

# Existence of an inverse integrating factor, center problem and integrability of a class of nilpotent systems

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## Abstract

We characterize the nilpotent systems whose lowest degree quasi-homogeneous term is  $(y, \sigma x^n)^T$ ,  $\sigma = \pm 1$ , having a formal inverse integrating factor. We prove that, for  $n$  even, the systems with formal inverse integrating factor are formally orbital equivalent to  $(\dot{x}, \dot{y})^T = (y, x^n)^T$ . In the case  $n$  odd, we give a formal normal form that characterizes them. As a consequence, we give the link among the existence of formal inverse integrating factor, center problem and integrability of the considered systems.

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## 1. Introduction and results.

We consider the system of differential equations given by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T, \quad (1)$$

where  $\mathbf{F}$  is an analytic planar vector field defined in a neighborhood of the origin  $U \subset \mathbb{C}^2$ .

We recall that a function  $H$  is a first integral of (1) on  $U$  if  $H$  is a non-constant function on  $U$  which is constant on each solution curve of (1). Clearly, if  $H \in C^1(U)$  verifies  $L_{\mathbf{F}}H := \partial H/\partial x P + \partial H/\partial y Q \equiv 0$ . The integrability problem consists in determining if the planar vector field has a first integral. In a general framework, the integrability is an important question because the existence of a first integral determines completely its phase portrait. Among others applications, the existence of an analytic first integral defined in a neighborhood of the origin can be used for characterizing when a monodromic singular point (the orbits of the system close to the isolated singular point revolve around it) is a center or a focus, see [5].

A non-null  $C^1$  class function  $V$  is an inverse integrating factor of system (1) on  $U$  if satisfies the linear partial differential equation  $L_{\mathbf{F}}V = \text{div}(\mathbf{F})V$ , being  $\text{div}(\mathbf{F}) = \partial P/\partial x + \partial Q/\partial y$  the divergence of  $\mathbf{F}$ . We will say that  $V$  is a formal

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inverse integrating factor of system (1) if  $V \in \mathbb{C}[[x, y]]$  where  $\mathbb{C}[[x, y]]$  is the algebra of the power series in  $x$  and  $y$  with coefficients in  $\mathbb{C}$ , convergent or not.

This name for  $V$  comes from the fact that  $V^{-1}$  defines on  $U \setminus \{V = 0\}$  an integrating factor of system (1), i.e.  $\mathbf{F}/V$  is divergence-free. Concretely, if system (1) would have a formal inverse integrating factor  $V$  with  $V(0, 0) \neq 0$ , then  $V^{-1}$  would be a formal integrating factor and thus

$$H = - \int P/V dy + \int \left( Q/V + \frac{\partial}{\partial x} \int P/V dy \right) dx$$

would be a formal first integral defined in a neighborhood of the origin, i.e. system (1) is formally integrable. Summarizing, we have the following result:

- Theorem 1.** (i) *If system (1) has a formal inverse integrating factor  $V$  with  $V(0, 0) \neq 0$ , then system (1) is formally integrable. Moreover, in such a case, system (1) has an infinite number of inverse integrating factors which are zero at origin and has an infinite number of inverse integrating factors which are not zero at origin.*
- (ii) *Reciprocally, if system (1) is not formally integrable and it has a formal inverse integrating factor  $V$ , then  $V(0, 0) = 0$ .*

The second part of (i) of above theorem follows from the fact that the product of  $V$  with any first integral (null or not at origin) of the system, is also an inverse integrating factor.

There are other important reasons to study the existence of inverse integrating factors, among them, we cite the following ones:

The concept of inverse integrating factor plays an important role in the study of the existence of limit cycles of a vector field, because the zero-set  $\{V = 0\}$  is formed by orbits of system (1) and it contains the limit cycles of system (1) which are in  $U$ , whenever they exist. This interesting result is due to Giacomini *et al.* [21]. Analogously, in [15] and [19] is shown that any inverse integrating factor must vanish identically on the polycycles that are limit sets of its flow. This fact allows to know the number of limit cycles which bifurcate from periodic orbits of a center (Hamiltonian or not) and compute the shape of them, see [22, 23]. To do this, we develop the function  $V$  in a power series of the small perturbation parameter. A remarkable fact is that the first term of this expansion coincides with the first term non-identically zero of its Melnikov function (see [24]).

If  $V$  is an inverse integrating factor for system (1), in an appropriate coordinates system, it holds that

$$V(2\pi, \Pi(\sigma)) = V(0, \sigma)\Pi'(\sigma) \tag{2}$$

where  $\Pi$  is the Poincaré map corresponding to a limit cycle or a focus or a polycycle. As a consequence, the Poincaré map can be obtained from equation (2) by quadratures. This allows to analyze the cyclicity of a limit cycle, a focus or a polycycle by means of an inverse integrating factor, see [17, 18].

For instance, the function  $V = (x^2 + y^2)^2(1 - x^2 - y^2)$  is an inverse integrating factor of the system

$$(\dot{x}, \dot{y})^T = (-y, x)^T + (x^2 + y^2)(1 - x^2 - y^2)(x, y)^T. \quad (3)$$

We can deduce that  $\gamma \equiv x^2 + y^2 = 1$  is a hyperbolic limit cycle of system (3). Moreover, it is globally asymptotically stable. The origin is a weak focus with multiplicity two, i.e. it can bifurcate up two small amplitude limit cycles from it.

In [11], the authors give a complete description of the phase portrait of a family of differential systems in terms of the parameters by using the inverse integrating factor as a key tool. In [23] is shown the hyperbolicity of the limit cycle and the cyclicity of the origin through the expression of an inverse integrating factor.

An interesting problem is to determine sufficient conditions so that the inverse integrating factor  $V$  of a polynomial differential system be globally defined in the whole plane; for instance, sufficient conditions in order to that  $V$  be a polynomial. Thus, if  $V$  is globally defined in the whole plane, we have determined all the limit cycles of the system. Additionally, if  $V$  is polynomial, then the limit cycles are algebraic and its number can be bounded using the degree of  $V$ . On the other hand, while a polynomial first integral does not allow the existence of limit cycles, foci, nodes, cycles separatrices which are stable or unstable in one side, all these objects are allowed for a system having a polynomial inverse integrating factor, see for instance [11]. The fact that  $V$  is polynomial is also useful for finding isochronous centers, because if a polynomial system does not have a polynomial  $V$ , then it can not commute with any other polynomial system, see [2].

The expressions of  $V$  usually are simpler than the expressions of the first integrals. For instance, all the linear differential systems have a quadratic polynomial inverse integrating factor but the first integrals of these system are not so easy, see [12]. The systems (1) with  $P$  and  $Q$  quadratic polynomials, and the systems with  $P = -y + P_3$  and  $Q = x + Q_3$  ( $P_3, Q_3$  cubic homogeneous polynomials) and whose origin is a center, have polynomial inverse integrating factors and their first integrals and integrating factors are more complicated functions, see [9].

The domain of definition and the regularity of  $V$  usually are larger than the domain and the regularity of the first integral. For example, the system

$$\dot{x} = y + x^2(x^4 - 2y^2), \quad \dot{y} = x^3 + 2xy(x^4 - 2y^2),$$

does not have any analytic first integral in a neighborhood of the origin, see [5], but it easy to check that it has the simple inverse integrating factor  $V(x, y) = (x^4 - 2y^2)^2$ .

For polynomial vector fields, it has the following relations between their inverse integrating factors and their first integrals (for a detailed definition of the considered objects, see [28, 13]):

- (i) If (1) has a Liouvillian first integral, then it has a Darboux inverse integrating factor.
- (ii) If (1) has a Darboux first integral, then it has a rational inverse integrating factor.
- (iii) If (1) has a polynomial first integral, then it has a polynomial inverse integrating factor.

The statements were proved in [28, 26, 13], respectively. In [16] it is given an example of a polynomial vector field having a rational first integral and having neither a polynomial first integral nor a polynomial inverse integrating factor.

The existence of inverse integrating factors regards to the type of the singularity of the system also have been considered. Enciso and Peralta-Salas [15] study the existence of a smooth inverse integrating factor in a neighborhood of an arbitrary elementary singularity, i.e. the systems whose linear part at the origin has at least one eigenvalue different from zero. This extends previous results given in [12], [ [14], Theorem 5.2], where these authors considered elementary singularities that admit analytic orbital normalization.

For degenerate stationary points with nilpotent or vanishing linearization, we only know the results obtained by Walcher [30]. These are partial results and show that generically there will be no formal inverse integrating factor.

