

$$Cond_m = Cond_m^* + \gamma_1 f_1 + \gamma_2 f_2 + \cdots + \gamma_{m-1} f_{m-1},$$

with $Cond_m^*$ which does not depend of $\gamma_1, \ldots, \gamma_{m-1}$, and where $f_1, \ldots, f_{m-1}$ are polynomials in the coefficients of H.

We now see that these polynomials are nulls.
Suppose $a_m \neq 0$, if some f_i was not null, taking $\gamma_i = -\frac{1}{f_i}Cond_m^*$, $\gamma_j = 0, \forall j \neq i$, we arrive at $Cond_m = 0$ which contradicts to Theorem 3.2.55. Hence, $f_i = 0,\ i = 1 \ldots m-1$, so that $Cond_m = Cond_m^*$.

Conversely, if $a_m = 0$, and if there existed an f_i non null, we can take γ_i such that $Cond_m$ was non zero, therefore, we again arrive at a contradiction. ∎

3.3 Recursive algorithm

We already present a recursive algorithm that allows to compute the conditions of compatibility that arise for the existence of the commutator, in function of the coefficients of the system.
To apply the algorithm, before we must fix the corrange of L_{2m}. For simplicity in the operations, we have chosen $Cor(L_{2m}) = Ker(L_{2m})$.
We assume that the first ones $2m-2$ components homogeneous of the commutator K are known, which, obviously, satisfy the $m-1$ first conditions that are mentioned in Theorem 3.2.54.

The algorithm consists of two steps:
Step 1.- Computation of odd component, K_{2m-1}.
From Lemma 3.2.53, this component is determined for the term that appears in the right-hand side of (3.2.8), i.e., for the homogeneous polynomials of H and K of low degree to $2m-2$. In fact, we have,

Lemma 3.3.57. *Let $P_{2m-1}(z,\bar{z}) = \sum_{j=0}^{m-1} a_j z^j \bar{z}^{2m-j-1} + \bar{a}_j z^{2m-j-1}\bar{z}^j$. The equation $L_{2m-1}(K_{2m-1}) = P_{2m-1}$ has unique solution and it is given by*

$$K_{2m-1} = -\sum_{j=0}^{m-1} \frac{a_j}{2m-2j-1} i z^j \bar{z}^{2m-j-1} - \frac{\bar{a}_j}{2m-2j-1} i z^{2m-j-1} \bar{z}^j.$$

Step 2.- Compatibility's condition and computation of the even term, K_{2m}.
Let $P_{2m}(z,\bar{z})$ be, a homogeneous polynomial of degree $2m$. It will exists a homogeneous polynomial $K_{2m}(z,\bar{z})$ verifying $L_{2m}(K_{2m}) = P_{2m}$ if and only if P_{2m} is in $L_{2m}(\tilde{\mathcal{H}}_{2m})$. That is,

$$Proy_{Cor\,L_{2m}} P_{2m} = Proy_{Ker\,L_{2m}} P_{2m} = \{\mathbf{0}\}.$$

We now give a explicit form of the compatibility's condition and the value of K_{2m} which is obtained by straightforward computation.

Lemma 3.3.58. *Let $P_{2m}(z,\bar{z}) = \sum_{j=0}^{m} a_j z^j \bar{z}^{2m-j} + \bar{a}_j z^{2m-j} \bar{z}^j$ be. Then, $Proy_{Cor\,L_{2m}} P_{2m} = 0$ if and only if $a_m = 0$, i.e., $Cond_m = a_m$.*

Lemma 3.3.59. *Let $P_{2m}(z,\bar{z}) = P_{2m}(z,\bar{z}) = \sum_{j=0}^{m-1} a_j z^j \bar{z}^{2m-j} + \bar{a}_j z^{2m-j} \bar{z}^j$ be. The equation $L_{2m}(K_{2m}) = P_{2m}$ has solution and comes given by*

$$K_{2m} = -\sum_{j=0}^{m-1} \frac{a_j}{2m-2j} i z^j \bar{z}^{2m-j} - \frac{\bar{a}_j}{2m-2j} i z^{2m-j} \bar{z}^j + \gamma_m (z\bar{z})^m.$$

For the real case, we can write the Lemmas 3.3.57, 3.3.58 y 3.3.59 of the following way.

Lemma 3.3.60. *Let $P_{2m+1}(x,y) = \sum_{j=0}^{m} a_{m,j} x^{2m-2j+1} y^{2j} + b_{m,j} x^{2m-2j} y^{2j+1}$ be. The unique solution of the equation $L_{2m+1}^{(R)}(K_{2m+1}) = P_{2m+1}$ (where $L_{2m+1}^{(R)}$ denotes the map L_{2m+1} in the real case) is*

$$K_{2m+1}(x,y) = \sum_{j=0}^{2m+1} \alpha_j x^{2m-j+1} y^j,$$

with α_j verifying

$$\left\{ \begin{array}{ll} \alpha_{2m+1} = & b_{m,m} \\ \alpha_{2j+1} = & \frac{b_{m,j} + (2j+2)\alpha_{2j+3}}{2m-2j+1} \\ & j = m-1, \cdots, 0. \end{array} \right. \qquad \left\{ \begin{array}{ll} \alpha_0 = & -a_{m,0} \\ \alpha_{2j} = & \frac{-a_{m,j} + (2m-2j+2)\alpha_{2j-2}}{2j+1} \\ & j = 1, \cdots, m. \end{array} \right.$$

Lemma 3.3.61. *Let* $P_{2m}(x,y)=\sum_{j=0}^{m} a_{m,j}x^{2m-2j}y^{2j}+b_{m,j}x^{2m-2j-1}y^{2j+1}$ *be. The equation* $L_{2m}^{(R)}(K_{2m})=P_{2m}$ *(where* $L_{2m}^{(R)}$ *denotes the map* L_{2m} *in the real case) has solution if and only if*

$$a_{m,m}+a_{m,0}+\sum_{j=1}^{m-1}\frac{(2j-1)!!(2m-2j-1)!!}{(2m-1)!!}a_{m,m-j}=0.$$

In this case, it comes given by $K_{2m}(x,y)=\sum_{j=0}^{2m}\beta_j x^{2m-j}y^j$, *with* β_j *verifying*

$$\begin{cases}\beta_1 = & -a_{m,0}\\ \beta_{2j+1} = & -\frac{a_{m,j}-(2m-2j+1)\beta_{2j-1}}{2j+1}\\ & j=1,\cdots,m-1.\end{cases}\qquad\begin{cases}\beta_{2m} = & \gamma_m\\ \beta_{2m-2j} = & \frac{b_{m,m-j}+(2m-2j+2)\beta_{2m-2j+2}}{2j}\\ & j=1,\cdots,m.\end{cases}$$

3.4 Applications

3.4.1 Time-reversible isochronous uniformly centres

The described procedure allows us to give an easy proof of the following well-known result.

Theorem 3.4.62. *The time-reversible analytic vector fields with constant angular speed are centres.*

Proof.

A vector field is called time-reversible if it has a symmetric phase portrait respect to a straight line which passes through the origin, inverting the time. Up to a linear change, we can assume that above mentioned straight line is the y-axis, in this case, the time-reversible systems (3.2.3) respect to this straight line are those which verify $H(x,-y)=-H(x,y)$. In complex coordinates, this condition over H is equivalent to say that the coefficients of the monomials of H are imaginary.

Applying the recursive algorithm (Lemmas 3.3.57 and 3.3.59), we see that the coefficients of the monomials of K are real. Therefore, H_{2m} and

$K_j H_{2m-j}$ with $j = 1,\ ...,\ 2m-1$ have only imaginary coefficients, and since this expression is real, we deduce that the coefficient of $z^m \bar{z}^m$ is zero, that is, for each $m > 1$, the proyection over $Ker\, L_{2m}$ is null, therefore, Theorem 3.2.54 holds. ∎

3.4.2 A new family of isochronous uniformly centres

We next show the following family of centres.

Theorem 3.4.63. *The systems*

$$\dot{z} = iz + z \sum_{r \geq 1} H_{rm}$$

with $H_{rm} = \frac{a_r}{m} L_m(\beta_m)\beta_m^{r-1}$, *being* $\beta_m(z,\bar{z}) \in \tilde{\mathcal{H}}_m$ *and* a_r *any real number, have a centre at the origin.*

Proof.
The proof consists of proving that the system commutes with (3.2.6), being $K = 1 + \sum_{r \geq 1} K_{rm}$ with $K_{rm} = -a_r \beta_m^r$, That is, we show that the equations (3.2.8) hold.
We see that first component K_m verifies $L_m(K_m) =< 1, H_m >$. So,

$$L_m(K_m) = L_m(-a_1\beta_m) = -a_1 L_m(\beta_m) = -mH_m.$$

We see that for $r > 1$, (3.2.8) also holds.
On the one hand, using the fact that $L(M^h) = hM^{h-1}L(M)$, with h natural and M homogeneous polynomial in $z,\ \bar{z}$, the left-hand side of (3.2.8) becomes,

$$L_{rm}(K_{rm}) = L_{rm}(-a_r\beta_m^r) = -ra_r\beta_m^{r-1}L_m(\beta_m).$$

On the other hand, given $j \geq 0$ and $l \geq 1$, with $j + l = r$, we have

$$\begin{aligned}(l-j)mK_{jm}H_{lm} &= -(l-j)ma_ja_l\beta_m^j \tfrac{1}{m} L_m(\beta_m)\beta_m^{l-1} \\ &= (j-l)a_ja_l\beta_m^{r-1}L_m(\beta_m),\end{aligned}$$

therefore, the right-hand side of (3.2.8) turns out

$$-\frac{rm}{m}a_r\beta_m^{r-1}L_m(\beta_m)+\beta_m^{r-1}L_m(\beta_m)\sum_{h=1}^{r-1}(r-2h)a_h a_{r-h} = -ra_r\beta_m^{r-1}L_m(\beta_m).$$

Remarks.- The considered systems in Theorem 3.4.63, in cartesian coordinates, can be written as

$$\left\{\begin{array}{rcl} \dot{x} & = & -y + x\sum_{r\geq 1} H_{rm}(x,y), \\ \dot{y} & = & x + y\sum_{r\geq 1} H_{rm}(x,y), \end{array}\right. \tag{3.4.9}$$

where $H_{rm} = \frac{a_r}{m}(y\partial_x\beta_m - x\partial_y\beta_m)\beta_m^{r-1}$.

If the system is polynomial, then the commutator is polynomial, and both have the same degree. In chapter 2, it is proved that they are the only ones with constant angular speed which have polynomial commutator with non-null linear part. Also, it is characterized the case when the commutator has null linear part.

There are time-reversible fields which cannot be expressed in the form (3.4.9), for instance,

$$\left\{\begin{array}{rcl} \dot{x} & = & -y + ax^2 + bx^4, \\ \dot{y} & = & x + axy + bx^3y, \end{array}\right.$$

Besides, there are non time-reversible systems of the family (3.4.9), even non homogeneous, for instance,

$$\left\{\begin{array}{rcl} \dot{x} & = & -y + xH_3 + xH_3K_3, \\ \dot{y} & = & x + yH_3 + yH_3K_3, \end{array}\right.$$

with $H_3 = 2x^3 - 6x^2y + 2xy^2 + 2y^3$ and $K_3 = 2x^3 + 6x^2y - 6xy^2 + 6y^3$. Its commutator is

$$\left\{\begin{array}{rcl} U & = & x + xK_3 + xK_3^2, \\ V & = & y + yK_3 + yK_3^2. \end{array}\right.$$

We next see that the fields of this family contain all the polynomial systems (3.2.3) with centres that we know, except the time-reversible systems.

1. The homogeneous systems (3.2.3), $\dot{z} = iz + zH_m$, studied by Conti [39].
   If m is odd, $L_m(\tilde{\mathcal{H}}_m) = \tilde{\mathcal{H}}_m$, therefore it is isochronous, since always exists $\beta_m \in \tilde{\mathcal{H}}_m$ such that $L_m(\beta_m) = H_m$. Thus, it is a particular case of the mentioned family. If m is even, the first condition different from zero is $Cond_m = Proy_{Ker\, L_m}(-mH_m)$, i. e., we must impose that $H_m \in L_m(\tilde{\mathcal{H}}_m)$, and we again see that exists β_m such that $L_m(\beta_m) = H_m$, thus, it is a subfamily of the above theorem.

2. When $m = 1$ and $\beta_1 = y$, we obtain the fields of the form $(-y + x^2\sigma(y),\ x + xy\sigma(y))^T$ with $\sigma(y)$ polynomial in y, that were studied in [127].

3. In [127] it is proved that if H and K are conjugate harmonic and $H^2(x,y) + K^2(x,y)$ is a function of x^2+y^2, then the systems (3.2.3) and (3.2.4) commute.
   We know that they are of this family, since are polynomial systems (3.2.3) with polynomial commutators, being $H = H_m = \frac{a}{m}(y\partial_x\beta_m - x\partial_y\beta_m)$ and $K = -a\beta_m$, with β_m homogeneous polynomial of degree m, satisfying $(\partial_x\beta_m)^2 + (\partial_y\beta_m)^2 = P_m(x^2+y^2),\ H_x = -K_y,\ H_y = K_x$, being P_m any homogeneous polynomial of degree m.

3.4.3 Centres of the family $H = H_1 + H_m$.

Next we characterize the systems (3.2.5) with $H = H_1 + H_m$, being H_1 and H_m not null, which have a commutator.
Since $H_1 \neq 0$, making a rotation and a rescaling on the state variables into the field, we can assume, without loss of generality, $H_1 = iz - i\bar{z}$.
The first terms of a commutator (3.2.6) of (3.2.5) with $K = 1 + \sum_{j\geq 1} K_j$ come given by:
As $L_1(K_1) = -H_1$, then $K_1 = z + \bar{z}$.
As $L_r(K_r) = 0,\ 2 \leq r \leq m-1$, from Lemma 3.2.56, we can suppose that

$K_r = 0$.
K_m satisfies the equation $L_m(K_m) = -mH_m$.
K_{m+1} verifies $L_{m+1}(K_{m+1}) = (1-m)(K_1H_m - K_mH_1)$.
Each term K_r, with $m+2 \leq r \leq 2m$, must satisfy the equation

$$L_r(K_r) = (r-2)(iz - i\bar{z})K_{r-1}.$$

Evidently, these equations relate the coefficients of the consecutive terms of the commutator. Next we mention some properties.

Lemma 3.4.64. *Let $K_r = \sum_{j=0}^{r} q_j^r z^j \bar{z}^{r-j}$ be, with $r \geq 1$, the homogeneous part of K of degree r of a commutator of the form (3.2.6) of the system (3.2.5), with $H = H_1 + H_m,\ H_1H_m \neq 0$. Then,*

1. *For l such that $m+1 \leq 2l \leq 2m-2$,*

$$\begin{cases} q_0^{2l+1} &= \frac{2l-1}{2l+1} q_0^{2l}, \\ q_j^{2l+1} &= \frac{2l-1}{2l-2j+1}(q_{j-1}^{2l} - q_j^{2l}),\ 1 \leq j \leq l. \end{cases} \tag{3.4.10}$$

2. *For l such that $m+2 \leq 2l \leq 2m$, $Cond_l = -4(l-1)Im\, q_{l-1}^{2l-1}$.*
   If $Cond_l = 0$,

$$\begin{cases} q_0^{2l} &= \frac{2l-2}{2l} q_0^{2l-1}, \\ q_j^{2l} &= \frac{2l-2}{2l-2j}(q_{j-1}^{2l-1} - q_j^{2l-1}),\ 1 \leq j \leq l-1. \end{cases} \tag{3.4.11}$$

   From Lemma 3.2.56, we can take $q_l^{2l} = 0$.

3. *For l such that $m+2 \leq 2l \leq 2m$,*

$$\begin{cases} q_0^{2l+1} &= \frac{(2l-1)(2l-2)}{(2l+1)(2l)} q_0^{2l-1}, \\ q_1^{2l+1} &= -\frac{1}{l} q_0^{2l-1} + q_1^{2l-1}, \\ q_j^{2l+1} &= \frac{(2l-1)(2l-2)}{2l-2j+1}\Big[\frac{1}{2l-2j+2} q_{j-2}^{2l-1} - (\frac{1}{2l-2j+2} + \frac{1}{2l-2j}) q_{j-1}^{2l-1} \\ & \quad + \frac{1}{2l-2j} q_j^{2l-1}\Big], \qquad 2 \leq j \leq l-1, \\ q_l^{2l+1} &= (2l-1)(l-1)(q_{l-2}^{2l-1} - q_{l-1}^{2l-1}). \end{cases} \tag{3.4.12}$$

Lemma 3.4.65. *Let* $K_r = \sum_{j=0}^{r} q_j^r z^j \bar{z}^{r-j}$ *be, with* $r \geq 1$, *the homogeneous part of* K *of degree* r *of a commutator of the form (3.2.6) of the system (3.2.5), with* $H = H_1 + H_m$, $H_1 H_m \neq 0$ *and* $m = 2n$ ($or\ m = 2n-1$). *If* $Cond_{n+1} = \ ... = \ Cond_{n+j} = 0$, *with* $1 \leq j \leq n-1$ ($or\ 1 \leq j \leq n-2$), *then*

$$Cond_{n+j+1} = -4(n+j)\begin{pmatrix} 2n+2j-1 \\ 2j \end{pmatrix} Im\, q_{n-j}^{2n+1}.$$

Proof.
Using (3.4.10), (3.4.11) and (3.4.12) it is easy to verify that:
$Cond_{n+1} = 0$ arrives at $Im\, q_n^{2n+1} = 0$,
$Cond_{n+2} = 0$ implies that $Im\, q_{n+1}^{2n+3} = Im\, q_n^{2n+2} = Im\, q_{n-1}^{2n+1} = 0$,
$Cond_{n+3} = 0$ implies that

$$Im\, q_{n+2}^{2n+5} = Im\, q_{n+1}^{2n+4} = Im\, q_n^{2n+3} = Im\, q_{n-1}^{2n+2} = Im\, q_{n-2}^{2n+1} = 0,$$

and, finally,
$Cond_{n+j} = 0$ arrives at

$$Im\, q_{n+j-1}^{2(n+j-1)+1} = Im\, q_{n+j}^{2(n+j-1)} = \ \cdots = Im\, q_{n-j+1}^{2n+1} = 0.$$

By (3.4.11), we deduce that $Cond_{n+j+1} = -4(n+j) Im\, q_{n+j}^{2n+2j+1}$. Using (3.4.12), we have

$$\begin{aligned} Im\, q_{n+j}^{2n+2j+1} &= \tfrac{(2n+2j-1)(2n+2j-2)}{2} Im\, q_{n+j-2}^{2n+2j-1} \\ &= \tfrac{(2n+2j-1)(2n+2j-2)(2n+2j-3)(2n+2j-4)}{2.3.4} Im\, q_{n+j-4}^{2n+2j-3} \\ &\vdots \\ &= \begin{pmatrix} 2n+2j-1 \\ 2j \end{pmatrix} Im\, q_{n-j}^{2n+1}. \end{aligned}$$ ∎

Theorem 3.4.66. *Let the field (3.2.5) be with* $H = H_1 + H_m$ *and* $H_1 H_m \neq 0$, *where* $m = 2n$ ($or\ m = 2n-1$).
Then,

1. *(3.2.5) is centre if and only if it is time-reversible, that is, up to rotation and a scaling, the system (3.2.5) comes given by* $\dot{z} = -iz + z(iz - i\bar{z}) + iz\sum_{j=0}^{n-1} b_j z^j \bar{z}^{m-j} - b_j z^{m-j}\bar{z}^j$*, with* b_j *real coefficients.*

2. *The maximum order of a fine focus of the system* $\dot{z} = -iz + z(iz - i\bar{z}) + z\sum_{j=0}^{[\frac{m}{2}]}(a_j + ib_j)z^j\bar{z}^{m-j} + (a_j - ib_j)z^{m-j}\bar{z}^j$*, with* a_j*,* b_j *any real numbers is* $[\frac{m}{2}] + 1$*. Besides, it is of order* $[\frac{m}{2}] + 1$ *if and only if* $a_{j-1}a_j < 0$*, with* $|a_j| << |a_{j-1}|$*,* $j = 1, \ ..., \ [\frac{m}{2}]$*.*

Proof.
Given $P_m \in \tilde{\mathcal{H}}_m$, from now on, we will denote by P_m^L and P_m^R the homogeneous polynomials such that $P_m = P_m^L + P_m^R$, with $P_m^L = \sum_{k=0}^{m} a_k z^k \bar{z}^{m-k}$.
We have seen that applying a rotation and a rescaling, we can assume $H_1 = iz - i\bar{z}$, $K_1 = z + \bar{z}$ and $K_j = 0$, $2 \le j \le m-1$.
We distinguish two cases, according to the parities of m:

1. Let $m = 2n$. The term K_{2n} must be particular solution of the equation
$$L_{2n}(K_{2n}) = -2nH_{2n}.$$
Thus, it must verify that $Proy_{Ker\, L_{2n}} H_{2n} = 0$. From Lemma 3.3.58 it follows that H_{2n} does not have the monomial $z^n\bar{z}^n$.
Let H_{2n} be with $H_{2n}^L(z, \bar{z}) = \sum_{j=0}^{n-1} h_j z^j \bar{z}^{2n-j}$, applying Lemma 3.3.59 turn out $K_{2n}^L = \sum_{j=0}^{n-1} \frac{2n}{2n-2j} i h_j z^j \bar{z}^{2n-j}$.
The term K_{2n+1} is solution of the equation
$$L_{2n+1}(K_{2n+1}) = (1-2n)(K_1 H_{2n} - K_{2n}H_1)$$
and from Lemma 3.3.57 we deduce that
$$\begin{aligned} K_{2n+1}^L = \ & -\sum_{j=1}^{n-1} \frac{2n-1}{2n-2j+1} i \left(\frac{2n-j+1}{n-j+1} h_{j-1} + \frac{j}{n-j} h_j\right) z^j \bar{z}^{2n-j+1} \\ & -(2n-1)(n+1) i h_{n-1} z^n \bar{z}^{n+1}. \end{aligned}$$

From Lemma 3.4.65, we obtain that the vanishing of the expressions $Cond_{n+1}$, $Cond_{n+2}$, ..., $Cond_{2n}$ leads to that the coefficients of K_{2n+1} must be real, that is, the coefficients of H_{2n} must be imaginaries, thus, the system is time-reversible.

2. We now suppose $m = 2n-1$ and let H_{2n-1} be with $H^L_{2n-1}(z,\bar{z}) = \sum_{j=0}^{n-1} h_j z^j \bar{z}^{2n-j}$, from Lemma 3.3.57,

$$K^L_{2n-1} = \sum_{j=0}^{n-1} \frac{2n-1}{2n-2j-1} i h_j z^j \bar{z}^{2n-j-1}.$$

We now consider the equation

$$L_{2n}(K_{2n}) = -2(n-1)(K_1 H_{2n-1} - K_{2n-1} H_1).$$

It has solution if $Proy_{Ker\, L_{2n}}(K_1 H_{2n-1} - K_{2n-1} H_1) = 0$, thus, from Lemma 3.3.58, it is clear that $h_{n-1} + \bar{h}_{n-1} = 0$, so that h_{n-1} is an imaginary number. By Lemma 3.3.59, we have $K_{2n} = K^L_{2n} + K^R_{2n} + \gamma_n z^n \bar{z}^n$, with

$$K^L_{2n} = \textstyle\sum_{j=1}^{n-1} \frac{2n-2}{2n-2j} \left(\frac{4n-j-1}{2n-j} h_{j-1} - \frac{j}{2n-j-1} h_j \right) i z^j \bar{z}^{2n-j},$$

moreover, we can take $\gamma_n = 0$.
Notice that if $Im\, q^{2n}_n = 0$, then $Im\, q^{2n+1}_l = (2n-1) Im\, q^{2n}_{n-1}$, by (3.4.10).
Moreover, from the expression of K_{2n} also is deduced that if $Im\, q^{2n}_n = Im\, q^{2n}_{n-1} = \; ... = Im\, q^{2n}_j = 0$, with $1 \le j \le n-1$, then $Re\, h_{n-1} = Re\, h_{n-2} = \; ... = Re\, h_{j-1} = 0$, therefore, the coefficients must be imaginary, and in consequence, the system must be time-reversible.

We see the second part.
If $m = 2n$, of the expression of K_{2n+1} and by (3.4.10), we have

$$q^{2n+1}_{n-j} = -\frac{(2n-1)(n+j+1)}{(2j+1)(j+1)} i h_{n-j-1}, \quad 1 \le j \le n-1,$$

hence,

$$Cond_{n+j+1} = \frac{2(2n-1)(n+j+1)}{j+1}\begin{pmatrix} 2n+2j \\ 2j+1 \end{pmatrix} Re\, h_{n-j-1},\ \ 1 \le j \le n{-}1.$$

And if $m = 2n-1$, from the expression of K_{2n}, we have

$$q^{2n}_{n-j-1} = \frac{(2n-2)(3n+j)}{(2j+2)(n+j+1)} ih_{n-j-2},\ 1 \le j \le n-2.$$

thus, $Cond_{n+j+1}$ is

$$-\frac{4(2n-1)(3n+j)(n+j)(n-1)}{(j+1)(2n-2j+1)(n+j+1)}\begin{pmatrix} 2n+2j-1 \\ 2j \end{pmatrix} Re\, h_{n-j-2},$$

with $1 \le j \le n-2$. Therefore, if a_0, ..., $a_{[\frac{m}{2}]}$ satisfy $a_{j-1}a_j < 0$ with $|a_j| << |a_{j-1}|$ for $j = 1$, ..., $[\frac{m}{2}]$, the numbers $Cond_{n+j}$ with $j = 0$, ..., $[\frac{m}{2}]$ are a Sturm's sequence, thus we have a system with $[\frac{m}{2}]+1$ limit cycles. ■

3.4.4 Centres of the family $H = H_2 + H_{2n}$.

We now characterize the systems (3.2.5), with $H = H_2 + H_{2n}$ and $H_2 H_{2n} \neq 0$, which have commutator.
Applying a rotation and a scaling into the field, we can assume that $H_2 = -iz^2 + i\bar{z}^2$.
The first terms of a commutator (3.2.6) of (3.2.5) with $K = 1 + \sum_{j\ge 1} K_j$ are:
As $L_1(K_1) = 0$, we have $K_1 = 0$. In general, the terms of odd degree of a commutator are null, since are solution of $L_{2r+1}(K_{2r+1}) = 0$ and applying Lemma 3.3.57 we deduce that $K_{2r+1} = 0$.
$K_2 = -z^2 - \bar{z}^2$ comes from $L_2(K_2) = -2H_2$.
By $L_{2r}(K_{2r}) = 0,\ 2 \le r \le n-1$ (from Lemma 3.2.56, we can suppose null parameters) we can assume $K_{2r} = 0,\ 2 \le r \le n-1$.
K_{2n} is solution of $L_{2n}(K_{2n}) = -2nH_{2n}$.

K_{2n+2} must be solution of $L_{2n+2}(K_{2n+2}) = 2(1-n)(K_2H_{2n} - K_{2n}H_2)$.
Each term K_{2r+2}, with $n+1 \le r \le 2n-1$, satisfies

$$L_{2r+2}(K_{2r+2}) = -2(r-1)(iz^2 - i\bar{z}^2)K_{2r}.$$

Evidently, these equations relate the coefficients of the consecutive terms of the commutator. We arrive at the following result, which proof we omitted for being analogous to that of Lemma 3.4.65.

Lemma 3.4.67. *Let $K_{2n+2} = \sum_{j=0}^{2n+2} q_j^{2n+2} z^j \bar{z}^{2n+2-j}$ be, the homogeneous part of K of orden $2n+2$ of a commutator of the form (3.2.6) of the system (3.2.5), with $H = H_2 + H_{2n}$, $H_2H_{2n} \neq 0$.*
Then, if $Cond_{n+1} = \ ... = \ Cond_{n+j} = 0$, with $1 \le j \le n-1$, it follows that

$$Cond_{n+j+1} = 4(-1)^j(n+j-1)\begin{pmatrix} n+j-2 \\ j-1 \end{pmatrix} Im\, q_{n-j+1}^{2n+2}.$$

Using similar arguments that in the proof of Theorem 3.4.66, we obtain the following result.

