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Contribution to the qualitative theory of dynamical systems

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Contribution to the Qualitative Theory of Dynamical Systems

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Contribución a la Teoría Cualitativa de Sistemas Dinámicos

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Contribución a la Teoría Cualitativa de Sistemas Dinámicos

Memoria presentada por Isabel Checa Camacho para optar al grado de Doctor en Matemáticas por la Universidad de Huelva.

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Certificamos que la presente memoria ha sido realizada por Isabel Checa Camacho bajo nuestra supervisión y constituye su Tesis para aspirar al grado de Doctor en Matemáticas por la Universidad de Huelva.

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Resumen

En esta memoria abordamos tres problemas fundamentales dentro de la teoría cualitativa de sistemas dinámicos. En un sentido amplio, el objetivo de la teoría de sistemas dinámicos es determinar la estructura del conjunto de soluciones de sistemas que modelan la evolución a través del tiempo de determinados fenómenos.

En primer lugar, consideramos el problema de determinar la expresión analítica más simple en la que se puede transformar un sistema autónomo, mediante cambios en las variables de estado y en el tiempo y con ello calcular el conjunto de invariantes de un campo vectorial. Este concepto es referido por algunos autores como forma normal única (o hipernormal). Demostraremos que estas simplificaciones se pueden llevar a cabo mediante procedimientos lineales y proporcionamos un método para obtener mayores simplificaciones en la forma normal, llegando con ello a calcular la forma normal única.

Otro problema que hemos abordado en esta memoria es el problema de la reversibilidad orbital de un sistema. Consiste, a grandes rasgos, en determinar si existe un cambio de variables en las variables de estado y una reparametrización en el tiempo de modo que el sistema resultante sea reversible respecto de una involución lineal. Adaptamos las ideas de la teoría de formas normales bajo equivalencia para caracterizar la reversibilidad orbital. Centrándonos en sistemas planos, la idea es reducir el sistema a una pre-forma normal y después analizar la reversibilidad respecto de los ejes (axis-reversibilidad) módulo orbital equivalencia, a través de las propiedades de las curvas invariantes del campo vectorial. Estas técnicas son aplicadas para calcular familias de centros. Completamos algunas familias de centros nilpotentes que habían sido parcialmente caracterizadas en la literatura.

Por último, hemos abordado el problema de la integrabilidad analítica de un campo vectorial plano. Este problema, en campos planos cuya primera componente cuasi-homogénea es hamiltoniana y su función de Hamilton tiene factores simples en su factorización en $\mathbb{C}[x, y]$, está completamente resuelto en [12]. Sin embargo, cuando la función de Hamilton de la primera componente tiene factores múltiples en $\mathbb{C}[x, y]$ es un problema abierto. En este caso es donde se enmarca nuestro problema. Hemos

De conformidad con el artículo 34 del Reglamento de los estudios de doctorado de la Universidad de Huelva, toda tesis redactada en otro idioma distinto del español, deberá incluir un resumen en castellano.

considerado un sistema nilpotente degenerado, para el que obtenemos una orbital equivalente forma normal, que posteriormente transformamos en un sistema cuya primera componente cuasi-homogénea es irreducible. Proporcionamos condiciones necesarias para la integrabilidad del sistema.

Introducción

El estudio de los sistemas dinámicos es de gran importancia, ya que éstos están relacionados con infinidad de situaciones del mundo real. Existen diversas disciplinas, entre las que podemos citar la biología, la economía, la física, etc., en las cuales aparecen magnitudes que se usan para describir determinados fenómenos. Por medio de los sistemas dinámicos es posible describir el comportamiento de tales fenómenos; es decir, mediante el uso de determinados modelos matemáticos que aproximan la evolución a través del tiempo de un sistema observado. En un sentido amplio, el objetivo de la teoría de sistemas dinámicos es determinar la estructura del conjunto de soluciones de estos modelos.

Este estudio comenzó en el siglo XVII con los trabajos de Newton y Leibnitz en el entorno de ecuaciones diferenciales ordinarias; éstas fueron utilizadas por Newton para describir las leyes de la mecánica enunciadas por él. Las ecuaciones diferenciales ordinarias, en ocasiones, resultaban difíciles de resolver al aplicarlas a problemas específicos. Durante los siglos XVIII y XIX, fueron muchos los matemáticos que se preocuparon por encontrar soluciones a determinadas ecuaciones diferenciales. Fueron Cauchy y, posteriormente, Lipschitz quienes se preocuparon por la existencia y unicidad de soluciones. Sin embargo, en muchas ocasiones, en el estudio de ciertos problemas físicos, resulta de gran interés conocer ciertas propiedades de las soluciones del sistema que modela el fenómeno (tales como acotamiento, estabilidad, periodicidad, etc.) sin necesidad de encontrar expresiones analíticas para éstas. De este modo surge la teoría cualitativa de ecuaciones diferenciales.

El impulsor de la formulación y desarrollo de la teoría de sistemas dinámicos, tal y como la conocemos actualmente, fue Poincaré (1881), quien, en su estudio sobre el problema de los tres cuerpos de la mecánica celeste, introdujo un método que consiste básicamente en estudiar el comportamiento de las soluciones de un sistema sin tener el conocimiento explícito de éstas. Basándose en la descripción global de todas las soluciones (o retrato de fases), describió la utilidad del llamado mapa de Poincaré en el estudio de soluciones periódicas y en el efecto de pequeñas perturbaciones de las condiciones iniciales (estabilidad). Prácticamente en la misma época, Lyapunov (1892) hizo grandes aportaciones a la teoría cualitativa contribuyendo notablemente en el problema de la estabilidad de las soluciones. Posteriormente, Andronov y Pontriaguin (1937) trabajaron extensamente en la teoría de la estabilidad de sis-

temas dinámicos, introduciendo la idea de estabilidad estructural. Peixoto (1959) permitió expresar de forma clara la noción de perturbación de un sistema dinámico, dotando al conjunto de ecuaciones diferenciales de estructura topológica. Por su parte, matemáticos como Kolmogorov, Arnold y Moser (1954) estudiaron la estabilidad de los sitemas hamiltonianos integrables, dando lugar al famoso teorema KAM en el que se explica cómo se modifica el aspecto de las trayectorias de un sistema integrable bajo pequeñas perturbaciones. Finalmente, cabe mencionar a Lorenz y Smale quienes han enriquecido la teoría de sistemas dinámicos dando lugar a un renovado interés en la teoría del caos.