Before of showing our results, we recall the following concepts and definitions. Given  $\mathbf{t} = (t_1, t_2)$  non-null with  $t_1$  and  $t_2$  non-negative integer numbers without common factors, a function  $f$  of two variables is quasi-homogeneous of type  $\mathbf{t}$  and degree  $k$  if  $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y)$ . The vector space of quasi-homogeneous polynomials of type  $\mathbf{t}$  and degree  $k$  will be denoted by  $\mathcal{P}_k^{\mathbf{t}}$ . A vector field  $\mathbf{F} = (F_1, F_2)^T$  is quasi-homogeneous of type  $\mathbf{t}$  and degree  $k$  if  $F_1 \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$  and  $F_2 \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$ . We will denote  $\mathcal{Q}_k^{\mathbf{t}}$  the vector space of the quasi-homogeneous polynomial vector fields of type  $\mathbf{t}$  and degree  $k$ .

Any vector field can be expanded into quasi-homogeneous terms of type  $\mathbf{t}$  of successive degrees. Thus, the vector field  $\mathbf{F}$  can be written in the form

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots,$$

for some  $r \in \mathbb{Z}$ , where  $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$  and  $\mathbf{F}_r \neq \mathbf{0}$ . Such expansions will be expressed as  $\mathbf{F} = \mathbf{F}_r + \text{q-h.h.o.t.}$

If we select the type  $\mathbf{t} = (1, 1)$ , we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees.

Throughout the paper, we will denote by  $\mathbf{D}_0 = (t_1x, t_2y)^T \in \mathcal{Q}_0^{\mathbf{t}}$  (a dissipative quasi-homogeneous vector field) and by  $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$  (the Hamiltonian vector field associated to the polynomial  $h$ ). If  $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ , then  $\mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$ , where  $|\mathbf{t}| = t_1 + t_2$ . Moreover, it is proved that every  $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$  can be expressed as

$$\mathbf{F}_k = \mathbf{X}_h + \mu \mathbf{D}_0 \tag{4}$$

with  $h = (\mathbf{D}_0 \wedge \mathbf{F}_k)/(k + |\mathbf{t}|)$  and  $\mu = \text{div}(\mathbf{F}_k)/(k + |\mathbf{t}|)$ , where  $\mathbf{D}_0 \wedge \mathbf{F}_k \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$  is the product wedge of both vector fields and  $\text{div}(\mathbf{F}_k) \in \mathcal{P}_k^{\mathbf{t}}$  is the divergence

of  $\mathbf{F}_k$ , see [5]. This sum is known as the conservative-dissipative splitting of a quasi-homogeneous vector field.

Next, we show our results. We first prove the reciprocal property of Theorem 1.(i) in a particular case. To do that, we will use the following theorem proved by Algaba et al. [5].

**Theorem 2** ([5], **Theorem 3.19**). *Let us assume that the lowest-degree quasi-homogeneous term of system (1) is  $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^t$ , where the factorization of  $h \in \mathcal{P}_{r+|t|}^t$  only has simple factors in  $\mathbb{C}[x, y]$ . Then, the quoted system is formally integrable if and only if it is formally conjugated, via dissipative transformations, to a divergence-free system.*

So, if system (1) would be formally integrable, with  $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^t$  and  $h$  only has simple factors in its factorization in  $\mathbb{C}[x, y]$ , there would exist  $\Phi$  a formal diffeomorphism that transforms  $\mathbf{F}$  into the hamiltonian system whose hamiltonian is  $H$ . The function  $f(H)$  with  $f$  any scalar function, would be a formal inverse integrating factor of the transformed system and, thus,  $V = \det(D\Phi)^{-1} f(H \circ \Phi)$  would be a formal inverse integrating factor of (1). Therefore, there would be an infinite number of formal inverse integrating factors. Also, if  $f(0) \neq 0$ ,  $V$  would be a formal inverse integrating factor with  $V(0, 0) \neq 0$ . Summarizing, it has the following result:

**Theorem 3.** *Let us assume that the lowest-degree quasi-homogeneous term of system (1) is  $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^t$ , where the factorization of  $h \in \mathcal{P}_{r+|t|}^t$  only has simple factors in  $\mathbb{C}[x, y]$ . Then, system (1) is formally integrable if and only if it has a formal inverse integrating factor which does not vanish at the origin.*

This fact allows us to do the following question:

**Open question:** In general, if system (1) whose first quasi-homogeneous term is  $\mathbf{X}_h + \mu \mathbf{D}_0$  with  $\mu \neq 0$ , is formally integrable then does it have a formal inverse integrating factor which does not vanish at the origin?

From Theorem 1 (ii), we note that the formal inverse integrating factors  $V$  of a system non-formally integrable verify  $V(0, 0) = 0$ .

Moreover, if a system has a limit cycle  $\gamma$  into a  $U \subset \mathbb{C}^2$ , then it is not formally integrable in  $U$  since, otherwise the first integral would be constant in a neighbourhood of the limit cycle and this is not possible. So, if there exists a formal inverse integrating factor  $V$ , it verifies  $V(0, 0) = 0$ .

In this paper, we deal with the nilpotent systems whose quasi-homogeneous expansion is of the form

$$(\dot{x}, \dot{y})^T = \mathbf{F}(x, y) = (y, \sigma x^n)^T + \text{q-h.h.o.t.}, \quad (5)$$

with  $\sigma = \pm 1$ , i.e. nilpotent systems which can be considered as perturbations of a hamiltonian system. Here, we will focus in looking for the systems that have a formal inverse integrating factor  $V$  which can be zero at the origin.

Walcher [30] solved this problem only for systems (5) with  $n = 2$ . Concretely, he proved that there is not any formal inverse integrating factor of the systems

$$(\dot{x}, \dot{y})^T = (y, \sum_{j \geq 2} \alpha_j x^j + \beta_j x^{j-1} y)^T \quad (6)$$

with  $\alpha_2 \neq 0$  and  $\beta_2 \neq 0$ . So, our task is to extend the above result.

In our study, we will distinguish two families of systems (5), according to the evenness of  $n$ .

Fixed the type  $\mathbf{t} = (2, 2n + 1)$ , with  $n \geq 1$ , we consider the systems

$$(\dot{x}, \dot{y})^T = (y, x^{2n})^T + \text{q-h.h.o.t.} \quad (7)$$

Notice that first quasi-homogeneous term is  $\mathbf{F}_{2n-1} = \mathbf{X}_h$  with  $h = 2x^{2n+1} - (2n+1)y^2$ . The following result states that the systems (7) which admit a formal inverse integrating factor are formally orbital equivalent to their lowest-degree quasi-homogeneous vector fields.

**Theorem 4.** *System (7) has a formal inverse integrating factor  $V$  (which is zero or not at origin) if and only if is formally orbital equivalent to  $(\dot{x}, \dot{y})^T = (y, x^{2n})^T \in \mathcal{Q}_{2n-1}^{(2, 2n+1)}$ . Moreover, in such a case, system (7) is formally integrable and any formal inverse integrating factor which vanishes at origin is of the form  $V = (2x^{2n+1} - (2n+1)y^2)^m + \text{q-h.h.o.t.}$ , with  $m$  any natural number, up to a multiplicative constant.*

As a consequence, it has the result of Walcher [30]:

**Corollary 1.** *System (6) with  $\alpha_2, \beta_2 \neq 0$ , does not have a formal inverse integrating factor.*

The above theorem plus the result of Algaba *et al.* [7], Theorem 1.3 page 809, that characterizes the systems which are orbitally equivalent to its lowest degree quasi-homogeneous term allow us to relate the existence of an inverse integrating factor with the existence of a Lie symmetry of the vector field.

**Theorem 5.** *System (7) has a formal inverse integrating factor  $V$  (which is zero or not at origin) if and only if there exist a formal vector field  $\mathbf{G}$  whose lowest degree quasi-homogeneous term is  $\mathbf{G}_0 = (2x, (2n+1)y)^T \in \mathcal{Q}_0^{(2, 2n+1)}$  and a formal scalar function  $\mu$  with  $\mu(\mathbf{0}) = 2n - 1$  verifying  $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$  ( $[\mathbf{F}, \mathbf{G}]$  denotes the Lie bracket of both vector fields,  $[\mathbf{F}, \mathbf{G}] = D\mathbf{F} \cdot \mathbf{G} - D\mathbf{G} \cdot \mathbf{F}$ ).*

Theorems 4 and 5 also are true if we change formal by analytic.

Fixed the type  $\mathbf{t} = (1, n)$ , with  $n \geq 1$ , we now consider the systems

$$(\dot{x}, \dot{y})^T = (y, \sigma x^{2n-1})^T + \text{q-h.h.o.t.}, \quad (8)$$

with  $\sigma = \pm 1$ . Notice that first quasi-homogeneous term is  $\mathbf{F}_{n-1} = \mathbf{X}_h$  with  $h = \sigma x^{2n} - ny^2$ .

**Theorem 6.** *System (8) has a formal inverse integrating factor  $V$  (which is zero or not at origin) if and only if it is formally orbital equivalent to*

$$(\dot{x}, \dot{y})^T = (y, \sigma x^{2n-1})^T + x^{n-1} f(h)(x, ny)^T,$$

with  $f(h) = \sum_{i \geq 1} a_i h^i$ , where  $h = \sigma x^{2n} - ny^2 \in \mathcal{P}_{2n}^{(1,n)}$ .