Theorem 3.4.68. *Let field (3.2.5) be, with $H = H_2 + H_{2n}$ with $H_2H_{2n} \neq 0$. Then,*

1. *(3.2.5) is centre if and only if it is time-reversible, That is, up to a rotation and scaling, the system (3.2.5) comes given by $\dot{z} = -iz + iz(-z^2 + \bar{z}^2) + iz\sum_{j=0}^{n-1} b_j z^j \bar{z}^{2n-j} - b_j z^{2n-j}\bar{z}^j$, with b_j real coefficients.*

2. *The maximum order of a fine focus of the system $\dot{z} = -iz + iz(-z^2 + \bar{z}^2) + z\sum_{j=0}^{n}(a_j + ib_j)z^j\bar{z}^{2n-j} + (a_j - ib_j)z^{2n-j}\bar{z}^j$, with a_j, b_j real constants is $n+1$. Moreover, it is of orden $n+1$ if and only if $a_{j-1}a_j > 0$, with $|a_{j-1}| << |a_j|$, $j = 1, \ ..., \ n$.*

Remark. The Lemmas 3.4.65 and 3.4.67 give necessary conditions that guarantee us the existence of the commutator of the families (3.2.5) with $H = H_1 + H_m$ and $H = H_2 + H_{2n}$, respectively. In both cases, it arrives at the vanishing, term to term, of the real coefficients of the expression of H_m and H_{2n}, respectively, and it implies that the centres are the time-reversible systems. In the case of the family (3.2.5) with $H = H_2 + H_{2n+1}$, with $H_2 = -iz^2 + i\bar{z}^2$, the first ones conditions non null are $Cond_{2n+2}$, $Cond_{2n+3}$, ..., $Cond_{4n+2}$, which correspond to the conditions of compatibility of the equations $L(K_{4n+4}) = -2K_{2n+3}H_{2n+1}$, $L(K_{4n+6}) = -4K_{2n+5}H_{2n+1}$, ..., $L(K_{8n+4}) = -(4n+2)K_{6n+3}H_{2n+1}$, being K_{2n+1}, ..., K_{6n+3}, the solutions of the functional equations

$$
\begin{aligned}
&L(K_{2n+1}) = -(2n+1)H_{2n+1},\\
&L(K_{2n+3}) = (2n-1)(K_2H_{2n+1} - K_{2n+1}H_2),\\
&L(K_{2n+2l+1}) = (3-2(n+l))K_{2(n+l)-1}H_2,\ l = 2,\ ...,\ 2n+1,
\end{aligned}
$$

respectively. The main difference with the cases $H = H_1 + H_m$ and $H = H_2 + H_{2n}$ is that these conditions are not a linear expression of the coefficients $h_j = A_j + iB_j$ of H_{2n+1}. In this case, they are bilinears expressions of the form

$$\sum_{j=1}^{n}(\sum_{l=1}^{n}\lambda_{r,l}(j)A_l)B_j = 0,\ r = 2n+2,\ ...,\ 4n+2,$$

with $\lambda_{r,l}(j)$ real. We have analyzed, the subfamilies $H = H_2 + H_{2n+1}$ for $n = 1, 2, 3$ and we have obtained that the only centres are the time-reversible ones.

3.4.5 Quartic isochronous uniformly centres.

Next we derive the quartic systems (3.2.5), that is, $H(z,\bar{z}) = H_1(z,\bar{z}) + H_2(z,\bar{z}) + H_3(z,\bar{z})$, with $H_3(z,\bar{z}) \neq 0$, with at least one nonlinearity besides H_3.

Theorem 3.4.69. *Let (3.2.5) be with* $H = H_1 + H_2 + H_3,\ H_3 \neq 0$ *and* $H_1^2 + H_2^2 \neq 0$. *Then, the origin of (3.2.5) is a centre if and only if the system (3.2.5) is time-reversible.*

Proof.

The proof is assisted by an algebraic computer, continuing the recursive algorithm described in the above section.

The nonhomogeneous quartic systems (3.2.5) take the form

$$\dot{z} = iz + z[Az + \bar{A}\bar{z} + Bz^2 + 2(b_1 + b_3)z\bar{z} + \bar{B}\bar{z}^2 + Cz^3 + Dz^2\bar{z} + \bar{D}z\bar{z}^2 + \bar{C}\bar{z}^3],$$

being $A = \frac{1}{2}(a_1 - ia_2),\ B = \frac{1}{4}(b_1 + b_3 - ib_2),\ C = \frac{1}{8}(d_1 - id_2)$ and $D = \frac{1}{8}(d_3 - id_4)$,

with a_1, a_2, b_1, b_2 and b_3 real constants not all nulls, and d_1, d_2, d_3 and d_4 real constants not all nulls. Next we compute necessary conditions in order to the existence of a commutator of the system, i.e. we will vanish the compatibility conditions $Cond_i$ which have the form $l_i = 0$. The first one is $l_1 = b_1 - b_3 = 0$, that is $b_1 = b_3$.

If $H_1 \equiv 0$ and H_2 is different from zero, it is easily seen that, making a rotation, we can take, without loss of generality, $H_2 = -\frac{b_2 i}{4}(z^2 - \bar{z}^2)$, with $b_2 \neq 0$.

In this case, the first three conditions of compatibility are nulls. If the coefficient $d_4 = 0$, the fourth one turns out $l_4 = d_2 d_3 b_2 = 0$. Thus, it must be vanished d_3 or d_2. In both cases, by means of a rotation, we can transform it into a field (3.2.5) time-reversible and from Theorem 3.4.62 we know that it is centre.

If $d_4 \neq 0$, in having imposed $l_4 = 0$, it turns out $d_1 = -\frac{d_3(3d_4 - d_2)}{d_4}$, and the fifth condition becomes $l_5 = d_3 b_2^2(-2d_2 + 3d_4) = 0$. If $d_3 = 0$, we obtain again a time-reversible system, and if $d_3 \neq 0$, we must impose that $d_2 = \frac{3}{2}d_4$. In this case, the sixth one condition is not null, therefore the system does not has a centre.

Finally, if H_1 is not null, by means of a rotation we can do $a_2 = 0$ and with a homotecia can take $a_1 = 1$. The second condition of compatibility

is $l_2 = d_4 - b_3 = 0$, that is $d_4 = b_3$. Substituting, the third condition becomes $l_3 = 2b_3 + 2d_2 + 2b_3d_1 + b_2b_3 + b_2d_2 = 0$.
Let us suppose that $b_3 = 0$, in this case, $l_3 = d_2(b_2+2) = 0$, hence, $d_2 = 0$ or $b_2 = -2$. In the first one, $d_2 = 0$, we obtain that H has its real coefficients, thus, it is centre.
In second case, $b_2 = -2$, we have $l_4 = d_2(-35+4d_3) = 0$, that is, $d_3 = \frac{35}{4}$, and now we have $l_5 = d_2 = 0$, i.e. it corresponds to the case $d_4 = 0$.
We suppose now $b_3 \neq 0$. Let's notice that this condition implies that the system is not time-reversible.
Then, we have

$$d_1 = \frac{2d_3b_3 + b_3b_2 - d_2b_2 + 2b_3 - 2d_2}{2b_3}.$$

The condition $l_4 = 0$ is given by

$$l_4 = Curve_1(d_3, b_2, b_3)b_3 + Curve_2(d_3, b_2, b_3)d_2 = 0,$$

being

$$\begin{aligned} Curve_1(d_3, b_2, b_3) &= 90 - 2d_3^2 + 52d_3 + 24b_2 + 26d_3b_2 + 7b_2^2 - 26b_3^2, \\ Curve_2(d_3, b_2, b_3) &= -90 - 24b_2 + 8d_3 - 28b_3^2 - 7b_2^2. \end{aligned}$$

We distinguish the following situations separately:

I.- If $Curve_2(d_3, b_2, b_3) = 0$, in order to vanish l_4,it must verify that $Curve_1(d_3, b_2, b_3) = 0$. The resultant of the curves $Curve_1 = 0$ and $Curve_2 = 0$ with respect to b_3 is $4\Omega_1^2(b_2, d_3)$, with

$$\Omega_1(b_2, d_3) = 189b_2^2 + 648b_2 + 364d_3b_2 + 624d_3 + 2430 - 28d_3^2. \quad (3.4.13)$$

By means of the afine change of coefficients

$$b_2 = \frac{1}{196}(7\xi + \eta - 336), \quad d_3 = \frac{1}{392}(189\xi - \eta),$$

it turns out $\Omega_1(\xi, \eta) = 7\xi\eta + 13122 = 0$. Since $\eta \neq 0$, we have that $\xi = -\frac{13122}{7\eta}$. The vanishing of l_5, l_6 and l_7 are given by the vanishing of $l_5 = \frac{l_5^*}{b_3\eta^3}$, $l_6 = \frac{l_6^*}{b_3^2\eta^4}$ and $l_7 = \frac{l_7^*}{b_3^2\eta^5}$, being l_5^*, l_6^* and l_7^* polynomial expressions that are given in function of η, d_2 and b_3. The resultant of l_5^* and $Curve_2$, l_6^* and $Curve_2$ and l_7^* and $Curve_2$ with respect to b_3 are, respectively, $\eta^{12}r_5(\eta, d_2)$, $\eta^{22}r_6(\eta, d_2)$, $\eta^{24}r_7(\eta, d_2)$. In turn, the resultant of r_5 and r_6, and r_5 and r_7 with respect to η are, respectively, polynomial expressions of the form $d_2^{56}r_{56}(d_2)$ and $d_2^{56}r_{57}(d_2)$. Last on, the resultant of these with respect to d_2 is a number different from zero, therefore, we conclude that it can not be a centre.

II.- If $Curve_2 \neq 0$, in having imposed $l_4 = 0$, it turns out

$$d_2 = -\frac{b_3(b_3^2 + 90 + 22d_3 + 7b_2^2 + 24b_2 - d_3^2 + 13d_3b_2)}{(90 - 8d_3 + 28b_3^2 + 7b_2^2 + 24b_2)}.$$

Similarly to that we made in the previous paragraphs we obtain that in order that l_5, l_6, l_7 and l_8 are nulls, there must be vanished the polynomials l_5^*, l_6^*, l_7^* and l_8^*, which appear in the numerator of l_5, l_6, l_7 and l_8, respectively. The resultant of l_5^* and l_6^*, l_5^* and l_7^* and l_5^* and l_8^* with respect to b_3 are, respectively:

$$\begin{aligned}
R_1(l_5^*, l_6^*, b_3) &= d_3^4\Omega_1^{12}(b_2, d_3)\Omega_2^2(b_2, d_3)\lambda_1^2(b_2, d_3),\\
R_2(l_5^*, l_7^*, b_3) &= d_3^4\Omega_1^{12}(b_2, d_3)\Omega_2^2(b_2, d_3)\lambda_2^2(b_2, d_3),\\
R_3(l_5^*, l_8^*, b_3) &= d_3^4\Omega_1^{16}(b_2, d_3)\Omega_2^4(b_2, d_3)\lambda_3^2(b_2, d_3).
\end{aligned}$$

being $\Omega_2 = 2b_2 + 4d_3 - 31$ and Ω_1 the above expression gives in (3.4.13).
Therefore in order to vanish the resultants we have four options:

- II.a) If $d_3 = 0$, the coefficients l_5^*, l_6^* and l_7^* come given by

$$l_5^* = b_3^3F_1(b_2, b_3),\ l_6^* = b_3^3G_1(b_2, b_3),\ l_7^* = b_3^3H_1(b_2, b_3),$$

  and the resultants are

$$\begin{aligned}
R(F_1, G_1, b_2) &= b_3^{10}(16b_3^2 + 1225)\Xi_1(b_3),\\
R(F_1, H_1, b_2) &= b_3^{10}(16b_3^2 + 1225)\Gamma_1(b_3).
\end{aligned}$$

Since $b_3 \neq 0$, the first and second factors of the above resultants cannot be vanished. Moreover, $\Xi_1(b_3)$ and $\Gamma_1(b_3)$ are not vanished simultaneously, so its resultant with respect to b_3 is a constant differently from zero.

- II.b) If $\Omega_1 = 0$, making the same change of coefficients that in the paragraph I and similarity coming, we deduce that $l_5,\ l_6$ and l_7 cannot be vanished simultaneous.
- II.c) If $\Omega_2 = 0$, we have $d_3 = \frac{1}{4}(31 - 2b_2)$ and, in this case, we obtain $l_5^* = b_3 F_2(b_2, b_3),\ l_6^* = b_3 G_2(b_2, b_3)$ and $l_7^* = b_3 H_2(b_2, b_3)$, and the resultants are

$$R(F_2, G_2, b_2) = b_3^4(1486873b_3^2 + 442225)^2 \Xi_2(b_3),$$
$$R(F_2, H_2, b_2) = b_3^4(1486873b_3^2 + 442225)^2 \Gamma_2(b_3),$$

  And since the resultant of $\Xi_2(b_3)$ and $\Gamma_2(b_3)$ with respect to b_3 is a not null constant, we come to that cannot be vanished simultaneously.
- II.d) Finally, the resultant of $\lambda_1(b_2, d_3)$ and $\lambda_2(b_2, d_3)$ and it of $\lambda_1(b_2, d_3)$ and $\lambda_3(b_2, d_3)$ with respect to b_2 are polynomials in d_3, and the resultant of both polynomials is a constant differently from zero. Therefore, $\lambda_1,\ \lambda_2$ and λ_3 do not have common real roots. ■

3.4.6 Quintic isochronous uniformly centres.

The vector fields we are interested on can be expressed as

$$(\dot{x}, \dot{y})^T = (-y + x\sum_{i=1}^{4} H_i(x, y), x + y\sum_{i=1}^{4} H_i(x, y))^T, \qquad (3.4.14)$$

with H_i homogeneous polynomials of degree i. From now on, we assume that $H_4 \neq 0$ and H_1, H_2 and H_3 not all nulls. We consider the vector field written as follows:

$$\begin{aligned}
H_1(x,y) &= 2a_1x - 2a_2y,\\
H_2(x,y) &= (b_3+2b_1)x^2 - 4b_2xy + (-2b_1+b_3)y^2,\\
H_3(x,y) &= (2c_1+2c_3)x^3 + (-6c_2-2c_4)x^2y + (2c_3-6c_1)xy^2\\
&\quad +(-2c_4+2c_2)y^3,\\
H_4(x,y) &= (2d_3+2d_1+d_5)x^4 + (-4d_4-8d_2)x^3y + (2d_5-12d_1)x^2y^2\\
&\quad +(8d_2-4d_4)xy^3 + (2d_1-2d_3+d_5)y^4.
\end{aligned}$$

The following result characterizes the centres of (3.4.14).

Theorem 3.4.70. *Let the vector field (3.4.14) be with one nonlinearity besides H_4 only. Then, the origin is a centre of (3.4.14) if and only if it is time-reversible.*

Proof.

Throughout the proof, we shall denote by $R(p,q,x)$ the resultant of p and q respect to x and by l_i the compatibility condition $Cond_i$ before defined. Equalities involving resultants are not exact. Numbers that appear multiplying or factor degrees have been dropped, since they are not important for the study.

Case $H_2 = H_3 = 0$.

Without lost of generality we can suppose $a_1 = 1$ and $a_2 = 0$. Vanishing l_2, l_3 and l_4 leads us to $d_1 = d_3 = d_5 = 0$. The field is time-reversible.

Case $H_1 = H_3 = 0$.

Since $l_1 = b_3$, if $b_3 \neq 0$ then the origin will not be a centre. Assumed $b_3 = 0$, since $H_2 \neq 0$, we can suppose without lost of generality $|b_1| = 1$ and $b_2 = 0$. Then, vanishing l_2, l_3 and l_4 leads to $d_1 = d_4 = d_5 = 0$. That is, to a time-reversible field.

Case $H_1 = H_2 = c_1 = c_2 = 0$.

Without lost of generality we can suppose $c_3 = 1$ and $c_4 = 0$. Then l_2, l_3, l_4 y l_5 vanish altogether if and only if $d_3 = d_5 = 0$. In that case, the field is time-reversible.

Case $H_1 = H_2 = 0$, $(c_1, c_2) \neq (0,0)$.

Without lost of generality we can suppose $c_1 = 1$ and $c_2 = 0$. Moreover, since $l_2 = d_5$, it has to be $b_5 = 0$. Then we jump to l_5 and begin to distinguish different possibilities.

Case 1: $c_3 \neq 0$**.** This condition allows us to obtain d_3 from l_5. Substituting d_3 in l_8 it can take place two situations. Let $cond_1$ be $c_4(3c_3^2 + 3c_4^2 - 2)(3c_3^2 - c_4^2)$.

We distinguish two situations:

Case 1.A: $cond_1 \neq 0$**.** Under this condition, we can obtain d_2 from l_8 and substitute it in the rest of the normal form coefficients.

From now on, we are going to work with the numerators of l_7, l_9, l_{10} y l_{12}. Let them be nl_7, nl_9, nl_{10} and nl_{12}, respectively. We notice that $R(nl_7, nl_9, d_3)$, $R(nl_7, nl_{10}, d_3)$ and $R(nl_7, nl_{12}, d_3)$ share the same factors: $3c_3^2 - c_4^2$, c_3, d_4, c_4 and $3c_3^2 + 3c_4^2 - 2$. Moreover, each one has got another *big* factor which just depends on c_3 and c_4: f_{79}, f_{710} and f_{712}, respectively. $R(f_{79}, f_{710}, c_4)$ and $R(f_{710}, f_{712}, c_4)$ also share the same structure, $(3c_3^2 - 6c_3 + 2)(1 + 6c_3 + 6c_3^2)$, plus a particular third factor: q_{910} and q_{1012}, respectively. Since $R(q_{910}, q_{1012}, c_3) \neq 0$, we know that $q_{910} \neq 0$ or $q_{1012} \neq 0$.

Case 1.A.1: $q_{910} \neq 0$**.** Then $R(f_{79}, f_{710}, c_4)$ vanishes if and only if $(3c_3^2 - 6c_3 + 2)(1 + 6c_3 + 6c_3^2) = 0$. That is, if and only if $c_3 \in U = \{1 \pm \frac{\sqrt{3}}{3}, -\frac{1}{2} \pm \frac{\sqrt{3}}{6}\}$.

Case 1.A.1.a: $c_3 \notin U$**.** Then $f_{79} \neq 0$ or $f_{710} \neq 0$.

Case 1.A.1.a.i: $f_{79} \neq 0$**.** If $d_4 \neq 0$ then $R(nl_7, nl_9, d_3) \neq 0$ and, therefore, $nl_7 \neq 0$ or $nl_9 \neq 0$. The origin is not a centre. So, it must be $d_4 = 0$. Then nl_7, nl_9, nl_{10} and nl_{12} have got analogous shapes: $c_3 d_3 u$, $c_3 d_3 (3c_3^2 - c_4^2)(3c_3^2 + 3c_4^2 - 2)v$, $c_3 d_3 (3c_3^2 - c_4^2) w$ and $c_3 d_3 (3c_3^2 - c_4^2) z$, respectively. Here u, v, w, z are polynomials depending just on c_3 and c_4. Computing $R(u, v, c_4)$ and $R(w, z, c_4)$ we obtain again the same factors, besides particular ones that we shall call g_{uv} and g_{wx}, respectively. The common factors are c_3, $c_3 - 1$, $3c_3^2 - 6c_3 + 2$, $2c_3 + 1$, $1 + 6c_3 + 6c_3^2$ and

$54c_3^5-18c_3^4-45c_3^3+6c_3^2+9c_3+1$. Let pol_1 be this last polynomial. Since $R(g_{uv},g_{wz},c_3)\neq 0$, we've got $g_{uv}\neq 0$ or $g_{wz}\neq 0$. Let's suppose $g_{uv}\neq 0$ (case $g_{wz}\neq 0$ is analogous). Then, vanishing $R(u,v,c_4)$ only takes place if $c_3=1$, $c_3=-\frac{1}{2}$ or $pol_1=0$.

⇒ If $R(u,v,c_4)$ doesn't vanish, then $u\neq 0$ or $v\neq 0$. If $u\neq 0$, nl_7 won't vanish unless $d_3=0$. But it would lead us to a quartic field. If $v\neq 0$, we are in the same situation with nl_9 instead of nl_7.

⇒ If $c_3=1$, then $u=c_4\tilde{u}$ and $v=c_4\tilde{v}$. We've got $R(\tilde{u},\tilde{v},c_4)\neq 0$. Therefore, $\tilde{u}\neq 0$ or $\tilde{v}\neq 0$. Since $c_4\neq 0$, it implies $u\neq 0$ or $v\neq 0$. We finish as above. The case $c_3=-\frac{1}{2}$ is analogous, with the factor $4c_4^2-3$ instead of c_4 in u and v.

⇒ The case $pol_1=0$ is a bit more difficult, since we are not able to compute its roots. $R(u,pol_1,c_3)$ and $R(v,pol_1,c_3)$ only vanish simultaneously if $pol_2=972c_4^{10}-1512c_4^8+783c_4^6-144c_4^4+3c_4^2+1=0$. Notice that if one of those resultants didn't vanish, since $pol_1=0$ it would lead us to $u\neq 0$ or $v\neq 0$, and then it would have to be $d_3=0$ in order to vanish nl_7 or nl_9. That is, a quartic field. So, we have $pol_2=0$ and, therefore, it's enough to prove that $pol_1=0$ plus $pol_2=0$ plus $cond_1\neq 0$ implies $u\neq 0$ or $v\neq 0$. Making $c_4=\sqrt{\frac{2-3c_3^2}{3}+A}$, it has that $R(pol_1,pol_2,c_3)=Apol_3$ with $pol_3(0)\neq 0$. Since pol_1 and pol_2 have got common roots, this resultant has to vanish. That is, $A=0$ or $A\neq 0$ and $pol_3=0$. The first option is not possible as it would lead us to $cond_1=0$. So, it has to be $pol_3=0$. Let's now consider $R(u,v,c_3)$. It has got two factors, A and pol_4, with $pol_4(0)\neq 0$. We can see that $R(pol_3,pol_4,A)\neq 0$, which joint to $pol_3=0$ means $pol_4\neq 0$. Consequently $R(u,v,c_3)\neq 0$ and, of course, $u\neq 0$ or $v\neq 0$.

Case 1.A.1.a.ii: $f_{710}\neq 0$**.** It has got the same development.

Case 1.A.1.b: $c_3\in U$**.** There are four possibilities, but all of them can be developed in the following way. We substitute the value of c_3 both in f_{79} and in f_{710}. Most of the obtained factors can't vanish since $cond_1\neq 0$. We compare the other factors using resultants and we conclude they can't vanish simultaneously. Consequently, either $f_{79} \neq 0$ or $f_{710} \neq 0$.

$\Rightarrow$ If $f_{79} \neq 0$, vanishing $R(nl_7, nl_9, d_3)$ (what is a necessary condition for the origin to be a centre) only takes place if $d_4 = 0$. Assuming this value for d_4 and studying factorizations of nl_7 and nl_9 (or nl_7 and nl_{10}) directly, we conclude that they only can vanish simultaneously if $d_3 = 0$. But then the vector field has lost again one degree.

$\Rightarrow$ If $f_{710} \neq 0$, we operate as above.

Case 1.A.2: $q_{910} \neq 0$**.** We can study this case following the same steps than in the one above.

Case 1.B: $cond_1 = 0$**.** We distinguish three possible situations.

Case 1.B.1: $c_4 = 0$**.** In l_8, three different factors take part: d_3, c_3 and $3c_3^2 - 6c_3 + 2$. Only the first and the third ones can vanish. If $d_3 = 0$ the field is time-reversible. So, let's suppose $d_3 \neq 0$ and $3c_3^2 - 6c_3 + 2 = 0$, that is, $c_3 = 1 \pm \frac{\sqrt{3}}{3}$. If $c_3 = 1 + \frac{\sqrt{3}}{3}$ then vanishing a_7 leads us to $d_4 = \frac{11\sqrt{3}+5}{39} d_2$. It implies that nl_{10} vanishes if and only if $d_2 = 0$. After this substitution, we need $d_3 = 0$ to vanish l_{12}. But we've got then a quartic field. The study is analogous for $c_3 = 1 - \frac{\sqrt{3}}{3}$.

Case 1.B.2: $c_4 \neq 0$ **and** $c_4^2 - 3c_3^2 = 0$**.** Let's suppose $c_4 = \sqrt{3}c_3$. Then, the following four factors appear in l_8: $d_3 - \sqrt{3}d_4$, $6c_3 + 3 + \sqrt{3}$, $-6c_3 - 3 + \sqrt{3}$ and c_3. If $d_3 = \sqrt{3}d_4$ the field is time-reversible. Let's suppose $d_3 \neq \sqrt{3}d_4$ and $c_3 = -\frac{1}{2} - \frac{\sqrt{3}}{6}$. Then, vanishing l_7 allows us to get d_2. And then we can get d_3 from l_{10}. Finally, we see that d_4 had to be 0 (quartic field) to vanish l_{12}. Case $d_3 \neq \sqrt{3}d_4$ and $c_3 = -\frac{1}{2} + \frac{\sqrt{3}}{6}$ is analogous. The same steps have to be followed if we suppose $c_4 = -\sqrt{3}c_3$ at the beginning.

Case 1.B.3: $c_4 \neq 0$**,** $c_4^2 - 3c_3^2 \neq 0$ **and** $\widetilde{cond}_1 = 3c_3^2 + 3c_4^2 - 2 = 0$**.** The coefficient l_8 has got this structure: $cond_2 d_3$+something. Two factors, c_3 and pol_1, take part in $R(\widetilde{l_1}, cond_2, c_4)$, and only the second one can be null.

Case 1.B.3.a: $pol_1 \neq 0$**.** Then, since $\widetilde{l_1} = 0$, it has to be $cond_2 \neq 0$ and we can get d_3 from l_8. The coefficient nl_7 is given then by $d_4 c_4 (c_4^2 -$

$3c_3^2)p_7(c_3, c_4, d_2, d_4)$. If $d_4 = 0$, we need d_2 to vanish l_{10}. But the field would be quartic then. So it has to be $d_4 \neq 0$ and $p_7 = 0$. Therefore, $l_7 = cond_3 d_2$+something. In $R(\widetilde{cond}_1, cond_3, c_4)$ no factor can vanishes. So it has to be $cond_3 \neq 0$, what allows us to obtain d_2 from p_7. The coefficients nl_9 and nl_{10} share the same factors, c_4, d_4 and $c_4^2 - 3c_3^2$, beside two particular ones, p_9 and p_{10}, respectively. Only these last factors can be null. We compute $R(p_9, \widetilde{cond}_1, c_4)$ and $R(p_{10}, \widetilde{cond}_1, c_4)$. They vanish simultaneously if $c_3 = \pm\frac{1}{\sqrt{6}}$, but this leads c_4 to be $\pm\frac{1}{\sqrt{2}}$ and for both pairs of values $p_9 \neq 0$. That is, $nl_9 \neq 0$. There is left one way to vanish both resultants simultaneously. It consists on vanishing two different polynomials than appear in their factorizations: q_9 and q_{10} respectively. But $R(q_9, q_{10}, c_3) \neq 0$ and, consequently, at least one of them is non null. That is, $p_9 \neq 0$ or $p_{10} \neq 0$ and, therefore, nl_9 or $nl_{10} \neq 0$.

Case 1.B.3.b: $pol_1 = 0$. We can get c_4^2 from $\widetilde{cond}_1$ and substitute it in $cond_2$. We see then that it has got two factors, c_3 and pol_1. Therefore, $cond_2 = 0$ and nl_8 factors are d_4, c_4 and a polynomial p_8 on c_3 and c_4.

Case 1.B.3.b.i: $d_4 = 0$. Then $nl_7 = d_3 p_7(d_2, d_3, c_3, c_4)$. If $d_3 = 0$ then the origin is not a centre since it would have to be $d_2 = 0$ to vanish l_{10} (what would lead us to a quartic system). Let's suppose $d_3 \neq 0$ and $p_7 = 0$. This polynomial has got the structure $cond_3 d_2$+something. Since $R(\widetilde{cond}_1, cond_3, c_4) = (12c_3^2 - 4 - 3c_3)$ and $R(12c_3^2 - 4 - 3c_3, pol_1, c_3) \neq 0$ we conclude that $cond_3$ has to be non null. So, we can get d_2 from p_7. Factorizing then na_9 we obtain d_3, c_3, $cond_3$ and $p_9(c_3, c_4)$. Only the last factor can vanish. But that's impossible since $R(\widetilde{cond}_1, p_9, c_4) = \tilde{p_9}$ and $R(\tilde{p_9}, pol_1, c_3) \neq 0$.

Case 1.B.3.b.ii: $d_4 \neq 0$ **and** $p_8 = 0$. The origin is not a centre since $R(\widetilde{cond}_1, p_8, c_4) = c_3\tilde{p_8}$ and $R(\tilde{p_8}, pol_1, c_3) \neq 0$.

Case 2: $c_3 = 0$ The coefficient l_5 has got two factors, c_4 and $3d_3c_4 - 2d_4 - 3d_2$. So we've got two possibilities.

Case 2.A: $c_4 = 0$ Then l_7 has got an unique factor, $d_3d_3 + d_1d_4$.