Anteriormente hemos comentado que, en un sentido amplio, el objetivo de la teoría de sistemas dinámicos es determinar la estructura del conjunto de soluciones de la ecuación diferencial que modela un determinado fenómeno. En concreto, si denotamos el correspondiente estado del sistema en un cierto instante de tiempo t mediante una variable $\mathbf{x}(t)$ (que podría tomar valores vectoriales), la evolución con el tiempo de $\mathbf{x}(t)$ viene descrita por una ecuación diferencial ordinaria de la forma

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t, \mathbf{x}(t)).$$

En esta memoria, nos centraremos en sistemas autónomos, es decir, en aquellos sistemas en los que el segundo miembro no depende explícitamente de t; esto es,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t)).$$

Con frecuencia, las únicas soluciones que pueden ser encontradas con precisión en las aplicaciones son las soluciones constantes, dadas por $\frac{d\mathbf{x}(t)}{dt} = 0$, llamadas puntos fijos (estacionarios o de equilibrio). Otro tipo de soluciones de especial interés son las soluciones periódicas, es decir, aquellas que vuelven al punto inicial después de un cierto tiempo T (periodo). Las gráficas de las soluciones periódicas describen trayectorias que son curvas cerradas.

Determinar la estructura topológica global del conjunto de soluciones, en el espacio de las variables de estado, puede resultar inabordable en la práctica. Lo mismo ocurre cuando el sistema depende de parámetros y pretendemos analizar cómo cambia esa estructura al variar los parámetros, es decir, caracterizar los fenómenos de bifurcación. Es por ello que se limita el estudio de la dinámica de un sistema a un entorno de un cierto elemento crítico, generalmente un punto fijo. En cuanto al estudio de las bifurcaciones, éste se aborda desde el punto de vista local, considerando además que los parámetros varían localmente en un entorno de un cierto valor.

Otra técnica para abordar la dinámica global del sistema es la simplificación de la expresión analítica del campo vectorial $\mathbf{F}(\mathbf{x})$. La más importante es la reducción a forma normal. La idea básica en la teoría (clásica) de formas normales es simplificar tanto como sea posible la expresión del campo vectorial, mediante cambios de

variables. Tradicionalmente, estas simplificaciones se consiguen mediante cambios en las variables de estado ($\mathcal{C}^\infty$-conjugación). Sin embargo, en ciertos problemas, es posible encontrar mejores simplificaciones haciendo también transformaciones en el tiempo ($\mathcal{C}^\infty$-equivalencia). Con la reducción a forma normal, se pretenden eliminar aquellos términos que no son esenciales para la dinámica.

Después de esta breve introducción, pasamos a describir la organización y contenidos de la presente memoria.

Una vez introducido el concepto de forma normal, en el primer capítulo realizamos un breve resumen de las formas normales clásicas. La simplificación en los términos no lineales se alcanza grado a grado; si hacemos un cambio de variables no lineal, identidad más un polinomio homogéneo de grado k, veremos que los términos hasta grado $k-1$ de $\mathbf{F}(\mathbf{x})$ no se modifican, mientras que los de grado k cambian de forma lineal; esto nos permite construir el operador homológico, que se define mediante el corchete de Lie. Haciendo una elección adecuada del cambio, siempre es posible eliminar en los términos de grado k la parte que está en el rango del citado operador. Es decir, en este sentido, la simplificación consiste en reducir los términos de orden k de nuestro sistema a un subespacio complementario a la imagen (co-rango) del operador homológico. A continuación, describiremos las formas normales cuasi-homogéneas bajo $\mathcal{C}^\infty$-conjugación y $\mathcal{C}^\infty$-equivalencia. Para ello definiremos cuándo un campo vectorial es cuasi-homogéneo y enunciaremos algunas propiedades. Veremos que es posible reducir el dominio del operador homológico bajo $\mathcal{C}^\infty$-equivalencia. Por último, describiremos un procedimiento recursivo, a través del triángulo de Lie, para computar formas normales.

Siguiendo con formas normales, en el segundo capítulo abordaremos el problema de determinar la expresión analítica más simple que se puede obtener de un sistema autónomo, mediante transformaciones en las variables de estado y en el tiempo. Este concepto es referido por algunos autores como forma normal única (o hipernormal). Veremos que tales simplificaciones se pueden obtener mediante procedimientos lineales. Es decir, cuando aplicamos la teoría clásica de formas normales, aparecen términos no lineales que, en principio, no sabemos si se pueden simplificar pues están ligados a la resolución de ecuaciones no lineales. En este capítulo demostraremos que es posible usar un procedimiento lineal que resulta equivalente al procedimiento no lineal; es decir, con el lineal se consiguen (exactamente) las mismas simplificaciones que con el no lineal. Expondremos un método para obtener mayores simplificaciones en la forma normal, llegando a condiciones para la forma normal única. Por último, aplicaremos estas técnicas para el cálculo de la forma normal única de un caso particular de la singularidad Takens-Bogdanov.

Los capítulos tres y cuatro versan sobre reversibilidad orbital. En ambos se hace una introducción a este concepto, sin embargo podemos decir brevemente que el problema de la reversibilidad orbital consiste, a grandes rasgos, en determinar si existe un cambio de variables en las variables de estado y una reparametrización en

el tiempo de modo que el sistema resultante sea reversible. En este sentido, en el tercer capítulo adaptaremos las ideas de la teoría de formas normales bajo equivalencia para caracterizar la reversibilidad orbital. En primer lugar, introduciremos el concepto de N-orbital–reversibilidad y obtendremos algunas condiciones necesarias. Presentaremos un algoritmo en el que la idea, básicamente, es descartar los casos de no reversibilidad orbital. Es importante destacar que las transformaciones en el tiempo y en las variables de estado son no lineales; en este sentido, estaremos interesados en determinar los elementos que pertenecen al rango de ciertos operadores no lineales. Sin embargo, en el procedimiento de la forma normal para determinar la reversibilidad orbital, usaremos los operadores homológicos bajo $\mathcal{C}^\infty$-equivalencia que son lineales. Esto es posible, y lo demostraremos a lo largo del capítulo, pues el rango del operador no lineal es equivalente al rango del operador lineal. Por último, aplicaremos nuestros resultados a dos familias concretas, en las que mostraremos casos que no son reversibles pero sí orbital reversibles. En el capítulo cuatro nos centraremos en la reversibilidad orbital para sistemas planos. La idea en este capítulo es reducir el sistema a una pre-forma normal y después analizar la reversibilidad respecto de los ejes (axis-reversibilidad) módulo orbital equivalencia, a través de las propiedades de las curvas invariantes del campo vectorial. Estudiaremos la reversibilidad orbital para campos nilpotentes y en algunos campos con parte lineal nula, y aplicaremos los resultados en familias concretas. Este capítulo está publicado (ver [2]).