Moreover, if  $f \equiv 0$ , system (8) is formally integrable and a formal inverse integrating factor which vanishes at origin is of the form  $V = (\sigma x^{2n} - ny^2)^m + q$ -h.h.o.t., with  $m$  any natural number, up to a multiplicative constant. Otherwise, system (8) is not formally integrable and a formal inverse integrating factor which vanishes at origin is of the form  $V = (\sigma x^{2n} - ny^2)^{l+1} + q$ -h.h.o.t with  $l$  the smallest index such that  $a_l \neq 0$ , up to a multiplicative constant.

If the origin of system (8) is a weak saddle or non-degenerated center-focus, i.e. system (8) with  $\mathbf{t} = (1, 1)$ ,  $n = 1$  and  $\sigma = 1$  or  $-1$ , respectively, it is known that both are formally orbital equivalent to

$$(\dot{x}, \dot{y})^T = (y, \sigma x)^T + \sum_{j \geq 2} \alpha_j (\sigma x^2 - y^2)^j (x, y)^T,$$

with  $\sigma = 1$  or  $-1$ , respectively. Thus, from Theorem 6, these systems have a formal inverse integrating factor. These results were proved by [12, 15, 29].

On the other hand, the well-known result of Poincaré that states that all the analytic centers that are perturbations of linear centers, that is,  $(\dot{x}, \dot{y})^T = \mathbf{X}_h + q$ -h.h.o.t. with  $h = \frac{x^2}{2} + \frac{y^2}{2}$ , are analytically integrable and applying Theorem 3, we give a new proof of the known Reeb Criterium, see [27]:

**Theorem 7 (Reeb Criterium).** *A non-degenerated center-focus is a center if and only if it has an analytic inverse integrating factor which does not vanish at the origin.*

Figure 1 summarizes the possible situations.

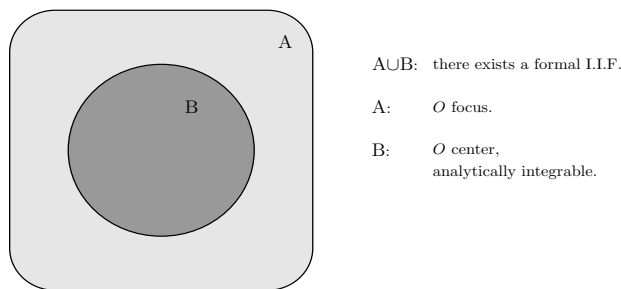


Figure 1: Non-degenerated center-focus systems

In general, for centers (degenerated or non-degenerated) it has the following result whose proof consists in an adaptation of Theorem 2.2 of Mazzi and Sabatini [25], by changing integrating factors by inverse integrating factors.

**Theorem 8.** *We assume that the origin of system (1) is a center. Then, there exists a inverse integrating factor  $V \in C^\infty(U)$ ,  $U \subset \mathbb{C}^2$  a neighborhood of the origin, with  $V(0, 0) = 0$ .*

Next, we study the link between the existence of an inverse integrating factor (it which can be zero at the origin) and the center and integrability problems of systems (8) with  $\sigma = -1$  (we note that if  $\sigma = 1$ , it is not a center).

We distinguish two situations according the evenness of  $n$ .  
For  $n$  even, it has the following result.

**Corollary 2.** *If system (8) with  $\sigma = -1$  and  $n$  even, admits a formal inverse integrating factor (which is zero or not at origin), then the origin is a center.*

We note that the reciprocal property is false, see Remark 1, page 15.

Concerning integrability problem, there are systems (8) with  $\sigma = -1$  and  $n$  even with a formal inverse integrating factor (that is, it is a center) and are not formally integrable. For example, the systems given in Theorem 10 case (iv).

Figure 2 shows all the possibilities.

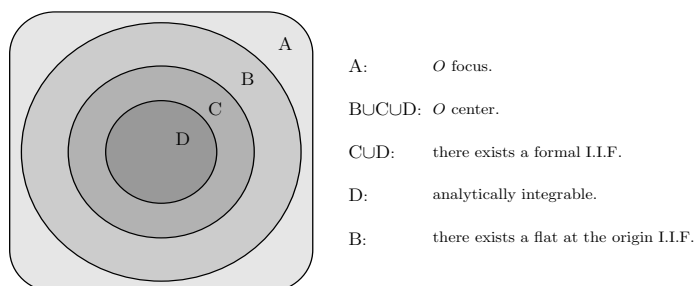


Figure 2: Systems  $(\dot{x}, \dot{y})^T = (y, -x^{2n-1})^T + \dots$  with  $n > 1$  even

Nevertheless, for  $n$  odd the situation is different since the following result shows that the centers of the systems (8) with  $\sigma = -1$  with a formal inverse integrating factor are formally integrable.

**Corollary 3.** *Let system (8) be with  $\sigma = -1$  and  $n$  odd. We assume that it has a formal inverse integrating factor (which is zero or not at origin). Then, the origin is a center if and only if system (8) is formally integrable.*

Moreover, there are centers which do not have a formal inverse integrating factor (see Remark 1) and from Corollary 3, they are not formally integrable.

And also there exist systems having a formal inverse integrating factor and are not center and not integrable, see Remark 2, page 15.

Figure 3 shows the link between center case, integrability and the existence of inverse integrating factor.

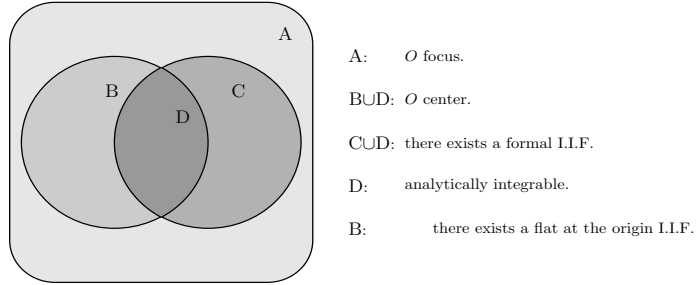


Figure 3: Systems  $(\dot{x}, \dot{y})^T = (y, -x^{2n-1})^T + \dots$  with  $n > 1$  odd

To prove our results, we have used the theory of quasi-homogeneous normal forms. These techniques also have been considered in the study of the center problem, reversibility, integrability, see [1, 3, 6].

As an application of our results, we study the systems

$$\dot{x} = y + a_1xy + a_2x^{n+1}, \quad \dot{y} = -x^{2n-1} + b_1y^2 + b_2x^ny + b_3x^{2n}, \quad (9)$$

with  $n \geq 2$  with a formal inverse integrating factor. These systems are centers perturbed by quasi-homogeneous vector fields of type  $(1, n)$ . Gasull and Torregrosa [20] found their centers and Chavarriga et al. [10] studied their integrability. Next, we determine the systems (9) with a formal inverse integrating factor.

**Theorem 9.** *System (9) with  $n > 2$ , has a formal inverse integrating factor (which is zero or not at origin) if and only if it satisfies one of the following series of conditions:*

- (i)  $a_2 = b_2 = 0$ ,
- (ii)  $b_2 + (n+1)a_2 = a_1 + 2b_1 = 0$ .

*In all the cases, the system is formally integrable.*

**Theorem 10.** *System (9) with  $n = 2$ , has a formal inverse integrating factor (which is zero or not at origin) if and only if it satisfies one of the following series of conditions:*

- (i)  $a_2 = b_2 = 0$ ,
- (ii)  $b_2 + 3a_2 = a_1 + 2b_1 = 0$ ,
- (iii)  $b_2 + 3a_2 = b_3 + b_1 - a_1 = (2b_1 - 3a_1)^2 - a_1^2 + 24a_2^2 = 0$ ,
- (iv)  $b_2 + 3a_2 = b_3 + 2a_1 = b_1 - 3a_1 = 0$ ,  $a_1a_2 \neq 0$ .

In the cases (i-iii), the system is formally integrable. The system in the case (iv) is not formally integrable.

The rest of the paper is organized as follows. Section 2 shows the technique of the theory of normal form for quasi-homogeneous expansions, we give a normal form under orbital equivalence for both  $n$  even or odd and we derive conditions preventing the existence of a formal inverse integrating factor. Section 3 contains the proofs of our main results.

## 2. Quasi-homogeneous Normal Forms and Inverse Integrating Factors.

In this section, we start by recalling some notions and results related to the quasi-homogeneous normal forms of systems (1). For more details, refer to Algaba *et al.* [5], where a proof of the auxiliary results can be seen.

The key in the problem of obtaining a normal form for the system (1) is to analyze the effect of a near-identity transformation  $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$  and a reparameterization of the time by  $\frac{dt}{dT} = 1 + \mu_k(\mathbf{x})$ , where  $\mathbf{P}_k \in \mathcal{Q}_k^t$  and  $\mu_k \in \mathcal{P}_k^t$ , with  $k \geq 1$ .