$\Rightarrow$ If $d_1 = 0$, vanishing l_7 implies two possible situations, $d_2 = 0$ or $d_2 \neq 0$

and $d_3 = 0$. The first one, after factorizing l_9 and assuming ($d_3 = 0$ or $d_3 \neq 0$ and $d_3^2 - 3d_4^2 = 0$) all different options, leads us to time-reversible fields. The second one gives directly a time-reversible field.

$\Rightarrow$ If $d_1 \neq 0$, vanishing l_7 implies $d_4 = -\frac{d_3 d_2}{d_1}$. Then na_9 has got three different factors, d_3, $d_1^2 - 3d_2^2$ and $-231d_1^2 + 144d_3d_1 + 140d_3^2$. Assuming $d_3 = 0$, we notice that it has to be $d_1^2 - 3d_2^2 = 0$ to vanish l_{12}, what means a time-reversible system. If fact this condition implies reversibility by itself. So, there's only one case left: $d_3 \neq 0$, $d_1^2 - 3d_2^2 \neq 0$ and $-231d_1^2 + 144d_3d_1 + 140d_3^2 = 0$. Let λ_1 and λ_2 be the roots of $-231x^2 + 144x + 140$. If $d_1 = \lambda_1 d_3$ then in nl_{12} appear two factors, d_3 and $70^2 d_3^2 - x_3^2 d_2^2$, with $x_3 = \sqrt{32031 - 216\sqrt{9381}}$. The first one is non null. Therefore, the second one has to vanish. That is, d_2 has to be $\pm\frac{70}{x_3}d_3$. And this is a time-reversible field. The case $d_1 = \lambda_2 d_3$ is analogous.

Case 2.B: $c_4 \neq 0$ **and** $d_2 = (3d_3c_4 - 2d_4)/2$**.** We have got two factors in nl_8, c_4 and $(2 - 3c_4^2)d_1 + (6c_4^2 - 2)d_3$.

Case 2.B.1: $3c_4^2 - 2 \neq 0$**.** We get d_1 from nl_8.

The resultants $R(nl_7/c_4, nl_9, d_3)$ and $R(na_{10}/c_4, nl_{12}, d_3)$ share two factors, d_4 and $3c_4^2 - 2$, but both have got a third one different from each other, $t_1(c_4)$ and $t_2(c_4)$, respectively. Since $R(t_1, t_2, c_4) \neq 0$ it has to be $t_1 \neq 0$ or $t_2 \neq 0$. If $t_1 \neq 0$, it has to be $d_4 = 0$ for the origin to be a centre. But substituting this value in nl_7 and nl_9 shows that at least one of then is non null. So, it's not a centre anyway. The same happens if $t_2 \neq 0$.

Case 2.B.2: $3c_4^2 - 2 = 0$**.** Let's suppose $c_4 = \sqrt{\frac{2}{3}}$ (case minus is analogous). Then vanishing l_8 implies $d_3 = 0$. The coefficient nl_9 has got two factors, d_4 and $2862d_4^2 - 1002d_1^2 + 589\sqrt{6}d_4d_1$. If $d_4 = 0$ it has to be $d_1 = 0$ to vanish l_{12}. It means a quartic system. So, let's suppose $d_4 \neq 0$ and $2862d_4^2 - 1002d_1^2 + 589\sqrt{6}d_4d_1 = 0$. If x_1 and x_2 are the roots of the polynomial $2862 - 1002x^2 + 589\sqrt{6}x = 0$, there are two possibilities, $d_1 = x_1d_4$ y $d_1 = x_2d_4$. In both cases, $l_{10} \neq 0$ and, therefore, the origin is not a centre. ∎

Chapter 4

The centres of a family of nilpotent systems.

4.1 Introduction

In the present chapter, fixed $p, q, n \in \mathbb{N}$ with $p \le q$, we consider the system of differential equations in the plane whose origin is a nilpotent singular point

$$\dot{x} = y + \sum_{i=1}^{\infty} P_{q-p+2is}(x,y), \qquad \dot{y} = \sum_{i=1}^{\infty} Q_{q-p+2is}(x,y), \tag{4.1.1}$$

where $s = (n+1)p - q > 0$ and $\mathbf{F}_i = (P_i, Q_i)^T$ is a vector field quasi-homogeneous of type (p,q) and degree i with $Q_{(2n+1)p-q}(1,0) < 0$. That is, according to the degree, $\mathbf{F}_{q-p} = (y,0)^T$ is the quasi-homogeneous component of minor degree, the second one is $\mathbf{F}_{(2n+1)p-q}$ which, among others, has the term $(0, -x^{2n+1})^T$, and the edges of its Newton diagram (see Bruno [34]) of the remaining components are parallel to the edge associated to $\mathbf{F}_{(2n+1)p-q}$ and the distance between two edges is multiple of $2s$. This class includes, among others, the nilpotent systems which are invariant to the change of variables $(x,y) \to (-x,-y)$. In particular, it includes the family $\dot{x} = y + X_{2n+1}(x,y)$, $\dot{y} = Y_{2n+1}(x,y)$ where X_{2n+1} and

Y_{2n+1} are homogeneous polynomials of degree $2n+1$ with $Y_{2n+1}(1,0) < 0$ (case $p = q = 1, P_{2n}(x,y) = X_{2n+1}(x,y),\ Q_{2n}(x,y) = Y_{2n+1}(x,y),\ P_{2i} = Q_{2i} = 0,\ i > n$ in (4.1.1)).

We detail the results obtained in this chapter. In the second section, we define the concepts of quasi-homogeneous function and vector field y show some properties which be used here and the chapters below. In the third section, we prove that for system (4.1.1) there exists a Lyapunov function of class $\mathcal{C}^\infty$ which can be formally expanded in the form $W = \frac{1}{2}y^2 + \sum_{l=1}^{\infty} W_{2q+2sl}$ where W_{2q+2sl} is a quasi-homogeneous function of type (p,q) and degree $2q+2sl,\ l \geq 1$. This result allows us to solve theoretically the centre problem for the system (4.1.1). In the fourth section, we give the Taylor expansion of the return map of the system (4.1.1). This result allow us to generate limit cycles bifurcating from the origin of the system. Finally, as an application, we characterize the centres of several families of (4.1.1) and we give the number of small amplitude limit cycles bifurcating from the origin. Concretely, we analyze the family

$$\begin{aligned} \dot{x} =& \ y + a_1x^5 + a_2x^2y + a_3x^7 + a_4x^4y + a_5xy^2, \\ \dot{y} =& \ \ -x^7 + b_1x^4y - a_2xy^2 + b_3x^6y + b_4x^3y^2 + b_5y^3, \end{aligned}$$

the families $(\dot{x}, \dot{y})^T = \mathbf{F}_2 + \mathbf{F}_i$ with $\mathbf{F}_2 = (y,0)^T,\ \mathbf{F}_i$ quasi-homogeneous vector fields of type $\mathbf{t} = (1, i-1)$ and degree i, with $i = 4, 6, 8$,
and the family $(\dot{x}, \dot{y})^T = \mathbf{F}_2 + \mathbf{F}_{52}$, where

$$\mathbf{F}_2 = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \mathbf{F}_{52} = \begin{pmatrix} a_1x^{15}y^2 + a_2x^{10}y^5 + a_3x^5y^8 + a_4y^{11} \\ -x^{19} + b_1x^{14}y^3 + b_2x^9y^6 + b_3x^4y^9 \end{pmatrix},$$

are quasi-homogeneous vector fields of type $\mathbf{t} = (3,5)$ and degree 2 and 52, respectively.
We find subfamilies of nilpotent centres which are neither a hamiltonian one, nor a time-reversible system (i.e. in this case, the system isn't invariant neither to the change $(x,y,t) \to (-x,y,-t)$ nor to $(x,y,t) \to (x,-y,-t)$).

Applying Proposition 4.5.81 we prove that the systems obtained have a centre at the origin. In particular, we conclude that all have a local analytic first integral at the origin and they can be written in the form

$$\dot{x} = y + v_y K(v, y^2) + y\Psi(v, y^2), \qquad \dot{y} = -v_x K(v, y^2)$$

where v, K, Ψ are analytic functions defined in a neighborhood of O with $\Psi(O) = 0$.

4.2 Quasi-homogeneous functions and vector fields

In order to express our results, we need to recall the concepts of quasi-homogeneous function and vector field and to show some properties of them.

Definition 4.2.71. *Given* $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$ *and* $k \in \mathbb{Z}$*, a function* f *is a quasi-homogeneous function of type* $\mathbf{t}$ *and degree* k *(also called* $\mathbf{t}$*-homogeneous function of degree* k*) if* $f(\varepsilon^{t_1}x_1, \ldots, \varepsilon^{t_n}x_n) = \varepsilon^k f(x_1, \ldots, x_n)$,

So, if f is a $\mathbf{t}$-homogeneous smooth function at the origin, its monomials $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} x_1^{a_1} \ldots x_n^{a_n}$ satisfy

$$a_1t_1 + a_2t_2 + \cdots + a_nt_n = k.$$

We will denote by $\mathcal{P}_k^{\mathbf{t}}$ the vector space of $\mathbf{t}$-homogeneous polynomials of degree k.

Definition 4.2.72. *Given* $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$ *and* $k \in \mathbb{Z}$*, a vector field* $\mathbf{F} = (F_1, \ldots, F_n)^T$ *is* $\mathbf{t}$*-homogeneous of degree* k *if each component* F_i *is a* $\mathbf{t}$*-homogeneous function of degree* $k + t_i$*,* $i = 1, \ldots, n$.

So, a polynomial vector field $\mathbf{F} = (F_1, \ldots, F_n)^T$ is $\mathbf{t}$-homogeneous of degree k if $F_i \in \mathcal{P}_{k+t_i}^{\mathbf{t}}$, $i = 1, \ldots, n$.

The vector space of $\mathbf{t}$-homogeneous polynomial vector fields of degree k will be denoted by $\mathcal{Q}_k^{\mathbf{t}}$.

In particular, the set $\mathcal{P}_k^{(1,\ldots,1)}$ is the vectorial space of the homogeneous polynomials of degree k and $\mathcal{Q}_k^{(1,\ldots,1)}$ is the vectorial space of the homogeneous vector fields of degree $k+1$.

Given a type $\mathbf{t} \in \mathbb{N}^n$, each smooth vector field $\mathbf{F}$, with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, can be written as $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$ where $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$. In some sense, $\mathbf{F}$ can be understood as a perturbation of $\mathbf{F}_r$ with higher-order $\mathbf{t}$-homogeneous terms. Thus, for any M we can define the $\mathbf{t}$-homogeneous M-jet of a smooth vector field F as $\mathcal{J}_{\mathbf{t}}^M \mathbf{F} = \sum_{j=r}^M \mathbf{F}_j$ with $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$.

For instance, the expansion in $(2,3)$-homogeneous terms of the vector field $\mathbf{F} = (\sum p_{ij} x^i y^j, \sum q_{ij} x^i y^j)^T$, is

$$\mathbf{F} = \underbrace{\begin{pmatrix} 0 \\ q_{10}x \end{pmatrix}}_{\mathbf{F}_{-1}} + \underbrace{\begin{pmatrix} p_{10}x \\ q_{01}y \end{pmatrix}}_{\mathbf{F}_0} + \underbrace{\begin{pmatrix} p_{01}y \\ q_{20}x^2 \end{pmatrix}}_{\mathbf{F}_1} + \underbrace{\begin{pmatrix} p_{20}x^2 \\ q_{11}xy \end{pmatrix}}_{\mathbf{F}_2} + \underbrace{\begin{pmatrix} p_{11}xy \\ q_{02}y^2 \end{pmatrix}}_{\mathbf{F}_3} + \cdots$$

Note that the degree of a $\mathbf{t}$-homogeneous polynomial is non-negative, nevertheless the degree of a $\mathbf{t}$-homogeneous polynomial vector field can be negative.

There is not loss of generality in assuming that $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$ with $1 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ since if $t_i > t_{i+j}$, we interchange the variables x_i and x_{i+j}.

We show several properties which can be obtained easily.

Lemma 4.2.73. *The following properties are satisfied:*

i) *If* $P \in \mathcal{P}_i^{\mathbf{t}}$ *and* $\mathbf{F} \in \mathcal{Q}_j^{\mathbf{t}}$, *then* $L_{\mathbf{F}} P \in \mathcal{P}_{i+j}^{\mathbf{t}}$, *where* $L_{\mathbf{F}} P$ *is the Lie derivative of* P *by* $\mathbf{F}$ *defined by* $L_{\mathbf{F}} P := \frac{\partial P}{\partial x_1} F_1 + \cdots + \frac{\partial P}{\partial x_n} F_n$.

ii) *Let* $\mathbf{D}_0(\mathbf{x}) = \operatorname{diag}(t_1, \ldots, t_n)\mathbf{x} \in \mathcal{Q}_0^{\mathbf{t}}$. *If* $P \in \mathcal{P}_i^{\mathbf{t}}$ *then* $L_{\mathbf{D}_0} P = iP$, *(Euler Theorem for quasi-homogeneous functions).*

iii) *If* $\mathbf{F} \in \mathcal{Q}_j^{\mathbf{t}}$ *and* $\mathbf{G} \in \mathcal{Q}_k^{\mathbf{t}}$, *then* $\mathbf{F} \wedge \mathbf{G} \in \mathcal{P}_{j+k+|\mathbf{t}|}^{\mathbf{t}}$, *with* $|\mathbf{t}| = t_1 + \cdots + t_n$ *being* $\mathbf{F} \wedge \mathbf{G}$ *the wedge product of* $\mathbf{F}$ *and* $\mathbf{G}$.

iv) *If* $\mathbf{F} \in \mathcal{Q}_i^{\mathbf{t}}$, *then* $[\mathbf{F}, \mathbf{D}_0] = i\mathbf{F}$.

In Algaba it et al. [4] and García [75] we can find several properties of the $\mathbf{t}$-homogeneous functions and the $\mathbf{t}$-homogeneous vector fields.

4.3 Generalized Liapunov function.

We consider the analytic system of differential equations

$$(\dot{x}, \dot{y})^T = \sum_{i=0}^{\infty} \mathbf{F}_{q-p+2is}, \tag{4.3.2}$$

where $p, q \in \mathbb{N}, p \leq q$ and without common factors, $s = (n+1)p - q > 0$, $n \in \mathbb{N}$ and $\mathbf{F}_i = (P_i, Q_i)^T$ are quasi-homogeneous vector fields of type $\mathbf{t} = (p, q)$ and degree i, with $\mathbf{F}_{q-p} = (y, 0)^T$ and $Q_{q-p+2s}(1, 0) < 0$ (without loss of generality, we can assume $Q_{q-p+2s}(1, 0) = -1$). In this system, this last condition implies that the germ is monodromic at O, see Andreev [22].

Note that if p or q is even, then the origin is a centre of (4.3.2). Indeed, we assume, for instance, p is even then q will be odd (since p and q have no common factors), in that case $P_{q-p+2is}(x, -y) = -P_{q-p+2is}(x, y)$ and $Q_{q-p+2is}(x, -y) = Q_{q-p+2is}(x, y)$ since $q + 2is$ is odd and $2q - p + 2is$ is even. The system (4.3.2) is time-reversible, i.e. has symmetrical phase portrait with regard to a straight line passing through the origin ($y = 0$, in this case), changing time direction. So, O is a centre, since it is monodromic.
In what follows, we assume that p and q are odd.

For all $k \geq 1$, we define the linear operator (see Lemma 4.2.73):

$$\begin{aligned} \ell_k \quad : \quad & \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{P}_{k+q-p}^{\mathbf{t}} \\ & U_k \to L_{\mathbf{F}_{q-p}} U_k. \end{aligned}$$

It is easy to prove that if k can be expressed as $k = k_3pq + k_2q + k_1p$ with $0 \leq k_1 < q,\ 0 \leq k_2 < p$, then the set $B_k^{\mathbf{t}} = \{x^{qi+k_1} y^{p(k_3-i)+k_2}, 0 \leq i \leq k_3\}$ is a base of $\mathcal{P}_k^{\mathbf{t}}$. Otherwise, $\mathcal{P}_k^{\mathbf{t}} = \{0\}$.

We prove the following result, which we will use later on.

Lemma 4.3.74. *For* $k \geq 1,\ k = k_3pq + k_2q + k_1p$ *with* $0 \leq k_1 < q,\ 0 \leq k_2 < p,\ k_3 \geq -1,$ *it holds:*
i) *If* $k_2 = p-1$*, a complementary subspace to the range (co-range) of* ℓ_k *is* $\mathrm{Cor}(\ell_k) = \mathrm{span}\{x^{q(k_3+1)+k_1-1}\}$*. Otherwise,* $\mathrm{Cor}(\ell_k) = \{0\}$.
ii) *If* $k_1 = 0$*, the kernel of the linear operator* ℓ_k *is* $\mathrm{Ker}(\ell_k) = \mathrm{span}\{y^{pk_3+k_2}\}$*. Otherwise,* $\mathrm{Ker}(\ell_k) = \{0\}$.

Proof.
Let $U_k = \sum_{i=0}^{k_3} \alpha_i^{(k)} x^{qi+k_1} y^{p(k_3-i)+k_2} \in \mathcal{P}_k^{\mathbf{t}}$.
We first assume that $k_1 > 0$. If $k_2 < p-1$, for $k_3 = -1$ it has $\mathcal{P}_{k+q-p}^{\mathbf{t}} = \mathcal{P}_k^{\mathbf{t}} = \{0\}$, and if $k_3 > -1$ a basis of $\mathcal{P}_{k+q-p}^{\mathbf{t}}$ is

$$B_{k+q-p}^{\mathbf{t}} = \{x^{qi+k_1-1} y^{p(k_3-i)+k_2+1}, 0 \leq i \leq k_3\},$$

that is $\dim(\mathcal{P}_k^{\mathbf{t}}) = \dim(\mathcal{P}_{k+q-p}^{\mathbf{t}}) = k_3 + 1$. Moreover,

$$\ell_k(U_k) \;=\; \frac{\partial U_k}{\partial x}(x,y)y = \sum_{i=0}^{k_3} \alpha_i^{(k)} (qi+k_1) x^{qi+k_1-1} y^{p(k_3-i)+k_2+1}$$

So, it deduces that $\mathrm{Ker}(\ell_k) = \{0\}$ and $\mathrm{Cor}(\ell_k) = \{0\}$.
If $k_2 = p-1$, a basis of $\mathcal{P}_{k+q-p}^{\mathbf{t}}$ is

$$B_{k+q-p}^{\mathbf{t}} = \{x^{qi+k_1-1} y^{p(k_3+1-i)}, 0 \leq i \leq k_3+1\},$$

therefore $\dim(\mathcal{P}_k^{\mathbf{t}}) = k_3 + 1$ and $\dim(\mathcal{P}_{k+q-p}^{\mathbf{t}}) = k_3 + 2$. The operator ℓ_k has the form

$$\ell_k(U_k) \;=\; \frac{\partial U_k}{\partial x}(x,y)y = \sum_{i=0}^{k_3} \alpha_i^{(k)} (qi+k_1) x^{qi+k_1-1} y^{p(k_3+1-i)}.$$

Therefore, in such a case, it has that

$$\mathrm{Ker}(\ell_k) = \{0\}, \qquad \mathrm{Cor}(\ell_k) = \mathrm{span}\{x^{q(k_3+1)+k_1-1}\}.$$

For $k_1 = 0$, we have $k_3 > -1$ since otherwise $k+q-p < 0$. Using an argument similar to that given above, it has that

$\mathrm{Ker}(\ell_k) = \mathrm{span}\{y^{pk_3+k_2}\},\ \mathrm{Cor}(\ell_k) = \{0\}$ if $k_2 < p-1$
and
$\mathrm{Ker}(\ell_k) = \mathrm{span}\{y^{p(k_3+1)-1}\},\ \mathrm{Cor}(\ell_k) = \mathrm{span}\{x^{q(k_3+1)-1}\}$ if $k_2 = p-1$. ■

Now we prove our main result of this section.

Theorem 4.3.75. *For system (4.3.2) with p and q odd, there exists W a $\mathcal{C}^\infty$-function in a neighborhood of the origin whose $2(q+Ns)$- homogeneous jet of type $\mathbf{t} = (p,q)$ at origin, $N \geq 0$, is*

$$\mathcal{J}_{\mathbf{t}}^{2(q+Ns)} W = \sum_{l=0}^{N} W_{2(q+sl)}$$

where $W_{2(q+sl)}$ is a $\mathbf{t}$-homogeneous polynomial of degree $2(q+sl),\ l \geq 0$, with $W_{2q}(x,y) = \frac{1}{2}y^2$ and $W_{2(q+s)}(1,0) = \frac{1}{2(q+s)}$ such that the derivative of W along the trajectories of the system (4.3.2) has the form

$$\dot{W} = x^{2m} \sum_{i=1}^{\infty} f_i x^{2is} + \tau(x,y)$$

where $m \in \mathbb{N}$ and $f_i,\ i \geq 1$, are polynomials in the coefficients of the right-hand sides of (4.3.2) and τ is a flat function at the origin.

Proof.
We first consider a formal series

$$U = \sum_{l=0}^{\infty} U_{2(q+sl)}$$

with $U_{2q}(x,y) = \frac{1}{2}y^2$, where $U_{2(q+sl)}$ is a $\mathbf{t}$-homogeneous function of degree $2(q+sl),\ l \geq 0$.
From Lemma 4.2.73, the $\mathbf{t}$-homogeneous expansion of the derivative of

U along the trajectories of the system (4.3.2) is given by

$$\begin{aligned}
\dot{U} &= L_{\mathbf{F}_{q-p}} U + \sum_{i=1}^{\infty} L_{\mathbf{F}_{q-p+2si}} U \\
&= \sum_{l=1}^{\infty} L_{\mathbf{F}_{q-p}} U_{2(q+sl)} + \sum_{l=0}^{\infty} \sum_{i=1}^{\infty} L_{\mathbf{F}_{q-p+2si}} U_{2(q+sl)} \\
&= \sum_{l=1}^{\infty} \left[L_{\mathbf{F}_{q-p}} U_{2(q+sl)} + \sum_{j=1}^{l} L_{\mathbf{F}_{q-p+2sj}} U_{2(q+sl-sj)} \right] \\
&= \sum_{l=1}^{\infty} \left[\ell_{2(q+sl)}(U_{2(q+sl)}) - A_{3q-p+2sl} \right]
\end{aligned}$$

being $A_{3q-p+2sl} = -\sum_{j=1}^{l} L_{\mathbf{F}_{q-p+2sj}} U_{2(q+sl-sj)} \in \mathcal{P}^{\mathbf{t}}_{3q-p+2sl}$. By breaking down $A_{3q-p+2sl} = R_{3q-p+2sl} + C_{3q-p+2sl}$, where

$$R_{3q-p+2sl} \in \mathrm{Range}(\ell_{2(q+sl)}) \quad \text{and } C_{3q-p+2sl} \in \mathrm{Cor}(\ell_{2(q+sl)}),$$

and by choosing $U_{2(q+sl)}$ such that $\ell_{2(q+sl)}(U_{2(q+sl)}) = R_{3q-p+2sl}$, for all $l \geq 1$, from Lemma 4.3.74, it is possible by means of a recursive procedure to obtain U such that the $\mathbf{t}$-homogeneous terms of the derivative of U along the trajectories of (4.3.2) are of the form $x^{\tilde{m}}$ with $\tilde{m}p = 3q-p+2sl$. As $3q-p+2sl = [2(n+1)l-1]p+(3-2l)q$ and p and q have no common factors, it must be $2l-3$ multiple of p, that is $2l = (2k-1)p+3$. So, $3q-p+2sl = [3n+2+(2k-1)s]p$. Concretely, if $k_0 = \min\{k \in \mathbb{Z},\ 3n+2+(2k-1)s > 0\}$ we have that

$$\dot{U} = \sum_{i=0}^{\infty} f_i x^{3n+2+[2(k_0+i)-1]s} = x^{3n+2+(2k_0-1)s} \sum_{i=0}^{\infty} f_i x^{2is}.$$

Moreover, as p and q are odd, it is easy to prove that $3n+2+(2k_0-1)s$ is even, since n is even (odd) if and only if s is even (odd).
Also, $A_{3q-p+2s} = -L_{\mathbf{F}_{q-p+2s}} U_{2q} = -yQ_{q-p+2s}(x,y) \in \mathrm{Range}(\ell_{2(q+s)})$, therefore, $f_0 = 0$.
We now see that $U_{2(q+s)}(1,0) = \frac{1}{2(n+1)}$. As $2(q+s) = 2(n+1)p$, the

polynomial $U_{2(q+s)}$ has the form $U_{2(q+s)}(x,y) = \alpha_{2(n+1)}^{(2(q+s))} x^{2(n+1)} + \cdots$ and as $\ell_{2(q+s)}(U_{2(q+s)}) = A_{3q-p+2s}$, we have that

$$\frac{\partial U_{2(q+s)}}{\partial x}(x,y)y + yQ_{q-p+2s}(x,y) = 0,$$

thus, $\frac{\partial U_{2(q+s)}}{\partial x}(x,y) + Q_{q-p+2s}(x,y) = 0$. For $(x,y) = (1,0)$, we get

$$2(n+1)\alpha_{2(n+1)}^{2(q+s)} = -Q_{q-p+2s}(1,0) = 1.$$

Thereby, $\alpha_{2(n+1)}^{2(q+s)} = \frac{1}{2(n+1)}$.
Lastly, from the Borel's Lemma, see Hartman [100], there exists W a $\mathcal{C}^\infty$-function in a neighborhood of the origin such that $\mathcal{J}_{\mathbf{t}}^n W = \mathcal{J}_{\mathbf{t}}^n U$, for all $n \geq 1$; thus, the result is proved. ∎

Note that, in general, from Lemma 4.3.74, the terms $U_{2(q+sl)}, l \geq 0$, are not unique and, as consequence, the constants $f_i, i \geq 1$, are not unique either. Nevertheless, by imposing that $U_{2(q+sl)}(1,0) = 0$, for all $l \geq 0$, it arrives at the uniqueness of the formal series U and of the constants f_i. Throughout the following, the f_i are referred to as the *focus quantities* of the singular point O of the system (4.3.2). The above recursive procedure will allow us to compute the first quantities focus of a family given.

Lemma 4.3.76. *In the conditions of Theorem 4.3.75, the locus of points satisfying $W(x,y) = C$ =constant are closed curves for different values of $C > 0$ encircling O with $W(O) = 0$ and $W(x,y) > 0$ in a punctured neighborhood of the origin.*

Proof.
It is enough to prove that the origin of the hamiltonian system

$$\dot{x} = \frac{\partial W}{\partial y}(x,y), \quad \dot{y} = -\frac{\partial W}{\partial x}(x,y), \tag{4.3.3}$$

is a centre.
The $\mathbf{t}$-homogeneous principal part of the system (4.3.3) is $(y, -x^{2n+1})^T$,

see Brunella & Miari [32], which does not have curves that arrive at O with defined direction. From Andreev [22], by using a $\mathbf{t}$-homogeneous blow up, the system (4.3.3) is monodromic and as it is hamiltonian, it follows that O is a centre. As a consequence, the curves $W(x, y) = C > 0$, are closed and fill a punctured neighborhood of the origin. ■

Theorem 4.3.77. *In the conditions of Theorem 4.3.75, the origin is a centre of (4.3.2) if and only if $f_i = 0$, for all $i \geq 1$.*

Proof.
If there exists $M > 0$ such that $f_i = 0,\ 1 \leq i \leq M-1$ and $f_M \neq 0$, the $\mathcal{C}^\infty$-function W verifies $\dot{W} = f_M x^{2(m+M)s} + \mathcal{O}(x^{2(m+M)s})$. So, there exists a neighborhood of the origin where $\dot{W}$ doesn't change its sign, and from Lemma 4.3.76 the curves $W(x, y)$ =constant are closed; therefore W is a Lyapunov function, thus, O is a focus. Concretely, if $f_M < 0$, O is asymptotically stable, otherwise O is asymptotically unstable.
On the other hand, if $f_i = 0$, for all $i \geq 1$, then O is a focus of infinite order. In the case of nilpotent monodromic fields, there exists a Poincaré map which is analytic, see Lyapunov [111]. Concretely, we can choose a section transversal to the field and a parametrization of this one, such that the Poincaré map is analytic. Therefore, a focus of infinity order is a centre. Consequently, if $f_i = 0$ for all $i \geq 1$, it follows that O is a centre. ■

4.4 Poincaré map near the origin.

We introduce the generalized polar coordinates. Given any natural number $n \in \mathbb{N}$, it defines the *generalized trigonometric functions,* $x(\theta) = Cs(\theta),\ y(\theta) = Sn(\theta)$, as the unique solution of the Cauchy problem

$$\frac{dx}{d\theta} = -y, \qquad \frac{dy}{d\theta} = x^{2n+1},$$

with $x(0) = 1,\ y(0) = 0$.
These functions are T-periodic with

$$T := 2\sqrt{\frac{\pi}{n+1}}\frac{\Gamma(\frac{1}{2n+2})}{\Gamma(\frac{n+2}{2n+2})},$$

and they verify the equality $Cs^{2n+2}(\theta) + (n+1)Sn^2(\theta) = 1$, see [111] for a proof.
We define the *generalized polar coordinates*, r and θ of the real plane $(x, y) \in \mathbb{R}^2$, as

$$x = rCs(\theta), \qquad y = r^{n+1}Sn(\theta). \tag{4.4.4}$$

Furthermore, the following equalities hold

$$\dot{r} = \frac{x^{2n+1}\dot{x} + y\dot{y}}{r^{2n+1}}, \qquad \dot{\theta} = \frac{x\dot{y} - (n+1)y\dot{x}}{r^{n+2}}.$$

The return map of (4.3.2) is analytic, see [111]. Now we provide a expression of the Taylor expansion of this return map.