En el capítulo quinto estudiaremos la integrabilidad analítica en una familia de campos nilpotentes degenerados. En este capítulo hacemos una extensa introducción al problema de la integrabilidad y la evolución que ha sufrido a lo largo de los estudios de muchos investigadores. Se puede decir que el problema de la integrabilidad para campos planos cuya primera componente hamiltoniana tiene factores simples en su factorización en $\mathbb{C}[x, y]$ está completamente resuelto en [12]. Sin embargo, cuando la función de Hamilton de la primera componente tiene factores múltiples en $\mathbb{C}[x, y]$ es un problema abierto. En este caso es donde se enmarca nuestro problema. Consideraremos un sistema nilpotente degenerado y obtendremos una orbital equivalente forma normal, que transformaremos en un sistema cuya primera componente cuasi-homogénea será irreducible. Tras poner de manifiesto la dificultad del problema de la integrabilidad, proporcionaremos condiciones necesarias para la integrabilidad de nuestro sistema. Aplicaremos nuestros resultados a dos familias concretas, para las que la integrabilidad está completamente resuelta. Los resultados obtenidos en este capítulo han dado lugar a una publicación (ver [1]). Aunque en el artículo se estudia la integrabilidad analítica en una familia de centros degenerados, en este capítulo se generaliza para una familia de campos nilpotentes degenerados.

Chapter 1

Theory of Quasi-homogeneous Normal Forms

1.1 Introduction

The normal form theory lies in the basis of the study of several problems in differential equations: bifurcation, stability analysis,... The idea underlying in this theory is the use of near-identity transformations to remove, in the analytic expression of the vector field, the terms that are unessential in the local dynamical behavior. In this way, we can determine canonical systems that represents a wide class, by discarding these unessential terms.

The classical normal form theorem determines how it is possible to simplify the analytic expression of a vector field, taking into account the linear part of the vector field. Our goal is to show how to obtain further simplifications on the classical normal form, considering the nonlinear terms of the vector field.

Let us consider an autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \tag{1.1.1}$$

having an equilibrium point at the origin (i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$). Usually, the normal form techniques are used to simplify the vector field $\mathbf{F}$ degree by degree. This requires that the vector field is written in homogeneous components, by its Taylor expansion.

Before to present the classical theory of normal forms we are going to make some definitions and useful concepts.

1.2 Preliminaries

We define the *Lie bracket* of two smooth vector fields $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ as:

$$[\mathbf{F}, \mathbf{G}](\mathbf{x}) \quad = \quad D\mathbf{F}(\mathbf{x}) \cdot \mathbf{G}(\mathbf{x}) - D\mathbf{G}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^n,$$

which has the next properties:

- $[a_1\mathbf{F}_1 + a_2\mathbf{F}_2, b_1\mathbf{G}_1 + b_2\mathbf{G}_2] = a_1b_1[\mathbf{F}_1, \mathbf{G}_1] + a_1b_2[\mathbf{F}_1, \mathbf{G}_2] + a_2b_1[\mathbf{F}_2, \mathbf{G}_2] + a_2b_2[\mathbf{F}_2, \mathbf{G}_2]$.
- $[\mathbf{F}, \mathbf{G}] = -[\mathbf{G}, \mathbf{F}]$.
- $[[\mathbf{F}, \mathbf{G}], \mathbf{H}] + [[\mathbf{G}, \mathbf{H}], \mathbf{F}] + [[\mathbf{H}, \mathbf{F}], \mathbf{G}] = 0$.
- If $\mathbf{F} \in \mathcal{H}_k^n$ and $\mathbf{G} \in \mathcal{H}_l^n$, then $[\mathbf{F}, \mathbf{G}] \in \mathcal{H}_{k+l-1}^n$, where $\mathcal{H}_j^n$ is the vector space of vector fields with n polynomial homogeneous components in n variables of degree j.

For more details, we can find these properties in [49], [62], etc.

We call *germen* to any function $\mathbf{F} \in \mathcal{C}^\infty(\mathcal{V}_0; \mathbb{R}^n)$ in some neighborhood of the origin of $\mathbb{R}^n$, $\mathcal{V}_0$, and we denote by V_0 the space of the germens which vanish at $\mathbf{0}$; that is, $\mathbf{F} \in V_0$ if there exists a neighborhood $\mathcal{V}_0$ of the origin of $\mathbb{R}^n$ such that $\mathbf{F} \in \mathcal{C}^\infty(\mathcal{V}_0; \mathbb{R}^n)$ and $\mathbf{F}(\mathbf{0}) = \mathbf{0}$.

Let $\mathbf{F}$ be a germen, we call *k-jet* of $\mathbf{F}$ in $\mathbf{0}$, and we denote it by $\mathcal{J}^k\mathbf{F}$, to the Taylor polynomial of $\mathbf{F}$ around $\mathbf{x} = \mathbf{0}$ of degree k. And we denote by $\mathbf{F}_k(\mathbf{x}) = \mathcal{J}^k\mathbf{F}(\mathbf{x}) - \mathcal{J}^{k-1}\mathbf{F}(\mathbf{x}) \in \mathcal{H}_k^n$ to the homogeneous part of degree k.

Next, we are going to define the concepts of $\mathcal{C}^\infty$-conjugation and $\mathcal{C}^\infty$-equivalence, that are basic along this memory.

We say that $\mathbf{F}, \mathbf{G} \in V_0$ are $\mathcal{C}^\infty$-conjugated if there exists a diffeomorphism $\Phi \in V_0$ which transforms the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ into the system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$, that is, $\mathbf{G}(\mathbf{y}) = [D\Phi(\mathbf{y})]^{-1} \cdot \mathbf{F}(\Phi(\mathbf{y}))$. In this case, we write $\mathbf{G} = \Phi_*\mathbf{F}$.

Analogously, $\mathbf{F}, \mathbf{G} \in V_0$ are $\mathcal{C}^\infty$-equivalent (orbital–equivalent) if there exist a diffeomorphism $\Phi \in V_0$ and a scalar germen μ, which is non null in any neighborhood of the origin, such that $\mathbf{G} = \Phi_*(\mu\mathbf{F})$. That is, the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is transformed into the system $\mathbf{y}' = \mathbf{G}(\mathbf{y})$, through the change of variables $\mathbf{x} = \Phi(\mathbf{y})$ and the change of time $dT/dt = 1/\mu(\mathbf{y})$. Obviously, the local dynamical of two $\mathcal{C}^\infty$-equivalent systems is the same.