The quasi-homogeneous terms of the transformed system  $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$  agree with the original ones up to degree  $r + k - 1$  and for the degree  $r + k$  it has

$$\begin{aligned} \mathbf{G}_{r+k} &= \mathbf{F}_{r+k} - (D\mathbf{P}_k\mathbf{F}_r - D\mathbf{F}_r\mathbf{P}_k) + \mu_k\mathbf{F}_r = \mathbf{F}_{r+k} - [\mathbf{P}_k, \mathbf{F}_r] + \mu_k\mathbf{F}_r \\ &= \mathbf{F}_{r+k} - \mathcal{L}_k(\mathbf{P}_k, \mu_k) \end{aligned}$$

where we have introduced the homological operator under formal orbital equivalence:

$$\begin{aligned} \mathcal{L}_k &: \mathcal{Q}_k^t \times \mathcal{P}_k^t \longrightarrow \mathcal{Q}_{r+k}^t \\ (\mathbf{P}_k, \mu_k) &\rightarrow \mathcal{L}_k(\mathbf{P}_k, \mu_k) = [\mathbf{P}_k, \mathbf{F}_r] - \mu_k\mathbf{F}_r. \end{aligned} \quad (10)$$

Following the ideas of the conventional normal form theory, it is enough to choose  $(\mathbf{P}_k, \mu_k) \in \mathcal{Q}_k^t \times \mathcal{P}_k^t$  adequately in order to simplify the  $(r + k)$ -degree quasi-homogeneous term in system (1), by annihilating the part belonging to the range of the linear operator  $\mathcal{L}_k$ . In other words, we can achieve that  $\mathbf{F}_{r+k}$  belongs to a complementary subspace to the range of  $\mathcal{L}_k$ . When this has been done, we say that the corresponding term has been reduced to normal form under orbital equivalence. So, by means of a sequence of time-reparametrizations and near identity transformations (by performing the procedure for  $k = 1$ , then for  $k = 2$  and so on) system (1) can be formally reduced to normal form under orbital equivalence.

There are elements of  $\mathcal{Q}_{r+k}^t$  in the expression of the transformed vector field which can be annihilated with transformations in the state variables as well as in the variable time. It is important to segregate the simplifications achieved by means of changes in the state variables with those that are consequence of

the change in the time. To study this homological operator, we define the linear operator

$$\begin{aligned}\ell_{k-r} &: \mathcal{P}_{k-r}^{\mathbf{t}} \longrightarrow \mathcal{P}_k^{\mathbf{t}} \\ \mu_{k-r} &\rightarrow L_{\mathbf{F}_r} \mu_{k-r},\end{aligned}$$

i.e. the Lie derivative of the lowest degree quasi-homogeneous term of  $\mathbf{F}$ .

Any function  $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$  we can express it as  $\mu_k = \nu_k + \mu_k^r$  with  $\nu_k \in \text{Cor}(\ell_{k-r})$ , being  $\text{Cor}(\ell_{k-r})$  a complementary subspace to the range of the linear operator  $\ell_{k-r}$ , and  $\mu_k^r \in \text{Range}(\ell_{k-r})$ , i.e. there exists  $\mu_{k-r} \in \mathcal{P}_{k-r}^{\mathbf{t}}$  such that  $\ell_{k-r}(\mu_{k-r}) = \mu_k^r = L_{\mathbf{F}_r} \mu_{k-r}$ . Also, for any smooth scalar function  $\mu$  and smooth vector fields  $\mathbf{F}, \mathbf{G}$ , it has that  $[\mu \mathbf{F}, \mathbf{G}] = (L_{\mathbf{G}} \mu) \mathbf{F} + \mu [\mathbf{F}, \mathbf{G}]$ . So,

$$\begin{aligned}\mathcal{L}_k(\mathbf{P}_k, \mu_k) &= [\mathbf{P}_k, \mathbf{F}_r] - \nu_k \mathbf{F}_r - (L_{\mathbf{F}_r} \mu_{k-r}) \mathbf{F}_r \\ &= [\mathbf{P}_k, \mathbf{F}_r] - \nu_k \mathbf{F}_r - [\mu_{k-r} \mathbf{F}_r, \mathbf{F}_r] + \mu_{k-r} [\mathbf{F}_r, \mathbf{F}_r] \\ &= [\mathbf{P}_k - \mu_{k-r} \mathbf{F}_r, \mathbf{F}_r] - \nu_k \mathbf{F}_r.\end{aligned}$$

Therefore, the operator  $\mathcal{L}_k$  restricted to  $\mathcal{Q}_k^{\mathbf{t}} \times \text{Cor}(\ell_{k-r})$  and the operator  $\mathcal{L}_k$  have the same range. In this way, by keeping the notation, we consider to  $\mathcal{L}_k$  restricted to  $\mathcal{Q}_k^{\mathbf{t}} \times \text{Cor}(\ell_{k-r})$  as the homological operator under equivalence.

We define the following subspaces of  $\mathcal{Q}_k^{\mathbf{t}}$  which will use for the study of the homological operator  $\mathcal{L}_k$ :

$$\mathcal{D}_k^{\mathbf{t}} := \{\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}} : \mathbf{P}_k = \mu_k \mathbf{D}_0, \mu_k \in \mathcal{P}_k^{\mathbf{t}}\}, \quad \mathcal{C}_k^{\mathbf{t}} := \{\mathbf{X}_g \in \mathcal{Q}_k^{\mathbf{t}} : g \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}\}.$$

By the uniqueness of conservative-dissipative splitting (4) of a quasi-homogeneous vector field, it has  $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{C}_k^{\mathbf{t}} \oplus \mathcal{D}_k^{\mathbf{t}}$ , for all  $k \in \mathbb{N}$ .

We give a number of properties related to the action of the Lie product on both subspaces.

**Lemma 11.** *Let us consider  $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ ,  $g \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$  and  $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$ . It holds:*

- (i)  $[\mathbf{X}_h, \mathbf{X}_g] = \mathbf{X}_f \in \mathcal{C}_{r+k}^{\mathbf{t}}$ , where  $f = -L_{\mathbf{X}_h} g$ .
- (ii)  $(r+k+|\mathbf{t}|)\mu_k \mathbf{X}_h = (r+|\mathbf{t}|)\mathbf{X}_{\mu_k h} + L_{\mathbf{X}_h} \mu_k \mathbf{D}_0 \in \mathcal{C}_{r+k}^{\mathbf{t}} \oplus \mathcal{D}_{r+k}^{\mathbf{t}}$ .
- (iii)  $[\mathbf{X}_h, \mu_k \mathbf{D}_0] = r \mathbf{X}_{\mu_k h} - \frac{k+|\mathbf{t}|}{r+k+|\mathbf{t}|} \ell_k(\mu_k) \mathbf{D}_0 \in \mathcal{C}_{r+k}^{\mathbf{t}} \oplus \mathcal{D}_{r+k}^{\mathbf{t}}$ .

From now on, we assume that  $\mathbf{F}_r$  is a hamiltonian quasi-homogeneous vector field,  $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$  with  $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ .

From Lemma 11, we can define the homological operator under equivalence as

$$\mathcal{L}_k : \mathcal{C}_k^{\mathbf{t}} \times \mathcal{D}_k^{\mathbf{t}} \times \text{Cor}(\ell_{k-r}) \longrightarrow \mathcal{C}_{r+k}^{\mathbf{t}} \times \mathcal{D}_{r+k}^{\mathbf{t}},$$

with

$$\begin{aligned}\mathcal{L}_k(\mathbf{X}_g, \mu_k \mathbf{D}_0, \nu_k) &= \left( r \mathbf{X}_{\mu_k h} + \frac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|} \mathbf{X}_{\nu_k h} - \mathbf{X}_{\ell_{k+|\mathbf{t}|}(g)}, \right. \\ &\quad \left. \ell_k(\mu_k) \mathbf{D}_0 + \frac{1}{r+k+|\mathbf{t}|} \ell_k(\mu_k) \mathbf{D}_0 \right).\end{aligned}$$

Hence, taking adequate bases, we obtain a triangular-block matrix for the above linear operator:

$-\mathbf{X}_{\ell_{k+ t }(g)}$	$r\mathbf{X}_{\mu_k h}$	$\frac{r+ t }{r+k+ t }\mathbf{X}_{\nu_k h}$	$\mathcal{C}_{r+k}^t$
0	$\ell_k(\mu_k)\mathbf{D}_0$	$\frac{1}{r+k+ t }\ell_k(\nu_k)\mathbf{D}_0$	$\mathcal{D}_{r+k}^t$
$\mathbf{X}_g \in \mathcal{C}_k^t$	$\mu_k\mathbf{D}_0 \in \mathcal{D}_k^t$	$\nu_k \in \text{Cor}(\ell_{k-r})$	

For systems (1) with  $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^t$  where  $h \in \mathcal{P}_{r+|t|}^t$  only has simple factors in its factorization on  $\mathbb{C}[x, y]$ , it has the following result which shows that a complementary subspace to the range of the operator  $\mathcal{L}_k$  can be expressed in terms of a complementary subspace to the range of the operator  $\ell_k$ , which acts on scalar functions, instead of vector fields.