Theorem 4.4.78. *Let system (4.3.2) with p and q odd. The return map of system (4.3.2) has the form*

$$P(u) = u - \frac{1}{2(n+1)}\sum_{l=1}^{\infty}(u^{n+(2l-1)s+1}f_l\int_0^T Cs^{3n-s+2ls+2}(\theta)d\theta)(1 + \mathcal{O}(u)) \ .$$

where $f_l \in \mathbf{R},\ l \geq 1$, *are polynomials in the coefficients of the right-hand sides of (4.3.2) (we will call* f_k *the* focus quantities *of the singular point* O *of the system (4.3.2)).*

Proof.
By making the change (4.4.4), after omitting a common factor r^n, the system (4.3.2) takes the form

$$\dot{r} = rf(r, \theta), \quad \dot{\theta} = -1 + rg(r, \theta) \tag{4.4.5}$$

with f and g analytic functions and $f(0, \theta) = g(0, \theta) = 0$, for all θ.
We now define the variable u verifying

$$u^{2(n+1)} \;=\; W\left(rCs(\theta), r^{n+1}Sn(\theta)\right) \tag{4.4.6}$$

being W a $\mathcal{C}^\infty$-function such that its derivative along the trajectories of the system (4.3.2) has the form

$$\dot{W} = x^{3n-s+2}\sum_{l=1}^{\infty} f_l x^{2ls} + \tau(x,y)$$

where $f_l,\ l \geq 1$, are polynomials in the coefficients of the right-hand sides of (4.3.2) and τ is a flat function at the origin, which exists from Theorem 4.3.75.
Let suppose $f_l = 0$ for $l = 1, \cdots, m-1$ and $f_m \neq 0$. The expression (4.4.6) is valid for $r > 0$ and for all θ. Furthermore, as

$$u^{2(n+1)} = W\left(rCs(\theta), r^{n+1}Sn(\theta)\right) = r^{2(n+1)} + \mathcal{O}(r^{2n+3},\theta) \quad (4.4.7)$$

from inverse function theorem, it has that $r = u + \mathcal{O}(u^2,\theta)$.
Next, we express (4.3.2) in the new coordinates system (u,θ)

$$\dot{u} = \frac{1}{2(n+1)u^{2n+1}}\dot{W} \quad (4.4.8)$$

$$= \frac{f_m}{2(n+1)}Cs^{3n-s+2+2ms}(\theta)u^{2ms+n-s+1}(1+\mathcal{O}(u,\theta)), \quad (4.4.9)$$

$$\dot{\theta} = -1 + \mathcal{O}(u,\theta) \quad (4.4.10)$$

whose differential equation associated can be written as

$$\frac{du}{d\theta} = -\frac{f_m}{2(n+1)}Cs^{3n-s+2+2ms}(\theta)u^{2ms+n-s+1}(1+\mathcal{O}(u,\theta))(4.4.11)$$

We write the solution of (4.4.11) starting at $u = u_0$ when $\theta = 0$ as

$$u(\theta,u_0) = \sum_{i=1}^{\infty} a_i(\theta)u_0^i + \tau(\theta,u_0), \quad (4.4.12)$$

with $a_1(0) = 1,\ a_i(0) = 0$ for $i \geq 2,\ \tau(0,u_0) = 0$ and τ flat at $u_0 = 0$. Hence the Poincaré return map from the section $\{(u,\theta) = (u_0,0),\ u_0 > 0\}$ to itself is given by the series

$$P(u_0) = a_1(T)u_0 + a_2(T)u_0^2 + ... \quad (4.4.13)$$

By replacing (4.4.12) in the equation differential (4.4.11) we obtain

$$a_1(\theta) \equiv 1,\ a_i(\theta) \equiv 0,\ \text{for } i = 2, ..., n + (2m-1)s$$

and

$$a_{n+(2m-1)s+1}(T) = -\frac{f_m}{2(n+1)} \int_0^T Cs^{3n-s+2+2ms}(\theta)d\theta. \blacksquare$$

Remark. In the classic sense, the $\mathcal{C}^{\infty}$-function W above defined in Theorem 4.3.75 is not a Lyapunov function, since it is not a defined positive function in a neighborhood of the origin. Therefore, it can not be used for looking for limit cycles which bifurcate from the origin.

As a consequence, the only significative constant f_l is the first one different from zero. It does that the return map differs from the identity map, and it determines the stability of the origin. Also, we note that the origin is a centre if and only if $P(u) \equiv u$. So, we have the following results: the first one characterizes the centres of system (4.3.2) and the second result is related to the number of small amplitude limit cycles which can bifurcate from the origin.

Corollary 4.4.79. *The origin of (4.3.2) with p and q odd is a centre if and only if $f_l = 0$, for all $l \geq 1$.*

Proof.
As p and q are odd and $s = (n+1)p - q$, it implies that s is even (odd) if and only if n is even (odd). Thereby, $3n - s + 2 + 2ms$ is even. So, $\int_0^T Cs^{3n-s+2+2ms}(\theta)d\theta$ is a positive value. The result is followed as a consequence. $\blacksquare$

If we want to find the systems with a centre of a polynomial family $\mathbf{X}(\lambda)$, $\lambda \in \mathbb{R}^m$, of systems (4.3.2) with p and q odd, we calculate recursively the sets on $\mathbb{R}^m$:

$$\Omega_1 = \{\lambda \in \mathbb{R}^m,\ f_1(\lambda) = 0\},\ \Omega_k = \{\lambda \in \Omega_{k-1},\ f_k(\lambda) = 0\},\ \text{for } k \geq 2.$$

By Hilbert Basis Theorem, we know that there is a M such that

$$\Omega_1 \supset \Omega_2 \supset \cdots \Omega_M \supset \Omega_{M+1} = \Omega_{M+2} = \cdots .$$

So, the systems $\mathbf{X}(\lambda^*)$ with $\lambda^* \in \Omega_{M+1}$ have a centre at the origin. Also, in such a case, it is said that M is the order of the family $\mathbf{X}(\lambda)$.

The focus quantities of system (4.3.2) can also be used to prove the existence of a certain number of small amplitude limit cycles bifurcating from the nilpotent critical point of a family of systems (4.3.2). Next result is used in order to study the degenerate Andronov-Hopf bifurcation, i.e. we analyze the existence of limit cycles which can bifurcate from the origin of $\mathbf{X}(\lambda)$ under variations of the parameters λ.

Corollary 4.4.80. *Let $\mathbf{X}(\lambda)$ be a family of systems (4.3.2) with p and q odd, depending on some parameters $\lambda \in \mathbb{R}^m$. We assume that $\lambda^* \in \Omega_r \setminus \Omega_{r+1}$, (i.e. O is a weak focus of order r of $\mathbf{X}(\lambda^*)$). If there exists $\bar{\lambda}$ enough close to λ^* such that $f_1(\bar{\lambda})$, $f_2(\bar{\lambda})$, ..., $f_{r-1}(\bar{\lambda})$, $f_r(\bar{\lambda})$ alternate sign and*

$$0 < |f_1(\bar{\lambda})| << |f_2(\bar{\lambda})| << ... << |f_{r-1}(\bar{\lambda})| << |f_r(\bar{\lambda})| << 1$$

then system $\mathbf{X}(\bar{\lambda})$ has exactly r limit cycles in a neighborhood of the origin.

Proof.
By Theorem 4.4.78, the Taylor expansion of the Poincaré return map of $\mathbf{X}(\bar{\lambda})$ has the form

$$P(u) = u - w_1 f_1(\bar{\lambda})(1 + u h_1(u))u^{j_1} - w_2 f_2(\bar{\lambda})(1 + u h_2(u))u^{j_2} - ...$$

where $w_m = \frac{1}{2(n+1)} \int_0^T C s^{3n-s+2+2ms}(\theta) d\theta > 0$, $j_m = n + (2m-1)s + 1$ and h_m are analytic functions at the origin.

Each small limit cycle around the origin corresponds to each positive fixed point of the Poincaré return map of $\mathbf{X}(\bar{\lambda})$, i.e. positive zeros of the function

$$F(u) = u - P(u) = \sum_{m=1}^{r+1} (1 + u h_m(u)) w_m f_m(\bar{\lambda}) u^{j_m} + \mathcal{O}(j_{r+1} + 1).$$

By writing $1+uh_m(u) = (1+uh_1(u))(1+\bar{h}_m(u))$, where $\bar{h}_m$ are analytic functions at the origin, it has $F(u) = (1+uh_1(u))u^{j_1}F_0(u)$ where

$$F_0(u) = w_1 f_1(\bar{\lambda}) + \sum_{m=2}^{r+1} w_m f_m(\bar{\lambda})(1+u\bar{h}_m(u))u^{j_m-j_1} + \mathcal{O}(j_{r+1}-j_1+1).$$

We must look for positive zeros of F_0.
By differentiating, we have

$$F_0'(u) = \sum_{m=2}^{r+1}(1+u\hat{g}_m(u))(j_m-j_1)w_m f_m(\bar{\lambda})u^{j_m-j_1-1} + \mathcal{O}(j_{r+1}-j_1)$$

where

$$(j_m-j_1)(1+u\hat{g}_m(u)) = (j_m-j_1)(1+u\bar{h}_m(u)) + u(\bar{h}_m(u)+u\bar{g}_m'(u)).$$

By writing $1+u\hat{g}_m(u) = (1+u\hat{g}_1(u))(1+\tilde{h}_m(u))$, where $\tilde{h}_m$ are analytic functions at the origin, F_0' has the form $F_0'(u) = (1+u\tilde{h}_m(u))u^{j_2-j_1-1}F_1$ where

$$\begin{aligned} F_1(u) = \quad & (j_2-j_1)w_2 f_2(\bar{\lambda}) + \textstyle\sum_{m=3}^{r+1}(j_m-j_1)w_m f_m(\bar{\lambda})(1+u\tilde{h}_m(u))u^{j_m-j_2} \\ & +\mathcal{O}(j_{r+1}-j_2+1). \end{aligned}$$

Now, the number of positive zeroes of F_0 can not exceed the number of positive zeroes of F_1 by more than unity. By continuing this process a further step we obtain a function F_2 such that the number of positive zeroes of F_1 cannot exceed the number of positive zeroes of F_1 by more than unity. So, the number of positive zeroes of F_0 can not exceed the number of positive zeroes of F_2 by more than two.
We finish this process at the r^{th} step, when we obtain a function F_r of the form

$$F_r(u) = (j_{r+1}-j_1)w_{r+1}f_{r+1}(\bar{\lambda}) + \mathcal{O}(1),$$

which does not have zeros in a neighborhood of origin, since $f_{r+1}(\bar{\lambda})$ is close to $f_{r+1}(\lambda^*) \neq 0$, by continuity. Therefore, F can not have more than r positive zeros. Moreover, as $f_1(\bar{\lambda})$, $f_2(\bar{\lambda})$, ..., $f_{r-1}(\bar{\lambda})$ and $f_r(\bar{\lambda})$

alternate sign and satisfy $0 < |f_1(\bar{\lambda})| << |f_2(\bar{\lambda})| << ... << |f_{r-1}(\bar{\lambda})| << |f_r(\bar{\lambda})|$, we can assure the existence of r limit cycles of small amplitude bifurcating of the origin of $\mathbf{X}(\lambda^*)$. ∎

4.5 Centres and cyclicity of several families

In this section, we compute the first focus quantities of some subfamilies of the system (4.3.2) by means of the recursive procedure developed in Theorem 4.3.75. These have the form

$$f_1 = \alpha_1 g_1, \quad f_i = \alpha_i g_i + \sum_{j=1}^{i-1} \beta_{i,j} f_j, \ i \geq 2,$$

with α_i positive constants and $\beta_{i,j}$ polynomials in the coefficients of the right-hand sides of (4.3.2).
We give the following result which we will use in order to prove the analytic integrability of the centres of several families of (4.3.2). We recall that if the system (4.3.2) is monodromic, the existence of an analytic first integral is a sufficient condition so that the origin be a centre.

Proposition 4.5.81. *The nilpotent systems*

$$\dot{x} = y + v_y K(v, y^2) + y\Psi(v, y^2), \qquad \dot{y} = -v_x K(v, y^2), \tag{4.5.14}$$

where v, K, Ψ *are analytic functions in a neighborhood of the origin with* $\Psi(O) = 0$, *are analytically integrable in a neighborhood of the origin.*

Proof.
Doing the change of variables $u = y^2$, $v = v(x, y)$, by redefining the variable time by $d\tau = yv_x(1 + \Psi(x, y))dt$ and by denoting $\frac{d}{d\tau} ='$, the system (4.5.14) becomes

$$u' = -\frac{2K(u, v)}{1 + \psi(u, v)}, \qquad v' = 1. \tag{4.5.15}$$

From Cauchy-Arnold's Theorem (see Bruno [34], page 98), the system (4.5.15) has got an analytic first integral $H(u, v) = cte$ defined in a neighborhood of O. Undoing the change of variable, (4.5.14) has a first integral of (4.5.14), $\tilde{H}(x, y) = H(y^2, v(x, y)) = cte$, which is analytic in a neighborhood N, since it is a composition of analytic functions. Also, if we denote $\mathbf{X}$ the vector field associated to (4.5.14), it has $\nabla\tilde{H}\cdot\mathbf{X} = 0$, for all $(x, y) \in N \setminus \{\nabla v \cdot \mathbf{X} \neq 0\}$. So, by continuity, $\nabla\tilde{H}\cdot\mathbf{X} = 0$, for all $(x, y) \in N$, that is $\tilde{H}$ is a local analytic first integral of (4.5.14). ∎

Remarks.

• For $v(x, y) = x$ the systems (4.5.14) turn out

$$\dot{x} = y + y\bar{\Psi}(x, y^2), \qquad \dot{y} = \bar{K}(x, y^2), \tag{4.5.16}$$

that is, the family of nilpotent systems reversible under the change of variables $(x, y, t) \to (x, -y, -t)$.

The analytic integrability of the nilpotent systems (4.5.16) is one of the main results of Chavarriga et al. [42].

• For

$$v = v_{2m},\ K(y^2, v) = v_{2m}^p,\ \Psi(y^2, v) = \sum_{k=0}^{p} a_k v_{2m}^{p-k} y^{2m(k+1)-2}$$

where $p \geq 0$, v_{2m} is a homogeneous polynomial of degree $2m$ and a_k arbitrary constants, the systems (4.5.14) come given by

$$\left\{\begin{array}{l} \dot{x} = y + \frac{\partial v_{2m}}{\partial y} v_{2m}^p + \sum_{k=0}^{p} a_k v_{2m}^{p-k} y^{2m(k+1)-1}, \\ \dot{y} = -\frac{\partial v_{2m}}{\partial x} v_{2m}^p. \end{array}\right. \tag{4.5.17}$$

These systems include the nilpotent family analytically integrable given in Andreev *et al.* [23], Lemma 2.

We show several applications of our research. Firstly, we study the problem of centre of the family

$$\begin{array}{rl} \dot{x} = & y + a_1x^5 + a_2x^2y + a_3x^7 + a_4x^4y + a_5xy^2, \\ \dot{y} = & -x^7 + b_1x^4y - a_2xy^2 + b_3x^6y + b_4x^3y^2 + b_5y^3. \end{array} \tag{4.5.18}$$

This system is a subfamily of (4.3.2) given by

$$(\dot{x}, \dot{y})^T = \mathbf{F}_2 + \mathbf{F}_4 + \mathbf{F}_6,$$

with $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}},\ i = 2, 4, 6,\ \mathbf{t} = (1, 3)$, and

$$\mathbf{F}_2 = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \mathbf{F}_4 = \begin{pmatrix} a_1 x^5 + a_2 x^2 y \\ -x^7 + b_1 x^4 y + b_2 x y^2 \end{pmatrix},$$

$$\mathbf{F}_6 = \begin{pmatrix} a_3 x^7 + a_4 x^4 y + a_5 x y^2 \\ b_6 x^9 + b_3 x^6 y + b_4 x^3 y^2 + b_5 y^3 \end{pmatrix}$$

with $b_6 = 0$ and $b_2 = -a_2$.

The following result characterizes the centres of the system (4.5.18).

Theorem 4.5.82. *The origin of the system (4.5.18) is a centre if and only if one of the following three series is satisfied:*

i) $5a_1 + b_1 = 7a_3 + b_3 = 2a_4 + b_4 = a_5 + 3b_5 = 0$, *(Hamiltonian system).*

ii) $a_1 = a_3 = a_5 = b_1 = b_3 = b_5 = 0$, *(Time-reversible system).*

iii) $a_2 = 4a_1^2,\ b_1 = -5a_1,\ b_5 = a_1 b_4,\ a_5 = a_1(4a_4 - b_4),\ a_3 = b_3 = 0.$

Moreover, each one of them has a local analytic first integral.

Proof.

Taking $\mathbf{t} = (1, 4)$, (that is, the type of the vector field $(y, -x^7)^T$, i.e. the $\mathbf{t}$-homogeneous principal part of the system (4.5.18)) the system (4.5.18) comes given by $(\dot{x}, \dot{y})^T = (h_x + x\mu(x, y), -h_y + 4y\mu(x, y))^T$ being h the defined positive function

$$h(x, y) = \tfrac{1}{8}(x^8 + 4y^2) - \tfrac{1}{9}c_1 x^5 y - \tfrac{1}{10}c_2 x^2 y^2 - \tfrac{1}{11}c_3 x^7 y$$
$$- \tfrac{1}{12}c_4 x^4 y^2 - \tfrac{1}{13}c_5 x y^3,$$

and

$$\mu(x, y) = \tfrac{1}{9}d_1 x^4 + \tfrac{1}{11}d_3 x^6 + \tfrac{1}{12}d_4 x^3 y + \tfrac{1}{13}d_5 y^2$$

where

$$c_1 = b_1 - 4a_1, \quad d_1 = 5a_1 + b_1, \quad c_2 = -5a_2,$$
$$c_3 = b_3 - 4a_3, \quad d_3 = 7a_3 + b_3, \quad c_4 = b_4 - 4a_4, \quad d_4 = 4a_4 + 2b_4,$$
$$c_5 = b_5 - 4a_5 \quad d_5 = a_5 + 3b_5,$$

The first nine constants $g_i,\ i = 1, ..., 9$ have the form

$$
\begin{aligned}
g_1 &= d_1, \quad g_2 = d_3, \quad g_3 = d_5 + \tfrac{12}{13}c_1 d_4,\\
g_4 &= d_4 \left[c_3 + 2c_1(c_2 + 2c_1^2)\right],\\
g_5 &= d_4 \left[c_5 + 4c_1(c_4 + \tfrac{1}{13}d_4) - \tfrac{100}{3}c_1^3(c_2 + 2c_1^2)\right],\\
g_6 &= -d_4 c_1(c_2 + 2c_1^2)\left[c_4 + \tfrac{1}{2}d_4 - \tfrac{62}{3}c_1^2(c_2 + 2c_1^2)\right],\\
g_7 &= d_4 c_1(c_2 + 2c_1^2)\left[d_4 - \tfrac{24}{5}c_2^2 - \tfrac{748}{15}c_2 c_1^2 - \tfrac{1408}{15}c_1^4\right],\\
g_8 &= -d_4 c_1(c_2 + 2c_1^2)(774c_2^2 - 4681c_2c_1^2 + 6641c_1^4),\\
g_9 &= -d_4 c_1(c_2 + 2c_1^2)(381374c_2 + 859813c_1^2).
\end{aligned}
$$

First, we suppose that $d_4 = 0$. Imposing $g_1 = g_2 = g_3 = 0$, we have that $d_1 = d_3 = d_5 = 0$, i.e. $5a_1 + b_1 = 7a_3 + b_3 = 2a_4 + b_4 = a_5 + 3b_5 = 0$.
In this case, (4.5.18) is a hamiltonian system whose Hamilton's function is

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{8}x^8 + a_1x^5y + \frac{1}{2}a_2x^2y^2 + a_3x^7y + \frac{1}{2}a_4x^4y^2 - b_5xy^3$$

and therefore, O is a centre.
In particular, H is a local analytic first integral defined at the origin.
If we suppose that $c_1 = 0$ and $d_4 \neq 0$, from $g_i = 0,\ i = 1, ..., 5$, it successively has that $d_1 = d_3 = d_5 = c_3 = c_5 = 0$, i.e. $a_1 = a_5 = a_3 = b_1 = b_3 = b_5 = 0$. Thereby, the singular point O is a centre, since the direction field of the system (4.5.18) is symmetric around $y = 0$, i.e. the system is invariant to the change $(x, y, t) \to (x, -y, -t)$. Therefore, the system has a local analytic first integral, see Chavarriga et. al. [42].
Finally, we suppose that $c_1d_4 \neq 0$. If g_i are zero, $i = 1, 2, 3$ then it arrives at $d_1 = d_3 = 0, d_5 = -\frac{12}{13}c_1d_4$.
If $c_2 = -2c_1^2$, from $g_4 = 0$, we obtain $c_3 = 0$, and from $g_5 = 0$, it follows that $c_5 = -4c_1(c_4 + \frac{1}{13}d_4)$. Therefore, by substituting we have

$$a_2 = 4a_1^2,\ b_1 = -5a_1,\ b_5 = a_1b_4,\ a_5 = a_1(4a_4 - b_4),\ a_3 = b_3 = 0.$$

In this case, the system (4.5.18) has the form

$$\begin{aligned} \dot{x} &= y + \tfrac{1}{4}(g(x,y) - b_4 y^2)\tfrac{\partial g}{\partial y}(x,y) + a_4 y g(x,y), \\ \dot{y} &= \tfrac{1}{4}(b_4 y^2 - g(x,y))\tfrac{\partial g}{\partial x}(x,y), \end{aligned} \tag{4.5.19}$$

where $g(x,y) = x^4 + 4a_1xy$.

This system belongs to the family (4.5.14) given in Proposition 4.5.81 with

$$v = g(x,y),\ K(v,y^2) = \tfrac{1}{4}(g(x,y) - b_4y^2),\ \Phi(v,y^2) = a_4 g(x,y)$$

and, therefore (4.5.19) has an analytic first integral at O, it follows that O is a centre.
Lastly, if $(c_2+2c_1^2)c_1d_4 \neq 0$, g_8 and g_9 are not zero simultaneously. Therefore, the origin of system (4.5.18) is a focus. ∎

We already get a lower bound for its cyclicity.

Theorem 4.5.83. *Under perturbations of the parameters of the system (4.5.18), it has:*
a) *if* $2a_4 + b_4 = 0$, *it can bifurcate 0,1 or 2 limit cycles from the origin.*
b) *if* $2a_4+b_4 \neq 0$ *and* $(-5a_2+2(b_1-4a_1)^2)(b_1-4a_1) = 0$, *it can bifurcate 0,1,2,3 or 4 limit cycles around the origin.*
c) *if* $2a_4+b_4 \neq 0$ *and* $(-5a_2+2(b_1-4a_1)^2)(b_1-4a_1) \neq 0$, *it can bifurcate 0,1,2,3,4,5,6,7 or 8 limit cycles around the origin.*

Proof.
First, we assume that $2a_4 + b_4 = 0$, that is $d_4 = 0$. The only g_i different from zero are $g_1 = d_1,\ g_2 = d_3,\ g_3 = d_5$. Therefore, if $d_1 \neq 0$ there is a neighborhood of the origin where the system (4.5.18) hasn't limit cycle around the origin. If d_1 is close to zero and $d_3 \neq 0$, it can exist, at least, 1 limit cycle. If $d_1d_3 < 0$ with $0 < |d_1| << |d_3|$ and also $d_5 \neq 0$, then there are 2 limit cycles of small amplitude.

We now assume that $2a_4+b_4 \neq 0$ and $(-5a_2+2(b_1-4a_1)^2)(b_1-4a_1)=0$, i.e. $d_4 \neq 0$ and $(c_2+2c_1^2)c_1=0$. In this case,

$$g_1 = d_1, \quad g_2 = d_3, \quad g_3 = d_5 + \tfrac{12}{13}c_1 d_4, \quad g_4 = d_4 c_3,$$
$$g_5 = d_4(c_5 + 4c_1(c_4 + \tfrac{1}{13}d_4)).$$

and the remain are zero.
So, if $d_1 \neq 0$ there isn't limit cycles around the origin. If d_1 is close to zero and $d_3 \neq 0$, can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0 < |d_1| << |d_3|$ and also d_5 different from $d_5^* = -\frac{12}{13}c_1 d_4$, then there are 2 small amplitude limit cycles. If we now take d_5 such that d_3 and d_5 alternate sign and $|d_3| << |d_5|$ and also $c_3 \neq 0$, then there exist 3 limit cycles around the origin. If we also choose c_3 close to 0 such that $d_5 c_3 < 0$ and $|d_5| << |c_3|$ and take $c_5 \neq -4c_1(c_4 + \frac{1}{13}d_4)$ then there exist 4 limit cycles at least, bifurcating from the origin.
Lastly, if $2a_4+b_4 \neq 0$ and $(-5a_2+2(b_1-4a_1)^2)(b_1-4a_1) \neq 0$, by denoting $q = d_4 c_1(c_2+2c_1^2) \neq 0$, the first nine constants $g_i,\ i=1,...,9$ of (4.5.18) are

$$g_1 = d_1, \quad g_2 = d_3, \quad g_3 = d_5 + \tfrac{12}{13}c_1 d_4,$$
$$g_4 = d_4\left[c_3 + 2c_1(c_2+2c_1^2)\right],$$
$$g_5 = d_4\left[c_5 + 4c_1(c_4 + \tfrac{1}{13}d_4) - \tfrac{100}{3}c_1^3(c_2+2c_1^2)\right],$$
$$g_6 = -q\left[c_4 + \tfrac{1}{2}d_4 - \tfrac{62}{3}c_1^2(c_2+2c_1^2)\right],$$
$$g_7 = q\left[d_4 - \tfrac{24}{5}c_2^2 - \tfrac{748}{15}c_2c_1^2 - \tfrac{1408}{15}c_1^4\right],$$
$$g_8 = -q\left[1548c_2 - (4681+5\sqrt{54049})c_1^2\right]\left[1548c_2 - (4681-5\sqrt{54049})c_1^2\right],$$
$$g_9 = -q(381374c_2 + 859813c_1^2).$$

We can choose $d_1, d_3, d_5, c_3, c_5, c_4, d_4$ and c_2 adequately such that $g_i g_{i+1}$ is negative, g_9 different from zero and

$$0 < |g_1| << |g_2| << |g_3| << |g_4| << |g_5| << |g_6| << |g_7| << |g_8|.$$

Applying Corollary 4.4.80, for $r = 1,2,3,4,5,6,7$ and 8, there are regions of the parameters where the system has 0,1,2,3,4,5,6,7 or until 8 limit cycles around the origin. ∎

We now give necessary and sufficient conditions for what the origin of the families

$$(\dot{x},\dot{y})^T = \mathbf{F}_{i-2} + \mathbf{F}_i, \ i = 4,6,8$$

with $\mathbf{F}_{i-2} = (y,0)^T \in \mathcal{Q}^{\mathbf{t}}_{i-2}$, $\mathbf{F}_i \in \mathcal{Q}^{\mathbf{t}}_i$, $i = 4,6,8$, $\mathbf{t} = (1, i-1)$, be a centre.
In the first two, we prove that O is a centre if and only if the system is either hamiltonian or time-reversible. Nevertheless, in the third family there are centres which have got an analytic first integral and they are not hamiltonian nor time-reversible system.

Theorem 4.5.84. *The origin of the system*

$$\begin{aligned} \dot{x} = & \quad y + a_1x^5 + a_2x^2y, \\ \dot{y} = & \quad -x^7 + b_1x^4y + b_2xy^2. \end{aligned} \tag{4.5.20}$$

is a centre if and only if one of the following two series is satisfied:
i) $b_1 + 5a_1 = a_2 + b_2 = 0$, *(Hamiltonian system).*
ii) $a_1 = b_1 = 0$, *(Time-reversible system).*
Moreover, each one of them has a local analytic first integral.