Notice that we denote $d/dt = \dot{}$ and $d/dT = '$.

1.3 Classical Normal Forms

The classical normal form theory is applied to systems with non null linear part and it uses near-identity transformations to vanish the unessential terms.

Suppose the vector field is written in homogeneous terms, by the Taylor expansion. If we denote by $A = D\mathbf{F}(\mathbf{0})$, we can write the system (1.1.1) of the form:

$$\dot{\mathbf{x}} = A\mathbf{x} + \sum_{k\geq 2} \mathbf{F}_k(\mathbf{x}), \tag{1.3.1}$$

where $\mathbf{F}_j \in \mathcal{H}_j^n$ is the term of degree j in the Taylor expansion of the vector field $\mathbf{F}$.

Consider the change of variables

$$\mathbf{x} = \Phi(\mathbf{y}).$$

We suppose $\Phi(\mathbf{0}) = \mathbf{0}$, to preserve the equilibrium point at the origin, and we denote by $D\Phi(\mathbf{0}) = B$ the linear part. So, the transformed vector field is

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}) = \Phi_* \mathbf{F}(\mathbf{y}) = (D\Phi(\mathbf{y}))^{-1} \mathbf{F}(\Phi(\mathbf{y})),\ \mathbf{y} \in \mathbb{R}^n. \tag{1.3.2}$$

It is easy to check that $\mathbf{G}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{G}(\mathbf{0}) = B^{-1}AB$.

The simplification procedure (also called normalization) is degree by degree. The first step in this procedure is over the lowest degree and uses linear transformations $\mathbf{x} = \Phi(\mathbf{y}) = B\mathbf{y}$, so that the linealization matrix A is reduced to the Jordan normal form. To simplify the terms of degree $k \geq 2$ in (1.3.1), we consider the near-identity transformation

$$\mathbf{x} = \Phi(\mathbf{y}) = \mathbf{y} + \mathbf{P}_k(\mathbf{y}),$$

with $\mathbf{P}_k \in \mathcal{H}_k^n$. The transformed system is

$$\begin{aligned} \dot{\mathbf{y}} &= (I + D_{\mathbf{y}}\mathbf{P}_k(\mathbf{y}))^{-1} A\,(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) + \sum_{k\geq 2} (I + D_{\mathbf{y}}\mathbf{P}_k(\mathbf{y}))^{-1} \mathbf{F}_k\,(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) \\ &= \mathbf{G}(\mathbf{y}) = A\mathbf{y} + \sum_{k\geq 2} \mathbf{G}_k(\mathbf{y}). \end{aligned} \tag{1.3.3}$$

It is easy to check (see [64]) that the vector field does not change up to order $k-1$ (i.e., $\mathcal{J}^{k-1}\mathbf{G} = \mathcal{J}^{k-1}\mathbf{F}$) and to order k is

$$\mathbf{G}_k(\mathbf{y}) = \mathbf{F}_k(\mathbf{y}) - \left(D_{\mathbf{y}}\mathbf{P}_k(\mathbf{y})A\mathbf{y} - A\mathbf{P}_k(\mathbf{y})\right).$$

This expression suggests to define the *homological operator*:

$$\begin{aligned} \mathbf{L}_k :\quad & \mathcal{H}_k^n \longrightarrow \mathcal{H}_k^n \\ & \mathbf{P}_k \longmapsto \mathbf{L}_k(\mathbf{P}_k), \end{aligned} \tag{1.3.4}$$

where $\mathbf{L}_k\left(\mathbf{P}_k(\mathbf{x})\right) = D_{\mathbf{x}}\mathbf{P}_k(\mathbf{x})A\mathbf{x} - A\mathbf{P}_k(\mathbf{x})$.

The application $\mathbf{L}_k$ is linear and its expression in terms of the Lie bracket is given by

$$\mathbf{L}_k\left(\mathbf{P}_k\right) = [\mathbf{P}_k, A].$$

We fix a complementary subspace $\mathrm{Cor}\left(\mathbf{L}_k\right)$ to the range, $\mathrm{Range}\left(\mathbf{L}_k\right)$, of the linear operator $\mathbf{L}_k$; that is, $\mathcal{H}_k^n = \mathrm{Range}\left(\mathbf{L}_k\right) \bigoplus \mathrm{Cor}\left(\mathbf{L}_k\right)$. So, we can decompose

$\mathbf{F}_k = \mathbf{F}_k^{\mathtt{r}} + \mathbf{F}_k^{\mathtt{c}}$, where $\mathbf{F}_k^{\mathtt{r}} \in \mathrm{Range}\,(\mathbf{L}_k)$ and $\mathbf{F}_k^{\mathtt{c}} \in \mathrm{Cor}\,(\mathbf{L}_k)$. Then, there exists $\mathbf{P}_k \in \mathcal{H}_k^n$ verifying the homological equation

$$\mathbf{L}_k\,(\mathbf{P}_k) = \mathbf{F}_k^{\mathtt{r}}. \tag{1.3.5}$$

And then we obtain

$$\mathbf{G}_k = \mathbf{F}_k - \mathbf{L}_k\,(\mathbf{P}_k) = \mathbf{F}_k^{\mathtt{c}} \in \mathrm{Cor}\,(\mathbf{L}_k)\,.$$

That is, we have simplified $\mathbf{F}_k$ eliminating the range of the homological operator. Reasoning analogously for $k = 2, 3, \ldots$, we can obtain $\mathbf{G}_k \in \mathrm{Cor}\,(\mathbf{L}_k)$, for all $k \geq 2$. We can enunciate the classical normal form theorem (see [93]):

Theorem 1. *There exists a $\mathcal{C}^\infty$-diffeomorphism Φ, with $\Phi(\mathbf{0}) = \mathbf{0}$ and $D\Phi(\mathbf{0}) = I$, such that the change of variables $\mathbf{x} = \Phi(\mathbf{y})$ transforms (1.3.1) into (1.3.3) where $\mathbf{G}_k \in \mathrm{Cor}\,(\mathbf{L}_k)$, for all $k \geq 2$.*

In this case, system (1.3.3) is a classical normal form of system (1.3.1).

The homological equation (1.3.5) is based in the linear part of the vector field (see [47], [48], [52], [54], [64], [65], [88], [96]). This linear equation has, in general, a non-unique solution, that will depend on an arbitrary additive term belonging to the kernel of the homological operator. Hence, a number of arbitrary constants will appear in the expression of $\mathbf{P}_k$ and, consequently, in the normal form of order greater than k. These constants can be used, depending on the form of the nonlinear terms, to obtain further simplifications in the normal form (see [4], [29], [30], [49], [88], [89], [92], and other works devoted to special cases: see references in [5]).