**Proposition 12.** *Let us assume that  $\mathcal{P}_{k-r}^t \neq \{0\}$  and the lowest-degree quasi-homogeneous term of system (1) is  $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^t$ , where the factorization of  $h \in \mathcal{P}_{r+|t|}^t$  in  $\mathbb{C}[x, y]$  only has simple factors.*

*It has that  $h\text{Cor}(\ell_{k-r}) \subset \text{Cor}(\ell_{k+|t|})$  and a complementary subspace to the range of  $\mathcal{L}_k$  can be written as  $\text{Cor}(\mathcal{L}_k) = \mathbf{X}_S \oplus \text{Cor}(\ell_k)\mathbf{D}_0$ , being  $S$  a subspace verifying  $\text{Cor}(\ell_{k+|t|}) = S \oplus h\text{Cor}(\ell_{k-r})$ .*

### 2.1. Suitable normal form of nilpotent vector fields

We give the normal forms of the systems (7) and (8), i.e. nilpotent systems whose lowest-degree quasi-homogeneous term is hamiltonian.

#### 2.1.1. Normal form of systems (7)

For  $\mathbf{t} = (2, 2n + 1)$ , system (7) is a perturbation of the hamiltonian quasi-homogeneous vector field of degree  $2n - 1$ ,  $\mathbf{F}_{2n-1} = \mathbf{X}_h \in \mathcal{Q}_{2n-1}^t$  with  $h = 2x^{2n+1} - (2n + 1)y^2 \in \mathcal{P}_{4n+2}^t$ .

**Proposition 13.** *Let system (7) be. A complementary subspace to the range of  $\ell_k$  is*

$$\text{Cor}(\ell_k) = \begin{cases} \langle x^j \rangle & \text{if } k = 2j + 1 - 2n, n \leq j \leq 2n - 1, \\ \langle x^j h^l \rangle & \text{if } k = 2(2n + 1)l + 2j + 1 - 2n, 0 \leq j \leq 2n - 1, l \in \mathbb{N}, \end{cases}$$

*or the trivial space otherwise. Moreover,  $\text{Cor}(\ell_{k+2n+3}) \setminus h\text{Cor}(\ell_{k-2n+1}) = \{0\}$ .*

Propositions 12 and 13 yield the next theorem, which determines a formal normal form of system (7).

**Theorem 14.** *A formally orbital equivalent normal form for system (7) is*

$$(\dot{x}, \dot{y})^T = (y, x^{2n})^T + \sum_{j=n}^{2n-1} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{2n-1} \alpha_j^{(l)} x^j h^l \mathbf{D}_0, \quad (11)$$

*where  $h = 2x^{2n+1} - (2n + 1)y^2 \in \mathcal{P}_{4n+2}^t$  and  $\mathbf{D}_0 = (2x, (2n + 1)y)^T \in \mathcal{Q}_0^t$ .*

### 2.1.2. Normal form of systems (8)

For  $\mathbf{t} = (1, n)$ , system (8) is a perturbation of the hamiltonian quasi-homogeneous vector field of degree  $n - 1$ ,  $\mathbf{F}_{n-1} = (y, \sigma x^{2n-1})^T$ ,  $\sigma = \pm 1$ . So,  $\mathbf{F}_{n-1} = \mathbf{X}_h \in \mathcal{Q}_{n-1}^{\mathbf{t}}$  with  $h = \sigma x^{2n} - ny^2 \in \mathcal{P}_{2n}^{\mathbf{t}}$ .

**Proposition 15.** *Let system (8) be. A complementary subspace to the range of  $\ell_k$  is*

$$\text{Cor}(\ell_k) = \begin{cases} \langle x^j \rangle & \text{if } k = j - n + 1, n \leq j \leq 2n - 2, \\ \langle x^j h^l \rangle & \text{if } k = 2nl + j - n + 1, 0 \leq j \leq 2n - 2, l \in \mathbb{N}, \end{cases}$$

or the trivial space otherwise. Moreover,  $\text{Cor}(\ell_{k+n+1}) \setminus h\text{Cor}(\ell_{k-n+1}) = \{0\}$ .

From Propositions 12 and 15, we can state the following result.

**Theorem 16.** *A formally orbital equivalent normal form for system (8) is*

$$(\dot{x}, \dot{y})^T = (y, \sigma x^{2n-1})^T + \sum_{j=n}^{2n-2} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{2n-2} \alpha_j^{(l)} x^j h^l \mathbf{D}_0, \quad (12)$$

where  $h = \sigma x^{2n} - ny^2 \in \mathcal{P}_{2n}^{\mathbf{t}}$  and  $\mathbf{D}_0 = (x, ny)^T \in \mathcal{Q}_0^{\mathbf{t}}$ .

### 2.2. Inverse Integrating Factor

Now, we provide a series of properties of the inverse integrating factor of the system (1) in order to give conditions that guaranty its existence.

We start showing the following relationship between inverse integrating factors of conjugated vector fields. A proof can be seen in Enciso and Peralta-Salas [15].

**Proposition 17.** *Let  $\Phi$  be a diffeomorphism on  $U \subset \mathbf{R}^2$ . If  $V \in \mathbb{C}[[x, y]]$  is an inverse integrating factor of system (1), then  $\det(D\Phi)^{-1}V \circ \Phi \in \mathbb{C}[[x, y]]$  is an inverse integrating factor of  $\dot{\mathbf{y}} = \Phi_* \mathbf{F}(\mathbf{y}) := D\Phi(\mathbf{y})^{-1} \mathbf{F}(\Phi(\mathbf{y}))$ .*

The relationship between inverse integrating factors of orbitally equivalent vector fields is given by the following result.

**Proposition 18.** *Let  $\Phi$  be a diffeomorphism and  $\eta$  a function on  $U \subset \mathbf{R}^2$  such that  $\det D\Phi$  has no zero on  $U$  and  $\eta(\mathbf{0}) \neq 0$ . If  $V(\mathbf{x}) \in \mathbb{C}[[x, y]]$  is an inverse integrating factor of system (1), then  $\eta(\mathbf{y})(\det(D\Phi(\mathbf{y})))^{-1}V(\Phi(\mathbf{y}))$  is an inverse integrating factor of  $\dot{\mathbf{y}} = \Phi_*(\eta\mathbf{F})(\mathbf{y}) := D\Phi(\mathbf{y})^{-1}\eta(\mathbf{y})\mathbf{F}(\Phi(\mathbf{y}))$ .*

PROOF. We prove that  $\eta V$  is an inverse integrating factor of  $\eta\mathbf{F}$ . Indeed,

$$\begin{aligned} L_{\eta\mathbf{F}}(\eta V) &= \eta(\eta L_{\mathbf{F}}V + V L_{\mathbf{F}}\eta) = \eta(\eta V \text{div}(\mathbf{F}) + V L_{\mathbf{F}}\eta) \\ &= \eta V(\eta \text{div}(\mathbf{F}) + L_{\mathbf{F}}\eta) = \eta V \text{div}(\eta\mathbf{F}). \end{aligned}$$

Applying Proposition 17, the proof is completed.  $\diamond$

We relate the terms of a formal inverse integrating factor and give the shape of its lowest-degree term.

**Proposition 19.** *Let system (1) be with  $\mathbf{F} = \mathbf{X}_h + \mu \mathbf{D}_0$ ,  $\mu = \sum_{j \geq 0} \mu_{r+j}$  and we assume that the factorization of  $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$  in  $\mathbb{C}[x, y]$  only has simple factors. Then, if  $V = \sum_{j \geq 0} V_{i_0+j} \in \mathbb{C}[[x, y]]$  is a formal inverse integrating factor of system (1), for each  $k \in \mathbb{N}_0$  it holds*

$$L_{\mathbf{X}_h} V_{i_0+k} = - \sum_{l=0}^k (k + i_0 - r - |\mathbf{t}| - 2l) V_{i_0+k-l} \mu_{r+l}. \quad (13)$$

Furthermore,

- (i) if  $\mu_r \equiv 0$ , then  $V$  is of the form  $V = h^m + \sum_{j \geq 1} V_{m(r+|\mathbf{t}|)+j}$ ,
  - (ii) if  $\mu_r \equiv 0$  and  $\mu_{r+j} \in \text{Cor}(\ell_j)$ , for all  $j$ , then  $V = V(h) = h^m + \sum_{i > m} b_i h^i$ ,
- for a certain  $m$  natural number.