Proof.
As $g_1 = b_1 + 5a_1$ and $g_2 = (b_1 - 4a_1)(a_2 + b_2)$, from the vanishing of g_1 and g_2 the assertion follows.
On the other hand, if **i)** holds, its Hamiltonian is, in particular, an analytic first integral at O and if **ii)** holds, the system is time-reversible under the change $(x,y,t) \to (x,-y,-t)$, thereby, the system has a local analytic first integral, see Chavarriga et. al. [42]. ∎

Theorem 4.5.85. *The origin of the system*

$$\begin{aligned} \dot{x} = & \quad y + a_1x^7 + a_2x^4y + a_3xy^2, \\ \dot{y} = & \quad -x^9 + b_1x^6y + b_2x^3y^2 + b_3y^3. \end{aligned} \tag{4.5.21}$$

is a centre if and only if one of the following two series is satisfied:
i) $b_1 + 7a_1 = 2a_2 + b_2 = a_3 + 3b_3 = 0$, *(Hamiltonian system).*
ii) $a_1 = b_1 = a_3 = b_3 = 0$, *(Time-reversible system).*
Moreover, each one of them has a local analytic first integral.

Proof.
The system (4.5.21) has the form $(\dot{x}, \dot{y})^T = (h_x + x\mu(x,y), -h_y + 5y\mu(x,y))^T$ being

$$\begin{aligned} h(x,y) &= \tfrac{1}{10}(x^{10} + 5y^2) - \tfrac{1}{12}(b_1 - 5a_1)x^7 y - \tfrac{1}{14}(b_2 - 5a_2)x^4 y^2 \\ &\quad - \tfrac{1}{16}(b_3 - 5a_3)xy^3, \\ \mu(x,y) &= \tfrac{1}{12}(b_1 + 7a_1)x^6 + \tfrac{1}{14}(4a_2 + 2b_2)x^3 y + \tfrac{1}{16}(a_3 + 3b_3)y^2. \end{aligned}$$

Therefore, it is convenient to replace a_1, a_2, a_3, b_1, b_2 and b_3 by the coefficients c_1, c_2, c_3, d_1, d_2 and d_3 where

$$\begin{aligned} c_1 &= b_1 - 5a_1, \quad & c_2 &= b_2 - 5a_2, \quad & c_3 &= b_3 - 5a_3, \\ d_1 &= b_1 + 7a_1, \quad & d_2 &= 4a_2 + 2b_2, \quad & d_3 &= a_3 + 3b_3. \end{aligned}$$

It has that

$$\begin{aligned} & g_1 = d_1, \quad g_2 = c_1 d_2 + 12 d_3, \\ & g_3 = d_2(70c_1^3 + 45c_1 d_2 + 288 c_1 c_2 + 756 c_3), \qquad g_4 = -c_1^5 d_2. \end{aligned}$$

From $g_i = 0,\ i = 1, ..., 4$ follows the statement. ∎

Theorem 4.5.86. *The origin of the system*

$$\begin{aligned} \dot{x} &= \quad y + a_1 x^9 + a_2 x^6 y + a_3 x^3 y^2 + a_4 y^3, \\ \dot{y} &= \quad -x^{11} + b_1 x^8 y + b_2 x^5 y^2 + b_3 x^2 y^3. \end{aligned} \tag{4.5.22}$$

is a centre if and only if one of the following three series is satisfied:
i) $9a_1 + b_1 = b_2 + 3a_2 = a_3 + b_3 = 0$, *(Hamiltonian system).*
ii) $a_1 = b_1 = a_3 = b_3 = 0$, *(Time-reversible system).*
iii) $b_1 = -9a_1, a_3 = -a_1(b_2 - 6a_2 + 54a_1^2), b_3 = 3a_1(b_2 + 18a_1^2)$.
Moreover, each one of them has a local analytic first integral.

Proof.
This system has the form $(\dot{x},\dot{y})^T = (h_x + x\mu(x,y), -h_y + 6y\mu(x,y))^T$, being

$$h(x,y) = \tfrac{1}{12}(x^{12}+6y^2) - \tfrac{1}{15}c_1x^9y - \tfrac{1}{18}c_2x^6y^2 - \tfrac{1}{21}c_3x^3y^3 - \tfrac{1}{24}c_4y^4,$$
$$\mu(x,y) = \tfrac{1}{15}d_1x^8 + \tfrac{1}{18}d_2x^5y + \tfrac{1}{21}d_3x^2y^2,$$

where the new coefficients that appear are

$$c_1 = b_1 - 6a_1, \quad c_2 = b_2 - 6a_2, \quad c_3 = b_3 - 6a_3, \quad c_4 = -6a_4,$$
$$d_1 = b_1 + 9a_1, \quad d_2 = 6a_2 + 2b_2, \quad d_3 = 3a_3 + 3b_3,$$

the expressions of the first focus quantities are:

$$g_1 = d_1, \quad g_2 = c_1d_2 + 5d_3,$$
$$g_3 = d_2(42c_1^3 + 25c_1d_2 + 175c_1c_2 + 375c_3).$$

If $d_2 = 0$. For that $g_1 = g_2 = g_3 = 0$, it must be $d_1 = d_2 = d_3 = 0$, i.e. $9a_1 + b_1 = b_2 + 3a_2 = a_3 + b_3 = 0$. In this case, the system (4.5.22) is hamiltonian system and O is a centre. In particular, its Hamiltonian is a local analytic first integral defined at the origin.
We suppose that $c_1 = 0$, but $d_2 \neq 0$. From $g_i = 0$, $i = 1,2,3$, it has that $d_1 = c_1 = d_3 = c_3 = 0$, i.e. $a_1 = b_1 = a_3 = b_3 = 0$. The singular point O is a centre, since the system is time-reversible under the change $(x,y,t) \to (x,-y,-t)$. Also, by Chavarriga et. al. [42] the system has a local analytic first integral.
Now, we suppose that $c_1d_2 \neq 0$. Referring to the expressions given above for the first focus quantities, for a centre we have $d_1 = 0, d_3 = -\frac{1}{5}c_1d_2$ and $c_3 = -\frac{1}{375}(42c_1^3 + 25c_1d_2 + 175c_1c_2)$.
This implies that $b_1 = -9a_1, a_3 = -a_1(b_2 - 6a_2 + 54a_1^2), b_3 = 3a_1(b_2 + 18a_1^2)$. Taking $\lambda_1 = b_2 + 18a_1^2, \lambda_2 = 6a_1^2 - a_2$, in this case, the system (4.5.22) has the form

$$\dot{x} = y + \tfrac{1}{6}g(x,y)\tfrac{\partial g}{\partial y}(x,y) + \tfrac{1}{3}(\lambda_1 - 3\lambda_2)g(x,y)y + (a_4 - \lambda_1\lambda_2)y^3,$$
$$\dot{y} = -\tfrac{1}{6}g(x,y)\tfrac{\partial g}{\partial x}(x,y),$$

where $g(x,y) = x^6 + 6a_1x^3y - \lambda_1 y^2$.

This system is a system of the family (4.5.14) given in Proposition 4.5.81 where

$$\begin{array}{l} v = g(x,y),\ K(v,y^2) = \frac{1}{6}g(x,y), \\ \Phi(v,y^2) = \frac{1}{3}(\lambda_1 - 3\lambda_2)g(x,y) + (a_4 - \lambda_1\lambda_2)y^2 \end{array}.$$

So, from Proposition 4.5.81, this system has an analytic first integral in a neighborhood of the origin. Thus, it follows that the singular point O is a centre. ∎

Finally, we find the centres of the family

$$(\dot{x},\dot{y})^T = \mathbf{F}_2 + \mathbf{F}_{6n-2},\ 1 \le n \le 9, \tag{4.5.23}$$

with $\mathbf{t} = (3,5)$ and where $\mathbf{F}_2 = (y,0)^T \in \mathcal{Q}_2^{\mathbf{t}}$ and $\mathbf{F}_{6n-2} \in \mathcal{Q}_{6n-2}^{\mathbf{t}}$.
For $1 \le n \le 8$, the vanishing of the first focus quantities leads us to hamiltonian or time-reversible systems.
For $n = 9$, we have the system

$$(\dot{x},\dot{y})^T = \mathbf{F}_2 + \mathbf{F}_{52}, \tag{4.5.24}$$

with

$$\mathbf{F}_2 = \begin{pmatrix} y \\ 0 \end{pmatrix},\quad \mathbf{F}_{52} = \begin{pmatrix} a_1x^{15}y^2 + a_2x^{10}y^5 + a_3x^5y^8 + a_4y^{11} \\ -x^{19} + b_1x^{14}y^3 + b_2x^9y^6 + b_3x^4y^9 \end{pmatrix}.$$

In this case, we also find nontrivial centres (neither hamiltonian nor time-reversible system).

Theorem 4.5.87. *The origin of system (4.5.24) is a centre if and only if one of the following three series is satisfied:*
i) $5a_1 + b_1 = 3b_2 + 5a_2 = 5a_3 + 9b_3 = 0$, *(Hamiltonian system).*
ii) $a_1 = b_1 = a_3 = b_3 = 0$, *(Time-reversible system).*
iii) $10773a_3 + 2a_1(-150530a_2 + 17447b_2 + 26600a_1^2) = 0$, $b_1 + 5a_1 = 0$, *and* $96957b_3 - 10a_1(-138560a_2 + 24629b_2 + 26600a_1^2) = 0$.
Moreover, each one of them has a local analytic first integral.

Proof.
The system can be expressed as $(\dot{x}, \dot{y})^T = (h_x + x\mu(x,y), -h_y + 9y\mu(x,y))^T$ with

$$h(x,y) = \tfrac{1}{20}(x^{20} + 10y^2) - \tfrac{1}{45}c_1 x^{15}y^3 - \tfrac{1}{70}c_2 x^{10}y^6 - \tfrac{1}{95}c_3 x^5 y^9 - \tfrac{1}{10}c_4 y^{12},$$
$$\mu(x,y) = \tfrac{1}{45}d_1 x^{14}y^2 + \tfrac{1}{70}d_2 x^9 y^5 + \tfrac{1}{95}d_3 x^4 y^8,$$

where the new coefficients are

$$\begin{array}{llll} c_1 = b_1 - 5a_1, & c_2 = b_2 - 10a_2, & c_3 = b_3 - 10a_3, & c_4 = -10a_4, \\ d_1 = 15a_1 + 3b_1, & d_2 = 10a_2 + 6b_2, & d_3 = 5a_3 + 9b_3, & \end{array}$$

The expressions of the first focus quantities are

$$g_1 = d_1, \quad g_2 = c_1 d_2 + 9d_3,$$
$$g_3 = d_2[5103c_3 + (266c_1^2 + 405d_2 + 15395c_2)c_1].$$

If $d_2 = 0$. So that $g_1 = g_2 = g_3 = 0$, it must be $d_1 = d_2 = d_3 = 0$, i.e. it holds **i)**. The system (4.5.24) is hamiltonian system whose Hamilton's function is $h(x,y)$ and O is a centre.
We suppose that $c_1 = 0$ but $d_2 \neq 0$. From $g_i = 0,\ i = 1,2,3$, it has that $d_1 = c_1 = d_3 = c_3 = 0$, i.e. $a_1 = b_1 = a_3 = b_3 = 0$. Thereby, the singular point O is a centre, since the system is time-reversible.
Now, we suppose that $c_1 d_2 \neq 0$. Referring to the expressions given above for the first focus quantities, for a centre we have $d_1 = 0, d_3 = -\frac{1}{9}c_1 d_2$ and $c_3 = -\frac{1}{5103}(266c_1^2 + 405d_2 + 15395c_2)c_1$.
Substituting, it has **iii)**. In this case, the system is of the family (4.5.14) given in Proposition 4.5.81 with

$$v = g(x,y) = x^{10} + \tfrac{20}{9}a_1 x^5 y^3 - (b_2 + \tfrac{200}{81}a_1^2)y^6,$$
$$K(v, y^2) = \tfrac{1}{10}g(x,y), \qquad .$$
$$\Psi(v, y^2) = (a_2 + \tfrac{2}{5}b_2)g(x,y)y^4 - [a_4 + (b_2 + \tfrac{200}{81}a_1^2)(a_2 - \tfrac{40}{27}a_1^2)]y^{10}.$$

Thus, the singular point O is a centre also it is analytically integrable. ■

We now consider the 11-parameter nilpotent system

$$\begin{aligned} \dot{x} = &\ y + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_6xy^4 + a_7y^5, \\ \dot{y} = &\ \quad -x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + b_5xy^4 + b_6y^5. \end{aligned} \tag{4.5.25}$$

This is a subfamily of (4.3.2) given by $(\dot{x}, \dot{y})^T = \mathbf{F}_0 + \mathbf{F}_2 + \mathbf{F}_4$, with $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}}$, $i = 0, 2, 4$, $\mathbf{t} = (1, 1)$, and

$$\mathbf{F}_0 = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \mathbf{F}_2 = \begin{pmatrix} a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \\ -x^3 + b_1x^2y + b_2xy^2 + b_3y^3 \end{pmatrix},$$
$$\mathbf{F}_4 = \begin{pmatrix} a_6xy^4 + a_7y^5 \\ b_5xy^4 + b_6y^5 \end{pmatrix}.$$

The following result characterizes the centres of the systems (4.5.25).

Theorem 4.5.88. *The origin of the system (4.5.25) is a centre if and only if one of the following two series is satisfied:*
i) $3a_1 + b_1 = 2a_2 + b_2 = a_3 + 3b_3 = b_5 = a_6 + 5b_6 = 0$, *(Hamiltonian system).*
ii) $b_1 + 3a_1 = b_3 - a_1(b_2 + 2a_1^2) = a_3 + a_1(b_2 - 2a_2 + 6a_1^2) = b_6 - a_1b_5 = a_6 + a_1b_5 = 0$.
Moreover, each one of them has a local analytic first integral.

Proof.
Taking $\mathbf{t} = (1, 2)$, (that is, the type of the vector field $(y, -x^3)^T$, i.e. the $\mathbf{t}$-homogeneous principal part of the system (4.5.25)), the system (4.5.25) comes given by $(\dot{x}, \dot{y})^T = (h_x + x\mu(x, y), -h_y + 2y\mu(x, y))^T$ being h the defined positive function

$$\begin{aligned} h(x, y) = &\ \tfrac{1}{4}(2y^2 + x^4) - \tfrac{1}{5}c_1x^3y - \tfrac{1}{6}c_2x^2y^2 - \tfrac{1}{7}c_3xy^3 - \tfrac{1}{8}c_4y^4 - \tfrac{1}{10}c_5x^2y^4 \\ &- \tfrac{1}{11}c_6xy^5 - \tfrac{1}{12}c_7y^6, \end{aligned} \tag{4.5.26}$$

and $\mu(x, y) = \frac{1}{5}d_1x^2 + \frac{1}{6}d_2xy + \frac{1}{7}d_3y^2 + \frac{1}{10}d_5xy^3 + \frac{1}{11}d_6y^4$ where

$$\begin{array}{llll} c_1 = b_1 - 2a_1, & d_1 = 3a_1 + b_1, & c_2 = b_2 - 2a_2, & d_2 = 2a_2 + 2b_2, \\ c_3 = b_3 - 2a_3, & d_3 = a_3 + 3b_3, & c_4 = -2a_4, & d_5 = 4c_5 = 4b_5, \\ c_6 = 2a_6, & d_6 = a_6 + 5b_6, & c_7 = -2a_7. & \end{array}$$

The first four focus quantities are

$$\begin{aligned}
&g_1 = d_1, \quad g_2 = d_2c_1 + 5d_3,\\
&g_3 = d_2(25d_2c_1 + 175c_1c_2 + 42c_1^3 + 375c_3),\\
&g_4 = 4c_5c_1 + 5d_6.
\end{aligned}$$

In this point, we distinguish two cases depending on the coefficient d_2. If $d_2 \neq 0$, some necessary conditions for the origin to be a centre are $g_1 = g_2 = g_3 = g_4 = 0$, i.e. $d_1 = 0,\ d_3 = -\frac{1}{5}d_2c_1,\ c_3 = -\frac{1}{375}c_1(25d_2 + 175c_2 + 42c_1^2),\ d_6 = -\frac{4}{5}c_1c_5$. In such a case, it has $g_5 = d_2(3c_1c_5 + 5c_6)$. Hence, $c_6 = -\frac{3}{5}c_1c_5$. Thus, we arrive to the system given by **ii)**,

$$\begin{aligned}
\dot{x} = &\ y + a_1x^3 + a_2x^2y - a_1(b_2 - 2a_2 + 6a_1^2)xy^2 + a_4y^3 - a_1b_5xy^4 + a_7y^5,\\
\dot{y} = &\quad -x^3 - 3a_1x^2y + b_2xy^2 + a_1(b_2 + 2a_1^2)y^3 + b_5xy^4 + a_1b_5y^5.
\end{aligned} \tag{4.5.27}$$

This system belongs to the family (4.5.14) given in Proposition 4.5.81 where

$$\begin{aligned}
&v = \tfrac{1}{2}x^2 + a_1xy + (\tfrac{1}{2}a_2 - a_1^2)y^2,\\
&K(v, y^2) = 2h - (a_2 + b_2)y^2 - b_5y^4,\\
&\Phi(v, y^2) = [a_4 + (a_2 - 2a_1^2)(2a_1^2 + b_2)]y^2 + [a_7 + b_5(a_2 - 2a_1^2)]y^4
\end{aligned}$$

and, therefore, the origin is a centre, since the system has a local analytic first integral and O is a monodromic point.

If $d_2 = 0$, from the vanishing of the first four constants above, we have $d_1 = d_3 = 0$ and $d_6 = -\frac{4}{5}c_1c_5$. In this case, the next focus quantities are $g_5 = c_5(175c_1c_2 + 42c_1^3 + 375c_3)$, $g_6 \equiv 0$ and $g_7 = c_5(3c_1c_5 + 5c_6)$. If $c_5 = 0$, the necessary conditions to have a centre leads us to the hamiltonian system **i)** whose hamiltonian function is (4.5.26). We note that the curves $h(x, y) = cte$ are closed, therefore, it is a centre.

And if $c_5 \neq 0$, g_5 and g_7 must be zero; hence,

$$c_3 = -\frac{7}{375}c_1(25c_2 + 6c_1^2),\ c_6 = -\frac{3}{5}c_1c_5.$$

In such a case, the system is of the form (4.5.27) with $d_2 = 0$. So, O is a centre. ∎

Theorem 4.5.89. *Under perturbations of the parameters of the system (4.5.25), it has:*
a) *if $a_2 + b_2 \neq 0$ or $b_5 \neq 0$, it can bifurcate 0,1,2,3 or 4 limit cycles around the origin.*
b) *if $a_2 + b_2 = b_5 = 0$, it can bifurcate 0,1 or 2 limit cycles around the origin.*

Proof.
We fix the constants c_1, d_2, c_2, c_5 above defined and consider the critical values

$$d_1^* = 0,\ d_3^* = -\tfrac{1}{5}d_2c_1,\ c_3^* = -\tfrac{1}{375}c_1(25d_2 + 175c_2 + 42c_1^2),$$
$$d_6^* = -\tfrac{4}{5}c_1c_5,\ c_6^* = -\tfrac{3}{5}c_1c_5.$$

Firstly, we assume that $a_2 + b_2 \neq 0$. From the expression of g_1, g_2, g_3, g_4 and g_5, applying Corollary 4.4.80, it deduces the following one: if $d_1 \neq d_1^*$ there isn't limit cycles around the origin. If d_1 is near zero and $d_3 \neq d_3^*$, then it can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0 < |d_1| << |d_3|$ and also c_3 is different from c_3^*, then there are 2 small amplitude limit cycles. If also $d_3c_3 < 0$, $|d_1| << |d_3|$ and $d_6 \neq d_6^*$, then 3 limit cycles bifurcate of the origin. If we take d_1, d_3, c_3 and d_6 different from d_1^*, d_3^*, c_3^* and d_6^* but near each of them, respectively, and $c_6 \neq c_6^*$ such that it satisfy the hypothesis of Corollary 4.4.80 with $r = 4$, it has that we can bifurcate 4 limit cycles, at least.
Now, we assume that $a_2 + b_2 = 0$ and $b_5 \neq 0$. The first constants g_i different from zero are

$$g_1 = d_1,\ g_2 = d_3,\ g_4 = 4c_1c_5 + 5d_6,\ g_5 = c_5(175c_1c_2 + 42c_1^3 + 375c_3),$$
$$g_7 = c_5(3c_1c_5 + 5c_6).$$

Thus, if $d_1 \neq 0$ there isn't limit cycles around the origin. If d_1 is close to zero and $d_3 \neq 0$, can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0 < |d_1| << |d_3|$ and also d_6 is different from d_6^*, then there are 2 limit cycles of small amplitude. If we now take d_6 such that d_3 and d_6 alternate sign and $|d_3| << |d_6|$ and also c_3 is different from $c_3^* = -\frac{7}{375}c_1(25c_2 + 6c_1^2)$ then there exist 3 limit cycles around the origin. If we also choose c_3 near c_3^* such that $d_6 c_3 < 0$ and $|d_6| << |c_3|$ and take $c_6 \neq c_6^*$, then there exist, at least, 4 limit cycles around the origin.
Last on, if $a_2 + b_2 = b_5 = 0$, that is $d_2 = c_5 = 0$, the g_i different from zero are

$$g_1 = d_1, \ g_2 = d_3, \ g_4 = d_6.$$

Therefore, if $d_1 \neq 0$ there isn't limit cycles near the origin. If d_1 is close zero and $d_3 \neq 0$, it can exist, at least, 1 limit cycle. If d_1 and d_3 alternate sign and $0 < |d_1| << |d_3|$ and also d_6 is different from zero, then 2 small amplitude limit cycles bifurcate from the origin. ■

Last on, we study the systems

$$\dot{x} = y + \sum_{i=1}^{[\frac{2n+2}{5}]} a_{i,n} x^{2n+2-5i} y^{3i-1}, \quad \dot{y} = -x^{2n+1} + \sum_{i=1}^{[\frac{2n+1}{5}]} b_{i,n} x^{2n+1-5i} y^{3i}. \tag{4.5.28}$$

with $1 \leq n \leq 9$. ($[x]$ means integer part of x). These systems are of the family (4.3.2) given by $(\dot{x}, \dot{y})^T = \mathbf{F}_2 + \mathbf{F}_{6n-2}$, with $\mathbf{F}_i \in \mathcal{Q}_i^{\mathbf{t}}$, $i = 2, 6n-2$, $\mathbf{t} = (3, 5)$.
Analogously to the above application, we obtain the ciclicity of the origin of (4.5.28).

Theorem 4.5.90. *The system (4.5.28) with n equal to 1,2,3 or 4, does not have limit cycles of small amplitude surrounding the origin. There are systems inside this family for $n = 5$ or $n = 6$ with 0 or 1 limit cycle around the origin. For $n = 7$ or $n = 8$ there exist systems inside this* *family with 0,1,2 or 3 limit cycles around the origin. There exist systems (4.5.28) with $n = 9$ which have 0, 1 or 2 limit cycles around the origin.*

Proof.
The system (4.5.28) takes the form

$$(\dot{x}, \dot{y})^T = (h_x + x\mu(x, y), -h_y + (n+1)y\mu(x, y))^T$$

with

$h(x, y) = -\frac{1}{2n+2}(x^{2n+2} + (n+1)y^2) + \sum_{i=1}^{[\frac{2n+1}{5}]} \frac{1}{2n+2+(3n-2)i} c_{i,n} x^{2n+2-5i} y^{3i}$,
$\mu(x, y) = \sum_{i=1}^{[\frac{2n+1}{5}]} \frac{1}{2n+2+(3n-2)i} d_{i,n} x^{2n+1-5i} y^{3i-1}$,

where the new coefficients that appear are

$$c_{i,n} = (n+1)a_{i,n} - b_{i,n}, \qquad d_{i,n} = (2n+2-5i)a_{i,n} + 3ib_{i,n}$$

The expressions of the focus quantities different from zero are:

$n = 2,\;\; g_1 = d_{1,2},$

$n = 3,\;\; g_1 = d_{1,3},$

$n = 4,\;\; g_1 = d_{1,4},$

$n = 5,\;\; g_1 = d_{1,5}, g_2 = c_{1,5}d_{2,5}$

$n = 6,\;\; g_1 = d_{1,6},\; g_2 = c_{1,6}d_{2,6},$

$n = 7,\;\; g_1 = d_{1,7},\; g_2 = c_{1,7}d_{2,7} + 35d_{3,7},$

$g_3 = d_{2,7}(33075c_{3,7} + 1314c_{1,7}^3 + (2660d_{2,7} + 7665c_{2,7})c_{1,7}),$

$g_4 = -c_{1,7}^5 d_{2,7},$

$n = 8,\;\; g_1 = d_{1,8},\; g_2 = 3c_{1,8}d_{2,8} + 40d_{3,8},$

$g_3 = d_{2,8}(49600c_{3,8} + 2387c_{1,8}^3 + (3960d_{2,8} + 13440c_{2,8})c_{1,8}),$

$g_4 = -c_{1,8}^5 d_{2,8},$

$n = 9,\;\; g_1 = d_{1,9},\; g_2 = c_{1,9}d_{2,9} + 9d_{3,9},$

$g_3 = d_{2,9}(5103c_{3,9} + (266c_{1,9}^2 + 405d_{2,9} + 1539c_{2,9})c_{1,9}).$

Choosing adequately the constants $c_{i,n}$ and $d_{i,n}$ and by applying Corollary 4.4.80 it follows the result. ∎

Chapter 5

Rational integrability of two-dimensional quasi-homogeneous vector fields

5.1 Introduction

In this chapter, we deal with polynomial differential systems

$$(\dot{x}, \dot{y})^T = \mathbf{F}_r = (P, Q)^T, \tag{5.1.1}$$

where $\mathbf{F}_r$ is a quasi-homogeneous polynomial vector field of degree $r \geq 0$ with respect to the type $\mathbf{t} = (t_1, t_2)$ fixed, i.e. $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. In the particular case that $\mathbf{t} = (1, 1)$, (5.1.1) is a homogeneous polynomial differential system of degree $r + 1$.

A function H is a first integral of system (5.1.1) in an open subset U of $\mathbb{R}^2$ if H is a nonconstant function in U which is constant on each solution curve of system (5.1.1). If there exists a rational first integral of (5.1.1) it is said that (5.1.1) is rationally integrable. Clearly, if $H = \frac{f}{g}$, with f, g polynomials, is a first integral of system (5.1.1) then the Lie derivative of H by $\mathbf{F}_r$ is zero in the open subset $\Omega_g = \{(x,y) \in \mathbb{R}^2 : g(x,y) \neq 0\}$, i.e. $L_{\mathbf{F}_r}H := \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q \equiv 0$ in Ω_g.

The purpose of our approach is to know when (5.1.1) is a rationally integrable system and, in particular, when the origin has a polynomial first integral since, it is easy to prove that, for the systems (5.1.1), the analytic integrability is equivalent to the polynomial integrability. Indeed, H is an analytic first integral of (5.1.1), with $H = H_m + H_{m+1} + \cdots$ its expansion into quasi-homogeneous polynomials of degree $m+i$ with respect to a fixed type $\mathbf{t}$, if and only if each quasi-homogeneous part H_{m+i} is a first integral of system (5.1.1) for all i.

It is known that we can always calculate a first integral of (5.1.1) explicitly. It is enough to make the change of variables $(x,y) \to (u,v)$ according to $x = v^{t_1},\ y = v^{t_2}u$ which transforms the differential equation $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$ into a linear equation $\frac{v}{t_1}\frac{du}{dv} + \frac{t_2}{t_1}u = \frac{Q(1,u)}{P(1,u)}$ easy to integrate. But this first integral usually has a huge algebraic expression and, therefore, it is difficult to show whether it is rational or not.

Several authors have studied the integrability of quasi-homogeneous systems and its relation with both n-dimensional and planar systems, see, Furta [74], Goriely [94], Llibre & Zhang [112], Tsygvintsev [157] and Cairó & Llibre [36], among others. As far as we known, only degree one polynomially integrable plane systems have been calculated, see Tsygvintsev [157] and Llibre and Zhang [112].

The results obtained in this chapter are closely linked to the conservative-dissipative decomposition of (5.1.1), Lemma 5.2.92. In section below, in relation to the rational integrability, Theorems 5.2.96 and 5.2.98 give an easy characterization for the quasi-homogeneous polynomial systems with a rational first integral. In section 5.3, we calculate the Kowalevskaya exponents of system (5.1.1) and we show how the Kowalevskaya exponents are when system (5.1.1) has got a $\mathbf{t}$-homogeneous rational (polynomial) first integral, Theorem 5.3.101. Finally, in last section, as an application, we find the $(1,2)$-homogeneous polynomial systems of degree two having a rational first integral (Theorems 5.4.102 and 5.4.103).