Note this two-step procedure to obtain simplifications in the vector field: firstly, we use the *linear part* of $\mathbf{F}$ to determine the simplifications expressed by the normal form theorem, and later, taking into account the *nonlinear terms*, we obtain further reductions in the classical normal form.

With the approach we will adopt in next section, the above procedure can be unified, because the use of quasi-homogeneous expansions allows to manage at once linear terms and nonlinear terms of different degree (monomials with different degrees could have the same quasi-homogeneous degree).

1.4 Quasi-homogeneous Normal Forms

The tools used in the analysis of the determinacy of the topological type of a singularity are changes in the variables of the form $x_i \to \varepsilon^{t_i} x_i$ with $t_i \in \mathbb{N}$ (blow-up techniques: see, e.g., [39]). Applying this change, the original system is transformed into:

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})\varepsilon^r + \mathbf{F}_{r+1}(\mathbf{x})\varepsilon^{r+1} + \cdots$$

where the terms $\mathbf{F}_k(\mathbf{x})$ are quasi-homogeneous.

Our first goal is to extend the ideas of the normal form theory to simplify vector fields expanded in quasi-homogeneous components. Some papers have already used similar ideas, in the case of the Takens-Bogdanov singularity with $\mathcal{C}^\infty$-conjugation (see [31], [33], [66], [97]). In these papers, the normal forms are obtained by means of near-identity transformations in the state variables ($\mathcal{C}^\infty$-conjugation).

The results in this section are published (see [6]), so we only present the results without the proofs. Here, we develop the normal forms with $\mathcal{C}^\infty$-conjugation and $\mathcal{C}^\infty$-equivalence.

1.4.1 Notations and Preliminaries

We will consider a fixed *type* $\mathbf{t} = (t_1, t_2, \ldots, t_n)$, with $t_i \in \mathbb{N}$ (here, $\mathbb{N}$ is the set of natural numbers not including zero, whereas $\mathbb{N}_0$ will denote the set of natural numbers including zero).

We will use standard multi-index notations: a *multi-index* is an element $\mathbf{a} = (a_1, \ldots, a_n) \in {\mathbb{N}_0}^n$. Moreover, we will write the monomials as $\mathbf{x}^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n}$. Its degree is $\deg \mathbf{x}^\mathbf{a} = |\mathbf{a}| = a_1 + \cdots + a_n$. The set of multi-indices is denoted by $\mathcal{I}$.

A scalar function f is said *quasi-homogeneous* of type $\mathbf{t}$ and degree k if its monomials $\mathbf{x}^\mathbf{a}$ satisfy

$$a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = k.$$

The vector space of quasi-homogeneous polynomials in n variables of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^\mathbf{t}$.

A vector field $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is said *quasi-homogeneous* of type $\mathbf{t}$ and degree k if its components $F_j \in \mathcal{P}_{k+t_j}^\mathbf{t}$ for all $j = 1, 2, \ldots, n$. We will denote $\mathcal{Q}_k^\mathbf{t}$ the vector space of quasi-homogeneous vector fields of type $\mathbf{t}$ and degree k.

There is an alternative characterization for quasi-homogeneous functions and vector fields. Let denote:

$$E = \begin{pmatrix} \varepsilon^{t_1} & 0 & \cdots & 0 \\ 0 & \varepsilon^{t_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon^{t_n} \end{pmatrix}.$$

Proposition 2. **(a)** *The function f is quasi-homogeneous of type $\mathbf{t}$ and degree k if and only if*

$$f(E\mathbf{x}) = \varepsilon^k f(\mathbf{x}). \tag{1.4.1}$$

(b) *The vector field $\mathbf{F}$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if and only if*

$$\mathbf{F}(E\mathbf{x}) = \varepsilon^k E\mathbf{F}(\mathbf{x}). \tag{1.4.2}$$

There exists a geometrical representation of the vector space $\mathcal{Q}_k^{\mathbf{t}}$ (see [38], [40]).

Although the type $\mathbf{t}$ can be selected arbitrarily, there are some cases that deserve special attention. The most usual case corresponds to $\mathbf{t} = (1, \ldots, 1)$, because $\mathcal{Q}_k^{\mathbf{t}}$ agrees with the space of homogeneous vector fields of degree $k-1$; that is, $\mathcal{Q}_k^{(1,\ldots,1)} = \mathcal{H}_{k-1}^n$.

There is another selection that corresponds to taking the type $\mathbf{t}$ of the principal quasi-homogeneous component of the vector field (see [40], [51]).

Recall that the *Newton polyhedron* of $\mathbf{F}$ is the convex hull of the indices $\mathbf{a} \in \mathcal{I}$ that correspond to monomials appearing in the Taylor expansion of $\mathbf{F}$ (see, for instance, [41]).

The *Newton diagram* of $\mathbf{F}$ is the union of the compact faces of the Newton polyhedron, and the principal quasi-homogeneous component of a vector field $\mathbf{F}$ is just the part of the Taylor expansion of $\mathbf{F}$ that involves those monomials that are mapped into the Newton diagram of the vector field. A type $\mathbf{t}$ of the principal quasi-homogeneous component is any normal vector to the convex hull.

Next, we summarize some properties we will use later.

1.4.2 Properties

Lemma 3. *Let $f(\mathbf{x}) \in \mathcal{P}_k^{\mathbf{t}}$ and $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$. Then:*

- $\nabla f(E\mathbf{x}) = \varepsilon^k E^{-1} \nabla f(\mathbf{x})$.
- $D\mathbf{F}(E\mathbf{x}) = \varepsilon^k E D\mathbf{F}(\mathbf{x}) E^{-1}$. *Moreover, the j-th column of the matrix $D\mathbf{F}$ is a quasi-homogeneous vector field of type $\mathbf{t}$ and degree $k - t_j$.*

Lemma 4. *Let $\mathbf{F} \in \mathcal{Q}_l^{\mathbf{t}}$, $\mathbf{G} \in \mathcal{Q}_k^{\mathbf{t}}$ and $\mu \in \mathcal{P}_k^{\mathbf{t}}$. Then:*

- $[\mathbf{F}, \mathbf{G}] \in \mathcal{Q}_{k+l}^{\mathbf{t}}$.
- $\mu\mathbf{F} \in \mathcal{Q}_{k+l}^{\mathbf{t}}$.
- $\nabla\mu \cdot \mathbf{F} \in \mathcal{P}_{k+l}^{\mathbf{t}}$.
- $\operatorname{div}(\mathbf{F}) \in \mathcal{P}_l^{\mathbf{t}}$.
- $\nabla\mu \cdot \mathbf{D}_0 = k\,\mu$ *(Euler's Theorem), where we have introduced the radial quasi-homogeneous vector field of type $\mathbf{t}$ and degree zero:* $\mathbf{D}_0 = (t_1 x_1, t_2 x_2, \ldots, t_n x_n)^T$.