PROOF. Let denote  $\tilde{\mu} = \text{div}(\mu \mathbf{D}_0) = \sum_{j \geq 0} (r + j + |\mathbf{t}|) \mu_{r+j}$  and  $\tilde{V} = L_{\mathbf{D}_0} V = \sum_{j \geq 0} (i_0 + j) V_{i_0+j}$ . So, if  $V$  is a formal inverse integrating factor of system (1), it has that  $L_{\mathbf{F}} V - V \text{div}(\mathbf{F}) = 0$ , thus

$$L_{\mathbf{X}_h + \mu \mathbf{D}_0} V - V \text{div}(\mathbf{X}_h + \mu \mathbf{D}_0) = L_{\mathbf{X}_h} V + \mu L_{\mathbf{D}_0} V - V \text{div}(\mu \mathbf{D}_0) = 0,$$

i.e.  $L_{\mathbf{X}_h} V = V \tilde{\mu} - \mu \tilde{V}$  with  $L_{\mathbf{X}_h} V = \sum_{k=0}^{\infty} L_{\mathbf{X}_h} V_{i_0+k}$  and

$$\begin{aligned} V \tilde{\mu} - \mu \tilde{V} &= \left( \sum_{j \geq 0} V_{i_0+j} \right) \left( \sum_{j \geq 0} (r + j + |\mathbf{t}|) \mu_{r+j} \right) - \left( \sum_{j \geq 0} \mu_{r+j} \right) \left( \sum_{j \geq 0} (i_0 + j) V_{i_0+j} \right) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^k (r + |\mathbf{t}| + l) V_{i_0+k-l} \mu_{r+l} - \sum_{k=0}^{\infty} \sum_{l=0}^k (i_0 + k - l) V_{i_0+k-l} \mu_{r+l} \\ &= - \sum_{k=0}^{\infty} \sum_{l=0}^k (k + i_0 - r - |\mathbf{t}| - 2l) V_{i_0+k-l} \mu_{r+l}, \end{aligned}$$

which establishes the formula (13).

We see the second part. If  $\mu_r \equiv 0$  and  $V = \sum_{j \geq 0} V_{i_0+j}$  is a formal inverse integrating factor of system (1), then  $L_{\mathbf{F}} V - V \text{div}(\mathbf{F}) = 0$ . In particular, the lowest degree term of the equation (13) is null, i.e.  $L_{\mathbf{X}_h} V_{i_0} = 0$ . So,  $V_{i_0}$  is a quasi-homogeneous polynomial first integral of  $\mathbf{X}_h$ , therefore  $V_{i_0} = h^m$  with  $i_0 = m(r + |\mathbf{t}|)$ , since the origin is an isolated point of  $\mathbf{X}_h$ .

Now we suppose that  $\mu_r \equiv 0$  and  $\mu_{r+j} \in \text{Cor}(\ell_j)$ , for all  $j$ . We now prove by induction that  $V$  is function of  $h$ . Indeed, we know that the inverse integrating factor is of the form  $V = h^m + \sum_{j=1}^{\infty} V_{(r+|\mathbf{t}|)m+j}$ , for some  $m \in \mathbb{N}$ , hence the lowest-degree term of  $V$  satisfies (ii).

We now assume that  $V_j = b_l h^l$  for  $j = l(r + |\mathbf{t}|)$  and null for  $m(r + |\mathbf{t}|) \leq j \leq m(r + |\mathbf{t}|) + k - 1$ . From (13),  $V_{m(r+|\mathbf{t}|)+k}$  holds

$$\ell_{k+m(r+|\mathbf{t}|)}(V_{m(r+|\mathbf{t}|)+k}) = - \sum_{l=0}^k [k + (m-1)(r + |\mathbf{t}|) - 2l] V_{m(r+|\mathbf{t}|)+k-l} \mu_{r+l}$$

From Proposition 12, it has that  $h^n \mu_{r+l} \in \text{Cor}(\ell_{l+n(r+|\mathbf{t}|)})$ ,  $n \in \mathbb{N}$ , and so  $\ell_{k+m(r+|\mathbf{t}|)}(V_{m(r+|\mathbf{t}|)+k}) = 0$ . Therefore,  $V_{m(r+|\mathbf{t}|)+k}$  is a quasi-homogeneous polynomial first integral of  $\mathbf{X}_h$ , i.e. it is a function of  $h$ .  $\diamond$

We show a class of systems (1) having a formal inverse integrating factor.

**Proposition 20.** *Let system (1) be with  $\mathbf{F} = \mathbf{X}_h + \lambda g(h)\mathbf{D}_0$ ,  $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ ,  $\lambda \in \mathcal{P}_{r+n(r+|\mathbf{t}|)}^{\mathbf{t}}$  and  $g(h) = 1 + \sum_{j \geq 1} a_j h^j$ , with  $n$  non-negative integer and  $a_j$  real numbers. The function  $V(h) = h^{n+1}g(h)$  is a formal inverse integrating factor of (1).*

PROOF. Applying the Euler theorem for quasi-homogeneous function, i.e.  $L_{\mathbf{D}_0}f = sf$  with  $f \in \mathcal{P}_s^{\mathbf{t}}$ , it has that

$$L_{\mathbf{F}}V = V'(h)L_{\mathbf{F}}h = (r + |\mathbf{t}|)\lambda hg(h)V'(h),$$

and

$$\begin{aligned} \text{div}(\mathbf{F}) &= \text{div}(\lambda g(h)\mathbf{D}_0) \\ &= g(h)L_{\mathbf{D}_0}\lambda + \lambda g'(h)L_{\mathbf{D}_0}h + |\mathbf{t}|\lambda g(h) \\ &= (n+1)(r + |\mathbf{t}|)\lambda g(h) + (r + |\mathbf{t}|)\lambda g'(h)h \\ &= (r + |\mathbf{t}|)((n+1)\lambda g(h) + \lambda g'(h)h). \end{aligned}$$

Therefore,  $V \text{div}(\mathbf{F}) = (r + |\mathbf{t}|)((n+1)h^n \lambda g(h) + \lambda g'(h)h^{n+1})hg(h) = (r + |\mathbf{t}|)\lambda hg(h)V'(h)$ . So,  $L_{\mathbf{F}}V - V \text{div}(\mathbf{F}) = 0$ . This completes the proof.  $\diamond$

The following result will be used to prove the sufficiency of Theorems 4 and 6.

**Proposition 21.** *Let  $\mathbf{F} = \mathbf{X}_h + (\lambda f(h) + \nu)\mathbf{D}_0$ , with  $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ ,  $\lambda \in \mathcal{P}_r^{\mathbf{t}}$ ,  $f(h) = \sum_{i \geq 1} a_i h^i$  and  $\nu = \sum_{j > 0} \nu_{r+j}$ ,  $\nu_{r+j} \in \text{Cor}(\ell_j)$ ,  $\nu_{r+l(r+|\mathbf{t}|)} \equiv 0$  for all  $l \in \mathbb{N}$ . Then, system  $\dot{\mathbf{x}} = \mathbf{F}$  has a formal inverse integrating factor if and only if  $\nu \equiv 0$ .*

PROOF. Proposition 20 proves the sufficiency for  $f \not\equiv 0$ . Otherwise, the system  $\dot{\mathbf{x}} = \mathbf{X}_h$  has the inverse integrating factor  $h$ .

To prove the necessary condition, we assume that  $\nu_{r+j} \equiv 0$  for  $j = 1, \dots, j_0 - 1$ . If  $V \in \mathbb{C}[[x, y]]$  is a formal inverse integrating factor, from Proposition 19, there exists  $m \in \mathbb{N}$  such that  $V = V(h) = \sum_{j \geq m} b_j h^j$ ,  $b_m = 1$ ,  $b_j \in \mathbb{R}$ . Taking into account the expressions of  $\nu$ ,  $\lambda f(h)$  and  $V(h)$ , the equation (13) for  $k = j_0$ , becomes

$$L_{\mathbf{X}_h}V_{m(r+|\mathbf{t}|)+j_0} = -[(m-1)(r + |\mathbf{t}|) - j_0]h^m \nu_{r+j_0} \in \text{Cor}(\ell_{m(r+|\mathbf{t}|)+j_0}).$$

By hypothesis,  $\nu_{r+l(r+|\mathbf{t}|)} \equiv 0$  for all  $l \in \mathbb{N}$ . Therefore, if  $j_0$  is a multiple of  $r + |\mathbf{t}|$  then  $\nu_{r+j_0} = 0$ . Otherwise,  $(m-1)(r + |\mathbf{t}|) - j_0$  is different from zero. So, it has that  $h^m \nu_{r+j_0} \in \text{Cor}(\ell_{m(r+|\mathbf{t}|)+j_0}) \cap \text{Range}(\ell_{m(r+|\mathbf{t}|)+j_0})$ , thus  $\nu_{r+j_0} = 0$ .  $\diamond$

**Remark 1.** The vector field  $\mathbf{F} = (y, -x^7)^T + x^5(x, 4y)^T$  is of the form  $\mathbf{X}_h + (\lambda f(h) + \nu)\mathbf{D}_0$  with  $\mathbf{t} = (1, 4)$ ,  $r = 3$ ,  $h = x^8/8 + y^2/2$ ,  $f \equiv 0$  and  $\nu = x^5 \in \text{Cor}(\ell_2)$ . From Proposition 21, the system  $\dot{\mathbf{x}} = \mathbf{F}$  does not have a formal inverse integrating factor. Nevertheless, it is a center since the origin is monodromic and the system is invariant to the change  $(x, y, t) \rightarrow (-x, y, -t)$ . Analogously, the system  $(\dot{x}, \dot{y})^T = (y, -x^5)^T + x^3(x, 3y)^T$  is a center and from Proposition 21, the system does not have a formal inverse integrating factor.