5.2 Characterization of the rationally integrable quasi-homogeneous vector fields

The following lemma shows that if system (5.1.1) has a rational first integral, then the existence of a $\mathbf{t}$-rational function first integral of (5.1.1) is necessary, that is, a first integral which is a quotient of two $\mathbf{t}$-homogeneous polynomials.

Lemma 5.2.91. *Let* $f = \sum_{j=m_0}^{M} f_j$ *and* $g = \sum_{j=n_0}^{N} g_j$ *with* $f_j, g_j \in \mathcal{P}_j^{\mathbf{t}}$ *(their expansions into* $\mathbf{t}$*-homogeneous polynomials, respectively). If* $\frac{f}{g}$ *is a first integral of system (5.1.1) in* Ω_g*, then* $\frac{f_{m_0}}{g_{n_0}}$ *is a first integral of system (5.1.1) in* $\Omega_{g_{n_0}}$*.*

Proof.
If $\frac{f}{g}$ is a first integral of system (5.1.1) in Ω_g it has that $L_{\mathbf{F}_r}\frac{f}{g} \equiv 0$ in Ω_g and therefore $0 \equiv gL_{\mathbf{F}_r}f - fL_{\mathbf{F}_r}g$. In particular its first $\mathbf{t}$-homogeneous term is also zero. So,

$$0 \equiv g_{n_0}L_{\mathbf{F}_r}f_{m_0} - f_{m_0}L_{\mathbf{F}_r}g_{n_0} = g_{n_0}^2 L_{\mathbf{F}_r}\frac{f_{m_0}}{g_{n_0}} \text{ in } \Omega_g \cap \Omega_{g_{n_0}}.$$

By continuity, it extends to $\Omega_{g_{n_0}}$. ■

We will assume that system (5.1.1) is a irreducible system (i.e. P and Q are coprime and $PQ \not\equiv 0$) since, otherwise, if $PQ \equiv 0$ then x or y are first integrals of system (5.1.1); so system (5.1.1) is polynomially integrable. And if P, Q are no coprime, it has that $P = fP'$, $Q = fQ'$ where $\mathbf{F}_{r'} = (P', Q')^T$ is a $\mathbf{t}$-homogeneous polynomial vector field of degree $r' < r$. It is easy to prove that H is a first integral of (5.1.1) if and only if H is a first integral of $(\dot{x}, \dot{y})^T = \mathbf{F}_{r'}$. Therefore, it is sufficient to study the integrability of the second system.

Next, we give the decomposition of a $\mathbf{t}$-homogeneous vector field as a sum of two $\mathbf{t}$-homogeneous vector fields, a conservative one (having zero-divergence) and a dissipative one that will be useful in what follows. Throughout this chapter, we denote by $\mathbf{X}_h$ the hamiltonian system associated to h, i.e. $\mathbf{X}_h = (-\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x})^T$, denote $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, (the dissipative vector field of degree 0 with respect to the type $\mathbf{t}$), and we also denote $|\mathbf{t}| = t_1 + t_2$.

Lemma 5.2.92. *Every $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$ can be expressed as*

$$\mathbf{F} = \frac{1}{k + |\mathbf{t}|}[\mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}} + \operatorname{div}(\mathbf{F})\mathbf{D}_0]. \tag{5.2.2}$$

Furthermore, a such decomposition is unique.

Proof.
Let $\mathbf{F} = (P, Q)^T \in \mathcal{Q}_r^{\mathbf{t}}$ be. It is straightforward to show that

$$-\tfrac{\partial \mathbf{D}_0 \wedge \mathbf{F}}{\partial y} + t_1 x \operatorname{div}(\mathbf{F}) = (t_1 x \tfrac{\partial P}{\partial x} + t_2 y \tfrac{\partial P}{\partial y}) + t_2 P,$$
$$\tfrac{\partial \mathbf{D}_0 \wedge \mathbf{F}}{\partial x} + t_2 y \operatorname{div}(\mathbf{F}) = (t_1 x \tfrac{\partial Q}{\partial x} + t_2 y \tfrac{\partial Q}{\partial y}) + t_1 Q.$$

As $P \in \mathcal{P}_{r+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{r+t_2}^{\mathbf{t}}$, from Euler Theorem for $\mathbf{t}$-homogeneous polynomial.
We see the second part. For any $h \in \mathcal{P}_{r+t_1+t_2}^{\mathbf{t}}$, $\mu \in \mathcal{P}_r^{\mathbf{t}}$ it has that

$$\operatorname{div}(\mathbf{X}_h) = 0,$$
$$\operatorname{div}(\mu \mathbf{D}_0) = \nabla \mu \cdot \mathbf{D}_0 + (t_1 + t_2)\mu = (r + t_1 + t_2)\mu.$$

Therefore, if h, μ verify (5.2.2), it has

$$\operatorname{div}(\mathbf{F}) = \tfrac{1}{r+t_1+t_2}(\operatorname{div}(\mathbf{X}_h) + \operatorname{div}(\mu \mathbf{D}_0)) = \mu,$$
$$\mathbf{D}_0 \wedge \mathbf{F} = \tfrac{1}{r+t_1+t_2}\mathbf{D}_0 \wedge \mathbf{X}_h = \tfrac{1}{r+t_1+t_2}\nabla h \cdot \mathbf{D}_0 = h.$$

it follows the result. ∎

Let note that if $h \equiv 0$, then system (5.1.1) is not irreducible. Moreover, in such a case, any $\mathbf{t}$-rational of the form $\frac{p}{q}$ with $p, q \in \mathcal{P}^{\mathbf{t}}_{m_0}$ is a first integral in Ω_q, since by Lemma 5.2.92, it has that

$$L_{\mathbf{F}_r}\tfrac{p}{q} = \frac{1}{q^2}(qL_{\mathbf{F}_r}p - pL_{\mathbf{F}_r}q) = \frac{1}{(r+|\mathbf{t}|)q^2}\mathrm{div}(\mathbf{F}_r)(qL_{\mathbf{D}_0}p - pL_{\mathbf{D}_0}q),$$

in Ω_q, and from Euler Theorem for $\mathbf{t}$-homogeneous polynomial, $L_{\mathbf{D}_0}p = m_0 p$, $L_{\mathbf{D}_0}q = m_0 q$, therefore $L_{\mathbf{F}_r}\frac{p}{q} \equiv 0$ in Ω_q.

The following properties characterize the $\mathbf{t}$-homogeneous polynomial systems having a $\mathbf{t}$-homogeneous rational first integral, and they give conditions on h.

Lemma 5.2.93. *An irreducible system (5.1.1) has got a* $\mathbf{t}$*-rational first integral* $\frac{p}{q}$*, with* $p \in \mathcal{P}^{\mathbf{t}}_{m_0}$ *and* $q \in \mathcal{P}^{\mathbf{t}}_{n_0}$*, if and only if* $L_{\mathbf{X}_h}\frac{p}{q} = (n_0 - m_0)\mathrm{div}(\mathbf{F}_r)\frac{p}{q}$*, in* Ω_q*. Moreover, in such a case,* m_0 *must be different from* n_0.

Proof.
If $\mathrm{div}(\mathbf{F}_r)$ is identically zero, then system (5.1.1) is a hamiltonian system and h is a polynomial first integral. In such a case, $p = h$, $q \equiv 1$, $n_0 = 0$, $m_0 = r + |\mathbf{t}|$.
If $\mathrm{div}(\mathbf{F}_r) \not\equiv 0$, by Lemma 5.2.92 and Euler Theorem for $\mathbf{t}$-homogeneous polynomial it arrives to

$$(r+|\mathbf{t}|)L_{\mathbf{F}_r}\tfrac{p}{q} = L_{\mathbf{X}_h}\tfrac{p}{q} + \mathrm{div}(\mathbf{F}_r)L_{\mathbf{D}_0}\tfrac{p}{q} = L_{\mathbf{X}_h}\tfrac{p}{q} + (m_0 - n_0)\mathrm{div}(\mathbf{F}_r)\tfrac{p}{q},$$

in Ω_q. We finish this proof, showing that $m_0 \neq n_0$. On the one hand, if $\frac{p}{q}$ is a first integral of $\mathbf{F}_r$, it holds $L_{\mathbf{F}_r}\frac{p}{q} = 0$ in Ω_q, i.e. $P(qp_x - pq_x) + Q(qp_y - pq_y) = 0$ in Ω_q, where p_x, q_x and p_y, q_y it denotes partial derivative respect to x of p and q and y, respectively. On the other hand, as the components of $\mathbf{F}_r$ haven't got common factors, it has that there exists $k \in \mathcal{P}^{\mathbf{t}}_{m_0+n_0-r-|\mathbf{t}|}$, such that

$$-(q_y p - p_y q) = kP, \qquad (q_x p - p_x q) = kQ,$$

thus, it follows that $k\mathbf{F}_r = q\mathbf{X}_p - p\mathbf{X}_q = q^2\mathbf{X}_{\frac{p}{q}}$ in Ω_q ($k \not\equiv 0$ since $P.Q \not\equiv 0$). It has that

$$(r + |\mathbf{t}|)kh = \mathbf{D}_0 \wedge (k\mathbf{F}_r) = \mathbf{D}_0 \wedge (q^2\mathbf{X}_{\frac{p}{q}}) = \mathbf{D}_0 \wedge (q\mathbf{X}_p - p\mathbf{X}_q)\,,$$

and from Euler Theorem,

$$(r + |\mathbf{t}|)kh = (m_0 - n_0)pq, \text{ in } \Omega_q, \tag{5.2.3}$$

and this equality can be extend to $\mathbb{R}^2$. So, $m_0 \neq n_0$ and it follows the result. ∎

Lemma 5.2.94. *Let $\frac{p}{q}$ be an irreducible* $\mathbf{t}$*-rational first integral of an irreducible system (5.1.1). Then, any irreducible factor of p or q on $\mathbb{K}[x,y]$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) is a factor of h on $\mathbb{K}[x,y]$.*
Reciprocally, any irreducible factor of h on $\mathbb{K}[x,y]$ is either a factor of p or a factor of q.

Proof.
From Lemma 5.2.93, $\frac{p(x,y)}{q(x,y)} = 0$ is an invariant rational curve of $\mathbf{X}_h$ in Ω_q and as $\frac{p}{q}$ is irreducible, then $\Omega_p \subset \mathbb{R}^2 \setminus \Omega_q$. Therefore, $p(x,y) = 0$ is a polynomial invariant curve of $\mathbf{X}_h$. On the other hand, the unique irreducible invariant curves of $\mathbf{X}_h$ are the irreducible factors of h. Thus, it follows that any irreducible factor of p is a factor of h.
Note that if $\frac{p}{q}$ is a first integral of $\mathbf{F}_r$, then $\frac{q}{p}$ also it is. Applying the same reasoning for the first integral $\frac{q}{p}$, it has that any irreducible factor of q is also a factor of h.
Finally, it follows by (5.2.3) that every irreducible factor of h must be either a factor of p or a factor of q. ∎

Lemma 5.2.95. *If an irreducible system (5.1.1), with* $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$*, has got a* $\mathbf{t}$*-rational first integral, then h has at least two irreducible factors on $\mathbb{K}[x,y]$ and all of them are distinct (i.e. the factors of h are simple).*

Proof.
On the one hand, if h had the form f^m with $m \geq 1$, then there would be a integer number non-zero n such that $\frac{p}{q} = f^n$, from Lemma 5.2.94. Applying Lemma 5.2.93, it has that $L_{\mathbf{X}_{f^m}} f^n = 0 = (n_0 - m_0)\mathrm{div}(\mathbf{F}_r) f^n$, it would arrive at $\mathrm{div}(\mathbf{F}_r) \equiv 0$. Thus, the assumption leads us to a contradiction.
On the other hand, if $h = \prod_{l=1}^{k} f_l^{m_l}$, with some $m_j > 1$, $1 \leq j \leq k$, then f_j would be a factor of $\mathbf{X}_h$. From Lemma 5.2.94, it has that $\frac{p}{q} = \prod_{l=1}^{k} f_l^{n_l}$, with n_j integers numbers non all zero, and in such a case, it is easy to check that the Lie derivative of $\frac{p}{q}$ by $\mathbf{X}_h$ is given by

$$L_{\mathbf{X}_h} \tfrac{p}{q} = \tfrac{p}{q} h \sum_{i=1}^{k} \sum_{j=i+1}^{k} (n_i m_j - n_j m_i) \frac{1}{f_i f_j} L_{\mathbf{X}_{f_j}} f_i. \qquad (5.2.4)$$

So, applying Lemma 5.2.93 and by cancelling, it would have that

$$h \sum_{i=1}^{k} \sum_{j=i+1}^{k} (n_i m_j - n_j m_i) \tfrac{1}{f_i f_j} L_{\mathbf{X}_{f_j}} f_i = (n_0 - m_0)\mathrm{div}(\mathbf{F}_r).$$

Thus, f_j would be a factor of both, $\mathbf{X}_h$ and $\mathrm{div}(\mathbf{F}_r)$, which would be in contradiction with the fact that the components of $\mathbf{F}_r$ are coprime. ■

From now, given a $\mathbf{t}$-homogeneous function f, we will denote by $\deg_{\mathbf{t}}(f)$ the degree of f with respect to the type $\mathbf{t}$.

The following result gives a link between the existence of a $\mathbf{t}$-rational first integral of system (5.1.1) and the conservative and dissipative terms of the vector field $\mathbf{F}_r$.

Theorem 5.2.96. *An irreducible system (5.1.1) has got a* $\mathbf{t}$*-rational first integral if and only if* $\mathrm{div}(\mathbf{F}_r) \equiv 0$ *or* $h = \prod_{j=1}^{k} f_j$ *where* $f_1, ..., f_k$ *are* $\mathbf{t}$*-homogeneous polynomial irreducible on* $\mathbb{K}[x, y]$ *(where* $\mathbb{K}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$*),* $k \geq 2$ *and there exist* k *non-zero integer numbers,* $n_1, n_2, ..., n_k$*, such that* $n_1 deg_{\mathbf{t}}(f_1) + \cdots + n_k deg_{\mathbf{t}}(f_k) \neq 0$ *and*

$$\Big(\sum_{j=1}^{k} n_j deg_{\mathbf{t}}(f_j)\Big)\mathrm{div}(\mathbf{F}_r) = h \sum_{j=1}^{k} \sum_{l=j+1}^{k} (n_l - n_j) \tfrac{1}{f_j f_l} L_{\mathbf{X}_{f_l}} f_j. \qquad (5.2.5)$$

Moreover, in such a case, $\prod_{j=1}^{k} f_j^{n_j}$ *is a* $\mathbf{t}$*-rational first integral of (5.1.1).*

Proof.
If $\mathrm{div}(\mathbf{F}_r) \equiv 0$, the system is rationally integrable, from Lemma 5.2.93. We assume that $\mathrm{div}(\mathbf{F}_r) \not\equiv 0$, $h \not\equiv 0$ and $\mathbf{F}_r = (P, Q)^T$ has got a rational first integral, $\frac{p}{q}$. Then, from Lemma 5.2.95, $h = \prod_{j=1}^{k} f_j$ where $f_1, ..., f_k$ are $\mathbf{t}$-homogeneous polynomial irreducible on $\mathbb{K}[x, y]$, $k \geq 2$ and, from Lemma 5.2.94, there exist $n_1, n_2, \cdots, n_k$ non-zero integer numbers, such that $p = \prod_{n_i > 0} f_i^{n_i}$ and $q = \prod_{n_i < 0} f_i^{-n_i}$. Moreover, from Lemma 5.2.93, $\deg_{\mathbf{t}}(p) - \deg_{\mathbf{t}}(q) = \sum_{i=1}^{k} n_i \deg_{\mathbf{t}}(f_i) \neq 0$.

In order to prove the sufficient condition, it is enough to apply Lemma 5.2.93 for $h = \prod_{j=1}^{k} f_j$ and $\frac{p}{q} = \prod_{j=1}^{k} f_j^{n_j}$. ■

It is easy to prove that if k can be expressed as $k = k_3 t_1 t_2 + k_2 t_2 + k_1 t_1$ with $0 \leq k_1 < t_2$, $0 \leq k_2 < t_1$, then the set $B_k^{\mathbf{t}} = \{x^{t_2 i + k_1} y^{t_1(k_3 - i) + k_2}, 0 \leq i \leq k_3\}$ is a base of $\mathcal{P}_k^{\mathbf{t}}$. Otherwise, $\mathcal{P}_k^{\mathbf{t}} = \{0\}$. This allows us to write any non-vanishing $\mathbf{t}$-homogeneous polynomial of degree k as $p(x, y) = x^{k_1} y^{k_2} p_0(x^{t_2}, y^{t_1})$ with

$$p_0(x, y) = \sum_{j=0}^{k_3} \alpha_j x^{k_3 - j} y^j,$$

a homogeneous polynomial of degree k_3. Introducing the variable $v = \frac{y}{x}$ and denoting by s the higher index such that $\alpha_s \neq 0$ (i.e. $\alpha_{s+1} = \cdots = \alpha_{k_3} = 0$), we have that

$$p_0(x, y) = x^{k_3 - s} \sum_{j=0}^{s} \alpha_j v^j.$$

If $\lambda_j \in \mathbb{C}$ are the distinct roots of the polynomial $\sum_{j=0}^{s} \alpha_j v^j$, by abusing the notation we can write any $\mathbf{t}$-homogeneous polynomial in a compact form

$$p(x, y) = \alpha_s \prod_{j=1}^{d} f_j^{m_j}, \quad \text{where } f_j(x, y) = x, \ y \text{ or } y^{t_1} - \lambda_j x^{t_2}$$

with $\deg_{\mathbf{t}}(p) = \sum_{j=1}^{d} m_j \deg_{\mathbf{t}}(f_j)$.

Under the hypothesis of irreducibility of $\mathbf{F}_r$, the $\mathbf{t}$-homogeneous polynomials h and $\operatorname{div}(\mathbf{F}_r)$ have the following expressions, according to their degrees.

Lemma 5.2.97. *Let system (5.1.1) be an irreducible system with* $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$, *then* $r + |\mathbf{t}| = k_3 t_1 t_2 + \delta_y t_2 + \delta_x t_1$ *with* $\delta_x, \delta_y \in \{0, 1\}$ *and* $k_3 \geq 0$.
As a consequence, it has that
i) $h(x, y) = x^{\delta_x} y^{\delta_y} h_0(x^{t_2}, y^{t_1})$ *with* $h_0(x, y)$ *homogeneous polynomial of degree* k_3.
ii) $\operatorname{div}(\mathbf{F}_r)(x, y) = x^{(1-\delta_x)(t_2-1)} y^{(1-\delta_y)(t_1-1)} \mu_0(x^{t_2}, y^{t_1})$ *with* $\mu_0(x, y)$ *homogeneous polynomial of degree* $k_3 - (1 - \delta_x) - (1 - \delta_y)$.

Proof.
We assume that there exist $k_1,\ k_2$ and k_3 integer numbers, with $0 \leq k_1 < t_2,\ 0 \leq k_2 < t_1,\ k_3 \geq 0$ such that $\deg_{\mathbf{t}}(h) = r + |\mathbf{t}| = k_1 t_1 + k_2 t_2 + k_3 t_1 t_2$, since otherwise, $h \equiv 0$. It follows easily that $k_1 = \delta_x$ and $k_2 = \delta_y$, with $\delta_x, \delta_y \in \{0, 1\}$, since otherwise x or y would be common factors of the components of $\mathbf{F}_r$. Also, if $t_1 = 1$ we will assume $\delta_y = 0$, and if $t_2 = 1$, $\delta_x = 0$. So, taking into account these considerations, it has that the degree of $\operatorname{div}(\mathbf{F}_r)$ is given by

$$r = [k_3 - (1 - \delta_x) - (1 - \delta_y)] t_1 t_2 + (1 - \delta_x)(t_2 - 1) t_1 + (1 - \delta_y)(t_1 - 1) t_2,$$

with $k_3 - (1 - \delta_x) \geq 0$ and $k_3 - (1 - \delta_y) \geq 0$, since otherwise P or Q are null.
From the expression of $r + |\mathbf{t}|$ and r it follows **i)** and **ii)**. ∎
Next, we define the function $\eta(x, y) := \frac{\mu_0(x,y)}{x^{\delta_x} y^{\delta_y} h_0(x,y)}$ being $r + |\mathbf{t}| = \delta_x t_1 + \delta_y t_2 + k_3 t_1 t_2$. This function will play a important role in our research.

We now give a result which simplifies the conditions of rational integrability of a $\mathbf{t}$-homogeneous polynomial system. This provides an effective approach for computing rationally integrable systems, from the above rational function η.

Theorem 5.2.98. *An irreducible system (5.1.1) has got a rational (polynomial, resp.) first integral if and only if* $\mathrm{div}(\mathbf{F}_r) \equiv 0$*, or the two following properties hold:*
i) *The* $\mathbf{t}$*-homogeneous polynomial* h *has at least two irreducible factors on* $\mathbb{K}[x, y]$*, all of them are distinct, that is, it can be written as* $h = \prod_{j=1}^{m+2} f_j$*, where* $f_1 = x^{\delta_x}$*,* $f_2 = y^{\delta_y}$*,* $\delta_x, \delta_y \in \{0, 1\}$*,* $f_j = y^{t_1} - \lambda_j x^{t_2}$*, for* $j \geq 3$*, and* $m \geq 0$*,*
ii) *for any pole of* $\eta(1, y)$*, its residue is a rational number.*
Moreover, in such a case, by denoting the poles of $\eta(1, y)$ *by* $w_1 = \infty$*,* $w_2 = 0$*,* $w_j = \lambda_j$*,* $j = 3, \cdots, m + 2$ *and the rational numbers* r_j *by*

$$r_j = \tfrac{1}{r+|\mathbf{t}|}(1 - \frac{t_1 t_2}{deg_{\mathbf{t}}(f_j)} Res[\eta(1, y), w_j]), \; j = 1, \cdots, m + 2, \qquad (5.2.6)$$

it has that $\prod_{j=1}^{m+2} f_j^{n_j}$ *is a* $\mathbf{t}$*-rational (polynomial, resp.) first integral of degree* M *being* M *such that* $n_j = M r_j \in \mathbb{Z}\,(\mathbb{N} \cup \{0\}$*, resp.).*

Proof.
First, we prove the necessary condition. If $\mathrm{div}(\mathbf{F}_r) \equiv 0$ or $h \equiv 0$, the system is rationally integrable, from Lemma 5.2.93.
We assume that $\mathrm{div}(\mathbf{F}_r) \not\equiv 0$, $h \not\equiv 0$ and system (5.1.1) has a rational first integral. By Theorem 5.2.96, it has that h verifies **i)**.
Next, we prove the second property. By applying (5.2.5), with $M = \sum_{j=1}^{m+2} n_j \mathrm{deg}_{\mathbf{t}}(f_j)$ and h given by **i)**, it has that

$$\begin{aligned}\mathrm{div}(\mathbf{F}_r) = \;& \tfrac{1}{M}\Big((n_2 - n_1)\delta_x\delta_y \tfrac{h}{f_1 f_2} L_{\mathbf{X}_{f_2}} f_1 + \sum_{j=3}^{m+2}(n_j - n_1)\delta_x \tfrac{h}{f_1 f_j} L_{\mathbf{X}_{f_j}} f_1 \\ & + \sum_{j=3}^{m+2}(n_j - n_2)\delta_y \tfrac{h}{f_2 f_j} L_{\mathbf{X}_{f_j}} f_2 + \sum_{j=3}^{m+2}\sum_{l=j+1}^{m+2}(n_l - n_j)\tfrac{h}{f_l f_j} L_{\mathbf{X}_{f_l}} f_j\Big).\end{aligned} \qquad (5.2.7)$$

The above Lie derivatives is given by

$$\begin{aligned}
&L_{\mathbf{X}_{f_2}} f_1 = -1,\\
&L_{\mathbf{X}_{f_j}} f_1 = -t_1 y^{t_1-1},\ j = 3, \cdots, m+2\\
&L_{\mathbf{X}_{f_j}} f_2 = -t_2 \lambda_2 x^{t_2-1},\ j = 3, \cdots, m+2\\
&L_{\mathbf{X}_{f_l}} f_j = (\lambda_j - \lambda_l) t_1 t_2 x^{t_2-1} y^{t_1-1},\ j, l = 3, \cdots, m+2.
\end{aligned}$$

By Lemma 5.2.97, it has

$$\frac{\operatorname{div}(\mathbf{F}_r)(x,y)}{h(x,y)} = x^{t_2-1} y^{t_1-1} \eta(x^{t_2}, y^{t_1}), \tag{5.2.8}$$

thus, dividing by h in (5.2.7) and changing (x^{t_2}, y^{t_1}) by (x, y), it has the following expression of η,

$$\begin{aligned}
\eta(x,y) = \ & \tfrac{1}{M}\Big((n_1 - n_2)\delta_x \delta_y \tfrac{1}{xy} + \sum_{j=3}^{m+2} t_1 (n_1 - n_j) \delta_x \tfrac{1}{x(y-\lambda_j x)}\\
& + \sum_{j=3}^{m+2} t_2 \lambda_j (n_2 - n_j) \delta_y \tfrac{\lambda_j}{y(y-\lambda_j x)}\\
& + \sum_{j=3}^{m+2} \sum_{l=j+1}^{m+2} t_1 t_2 (n_l - n_j)(\lambda_j - \lambda_l) \tfrac{1}{(y-\lambda_j x)(y-\lambda_l x)}\Big).
\end{aligned}$$

Now, we prove that it holds

$$\operatorname{Res}[\eta(1,y), \infty] = -\delta_x \operatorname{Res}[\eta(x,1), 0], \tag{5.2.9}$$

where, by definition, $\operatorname{Res}[\eta(1,y),\infty] = \frac{1}{2\pi i}\oint_{\gamma^-} \eta(1,y)dy$ being γ^- any closed curve negatively oriented which contains in its interior all the poles of $\eta(1,y)$.

Actually, if $\delta_x = 0$, by Lemma 5.2.97,

$$\deg_{\mathbf{t}}(\eta(1,y)) = \deg_{\mathbf{t}}(\mu_0(1,y)) - \deg_{\mathbf{t}}(y^{\delta_y} h_0(1,y)) = 2,$$

therefore $\operatorname{Res}[\eta(1,y),\infty] = 0$.

If $\delta_x = 1$, the difference of both degrees is one, therefore $\operatorname{Res}[\eta(1,y),\infty] =$

$-\lim_{y\to\infty} y\eta(1,y)$, and expressing η in the form

$$\eta(x,y)=\frac{\sum_{j=0}^{m+1+\delta_y} d_j x^{m+1+\delta_y-j}y^j}{cxy^{\delta_y}\prod_{j=3}^{m+2}(y-\lambda_j x)},$$

then

$$\mathrm{Res}[\eta(x,1),0]=\lim_{x\to 0} x\eta(x,1)=\frac{d_{m+1+\delta_y}}{c}=\lim_{y\to\infty} y\eta(1,y)=-\mathrm{Res}[\eta(1,y),\infty],$$

thus, (5.2.9) holds. Now, we prove that

$$\mathrm{Res}[\eta(1,y),w_i]=\tfrac{1}{t_1t_2}(1-\tfrac{n_i(r+|\mathbf{t}|)}{M})\mathrm{deg}_{\mathbf{t}}(f_i),\ i=1,\cdots,m+2. \quad (5.2.10)$$

For $w_1=\infty$ (case $\delta_x=1$), we have that

$$\begin{aligned}\mathrm{Res}[\eta(1,y),w_1] &= -\mathrm{Res}[\eta(x,1),0]=-\lim_{x\to 0} x\eta(x,1)\\ &= -\tfrac{1}{M}[(n_1-n_2)\delta_y+\textstyle\sum_{j=3}^{m+2} t_1(n_1-n_j)]=-\tfrac{n_1(r+|\mathbf{t}|)-M}{t_2M}.\end{aligned}$$

Analogously, for $w_2=0$ ($\delta_y=1$), it holds

$$\mathrm{Res}[\eta(1,y),w_2]=\tfrac{1}{M}[(n_1-n_2)\delta_x-\sum_{j=3}^{m+2} t_2(n_2-n_j)]=-\tfrac{n_2(r+|\mathbf{t}|)-M}{t_1M}.$$

And for each $w_i=\lambda_i,\ i=3,\cdots,m+2$, it has that

$$\begin{aligned}\mathrm{Res}[\eta(1,y),w_i] &= \lim_{y\to\lambda_i}(y-\lambda_i)\eta(1,y)\\ &= \tfrac{1}{M}\Big(t_1(n_1-n_i)\delta_x+t_2(n_2-n_i)\delta_y+t_1t_2\textstyle\sum_{j=3}^{m+2}(n_j-n_i)\Big)\\ &= \tfrac{-n_i(r+|\mathbf{t}|)+M}{M}.\end{aligned}$$

Therefore, (5.2.10) is proved. As a consequence, it has **ii)**.
Also, defining $r_j=\frac{n_j}{M}\in\mathbb{Q}$ and by solving r_j in (5.2.10) it has (5.2.6). By Theorem 5.2.96, $\prod_{j=1}^{m+2} f_j^{n_j}$ with $n_j=Mr_j\in\mathbb{Z}$, is a rational first integral whose degree is $\sum_{j=1}^{m+2} n_j\mathrm{deg}_{\mathbf{t}}(f_j)=M\sum_{j=1}^{m+2} r_j\mathrm{deg}_{\mathbf{t}}(f_j)$. By (5.2.10), it has that

$$\sum_{j=1}^{m+2} r_j\mathrm{deg}_{\mathbf{t}}(f_j)=\sum_{j=1}^{m+2}\frac{\mathrm{deg}_{\mathbf{t}}(f_j)}{r+|\mathbf{t}|}-t_1t_2\sum_{j=1}^{m+2}\mathrm{Res}[\eta(1,y),w_j].$$

The first summand on the right-hand is 1 and the second is 0, from Residues Theorem. Thus, the degree of the first integral is M.