As a consequence of the above property, we obtain:

$$[\mathbf{F}, \mathbf{D}_0] = k\mathbf{F}, \quad \textit{for any} \quad \mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}. \tag{1.4.3}$$

In particular, if $\mathbf{F}_0 \in \mathcal{Q}_0^{\mathbf{t}}$ we get $[\mathbf{F}_0, \mathbf{D}_0] = 0$.

It is easy to show that

$$[\mu\mathbf{F}, \mathbf{G}] = (\nabla\mu \cdot \mathbf{G})\mathbf{F} + \mu\,[\mathbf{F}, \mathbf{G}], \tag{1.4.4}$$

for any smooth scalar function μ and vector fields $\mathbf{F}$, $\mathbf{G}$.

1.4.3 Normal Forms using $\mathcal{C}^\infty$-Conjugation

The normal form theory can be developed for vector fields expanded in quasi-homogeneous components of successive degrees of a given type $\mathbf{t}$. The key idea is summarized in the following commutative diagram:

$$\begin{array}{ccc}
\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) & \xrightarrow[\mathbf{x} = \mathbf{y} + \sum_{k\geq 1} \mathbf{P}_k(\mathbf{y})]{} & \dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}) \\
\Big\downarrow \mathbf{x} = E\mathbf{X} & & \Big\uparrow \mathbf{y} = E\mathbf{Y}, (\varepsilon = 1) \\
\dot{\mathbf{X}} = E^{-1}\mathbf{F}(E\mathbf{X}) = \sum_{j\geq r} \mathbf{F}_j(\mathbf{X})\varepsilon^j & \xrightarrow[\mathbf{X} = \mathbf{Y} + \sum_{k\geq 1} \mathbf{P}_k(\mathbf{Y})\varepsilon^k]{} & \dot{\mathbf{Y}} = \sum_{j\geq r} \mathbf{G}_j(\mathbf{Y})\varepsilon^j = E^{-1}\mathbf{G}(E\mathbf{Y})
\end{array}$$

To adapt the procedure to determine normal forms using quasi-homogeneous components of successive degrees, we first include a parameter ε by means of the rescaling $\mathbf{x} = E\mathbf{X}$, with $\mathbf{X} \in \mathbb{R}^n$. Then, we get the system $\dot{\mathbf{X}} = E^{-1}\mathbf{F}(E\mathbf{X})$. Expanding in ε, we can write this system in the form

$$\dot{\mathbf{X}} = \mathbf{F}_r(\mathbf{X})\varepsilon^r + \mathbf{F}_{r+1}(\mathbf{X})\varepsilon^{r+1} + \cdots, \tag{1.4.5}$$

where $\mathbf{F}_j \in \mathcal{Q}^{\mathbf{t}}_j$. Notice that, at $\varepsilon = 1$, we can expand the system (1.1.1) in terms respect the type $\mathbf{t}$:

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots. \tag{1.4.6}$$

Once this last system (1.4.5) has been reduced to normal form, it is enough to undo the rescaling by $\mathbf{y} = E\mathbf{Y}$, and set $\varepsilon = 1$ to recover the normal form.

We next describe how proceed to obtain a normal form for system (1.4.6). We will perform the near-identity transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$, where $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ with $k \geq 1$. The transformed system is

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}) = \Phi_* \mathbf{F}(\mathbf{y}) = (I + D\mathbf{P}_k(\mathbf{y}))^{-1} \sum_{i \geq r} \mathbf{F}_i \left(\mathbf{y} + \mathbf{P}_k(\mathbf{y})\right).$$

This transformed system can be expanded in quasi-homogeneous components:

$$\dot{\mathbf{y}} = \mathbf{G}_r(\mathbf{y}) + \mathbf{G}_{r+1}(\mathbf{y}) + \cdots, \tag{1.4.7}$$

being $\mathbf{G}_j(\mathbf{y}) \in \mathcal{Q}_j^{\mathbf{t}}$, for all $j \geq r$. The following result holds:

Proposition 5. *With the above notation:*

- $\mathbf{G}_j(\mathbf{y}) = \mathbf{F}_j(\mathbf{y})$, *for* $j = r, r+1, \ldots, r+k-1$,
- $\mathbf{G}_{r+k}(\mathbf{y}) = \mathbf{F}_{r+k}(\mathbf{y}) - (D\mathbf{P}_k(\mathbf{y})\mathbf{F}_r(\mathbf{y}) - D\mathbf{F}_r(\mathbf{y})\mathbf{P}_k(\mathbf{y}))$.

The above result suggests to define the homological operator:

$$\begin{array}{rccl} \mathbf{L}_k & : & \mathcal{Q}_k^{\mathbf{t}} & \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & & \mathbf{P}_k & \to \mathbf{L}_k(\mathbf{P}_k), \end{array}$$

where $\mathbf{L}_k(\mathbf{P}_k(\mathbf{y})) := D\mathbf{P}_k(\mathbf{y})\mathbf{F}_r(\mathbf{y}) - D\mathbf{F}_r(\mathbf{y})\mathbf{P}_k(\mathbf{y})$.

Notice that the homological operator only depends on the principal part $\mathbf{F}_r$, and can be expressed in terms of the Lie product: $\mathbf{L}_k(\mathbf{P}_k(\mathbf{y})) = [\mathbf{P}_k, \mathbf{F}_r](\mathbf{y})$. This expression shows that it is linear.