**Remark 2.** The system  $\dot{\mathbf{x}} = \mathbf{F}$  with  $\mathbf{F} = (y, -x^5)^T + x^2(x^6 + 3y^2)(x, 3y)^T$  has the formal inverse integrating factor  $h = x^6 + 3y^2$ . We note that the origin is not a center since  $h$  is a Lyapunov function of the origin with  $L_{\mathbf{F}}h > 0$  for  $\mathbf{x} \neq \mathbf{0}$ . Moreover, the system is a normal form which is not hamiltonian, see Theorem 16. Thus, from Algaba et al. [5], Theorem 3.19, it is not integrable.

### 3. Proofs of the main results.

PROOF OF THEOREM 4. We see the sufficiency. The polynomial  $h(x, y) = (2x^{2n+1} - (2n+1)y^2)^m$ ,  $m \in \mathbb{N}$ , is a formal inverse integrating factor of  $(\dot{x}, \dot{y})^T = (y, x^{2n})^T$ . Thus, from Proposition 18, system (7) has a formal inverse integrating factor and is of the form  $V(x, y) = (2x^{2n+1} - (2n+1)y^2)^m + \text{q-h.h.o.t.}$  We prove the necessity. Applying Theorem 14, system (11) is a formal normal form of system (7). We note that system (11) is of the form  $(\dot{x}, \dot{y})^T = \mathbf{X}_h + \nu\mathbf{D}_0$  with  $\nu$  satisfying hypothesis of Proposition 21. This completes the proof.  $\diamond$

PROOF OF COROLLARY 1. From Theorem 14, the system (6) is orbital equivalent to  $(\dot{x}, \dot{y})^T = (y, \alpha_2 x^2)^T + \beta_2 x(2x, 3y)^T + \text{q-h.h.o.t.}$  As  $\beta_2 \neq 0$ , the system (6) is not orbital equivalent to  $(\dot{x}, \dot{y})^T = (y, \alpha_2 x^2)^T$ , and by applying Theorem 4, it follows that it does not have any formal inverse integrating factor.  $\diamond$

PROOF OF THEOREM 6. We prove that the condition is sufficient. If  $f \equiv 0$ , the system is a hamiltonian system, hence it has a formal inverse integrating factor. Moreover, the polynomial  $(\sigma x^{2n} - ny^2)^m$ ,  $m \in \mathbb{N}$ , is a formal inverse integrating factor of  $(\dot{x}, \dot{y})^T = (y, \sigma x^{2n-1})^T$ . Therefore, from Proposition 18, the formal inverse integrating factors of system (8) are of the form  $(\sigma x^{2n} - ny^2)^m + \text{q-h.h.o.t.}$ , with  $m$  any natural number.

Otherwise, if  $l$  is the smallest index such that  $a_l \neq 0$ , the system  $(\dot{x}, \dot{y})^T = (y, \sigma x^{2n-1})^T + x^{n-1}f(h)(x, ny)^T$  with  $f(h) = \sum_{i \geq 1} a_i h^i$ , has the form given in Proposition 20 for  $h = \sigma x^{2n} - ny^2$ ,  $\lambda = a_l x^{n-1} h^l$  and  $g(h) = 1 + \sum_{i \geq 1} a_{l+i}/a_l h^{l+i}$ . Thus, applying Proposition 20, the system has the inverse integrating factor  $V(h) = h^{l+1}g(h)$ . Thus, from Proposition 18, system (8) has a formal inverse integrating factor and is of the form  $V = (\sigma x^{2n} - ny^2)^{l+1} + \text{q-h.h.o.t.}$

We now prove that the condition is necessary. Applying Theorem 16, system (12) is a formal normal form of system (8).

If  $\alpha_{n-1}^{(l)} = 0$ , for all  $l$ , applying Proposition 21 for  $f \equiv 0$ , it deduces that system (8) has a formal inverse integrating factor if and only if  $\alpha_j^{(l)} = 0$ , for all  $j$  and  $l$ . Otherwise, if  $l_0 = \min(\{l, \alpha_{n-1}^{(l)} \neq 0\})$ , system (12) is of the form  $(\dot{x}, \dot{y})^T =$

$\mathbf{F} = \mathbf{X}_h + (x^{n-1}f(h) + \nu)\mathbf{D}_0$ , with  $f(h) = \sum_{i \geq l_0} a_j h^j$ . From Proposition 21 for  $\lambda = \alpha_{n-1}^{(l_0)} x^{n-1}$ , it follows the result.  $\diamond$

PROOF OF COROLLARY 2. From Theorem 6, a formal normal form of systems (8) with  $n = 2m$ , admitting a formal inverse integrating factor is of the form  $(\dot{x}, \dot{y})^T = (y, -x^{4m-1})^T + x^{2m-1}f(h)(x, 2my)^T$ , which is time-reversible. Thus, the origin is a center since it is a monodromic point (it is surrounded by orbits of the system).  $\diamond$

PROOF OF COROLLARY 3. We prove the sufficient condition. If system (8) with  $n = 2m - 1$  and  $\sigma = -1$  would be formally integrable, from Algaba *et al.* [5], it will be orbitally equivalent to  $(\dot{x}, \dot{y})^T = (y, -x^{2n-1})^T$ , therefore, it is a center.

Reciprocally, we assume that the origin is a center and has a formal inverse integrating factor. From Theorem 6, a formal normal form of the system will be of the form  $(\dot{x}, \dot{y})^T = (y, -x^{4m-3})^T + x^{2(m-1)}f(h)(x, (2m-1)y)^T$  with  $f(h) = \sum_{j \geq 0} a_j h^j$ . From [5], it suffices to prove that  $f \equiv 0$ . Otherwise, if  $l$  is the smallest index such that  $a_l \neq 0$ , then system (8) is formal orbital equivalent to  $(\dot{x}, \dot{y})^T = (y, -x^{4m-3})^T + a_l x^{2(m-1)} h^l(x, (2m-1)y)^T + \text{q-h.h.o.t.}$  Performing the generalized polar coordinates change  $x = uCs(\theta)$ ,  $y = u^{2m-1}Sn(\theta)$  where  $Cs(\theta)$  and  $Sn(\theta)$  are the unique solution of the Cauchy problem  $(\dot{x}, \dot{y})^T = (y, -x^{4m-3})^T$ ,  $x(0) = 1$ ,  $y(0) = 0$ , it has that

$$\frac{du}{d\theta} = u^{2(l-1)(2m-1)+2(m-1)+1} \left[ a_l Cs^{2(m-1)}(\theta) + \mathcal{O}(u) \right].$$

As  $a_l \neq 0$  and  $\int_0^T Cs^{2(m-1)}(\theta)d\theta > 0$  (being  $T$  the period of  $Cs(\theta)$  and  $Sn(\theta)$ ), we conclude that the origin is a focus.  $\diamond$

The coefficients of the formal normal form (12) of system (9) have been obtained by using the procedure given in Algaba *et al.* [4]. This method consists in a recursive procedure to compute quasi-homogeneous normal form under equivalence, which uses the Lie triangle.

PROOF OF THEOREM 9. From Theorem 12, a formally orbital equivalent normal form of system (9) for  $n > 2$  is

$$(\dot{x}, \dot{y})^T = (y, -x^{2n-1})^T + (\alpha_n^{(0)}x^n + \alpha_{n+1}^{(0)}x^{n+1} + \dots)(x, ny)^T.$$

The first coefficients are

$$\begin{aligned} \alpha_n^{(0)} &= b_2 + (n+1)a_2, \\ \alpha_{n+1}^{(0)} &= a_2(2b_1 + a_1). \end{aligned}$$

From Theorem 6, in order to that system (9) to have a formal inverse integrating factor, both must be zero. So, it has the following situations:

- (i)  $a_2 = b_2 = 0$ . System (9) is invariant to the change  $(x, y, t) \rightarrow (x, -y, -t)$ . Performing the change  $u = x, v = y^2, dT = ydt$ , system (9) becomes

$$(u', v')^T = (1 + a_1u, -2u^{2n-1} + 2b_1v + 2b_3u^{2n})^T,$$

which is integrable by flow's box theorem. Therefore, system (9) is formally integrable. So, from Theorem 3, it has a formal inverse integrating factor.