We now see the sufficient condition. We assume that h verifies **i)** and $\eta(1,y)$ satisfies **ii)**. First, we check that η is univocally determined. It is easily followed because, making $v=\frac{y}{x}$, η can be expressed in the form

$$\eta(x,xv)=\tfrac{1}{x^{1+\delta_x}}\big(\tfrac{A_2\delta_y}{v}+\sum_{j=3}^{m+2}\tfrac{A_j}{v-\lambda_j}\big)$$

and $A_2, A_3, \cdots A_{m+2}$ are given by

$$A_j=\mathrm{Res}[\eta(1,v),w_j]=\tfrac{1}{t_1t_2}(1-\tfrac{n_j(r+|\mathbf{t}|)}{M})\deg_{\mathbf{t}}(f_j), j=2,\cdots,m+2.$$

Finally, we prove that $\mathbf{F}_r$ verifies (5.2.5). Let $\tilde{\mathbf{F}}_r=\frac{1}{r+|\mathbf{t}|}[\mathbf{X}_h+\mu\mathbf{D}_0]$, where μ is the $\mathbf{t}$-homogeneous polynomial such that $(\sum_{j=1}^{m+2} n_j\deg_{\mathbf{t}}(f_j))\mu$ is given by the right-side of (5.2.7). Trivially, $\tilde{\mathbf{F}}_r$ verifies (5.2.5), thus $\tilde{\mathbf{F}}_r$ has got a rational first integral. From the necessary condition, η holds **ii)** and therefore, $\mu=\mathrm{div}(\mathbf{F}_r)$. So, system (5.1.1) is rationally integrable.
■

5.3 Kowalevskaya exponents and rational integrability

The Kowalevskaya exponents arose from the study of the existence of particular solutions of the form $(x(t),y(t))=(c_1t^{-t_1},c_2t^{-t_2})$ of system (5.1.1), where the coefficients $\mathbf{c}=(c_1,c_2)\in\mathbb{C}^2\setminus\{O\}$ are given by the vectorial equation

$$\mathbf{F}_r(\mathbf{c})+\tfrac{1}{r}\mathbf{D}_0(\mathbf{c})=\mathbf{0}. \tag{5.3.11}$$

For a given type $\mathbf{t}$, there may exist different $\mathbf{c}$-s called system *balances*.

Now, for each balance $\mathbf{c}$, it defines the differential of $\mathbf{F}_r+\frac{1}{r}\mathbf{D}_0$ evaluated at $\mathbf{c}$, that is $K(\mathbf{c})=D\left(\mathbf{F}_r+\frac{1}{r}\mathbf{D}_0\right)(\mathbf{c})$. The eigenvalues of $K(\mathbf{c})$ are called the *Kowalevskaya exponents* of the balance $\mathbf{c}$, see Kowalevskaya [107]. It can be shown that there always exists a Kowalevskaya exponent equal to -1, see [157, 162].

Next, we calculate the balances of system (5.1.1), by showing the relation among them and the irreducible factors of h over $\mathbb{C}[x,y]$.

Proposition 5.3.99. *If* $\mathbf{c}$ *is a balance of (5.1.1), then* $h(\mathbf{c}) = 0$.
Furthermore, if P, Q *are coprime and* $PQ \not\equiv 0$, *it holds:*
i) *if* x *is a factor of* h, *then* $(0, c_2)$, *with* $c_2^{r/t_2} = -\frac{t_2}{rQ(0,1)}$, *is a balance of (5.1.1),*
ii) *if* y *is a factor of* h, *then* $(c_1, 0)$, *with* $c_1^{r/t_1} = -\frac{t_1}{rP(1,0)}$, *is a balance of (5.1.1),*
iii) *if* $y^{t_1} - \lambda x^{t_2}$ *is a factor of* h, *with* $\lambda \in \mathbb{C} \setminus \{0\}$, *then* (c_1, c_2) *with* $c_1 = u^{t_1}$, $c_2 = u^{t_2}\lambda^{1/t_1}$ *and* $u^r = -\frac{t_1}{rP(1,\lambda^{1/t_1})}$, *is a balance of (5.1.1).*

Proof.
Every balance $\mathbf{c} = (c_1, c_2)$ is a solution of an irreducible factor of h over $\mathbb{C}[x,y]$, since $h(\mathbf{c}) = (\mathbf{D}_0 \wedge \mathbf{F}_r)(\mathbf{c}) = \mathbf{D}_0(\mathbf{c}) \wedge [\mathbf{F}_r(\mathbf{c}) + \frac{1}{r}\mathbf{D}_0(\mathbf{c})] = 0$.

Let us assume that P, Q are coprime and $PQ \not\equiv 0$. From Lemma 5.2.97, we can assume that $r + |\mathbf{t}| = k_3 t_1 t_2 + \delta_y t_2 + \delta_x t_1$ with $\delta_x, \delta_y \in \{0, 1\}$ and $k_3 \geq 0$. Also, it follows easily that P and Q evaluated at $\mathbf{c}$ may be written as

$$P(\mathbf{c}) = c_1^{\delta_x} c_2^{(t_1-1)(1-\delta_y)} P_0(c_1^{t_2}, c_2^{t_1}), \quad Q(\mathbf{c}) = c_1^{(t_2-1)(1-\delta_x)} c_2^{\delta_y} Q_0(c_1^{t_2}, c_2^{t_1}),$$

where P_0 and Q_0 are homogeneous polynomials of degree $k_3 + \delta_y - 1$ and $k_3 + \delta_x - 1$, respectively.
The factors of h can be of three types: x, y or $y^{t_1} - \lambda x^{t_2}$ with $\lambda \in \mathbb{C} \setminus \{0\}$. If x is a factor of $h = t_1 x Q - t_2 y P$ then x is a factor of P (that is, $P(0, y) \equiv 0$) and $Q(0, 1) \neq 0$, since otherwise x would be a factor of both components of $\mathbf{F}_r$. We compute the balances $\mathbf{c} = (0, c_2)$ of system (5.1.1) associated to x. As $P(0, c_2) = 0$, the first equation of (5.3.11) evaluated at $(0, c_2)$ holds. On the other hand, we note that $(t_2 - 1)(1 - \delta_x) = 0$,

since if $\delta_x = 0$ and $t_2 > 1$, x would be factor of P and Q. Therefore, the second equation of (5.3.11) becomes

$$0 = c_2^{\delta_y} Q(0,1) c_2^{(k_3+\delta_x-1)t_1} + \tfrac{t_2 c_2}{r} = c_2 \left[c_2^{\delta_y - 1 + (k_3+\delta_x-1)t_1} Q(0,1) + \tfrac{t_2}{r} \right],$$

and as $r = t_2[t_1(k_3 - 1 + \delta_x) - (1 - \delta_y)]$, it has $c_2^{r/t_2} = -\frac{t_2}{rQ(0,1)}$, item **i)**.
If y is a factor of h, by the same reasoning we arrive at **ii)**.
If $y^{t_1} - \lambda x^{t_2}$ is a factor of h with $\lambda \in \mathbb{C} \setminus \{0\}$, there exist balances of the form $(c_1, c_2) = \left(u^{t_1}, u^{t_2}\lambda^{1/t_1}\right)$ with $u \neq 0$, since $h\left(u^{t_1}, u^{t_2}\lambda^{1/t_1}\right) = u^{r+|\mathbf{t}|} h\left(1, \lambda^{1/t_1}\right) = 0$.
In this case, the first equation of (5.3.11) is

$$0 = P(u^{t_1}, u^{t_2}\lambda^{1/t_1}) + \tfrac{t_1 u^{t_1}}{r} = u^{t_1}\left[u^r P(1, \lambda^{1/t_1}) + \tfrac{t_1}{r}\right].$$

Moreover $P(1, \lambda^{1/t_1}) \neq 0$, since otherwise $Q(1, \lambda^{1/t_1})$ would be zero and therefore $y^{t_1} - \lambda x^{t_2}$ would be a common factor of P and Q. Thus, it holds **iii)**. The second equation of (5.3.11), by replacing Q, becomes $h(c_1, c_2) + t_2 c_2 \left(P(c_1, c_2) + \frac{t_1 c_1}{r}\right) = 0$ which is true. ■
Next, we obtain the Kowalevskaya exponents associated to the balances of (5.1.1) through the rational function η defined in the previous section.

Proposition 5.3.100. *Let system (5.1.1) be an irreducible system with* $h = \prod_{j=1}^{m+2} f_j^{m_j} \not\equiv 0$. *If* $w_1 = \infty,\ w_2 = 0,\ w_j = \lambda_j,\ j = 3, \cdots, m+2$ *are the poles of* $\eta(1, y)$, *then* $\rho_i = 0$ *if* $m_i > 1$, *otherwise,*

$$\rho_i^{-1} = \tfrac{r}{r+|\mathbf{t}|}\left(1 - \frac{t_1 t_2}{deg_{\mathbf{t}}(f_i)} Res[\eta(1, y), w_i]\right),$$

where every $(-1, \rho_i)$ *is the Kowalevskaya exponents associated to the factor* f_i *of* h, *for* $i = 1, \cdots, m+2$.

Proof.
Throughout the demonstration, we will denote $\mu = \operatorname{div}(\mathbf{F}_r)$ and $h_x,\ h_y$ and $\mu_x,\ \mu_y$ the partial derivatives of h and μ respect to the variables x

and y, respectively.
From Lemma 5.2.92,

$$\mathbf{F}_r + \tfrac{1}{r}\mathbf{D}_0 = \tfrac{1}{r+|\mathbf{t}|}\left(\mathbf{X}_h + (\mu + \tfrac{r+|\mathbf{t}|}{r})\mathbf{D}_0\right).$$

The trace of its differential is

$$\begin{aligned}
&\tfrac{1}{r+|\mathbf{t}|}\left(\mu_x t_1 x + \mu t_1 + \tfrac{r+|\mathbf{t}|}{r}t_1 + \mu_y t_2 y + \mu t_2 + \tfrac{r+|\mathbf{t}|}{r}t_2\right)\\
&= \tfrac{1}{r+|\mathbf{t}|}\left(\nabla\mu\cdot\mathbf{D}_0 + (\mu + \tfrac{r+|\mathbf{t}|}{r})|\mathbf{t}|\right)\\
&= \tfrac{1}{r+|\mathbf{t}|}\left(r\mu + |\mathbf{t}|\mu + \tfrac{r+|\mathbf{t}|}{r}|\mathbf{t}|\right) = \mu + \tfrac{|\mathbf{t}|}{r}.
\end{aligned}$$

Then, for each balance $\mathbf{c}$, if $\rho(\mathbf{c})$ is the eigenvalue different from -1, it has that $\rho(\mathbf{c}) - 1 = \mathrm{Trace}(K(\mathbf{c})) = \mu(\mathbf{c}) + \frac{|\mathbf{t}|}{r}$, that is,

$$\rho(\mathbf{c}) = \mu(\mathbf{c}) + \tfrac{r+|\mathbf{t}|}{r}. \tag{5.3.12}$$

We compute the Kowalevskaya exponents different from -1 associated to the factors of h.
We suppose that x is a factor of h ($\delta_x = 1$), that is, there exist balances of the form $(0, c_2)$. From Lemma 5.2.92, $h_x = (r + |\mathbf{t}|)Q - yt_2\mu$. As μ is a $\mathbf{t}$-homogeneous polynomial of degree r with respect to $\mathbf{t}$, it has that $\mu(0, c_2) = c_2^{r/t_2}\mu(0, 1)$, and by applying (5.3.12) and Proposition 5.3.99, we have that

$$\begin{aligned}
h_x(0,1) &= (r + |\mathbf{t}|)Q(0,1) - t_2\mu(0,1)\\
&= (r + |\mathbf{t}| + r\mu(0,c_2))Q(0,1) = r\rho_1 Q(0,1),
\end{aligned}$$

with $Q(0,1) \neq 0$. Therefore, all the balances $(0, c_2)$ have the same eigenvalues. Moreover, $\rho_1 = 0$ if and only if $h_x(0,1) = 0$, i.e. $m_x > 1$. Otherwise,

$$\begin{aligned}
\tfrac{1}{\rho_1} &= \tfrac{r}{r+|\mathbf{t}|}\left(\tfrac{h_x(0,1)+t_2\mu(0,1)}{h_x(0,1)}\right)\\
&= \tfrac{r}{r+|\mathbf{t}|}\left(1 + t_2\tfrac{\mu(0,1)}{h_x(0,1)}\right) = \tfrac{r}{r+|\mathbf{t}|}\left(1 + t_2\mathrm{Res}[\eta(x,1),0]\right),
\end{aligned}$$

and by (5.2.9) it follows the result. If y is a factor of h, the reasoning is analogous. Finally, we compute the exponents ρ_i associated to the factors

$y^{t_1} - \lambda_i x^{t_2}$, where $\lambda_i \in \mathbb{C} \setminus \{0\},\ i = 3, \cdots, m+2$. From Lemma 5.2.92, $h_y = -(r + |\mathbf{t}|)P + xt_1\mu$. Also $\mu(c_1, c_2) = \mu(u^{t_1}, u^{t_2}\lambda_i^{1/t_i}) = u^r \mu(1, \lambda_i^{1/t_i})$. So, from Proposition 5.3.99,

$$\begin{aligned} h_y(1, \lambda_i^{1/t_i}) &= -(r + |\mathbf{t}|)P(1, \lambda_i^{1/t_i}) + t_1\mu(1, \lambda_i^{1/t_i}) \\ &= -(r + |\mathbf{t}| + r\mu(c_1, c_2))P(1, \lambda_i^{1/t_i}) = -r\rho_i P(1, \lambda_i^{1/t_i}), \end{aligned}$$

with $P(1, \lambda_i^{1/t_i}) \neq 0$. Thus, $\rho_i = 0$ if and only if $h_y(1, \lambda_i^{1/t_i}) = 0$, i.e. $m_i > 1$. Otherwise,

$$\begin{aligned} \tfrac{1}{\rho_i} &= -\tfrac{r}{r+|\mathbf{t}|} \left(\tfrac{-h_y(1,\lambda_i^{1/t_i}) + t_1\mu(1,\lambda_i^{1/t_i})}{h_y(1,\lambda_i^{1/t_i})} \right) \\ &= \tfrac{r}{r+|\mathbf{t}|} \left(1 - t_1 \tfrac{\mu(1,\lambda_i^{1/t_i})}{h_y(1,\lambda_i^{1/t_i})} \right) = \tfrac{r}{r+|\mathbf{t}|} \left(1 - \lim_{y \to \lambda_i^{1/t_1}} t_1 \frac{(y - \lambda_i^{1/t_1})\mu(1, y)}{h(1, y)} \right). \end{aligned}$$

As the above limit is $\mathrm{Res}[\eta(1, y), \lambda_i]$ since $lim_{y \to \lambda_i^{1/t_1}} \frac{y - \lambda_i^{1/t_1}}{y^{t_1} - \lambda_i} = t_1$ and $\frac{\mu(1,y)}{h(1,y)} = y^{t_1 - 1}\eta(1, y^{t_1})$, the result follows. ∎

Next, as a consequence of Theorem 5.2.98 and Proposition 5.3.100, we show how the Kowalevskaya exponents are when system (5.1.1) has got a $\mathbf{t}$-homogeneous rational (polynomial) first integral.

Theorem 5.3.101. *An irreducible system (5.1.1) has got a rational (polynomial, resp.) first integral if and only if* $\mathrm{div}(\mathbf{F}_r) \equiv 0$ *or the two following properties hold:*

i) *The $\mathbf{t}$-homogeneous polynomial h has at least two irreducible factors on $\mathbb{K}[x, y]$, and all of them are distinct, that is, it can be written as $h = \prod_{j=1}^{m+2} f_j$, where $f_1 = x^{\delta_x},\ f_2 = y^{\delta_y},\ \delta_x, \delta_y \in \{0, 1\},\ f_j = y^{t_1} - \lambda_j x^{t_2}$, for $j \geq 3$, and $m \geq 0$,*

ii) *for any $(-1, \rho_j)$ Kowalevskaya exponents, ρ_j is a rational number. Moreover, in such a case, if it denotes the rational numbers r_j by*

$$r_j = \tfrac{1}{r}\rho_j^{-1},\ j = 1, \cdots, m+2, \tag{5.3.13}$$

it has that $\prod_{j=1}^{m+2} f_j^{n_j}$ is a $\mathbf{t}$-rational (polynomial, resp.) first integral of degree M with M such that $n_j = Mr_j \in \mathbb{Z}\,(\mathbb{N} \cup \{0\}$, resp.).

Our results extend to those given by Tsygvintsev [157] and Llibre & Zhang [112]. In fact, in these papers, the relationship between polynomial integrability of vector fields (5.1.1) with $r = 1$ and their Kowalevskaya exponents is proved.

5.4 Rational integrability of the $(1,2)$-homogeneous polynomial systems of degree 2.

We illustrate our method by studying the rational integrability of the $(1,2)$-polynomial systems of degree 2, i.e.

$$\begin{aligned} \dot{x} = &\ a_1x^3 + a_2xy, \\ \dot{y} = &\ b_1x^4 + b_2x^2y + b_3y^2, \end{aligned} \tag{5.4.14}$$

with a_1, a_2, b_1, b_2 and b_3 real parameters with $b_3 \neq 0$ and $b_1a_2^2 - b_2a_1a_2 + b_3a_2^2 \neq 0$ (irreducibility of system (5.4.14)). The function h associated to (5.4.14) is $h(x,y) = \frac{1}{5}x[(b_3-2a_2)y^2+(b_2-2a_1)x^2y+b_1x^4]$. If $b_3-2a_2 = 0$, then x is a multiple factor of h and therefore systems (5.4.14) are not integrable. Otherwise, we can write h in the form $h(x,y) = \frac{1}{5}(b_3 - 2a_2)x[(y + Bx^2)^2 + Ax^4]$ with $A = \frac{4b_1(b_3-2a_2)-(b_2-2a_1)^2}{4(b_3-2a_2)^2}$, $B = \frac{b_2-2a_1}{2(b_3-2a_2)}$. If $A = 0$, systems (5.4.14) are not integrable, since h would have multiple factors. So, under the assumption of integrability, systems (5.4.14) can be transformed, by means of the change $u = \sqrt[4]{\mathrm{sig}(A)A}x,\ v = y + Bx^2$ into $(\dot{u}, \dot{v})^T = \tilde{\mathbf{F}}_2(u,v)$, with

$$\tilde{h}(u,v) = \frac{1}{5}cu(v^2 + \sigma u^4), \qquad \mathrm{div}(\tilde{\mathbf{F}}_2)(u,v) = \frac{1}{5}(d_1v + d_2u^2),$$

where $c \neq 0$ and $\sigma = \pm 1$. That is, the systems become

$$\begin{aligned} \dot{u} = &\ (-2c + d_1)uv + d_2u^3, \\ \dot{v} = &\ (c + 2d_1)v^2 + 2d_2u^2v + 5c\sigma u^4, \end{aligned} \tag{5.4.15}$$

with c, d_1 and d_2 real parameters and $c \neq 0$, $\sigma = \pm 1$.
The following result characterizes both rationally and polynomially integrable systems of the family (5.4.15).

Theorem 5.4.102. *A system (5.4.15) with $\sigma = -1$ is rationally integrable if and only if $\frac{d_1}{c}$, $\frac{d_1+d_2}{2c}$ and $\frac{d_2-d_1}{2c}$ are integers numbers.*
A such system (5.4.15) is polynomially integrable if and only if there is a natural number M such that $\frac{M}{5}(1+\frac{2d_1}{c})$, $\frac{M}{5}(1-\frac{d_1+d_2}{2c})$, $\frac{M}{5}(1-\frac{d_1-d_2}{2c})$ are natural numbers. In such a case

$$u^{\frac{M}{5}(1+\frac{2d_1}{c})}(v-u^2)^{\frac{M}{5}(1-\frac{d_1+d_2}{2c})}(v+u^2)^{\frac{M}{5}(1-\frac{d_1-d_2}{2c})},$$

is a $(1,2)$-polynomial first integral of degree M of the system (5.4.15).

Proof.
In this case, $\eta(u,v) = \frac{d_1 v + d_2 u}{cu(v-u)(v+u)}$. By Theorem 5.2.98, system (5.4.15) is rationally integrable if and only if

$$\begin{aligned}
\mathrm{Res}[\eta(1,v),\infty] &= -\mathrm{Res}[\eta(u,1),0] = -\tfrac{d_1}{c}, \\
\mathrm{Res}[\eta(1,v),1] &= \tfrac{d_1+d_2}{2c}, \\
\mathrm{Res}[\eta(1,v),-1] &= \tfrac{d_1-d_2}{2c}
\end{aligned}$$

are integer numbers. System (5.4.15) has a $(1,2)$-polynomial first integral of degree $M > 0$ if and only if

$$\frac{M}{5}(1+2\frac{d_1}{c}),\ \frac{M}{5}(1-\frac{d_1+d_2}{2c}),\ \frac{M}{5}(1+\frac{d_2-d_1}{2c})$$

are non-negative integer numbers. This completes the proof. ∎

Theorem 5.4.103. *A system (5.4.15) with $\sigma = 1$ is rationally integrable if and only if $d_2 = 0$ and $\frac{d_1}{2c}$ is an integer number.*
A such system (5.4.15) is polynomially integrable if and only if $d_2 = 0$ and there is a natural number M such that $\frac{M}{5}(1+\frac{2d_1}{c})$, $\frac{M}{5}(1-\frac{d_1}{2c})$ are natural numbers. In such a case

$$u^{\frac{M}{5}(1+\frac{2d_1}{c})}(v^2+u^4)^{\frac{M}{5}(1-\frac{d_1}{2c})},$$

is a $(1,2)$*-polynomial first integral of degree* M *of the system (5.4.15).*

Proof.

In this case, $\eta(u,v) = \frac{d_1 v + d_2 u}{cu(v-Iu)(v+Iu)}$. So,

$$\begin{aligned}
\mathrm{Res}[\eta(1,v),\infty] &= -\mathrm{Res}[\eta(u,1),0] = -\tfrac{d_1}{c},\\
\mathrm{Res}[\eta(1,v),I] &= \tfrac{d_1 I + d_2}{2Ic} = \tfrac{d_1}{2c} - \tfrac{d_2}{2c}I,\\
\mathrm{Res}[\eta(1,v),-I] &= \tfrac{d_1 I - d_2}{2Ic} = \tfrac{d_1}{2c} + \tfrac{d_2}{2c}I.
\end{aligned}$$

Therefore, system (5.4.15) is rationally integrable if and only if $d_2 = 0$ and $\frac{d_1}{2c}$ is an integer number. A system of (5.4.15) is polynomially integrable if and only if

$$\frac{M}{5}(1 + 2\frac{d_1}{c}),\ \frac{M}{5}(1 - \frac{d_1}{2c})$$

are non-negative integer numbers. ∎

Chapter 6

Quasi-homogeneous linearization of vector fields

6.1 Introduction

Given a type $\mathbf{t} \in \mathbb{N}^n$, each smooth vector field $\mathbf{F}$, with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, can be written as $\mathbf{F} = \sum_{j \geq r} \mathbf{F}_j$ where $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$. In some sense, $\mathbf{F}$ can be understood as a perturbation of $\mathbf{F}_r$ with higher-order $\mathbf{t}$-homogeneous terms. Thus, for any M we can define the $\mathbf{t}$-homogeneous M-jet of a smooth vector field F as $\mathcal{J}_{\mathbf{t}}^M \mathbf{F} = \sum_{j=r}^{M} \mathbf{F}_j$ with $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$. In this chapter, we give necessary and sufficient conditions so that a vector field has the same orbital structure as a quasi-homogeneous vector field, which is non-linear, in general. Concretely, given a type $\mathbf{t}$, we characterize the vector fields which are smoothly (analytically) conjugated to their $\mathbf{t}$-homogeneous part of minimum degree. As a consequence, it has the result given by Bambusi et al., for $\mathbf{t} = (1, \ldots, 1)$. Analogously, we generalize to a higher dimension the result given by Giné and Grau for smooth (analytic) orbital equivalence.

6.2 Conjugation and orbital equivalence

First we show several properties of the **t**-homogeneous change of variables.

Proposition 6.2.104. *Let ϕ a diffeomorphism with $D\phi(\mathbf{0}) = I$. For each type $\mathbf{t} \in \mathbb{N}^n$, there exist $\Psi_1 = \sum_{i\leq 0} \Psi_{1,i}$ and $\Psi_2 = id + \sum_{i>0} \Psi_{2,i}$ diffeomorphism with $D\Psi_1(\mathbf{0}) = D\Psi_2(\mathbf{0}) = I$ and $\Psi_{1,i}, \Psi_{2,i} \in \mathcal{Q}_i^{\mathbf{t}}$, such that $\phi = \Psi_1 \circ \Psi_2$.*

To proving the above result, we would make the change $\mathbf{x} = \epsilon^{\mathbf{t}}\mathbf{y}$ and would expand the diffeomorphism ϕ in a power series of ϵ. So, for example, for $n = 3$, by means of the change of variables $(x, y, z) \to (\epsilon_1^t x, \epsilon_2^t y, \epsilon_3^t z)$ we would have the following expansion of ϕ

$$\phi(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sum_{i=-m}^{0} \epsilon^i \begin{pmatrix} 0 \\ g_i^-(x) \\ h_i^-(x,y) \end{pmatrix} + \sum_{i=1}^{\infty} \epsilon^i \begin{pmatrix} f_i^+(x,y,z) \\ g_i^+(x,y,z) \\ h_i^+(x,y,z) \end{pmatrix},$$

with $g_i^-, h_i^-, f_i^+, g_i^+, h_i^+$, **t**-homogeneous functions of degree i. Next, it would be enough with to write Ψ_1 and Ψ_2 in the form

$$\Psi_1(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sum_{i=-m}^{0} \epsilon^i \begin{pmatrix} 0 \\ q_i^-(x) \\ r_i^-(x,y) \end{pmatrix}$$

and

$$\Psi_2(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sum_{i=1}^{\infty} \epsilon^i \begin{pmatrix} p_i^+(x,y,z) \\ q_i^+(x,y,z) \\ r_i^+(x,y,z) \end{pmatrix},$$

and to impose that $\phi = \Psi_1 \circ \Psi_2$ Following a triangular scheme, it would obtain the **t**-homogeneous functions $q_i^-, r_i^-, p_i^+, q_i^+$ and r_i^+. For the general case, we would reason in a similar sense.
Analogously, we can prove the following results.

Proposition 6.2.105. *Let Ψ be a diffeomorphism of the form $\Psi = id + \sum_{i\leq 0}\Psi_i$, with $\Psi_i \in \mathcal{Q}_i^{\mathbf{t}}$. Then, the inverse diffeomorphism $\xi = \Psi^{-1}$ is of the form $\xi = id + \sum_{i\leq 0}\xi_i,\ \xi_i \in \mathcal{Q}_i^{\mathbf{t}}$.*

Proposition 6.2.106. *Let Ψ be a diffeomorphism of the form $\Psi = id + \sum_{i>0}\Psi_i$ with $\Psi_i \in \mathcal{Q}_i^{\mathbf{t}}$. Then, the inverse diffeomorphism $\xi = \Psi^{-1}$ is of the form $\xi = id + \sum_{i>0}\xi_i,\ \xi_i \in \mathcal{Q}_i^{\mathbf{t}}$.*

The following result allows to consider only changes of variables with $\mathbf{t}$-homogeneous terms of degree greater than zero.