Proposition 5 affirms that the quasi-homogeneous components up to order $r + k - 1$ do not change, and the components of order $r + k$ in the transformed vector field are:

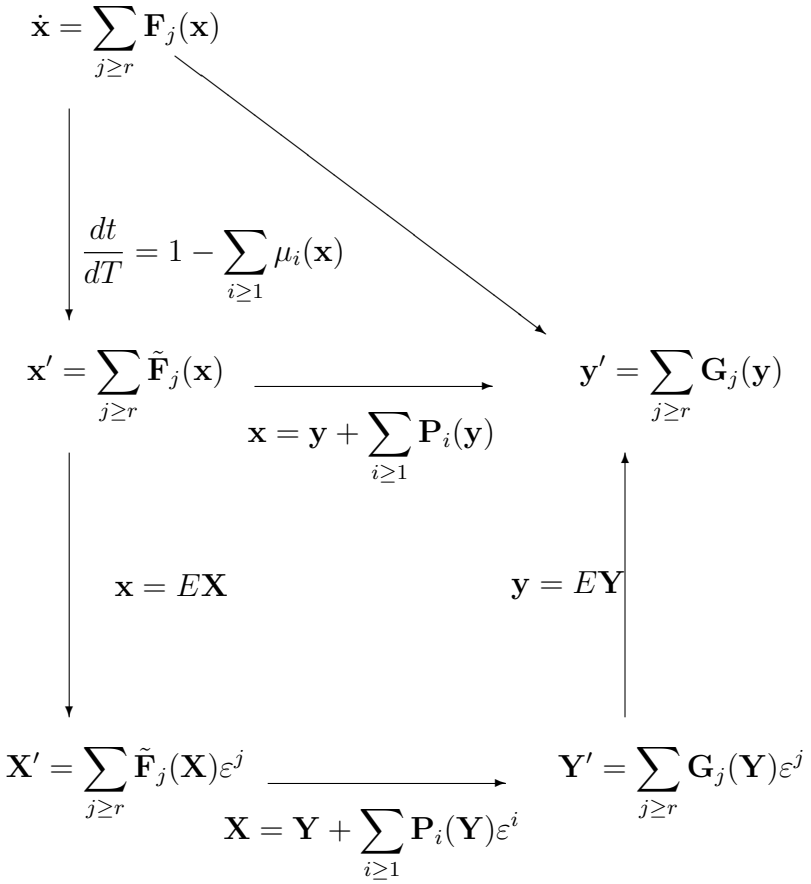
$$\mathbf{G}_{r+k} = \mathbf{F}_{r+k} - [\mathbf{P}_k, \mathbf{F}_r] = \mathbf{F}_{r+k} - \mathbf{L}_k(\mathbf{P}_k).$$

Following the same ideas that in the conventional normal form theory, it is possible to annihilate in these components the part belonging to the range of the linear operator $\mathbf{L}_k$, by selecting $\mathbf{P}_k$ adequately. So, the next result holds:

Theorem 6. *The system (1.4.6) is formally conjugate ($\mathcal{C}^\infty$-conjugate) to (1.4.7), where $\mathbf{G}_{r+k}$ belongs to a complementary subspace to the range of the linear operator $\mathbf{L}_k$, for all $k \geq 1$.*

1.4.4 Normal Forms using $\mathcal{C}^\infty$-Equivalence

Beyond the use of $\mathcal{C}^\infty$-conjugation (transformations in the state variables) we show the improvements that arise when we also use transformations in the time. As in the case of conjugation, our only hypothesis are based on the quasi-homogeneous principal part $\mathbf{F}_r$. The key idea is summarized in the following commutative diagram:



Let consider the system (1.4.6), and reparametrize the time by $\frac{dt}{dT} = 1 - \mu_k(\mathbf{x})$, with $\mu_k \in \mathcal{P}^{\mathbf{t}}_k$. We get the system

$$\tfrac{d\mathbf{x}}{dT} = \mathbf{x}' = \mathbf{F}_r(\mathbf{x}) + \cdots + \mathbf{F}_{r+k-1}(\mathbf{x}) + (\mathbf{F}_{r+k}(\mathbf{x}) - \mu_k(\mathbf{x})\,\mathbf{F}_r(\mathbf{x})) + \cdots,$$

From Lemma 4, we get that this system is expanded in quasi-homogeneous components of type $\mathbf{t}$ and successive degrees (as well as system (1.4.6)). Next, let consider a transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$ leading to normal form. Then, we obtain

$$\mathbf{y}' = \mathbf{G}_r(\mathbf{y}) + \cdots + \mathbf{G}_{r+k-1}(\mathbf{y}) + \mathbf{G}_{r+k}(\mathbf{y}) + \cdots, \tag{1.4.8}$$

where $\mathbf{G}_r = \mathbf{F}_r, \mathbf{G}_{r+1} = \mathbf{F}_{r+1}, \cdots, \mathbf{G}_{r+k-1} = \mathbf{F}_{r+k-1}$ and

$$\mathbf{G}_{r+k} = \mathbf{F}_{r+k} - \mu_k\,\mathbf{F}_r - [\mathbf{P}_k, \mathbf{F}_r] = \mathbf{F}_{r+k} - (\mu_k\,\mathbf{F}_r + \mathbf{L}_{r+k}(\mathbf{P}_k))\,.$$

This suggests to define the homological operator under $\mathcal{C}^\infty$-equivalence (orbital homological operator):

$$\begin{aligned}\mathcal{L}_k \;:\; & \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \mu_k) \to \mathcal{L}_k(\mathbf{P}_k, \mu_k) = \mu_k \, \mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r].\end{aligned}$$

To study this homological operator, we define the linear operator

$$\begin{aligned}M_k \;:\; & \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & \mu_k \to \mu_k \, \mathbf{F}_r.\end{aligned}$$

In this way, we can write:

$$\mathcal{L}_k(\mathbf{P}_k, \mu_k) = \mathbf{L}_k(\mathbf{P}_k) + M_k(\mu_k).$$

Using the linear operator:

$$\begin{aligned}\ell_{k-r} \;:\; & \mathcal{P}_{k-r}^{\mathbf{t}} \longrightarrow \mathcal{P}_k^{\mathbf{t}} \\ & f_{k-r} \to \nabla f_{k-r} \cdot \mathbf{F}_r,\end{aligned}$$

the following result holds:

Proposition 7.