- (ii)  $b_2 = -(n+1)a_2$ ,  $a_1 = -2b_1$ . In such a case, system (9) is a hamiltonian system whose Hamiltonian is

$$H(x, y) = -\frac{1}{2}y^2 - \frac{1}{2n}x^{2n} + b_1xy^2 - a_2x^{n+1}y + \frac{1}{2n+1}b_3x^{2n+1}.$$

Hence,  $V(x, y) = 1 + H(x, y)$  is a formal inverse integrating factor.  $\diamond$

PROOF OF THEOREM 10. From Theorem 12, a formally orbital equivalent normal form of system (9) for  $n = 2$  is

$$(\dot{x}, \dot{y})^T = (y, -x^3)^T + (\alpha_2^{(0)}x^2 + \alpha_0^{(1)}h(x, y) + \alpha_1^{(1)}xh(x, y) + \alpha_2^{(1)}x^2h(x, y) + \dots)(x, 2y)^T,$$

with  $h(x, y) = x^4 + 2y^2$ .

In this case, it has  $\alpha_2^{(0)} = b_2 + 3a_2$  and  $\alpha_0^{(1)} = a_2(2b_1 + a_1)(b_3 + b_1 - a_1)$ . Following a similar argument as in above theorem, it has the cases (i) and (ii).

We assume that  $b_2 = -3a_2$ ,  $b_3 = a_1 - b_1$  and  $a_2(2b_1 + a_1) \neq 0$ . It has that

$$\begin{aligned}\alpha_1^{(1)} &= a_2(2b_1 + a_1)[(2b_1 - 3a_1)^2 - a_1^2 + 24a_2^2], \\ \alpha_2^{(1)} &= a_2(2b_1 + a_1)[(2b_1 - 3a_1)^2 - a_1^2 + 24a_2^2](b_1 - 3a_1).\end{aligned}$$

We analyze the following cases separately:

- (iii)  $(2b_1 - 3a_1)^2 - a_1^2 + 24a_2^2 = 0$ . In this case,  $V^{1/A}$  with  $V(x, y) = 1 + A(a_1 + 2b_1)[x + a_2y + \frac{1}{2}(2Ab_1 + Aa_1 - a_1)x^2]$  and  $A = b_1/(a_1 + 2b_1)$ , is a formal inverse integrating factor of system (9). It is an integrable system, since  $V(0, 0) \neq 0$ .
- (iv)  $b_1 - 3a_1 = 0$ ,  $a_1a_2 \neq 0$  (i.e.  $\alpha_1^{(1)} \neq 0$ ). The system is not formally conjugated to an hamiltonian system and by Algaba et al. [5], it is not integrable. However, system (9) has a formal inverse integrating factor which vanishes at origin. Indeed, Gasull and Torregrosa [20] proved that this system is a center. So, it can be transformed, by means of a change of variables and reparameterizing the time, into a system of the form  $(\dot{x}, \dot{y})^T = (y, -x^3 + xyf(x^2))^T$ , see Berthier and Moussu [8]. If we perform the singular change  $u = x^2$ ,  $v = y$ ,  $dt = d\tau/x$ , the system becomes  $(u', v')^T = (2v, -u + vf(u))^T$  which is a weak focus, and by Enciso and Peralta-Salas [15], the system has a formal inverse integrating factor.  $\diamond$

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## References

- [1] A. ALGABA; C. GARCÍA; M.A. TEIXEIRA, *Reversibility and quasi-homogeneous normal forms of vector fields*. Nonlinear Analysis: Theory, Methods and Applications, **73**, (2010), 510-525.
- [2] A. ALGABA; E. FREIRE; E. GAMERO, *Isochronicity via normal form*. Qualitative Theory and Dynamical Systems, **1**, (2000), 133-156.
- [3] A. ALGABA; E. FREIRE; E. GAMERO; C. GARCÍA, *Monodromy, center-focus and integrability problems for quasi-homogeneous polynomial systems*. Nonlinear Analysis: Theory, Methods and Applications, **72**, (2010), 3-4, 1726-1736.
- [4] A. ALGABA; E. FREIRE; E. GAMERO; C. GARCÍA, *An algorithm for computing quasi-homogeneous formal normal forms under equivalence*. Acta Applicandae Mathematicae, **80**, (2004), 335-359.
- [5] A. ALGABA; E. GAMERO; C. GARCÍA, *The integrability problem for a class of planar systems*. Nonlinearity **22**, (2009), 2, 395-420.
- [6] A. ALGABA; C. GARCÍA; M. REYES, *Local bifurcation of limit cycles and integrability of a class of nilpotent systems of differential equations*. Applied Mathematics and Computation, **215**, (2009), 314-323.
- [7] A. ALGABA; C. GARCÍA; M. REYES, *Like-linearizations of vector fields*. Bulletin des Sciences Mathématiques, **133**, 806-816, (2009).
- [8] M. BERTHIER; R. MOUSSU, *Réversibilité et classification des centres nilpotents*, Ann. Inst. Fourier (Grenoble), **44**, 2, (1994), 465-494.
- [9] J. CHAVARRIGA, *Integrable systems in the plane with a center type linear part*, Appl. Math. (Warsaw) **22**, (1994), 285-309.
- [10] J. CHAVARRIGA; I. GARCÍA; J. GINÉ, *Integrability of centers perturbed by quasi-homogeneous polynomials*, J. Math. Anal. Appl. **210**, (1997), 268-278.
- [11] J. CHAVARRIGA; H. GIACOMINI; J. GINÉ, *On a new type of bifurcation of limit cycles for a planar cubic system*, Nonlinear Analysis **36**, (1999), 139-149.
- [12] J. CHAVARRIGA; H. GIACOMINI; J. GINÉ; J. LLIBRE, *On the integrability of two-dimensional flows*. J. Differential Equations **157**, (1999), 163-182.
- [13] J. CHAVARRIGA; H. GIACOMINI; J. GINÉ; J. LLIBRE, *Darboux integrability and the inverse integrating factor*. J. Differential Equations **194**, (2003), 116-139.
- [14] C. CHRISTOPHER; P. MARDESIC; C. ROUSSEAU, *Normalizable, integrable, and linearizable saddle points for complex quadratic systems in  $\mathbb{C}^2$* . J. Dyn. Control Syst. **9**, (2003), 311-363.

- [15] A. ENCISO; D. PERALTA-SALAS, *Existence and vanishing set of inverse integrating factors for analytic vector fields*. Bull. London Math. Soc. **41**, (2009), 1112-1124.
- [16] A. FERRAGUT; J. LLIBRE; A. MAHDI, *Polynomial inverse integrating factors for polynomial vector fields*. Discrete Contin. Dyn. Syst. **17**, (2006), 387-395.
- [17] I. GARCÍA; H. GIACOMINI; M. GRAU, *The inverse integrating factor and the Poincaré map*, Trans. Amer. Math. Soc., **362**, 7, 3591-3612,(2010).
- [18] I. GARCÍA; H. GIACOMINI; M. GRAU, *Generalized Hopf Bifurcation for planar vector fields via the inverse integrating factor*, J. Dynam. Differential. Equations, **23**, 2, 251-281,(2011).
- [19] I. GARCÍA; D. SHAFER, *Integral invariants and limit sets of planar vector fields*. J. Differential Equations **217**,(2005), 2, 363-376.
- [20] A. GASULL; J. TORREGROSA, *Center problem for several differential equations via Cherkas' method*, J. Math. Anal. Appl. **228**, (1998), 322-343.
- [21] H. GIACOMINI; J. LLIBRE; M. VIANO, *On the nonexistence, existence and uniqueness of limit cycles*, Nonlinearity **9**, (1996), 501-516.
- [22] H. GIACOMINI; J. LLIBRE; M. VIANO, *On the shape of limit cycles that bifurcate from Hamiltonian centers*, Nonlinear Analysis **41**, (2000), 523-537.
- [23] H. GIACOMINI; J. LLIBRE; M. VIANO, *On the shape of limit cycles that bifurcate from non-Hamiltonian centers*, Nonlinear Analysis **43**, (2001), 837-859.
- [24] M. VIANO; J. LLIBRE; H. GIACOMINI, *On the shape of limit cycles that bifurcate from non-Hamiltonian centers*, Nonlinear Analysis **48**, (2002), 117-136.
- [25] L. MAZZI; M. SABATINI, *A characterization of centers via first integrals*, J. Differential Equations, **76**, (1988),222-237.
- [26] M.J. PRELLE; M.F. SINGER, *Elementary first integrals of of differential equations*. Trans. Amer. Math. Soc. **279**, (1983), 215-229.
- [27] G. REEB, *Sur certaines propriétés topologiques des variétés feuilletés*, W.T. Wu, G. Reeb (Eds.), Paris, 1952.
- [28] M.F. SINGER, *Liouvillian first integrals of differential equations*. Trans. Amer. Math. Soc. **333**, (1992), 673-688.
- [29] S. WALCHER, *On the Poincaré problem*. J. Differential Equations **166**, (2000), 51-78.
- [30] S. WALCHER, *Local integrating factors*. J. Lie Theory **13**, (2003), 279-289.