Proposition 6.2.107. *Let $\mathbf{F}$ be a smooth (analytic) vector field with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $\mathcal{J}_{\mathbf{t}}^r\mathbf{F} = \mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. If $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) conjugated, then there exists a smooth (analytic) near-identity change of coordinates $\mathbf{x} = \phi(\mathbf{y}) = \sum_{i\geq 0}\phi_i(\mathbf{y})$, with $D\phi(\mathbf{0}) = I$ and $\phi_i \in \mathcal{Q}_i^{\mathbf{t}}$ which transforms the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ into $\dot{\mathbf{y}} = \mathbf{F}_r(\mathbf{y})$.*

Proof.
By hypothesis, there exists a smooth (analytic) near-identity change of coordinates $\mathbf{x} = \varphi(\mathbf{y}) = \mathbf{y} + h.o.t$, that transforms the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ into $\dot{\mathbf{y}} = \mathbf{F}_r(\mathbf{y})$. The diffeomorphism φ has the form $\varphi = \sum_{i\geq k}\varphi_i$ with k integer number, $\varphi_i \in \mathcal{Q}_i^{\mathbf{t}}$ and $D\varphi(\mathbf{0}) = I$. By proposition 6.2.104, φ can be decomposed as $\varphi = \Psi \circ \phi$ with $\Psi = \sum_{i=k}^{0}\Psi_i,\ \phi = \sum_{i\geq 0}\phi_i$ and $D\phi(\mathbf{0}) = D\Psi(\mathbf{0}) = I$, being Ψ_i and ϕ_i $\mathbf{t}$-homogeneous vector fields of degree i.
On the one hand, ϕ transforms $\mathbf{F}$ into $\mathbf{F}_r + H$ with H having $\mathbf{t}$-terms of order greater than r. On the other hand, φ brings $\mathbf{F}$ into $\mathbf{F}_r$ and Ψ^{-1} brings $\mathbf{F}_r$ into $\mathbf{F}_r + L$ with L having $\mathbf{t}$-terms of order smaller than r. As $\phi = \Psi^{-1} \circ \varphi$, it has that $G \equiv L \equiv 0$. So, ϕ transforms $\mathbf{F}$ into $\mathbf{F}_r$. ∎

Now, we prove that any smooth (analytic) perturbation of $\mathbf{D}_0$ with $\mathbf{t}$-homogeneous terms of degree greater than zero is smoothly (analytically) conjugated to $\mathbf{D}_0$.

Proposition 6.2.108. *Let* $\mathbf{G}$ *be a smooth (analytic) vector field, with* $\mathcal{J}_{\mathbf{t}}^0\mathbf{G} = \mathbf{D}_0$*. Then,* $\mathbf{G}$ *and* $\mathbf{D}_0$ *are smoothly (analytically) conjugated.*

Proof.
First, we prove that $\mathbf{D}_0$ and $\mathbf{G} = \sum_{j\geq 0}\mathbf{G}_j$ where $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{G}_0 = \mathbf{D}_0$ are smoothly (analytically) conjugated. To do that, we consider the linear map $\mathbf{L}_k : \mathcal{Q}_k^{\mathbf{t}} \to \mathcal{Q}_k^{\mathbf{t}}$ given by $\mathbf{L}_k(\mathbf{P}_k) = [\mathbf{P}_k, \mathbf{D}_0]$. As $\mathbf{L}_k(\mathbf{P}_k) = k\mathbf{P}_k$, it has that the range of the operator $\mathbf{L}_k$ is $\mathcal{Q}_k^{\mathbf{t}}$. Therefore, by Normal Form Theory, it follows that there exists a near-identity nonlinear change of coordinates which transforms $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ into $\dot{\mathbf{y}} = \mathbf{D}_0(\mathbf{y})$. By Chen [51], it has that $\mathbf{G}$ and $\mathbf{D}_0$ are smoothly conjugated.

We prove that if $\mathbf{G}$ is analytic then $\mathbf{G}$ and $\mathbf{D}_0$ are analytically conjugated. If the eigenvalues of the matrix of the linear part of $\mathbf{G}$ at the origin are resonant, i.e. there is $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $\sum_{j=1}^n \alpha_j \geq 2$ such that $t_j = \alpha\cdot\mathbf{t} = \sum_{i=1}^n \alpha_i t_i$ for some $j \in \{1, \ldots, n\}$, it is known that there exists an analytic change of variables which brings $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ to the Poincaré-Dulac normal form, that is $\dot{\mathbf{x}} = \mathbf{D}_0(\mathbf{x}) + h(\mathbf{x})$, where $h(\mathbf{x})$ has only resonant terms, i.e. terms $\mathbf{x}^\alpha\mathbf{e}_j$ (see Arnold [25]), and as $\mathbf{x}^\alpha \in \mathcal{P}_{\alpha\cdot\mathbf{t}}^{\mathbf{t}}$ then $\mathbf{x}^\alpha\mathbf{e}_j \in \mathcal{Q}_{\alpha\cdot\mathbf{t}-t_j}^{\mathbf{t}} = \mathcal{Q}_0^{\mathbf{t}}$. Applying Proposition 6.2.107, it has that $\mathbf{D}_0 + h = \mathbf{D}_0$, i.e. $h \equiv 0$. Therefore, there aren't any resonant terms, so $\mathbf{G}$ and $\mathbf{D}_0$ are $\mathcal{C}^\omega$ conjugated. If $\mathbf{t}$ is not resonant, the result is followed by applying the Poincaré Theorem since the eigenvalues of the linear approximation is in the Poincaré domain, see Chow *et al.* [58]. ∎

We denote as ϕ_* and ϕ^* the push-forward and pull-back defined by the diffeomorphism ϕ, respectively, that is $\phi_*\mathbf{F}(\mathbf{x}) = (D\phi(\mathbf{x}))^{-1}\mathbf{F}(\phi(\mathbf{x}))$, see [124].

In what follows, we suppose a type $\mathbf{t} = (t_1, \ldots, t_n)$ to be fixed.

Next, we prove that the existence of a vector field normalizator of $\mathbf{F}$ allows us to remove, under conjugation, the terms $\mathbf{t}$-homogeneous of degree greater than r.

Theorem 6.2.109 (Like-linearization under conjugation). *Let* $\mathbf{F}$ *be a smooth (analytic) vector field with* $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ *and* $\mathcal{J}_{\mathbf{t}}^r\mathbf{F} = \mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. *Then,* $\mathbf{F}$ *and* $\mathbf{F}_r$ *are smoothly (analytically) conjugated if and only if there exists a smooth (analytic) vector field* $\mathbf{G}$, *with* $\mathcal{J}_{\mathbf{t}}^0\mathbf{G} = \mathbf{D}_0$, *such that* $[\mathbf{F}, \mathbf{G}] = r\mathbf{F}$.
Moreover, there is a smooth (analytic) near-identity nonlinear change of coordinates which transforms $\mathbf{F}$ *into* $\mathbf{F}_r$ *and also linearizes to* $\mathbf{G}$.

Proof.
Necessary condition. From Proposition 6.2.107, if $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) conjugated then there exists a change of variables $\mathbf{x} = \phi(\mathbf{y}) = \sum_{i\geq 0}\phi_i(\mathbf{y})$, with $D\phi(\mathbf{0}) = I$ and $\phi_i \in \mathcal{Q}_i^{\mathbf{t}}$ which transforms the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ into $\dot{\mathbf{y}} = \mathbf{F}_r(\mathbf{y})$, i.e. $\phi_*\mathbf{F} = \mathbf{F}_r$. Also, as $[\mathbf{F}_r, \mathbf{D}_0] = r\mathbf{F}_r$ then it has that

$$[\mathbf{F}, \phi^*\mathbf{D}_0] = [\phi^*\mathbf{F}_r, \phi^*\mathbf{D}_0] = \phi^*[\mathbf{F}_r, \mathbf{D}_0] = r\phi^*\mathbf{F}_r = r\mathbf{F},$$

with $\phi^*\mathbf{D}_0 = \mathbf{D}_0 + \cdots$. Thus, taking $\mathbf{G} = \phi^*\mathbf{D}_0$ it arrives at $[\mathbf{F}, \mathbf{G}] = r\mathbf{F}$.
Sufficient condition. Let $\mathbf{G}$ be a smooth (analytic) vector field, with $\mathcal{J}_{\mathbf{t}}^0\mathbf{G} = \mathbf{D}_0$ with $[\mathbf{F}, \mathbf{G}] = r\mathbf{F}$. By the propositions 6.2.107 and 6.2.108, we can assert that there is a smooth (analytic) change of variables $\mathbf{x} = \phi(\mathbf{y}) = \sum_{i\geq 0}\phi_i(\mathbf{y})$, with $D\phi(\mathbf{0}) = I$ and $\phi_i \in \mathcal{Q}_i^{\mathbf{t}}$ such that $\phi_*\mathbf{G} = \mathbf{D}_0$. Let $\phi_*\mathbf{F} = \overline{\mathbf{F}} = \sum_{j\geq 0}\overline{\mathbf{F}}_{r+j}$ be with $\overline{\mathbf{F}}_r = \mathbf{F}_r$, then $[\overline{\mathbf{F}}, \mathbf{D}_0] = r\overline{\mathbf{F}}$, therefore

$$\sum_{j\geq 0} r\overline{\mathbf{F}}_{r+j} = r\overline{\mathbf{F}} = [\overline{\mathbf{F}}, \mathbf{D}_0] = \sum_{j\geq 0}(r+j)\overline{\mathbf{F}}_{r+j}.$$

So, it follows that $\overline{\mathbf{F}} = \mathbf{F}_r$.
Following the proof, we note that the diffeomorphism ϕ linearizes both ventor fields $\mathbf{F}$ and $\mathbf{G}$. ∎

Now, we give an auxiliary result.

Lemma 6.2.110. *Given a smooth (analytic) scalar function* η, *with* $\eta(\mathbf{0}) = 0$, *and a smooth (analytic) vector field* $\mathbf{G}$ *with* $\mathcal{J}_{\mathbf{t}}^0\mathbf{G} = \mathbf{D}_0$, *there* *exists a smooth (analytic) scalar function* λ *with* $\lambda(\mathbf{0}) = 0$, *such that*

$$L_{\mathbf{G}}\lambda(\mathbf{x}) = (1+\lambda(\mathbf{x}))\eta(\mathbf{x}). \tag{6.2.1}$$

Proof.
By Proposition 6.2.108, there exists a change of variables $\mathbf{x} = \phi(\mathbf{y})$, with ϕ smooth (analytic), which transforms $\mathbf{G}$ into $\mathbf{D}_0$. In these coordinates, equation (6.2.1) becomes

$$L_{\mathbf{D}_0}\bar{\lambda}(\mathbf{y}) = (1+\bar{\lambda}(\mathbf{y}))\bar{\eta}(\mathbf{y})$$

with $\bar{\eta}(\mathbf{0}) = 0$, or equivalently, $L_{\mathbf{D}_0}\mathrm{Ln}(1+\bar{\lambda})(\mathbf{y}) = \bar{\eta}(\mathbf{y})$. We now solve this equation. It is sufficient to show that

$$\begin{array}{rcl} \mathrm{Ln}(1+\bar{\lambda}(\mathbf{y})) & = & \int_{-\infty}^{0} \frac{d}{dt}\mathrm{Ln}(1+\bar{\lambda}(e^{At}\mathbf{y}))dt \\ & = & \int_{-\infty}^{0} D(\mathrm{Ln}(1+\bar{\lambda}))(e^{At}\mathbf{y}) \cdot Ae^{At}\mathbf{y}dt \\ & = & \int_{-\infty}^{0} L_{\mathbf{D}_0}\mathrm{Ln}(1+\bar{\lambda})(e^{At}\mathbf{y})dt \\ & = & \int_{-\infty}^{0} \bar{\eta}(e^{At}\mathbf{y})dt, \end{array}$$

being $A = diag(t_1, \ldots, t_n)$. Therefore, the solution $\bar{\lambda}(\mathbf{y})$ of the equation in partial derivatives with $\bar{\lambda}(\mathbf{0}) = 0$ is

$$\bar{\lambda}(\mathbf{y}) = \exp\left(\int_{-\infty}^{0} \bar{\eta}(e^{At}\mathbf{y})dt\right) - 1 \ .$$

Moreover, $\bar{\lambda}$ and $\bar{\eta}$ have the same regularity. Undoing the change of variables, it gives the result. ∎

Theorem 6.2.111 (Like-linearization under orbital equivalence)**.** *Let* $\mathbf{F}$ *be a smooth (analytic) vector field with* $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ *and* $\mathcal{J}_{\mathbf{t}}^{r}\mathbf{F} = \mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. *Then,* $\mathbf{F}$ *and* $\mathbf{F}_r$ *are smoothly (analytically) orbital equivalents if and only if there exists a smooth (analytic) vector field* $\mathbf{G}$, *with* $\mathcal{J}_{\mathbf{t}}^{0}\mathbf{G} = \mathbf{D}_0$ *and a smooth (analytic) scalar function* μ *with* $\mu(\mathbf{0}) = r$ *verifying* $[\mathbf{F},\mathbf{G}] = \mu\mathbf{F}$. *Moreover, in such a case,* $\mathbf{F}$ *is smoothly (analytically) conjugated to* $(1+\lambda)\mathbf{F}$ *being* λ *a smooth (analytic) scalar function with* $\lambda(\mathbf{0}) = 0$ *such that* $L_{\mathbf{G}}\lambda = (1+\lambda)(r-\mu)$.

Proof.
Necessary condition. From Proposition 6.2.107, if $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) orbital equivalent, there exists a change of variables $\mathbf{x} = \phi(\mathbf{y}) = \sum_{i\geq 0} \phi_i(\mathbf{y})$, with $D\phi(\mathbf{0}) = I$, $\phi_i \in \mathcal{Q}_i^{\mathbf{t}}$ and a paremeterization by time $dt = \frac{d\tau}{1+f(\mathbf{y})}$ where f is a smooth (analytic) function, $f(\mathbf{0}) = 0$, that transforms the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ into $\mathbf{y}' = \frac{d\mathbf{y}}{d\tau} = \frac{\phi_*\mathbf{F}(\mathbf{y})}{1+f(\mathbf{y})} = \mathbf{F}_r(\mathbf{y})$. Therefore, $\phi_*\mathbf{F} = (1+f)\mathbf{F}_r$. So, we have that the vector field $\mathbf{F}$ is smoothly (analytically) conjugated to $(1+f)\mathbf{F}_r$.

Performing the inverse change ϕ^* to the system $\dot{\mathbf{y}} = \mathbf{F}_r(\mathbf{y})$, this is transformed into $\dot{\mathbf{x}} = \widetilde{\mathbf{F}}(\mathbf{x})$ with $\widetilde{\mathbf{F}} = (1+g)\mathbf{F}$ and a g smooth (analytic) function, $g(\mathbf{0}) = 0$. Thus, by Theorem 6.2.109, there exists a smooth (analytic) vector field $\mathbf{G} = \sum_{j\geq 0} \mathbf{G}_j$, $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{G}_0 = \mathbf{D}_0$, such that $[(1+g)\mathbf{F}, \mathbf{G}] = r(1+g)\mathbf{F}$. On the other hand, it has that

$$[(1+g)\mathbf{F}, \mathbf{G}] \;=\; (1+g)[\mathbf{F}, \mathbf{G}] + L_{\mathbf{G}} g \mathbf{F}.$$

Therefore, $[\mathbf{F}, \mathbf{G}] = \left(r - \frac{L_{\mathbf{G}} g}{1+g}\right)\mathbf{F}$. Taking $\mu = \left(r - \frac{L_{\mathbf{G}} g}{1+g}\right)$, the result follows.
Sufficient condition. We assume that there exists a smooth (analytic) vector field $\mathbf{G}$, with $\mathcal{J}_{\mathbf{t}}^0\mathbf{G} = \mathbf{D}_0$, verifying $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$ with μ a smooth (analytic) function, $\mu(\mathbf{0}) = r$. Let λ be a smooth (analytic) scalar function with $\lambda(\mathbf{0}) = 0$ such that

$$L_{\mathbf{G}}\lambda = (1+\lambda)(r-\mu)$$

which exists by Lemma 6.2.110 for $\eta = r - \mu$. Then, we have that

$$\begin{aligned} [(1+\lambda)\mathbf{F}, \mathbf{G}] &= (1+\lambda)[\mathbf{F}, \mathbf{G}] + L_{\mathbf{G}}\lambda\mathbf{F} \\ &= (1+\lambda)[\mathbf{F}, \mathbf{G}] + (1+\lambda)\,(r-\mu)\mathbf{F} \\ &= (1+\lambda)([\mathbf{F}, \mathbf{G}] + r\mathbf{F} - \mu\mathbf{F}) = r(1+\lambda)\mathbf{F}. \end{aligned}$$

By Theorem 6.2.109, $(1+\lambda)\mathbf{F}$ is smoothly (analytically) conjugated to $\mathbf{F}_r$ and thus $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) orbital equivalent. ∎

In particular, if we take $\mathbf{t} = (1, \ldots, 1)$ and $r = 0$, $\mathbf{F}_0 + \mathbf{F}_1 + \cdots$ is the expansion into homogeneous polynomial vector fields of $\mathbf{F}$, being $\mathbf{F}_0$ its linear part. So, as particular cases of Theorems 6.2.109 and 6.2.111 we obtain the following results which are the main results of the papers [91] and [2], respectively.

Corollary 6.2.112. *Let* $\mathbf{F}$ *be a smooth (analytic) vector field. It holds:*
i) $\mathbf{F}$ *is linearizable under smooth (analytic) conjugation if and only if there exists a smooth (analytic) vector field* $\mathbf{G}$ *with* $\mathbf{G}(\mathbf{x}) = \mathbf{x} + O(|\mathbf{x}|^2)$ *such that* $[\mathbf{F}, \mathbf{G}] = \mathbf{0}$.
ii) $\mathbf{F}$ *is linearizable under smooth (analytic) orbital equivalence if and only if there exists a smooth (analytic) vector field* $\mathbf{G}$ *with* $\mathbf{G}(\mathbf{x}) = \mathbf{x} + O(|\mathbf{x}|^2)$ *and a smooth (analytic) scalar function* μ *with* $\mu(\mathbf{0}) = 0$ *such that* $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$.

Corollary 6.2.113. *If* $\mathbf{F}$ *is an analytic two-dimensional vector field type centre-focus, i.e.* $\mathbf{F}(x, y) = (-y, x)^T + \cdots$, *it has:*
i) *the origin is an isochronous centre (all the closed orbits neighbouring* O *have the same period) of* $(\dot{x}, \dot{y})^T = \mathbf{F}(x, y)$ *if and only if there is an analytic vector field* $\mathbf{G}$ *with* $\mathbf{G}(x, y) = (x, y)^T + \cdots$ *such that* $[\mathbf{F}, \mathbf{G}] = \mathbf{0}$,
ii) *the origin is a centre of* $(\dot{x}, \dot{y})^T = \mathbf{F}(x, y)$ *if and only if there is an analytic vector field* $\mathbf{G}$ *with* $\mathbf{G}(x, y) = (x, y)^T + \cdots$ *and an analytic scalar function* μ *with* $\mu(\mathbf{0}) = 0$ *such that* $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$.

6.3 Several examples

In this section, we show some very simple examples. The first four examples are nilpotent vector fields, three of them are two-dimensional and the fourth is a three-dimensional system. The fifth example is a system with linear part null, the so-called generalized nilpotent system.

Example 6.3.114. *Let us consider the nilpotent system*

$$\begin{aligned} \dot{x} &= y + Ax^2 - 2y^2 - Ax^2y + y^3 + Bx^4, \\ \dot{y} &= -Bx^3 + Bx^3y, \end{aligned} \tag{6.3.2}$$

with $B \neq 0$.
The $(1,2)$*-homogeneous expansion of the vector field associated to (6.3.2) is given by* $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_3 + \mathbf{F}_5$ *where*

$$\begin{aligned} &\mathbf{F}_1 = (y + Ax^2, -Bx^3)^T, \\ &\mathbf{F}_3 = (-2y^2 - Ax^2y + Bx^4, Bx^3y)^T, \\ &\mathbf{F}_5 = (y^3, 0)^T. \end{aligned}$$

One can check that

$$\left[\mathbf{F}(x,y), (x - 2xy, 2y - 2y^2)^T\right] = (1 - 4y)\mathbf{F}(x,y).$$

Therefore, system (6.3.2) is orbitally equivalent to the system

$$(\dot{x}, \dot{y})^T = (y + Ax^2, -Bx^3)^T. \tag{6.3.3}$$

We now analyze system (6.3.3). The change of variables $x_1 = x, x_2 = y + \frac{1}{2}Ax^2$*, transforms (6.3.3) into*

$$\dot{x}_1 = x_2 + \frac{A}{2}x_1^2,\ \dot{x}_2 = -(B - \frac{A^2}{2})x_1^3 + Ax_1x_2.$$

Next, we perform the change to generalized polar coordinates

$$x_1 = uCs(\theta),\ x_2 = u^2Sn(\theta),\ dt = -\frac{4}{u}d\tau,$$

where $(Cs(\theta),\ Sn(\theta))$ *is the unique solution to the Cauchy problem*

$$\frac{dx}{d\theta} = -2y, \qquad \frac{dy}{d\theta} = 4x^3,$$

with $x(0) = 1,\ y(0) = 0$*. So, system (6.3.3) becomes*

$$\begin{aligned} &u' = -2uCs(\theta)\left[A - (B - \tfrac{A^2}{2} - 2)Cs^2(\theta)Sn(\theta)\right], \\ &\theta' = 2Sn^2(\theta) + (B - \tfrac{A^2}{2})Cs^4(\theta), \end{aligned} \tag{6.3.4}$$

where $'$ represents $\frac{d}{d\tau}$. Thus, if $A^2 - 2B < 0$, the origin of (6.3.3) is a monodromic point. Moreover, system (6.3.3) is reversible under the change of variables $(x, y, t) \to (-x, y, -t)$, therefore the origin is a centre. If $A^2 - 2B > 0$, system (6.3.4) has two hyperbolic equilibriums, hence the origin of (6.3.3) is a saddle point. And if $A^2 - 2B = 0$, the origin of (6.3.3) is a degenerate point.

Example 6.3.115. *Let us consider the nilpotent vector field*

$$\mathbf{F}(x, y) = (y - ax^2,\ x^2 + 2axy - 2a^2x^3)^T.$$

This vector field, for $\mathbf{t} = (2, 3)$, is expressed as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ where $\mathbf{F}_1(x, y) = (y, x^2)^T$ is its $(2, 3)$-homogeneous part of minimum degree and

$$\mathbf{F}_2(x, y) = a(-x^2, 2xy)^T, \qquad \mathbf{F}_3(x, y) = -2a^2(0, x^3)^T.$$

It can be checked that $\mathbf{G}(x, y) = (2x, 3y + ax^2)^T$, which has a linear part given by $(2x, 3y)^T$, verifies $[\mathbf{F}, \mathbf{G}] = \mathbf{F}$ and thus we conclude that $\mathbf{F}$ can be reduced to $(y, x^2)^T$ by a convergent transformation.

Example 6.3.116. *Let us consider the nilpotent vector field*

$$\mathbf{F}(x, y) = \begin{pmatrix} y(1 + y)^4 + (1 - c)x^3(1 + cy)^2 \\ x^2(1 + y)(1 + cy)^3 \end{pmatrix}, \tag{6.3.5}$$

whose $(2, 3)$-homogeneous expansion is of the form $(y, x^2)^T + \cdots$
For

$$\mathbf{G}(x, y) = (\frac{(2 + (5 - c)y + 2cy^2)x}{(1 + y)(1 + cy)}, 3y)^T,\ \ \mu(x, y) = \frac{1 + 4cy + 10y + 13cy^2}{(1 + y)(1 + cy)}$$

it has $[\mathbf{F}(x, y), \mathbf{G}(x, y)] = \mu(x, y)\mathbf{F}(x, y)$, with $\mathbf{G}(x, y) = (2x, 3y)^T + \cdots$ and $\mu(\mathbf{0}) = 1$, thus $\mathbf{F}$ is orbitally equivalent to $(y, x^2)^T$.
Also, it has that $\mathbf{F}$ is conjugated to $(1+\lambda(x, y))(y, x^2)^T$ where $1+\lambda(x, y) = \frac{1}{(1+y)(1+cy)}$.

Example 6.3.117. *Let us consider the three-dimensional system*

$$\begin{aligned}\dot{x} &= y + 2zT_1(x,y,z) + 2czT_2(x,y,z),\\ \dot{y} &= z - y^2 + c(x-z^2)^2,\\ \dot{z} &= T_1(x,y,z) + cT_2(x,y,z),\end{aligned} \tag{6.3.6}$$

with $T_1(x,y,z) = (x-z^2)^2 + 2y(z-y^2)$ *and* $T_2(x,y,z) = 2y(x-z^2)(x-z^2-1)$.

For $\mathbf{t} = (3,4,5)$, *the vector field associated to (6.3.6) has the form* $\mathbf{F}(x,y,z) = (y,z,x^2)^T + \tilde{\mathbf{F}}$, *where* $\tilde{\mathbf{F}}$ *has* $(3,4,5)$*-homogeneous parts of a higher degree. This vector field is conjugated to* $(y,z,x^2)^T$, *since* $[\mathbf{F},\mathbf{G}] = \mathbf{F}$, *where*

$$\mathbf{G}(x,y,z) = (3x + z(7z+6y^2) - 2cz(x-z^2)^2, 4y, 5z + 3y^2 - c(x-z^2)^2)^T.$$

Example 6.3.118. *Let us consider the family of vector fields*

$$\mathbf{F}(x,y) = (y^3 + P_5(x,y), Q_5(x,y))^T, \tag{6.3.7}$$

where P_5 *and* Q_5 *are quintic homogeneous polynomials with* $Q_5(1,0) \neq 0$. *Integrability and centres of these vector fields have been studied in Giné [89]. The* $(2,3)$*-homogeneous expansion of the vector fields of the family is given by* $\mathbf{F} = \mathbf{F}_7 + \cdots + \mathbf{F}_{13}$ *where*

$$\begin{aligned}&\mathbf{F}_7 = (y^3, cx^5)^T,\ \mathbf{F}_8 = (a_{50}x^5, b_{41}x^4y)^T,\ \mathbf{F}_9 = (a_{41}x^4y, b_{32}x^3y^2)^T,\\ &\mathbf{F}_{10} = (a_{32}x^3y^2, b_{23}x^2y^3)^T,\ \mathbf{F}_{11} = (a_{23}x^2y^3, b_{14}xy^4)^T,\\ &\mathbf{F}_{12} = (a_{14}xy^4, b_{05}y^5)^T,\quad \mathbf{F}_{13} = (a_{05}y^5, 0)^T,\end{aligned}$$

We look for systems of the family (6.3.7) which, by means of a change of variables, can be transformed into $(y^3, cx^5)^T, c \neq 0$. *It has the following result.*

Proposition 6.3.119. *No vector field of the family (6.3.7) is smoothly conjugated to* $(y^3, cx^5)^T, c \neq 0$, *except for itself.*

Proof.
As $(y^3, cx^5)^T$ *is an integrable vector field,* $\frac{1}{4}y^4 - \frac{c}{6}x^6$ *is an analytic first integral, the vector fields of the family (6.3.7) which could be smoothly conjugated to* $(y^3, cx^5)^T$ *should also be integrable. By Giné [89], these are*

(*Fam1*) $Q_5 \equiv 0$,
(*Fam2*) *Hamiltonian system,*
(*Fam3*) $P_5(x,y) = y^3(1 + a_{23}x^2),\ Q_5(x,y) = b_{50}x^5 + b_{14}xy^4$.

It is easy to check that there isn't any change in the form $x = x + \cdots,\ y = y + \cdots$ *which transforms a vector field of (Fam1) into* $(y^3, cx^5)^T,\ c \neq 0$. *If (6.3.7) is of (Fam2) then*

$$b_{41} = a_{50} = 2a_{41} + b_{32} = a_{32} + b_{23} = a_{23} + 2b_{14} = a_{14} + 5b_{05} = 0.$$

By imposing the existence of a vector field $\mathbf{G}(x,y) = (2x, 3y)^T + \mathbf{G}_1 + \cdots$, *such that* $[\mathbf{F}, \mathbf{G}] = 7\mathbf{F}$, *we obtain recursively the conditions* $a_{41} = 0,\ a_{23} = 0,\ a_{14} = 0,\ 3ca_{05} + a_{32}^2 = 0$ *and* $a_{32} = 0$, *that is, we get* $(y^3, cx^5)^T$. *Analogously, in the case (Fam3) we have the conditions* $6b_{14} + 5a_{23} = 0,\ a_{23} = 0$. *So, the result follows.* ∎

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