(a) $\mathrm{Range}(\mathbf{L}_k) = \mathrm{Range}(M_k \circ \ell_{k-r}) + \mathbf{L}_k(\mathrm{Cor}(M_{k-r}))$,

(b) $\mathrm{Range}(M_k) = \mathrm{Range}(M_k \circ \ell_{k-r}) + M_k(\mathrm{Cor}(\ell_{k-r}))$,

where $\mathrm{Cor}(\cdot)$ *denotes any complementary subspace to the range* $\mathrm{Range}(\cdot)$ *of the corresponding linear operator.*

Proof. **(a)** Consider $\mathbf{P}_{r+k} \in \mathrm{Range}(\mathbf{L}_k)$. Then, there exists $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ such that $\mathbf{P}_{r+k} = [\mathbf{P}_k, \mathbf{F}_r]$. We can write $\mathbf{P}_k = \mathbf{P}_k^{\mathbf{r}} + \mathbf{P}_k^{\mathbf{c}}$, where $\mathbf{P}_k^{\mathbf{r}} \in \mathrm{Range}(M_{k-r})$ and $\mathbf{P}_k^{\mathbf{c}} \in \mathrm{Cor}(M_{k-r})$, because $\mathcal{Q}_k^{\mathbf{t}} = \mathrm{Range}(M_{k-r}) \bigoplus \mathrm{Cor}(M_{k-r})$. Let be $\mu_{k-r} \in \mathcal{P}_{k-r}^{\mathbf{t}}$ such that $\mathbf{P}_k^{\mathbf{r}} = \mu_{k-r} \, \mathbf{F}_r$. Then:

$$\mathbf{P}_{r+k} = [\mathbf{P}_k^{\mathbf{r}}, \mathbf{F}_r] + [\mathbf{P}_k^{\mathbf{c}}, \mathbf{F}_r] = [\mu_{k-r} \, \mathbf{F}_r, \mathbf{F}_r] + [\mathbf{P}_k^{\mathbf{c}}, \mathbf{F}_r].$$

Applying previous properties, we can write:

$$\begin{aligned}\mathbf{P}_{r+k} &= (\nabla \mu_{k-r} \cdot \mathbf{F}_r) \, \mathbf{F}_r + \mu_{k-r} \, [\mathbf{F}_r, \mathbf{F}_r] + [\mathbf{P}_k^{\mathbf{c}}, \mathbf{F}_r] \\ &= (\nabla \mu_{k-r} \cdot \mathbf{F}_r) \, \mathbf{F}_r + [\mathbf{P}_k^{\mathbf{c}}, \mathbf{F}_r] = M_k(\ell_{k-r}(\mu_{k-r})) + \mathbf{L}_k(\mathbf{P}_k^{\mathbf{c}}).\end{aligned}$$

(b) We are going to prove $\text{Range}(M_k) \subseteq \text{Range}(M_k \circ \ell_{k-r}) + M_k(\text{Cor}(\ell_{k-r}))$. The other inclusion is trivial.

Consider $\mathbf{P}_{r+k} \in \text{Range}(M_k)$. Then, there exists $\mu_k \in \mathcal{P}_k^{\mathtt{t}}$ such that $\mathbf{P}_{r+k} = \mu_k \, \mathbf{F}_r$.

Using that $\mathcal{P}_k^{\mathtt{t}} = \text{Range}(\ell_{k-r}) \bigoplus \text{Cor}(\ell_{k-r})$, we can decompose $\mu_k = \mu_k^{\mathtt{r}} + \mu_k^{\mathtt{c}}$, where $\mu_k^{\mathtt{r}} \in \text{Range}(\ell_{k-r})$ and $\mu_k^{\mathtt{c}} \in \text{Cor}(\ell_{k-r})$. Let be $\mu_{k-r} \in \mathcal{P}_{k-r}^{\mathtt{t}}$ such that $\mu_k^{\mathtt{r}} = \nabla \mu_{k-r} \cdot \mathbf{F}_r$. Then:

$$\begin{aligned} \mathbf{P}_{r+k} &= \mu_k \, \mathbf{F}_r = \mu_k^{\mathtt{r}} \, \mathbf{F}_r + \mu_k^{\mathtt{c}} \, \mathbf{F}_r = (\nabla \mu_{k-r} \cdot \mathbf{F}_r) \, \mathbf{F}_r + \mu_k^{\mathtt{c}} \, \mathbf{F}_r \\ &= M_k(\ell_{k-r}(\mu_{k-r})) + M_k(\mu_k^{\mathtt{c}}). \end{aligned}$$

□

The range of the homological operator under $\mathcal{C}^\infty$-equivalence:

$$\text{Range}(\mathcal{L}_k) = \text{Range}(\mathbf{L}_k) + \text{Range}(M_k),$$

can be determined from the above Proposition. However, the above sum is not direct, i.e., there are elements in $\mathcal{Q}_{r+k}^{\mathtt{t}}$ which can be annihilated with transformations in the state variables as well as in time. It is important to segregate the simplifications achieved by means of changes in the state variables with those that are consequence of the change in the time.

Proposition 8. $\text{Range}(\mathcal{L}_k) = M_k(\text{Cor}(\ell_{k-r})) + \text{Range}(\mathbf{L}_k)$. *This sum is not direct.*

Proof. Consider $\mathbf{P}_{r+k} \in \text{Range}(\mathcal{L}_k)$. So, there exist $\mathbf{P}_k \in \mathcal{Q}_k^{\mathtt{t}}$ and $\mu_k \in \mathcal{P}_k^{\mathtt{t}}$ such that $\mathbf{P}_{r+k} = \mu_k \, \mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r]$.

Taking into account that $\mathcal{P}_k^{\mathtt{t}} = \text{Range}(\ell_{k-r}) \bigoplus \text{Cor}(\ell_{k-r})$, we can write $\mu_k = \mu_k^{\mathtt{r}} + \mu_k^{\mathtt{c}}$ where $\mu_k^{\mathtt{r}} \in \text{Range}(\ell_{k-r})$ and $\mu_k^{\mathtt{c}} \in \text{Cor}(\ell_{k-r})$. Let be $\mu_{k-r} \in \mathcal{P}_{k-r}^{\mathtt{t}}$ such that $\ell_{k-r}(\mu_{k-r}) = \nabla \mu_{k-r} \cdot \mathbf{F}_r = \mu_k^{\mathtt{r}}$. Then, it is possible to write:

$$\begin{aligned} \mathbf{P}_{r+k} &= \mu_k^{\mathtt{c}} \, \mathbf{F}_r + (\nabla \mu_{k-r} \cdot \mathbf{F}_r) \, \mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r] \\ &= \mu_k^{\mathtt{c}} \, \mathbf{F}_r + \mu_{k-r} \, [\mathbf{F}_r, \mathbf{F}_r] + [\mu_{k-r} \mathbf{F}_r, \mathbf{F}_r] + [\mathbf{P}_k, \mathbf{F}_r] \\ &= \mu_k^{\mathtt{c}} \, \mathbf{F}_r + [\mathbf{P}_k + \mu_{k-r} \, \mathbf{F}_r, \mathbf{F}_r] = M_k(\mu_k^{\mathtt{c}}) + \mathbf{L}_k(\mathbf{P}_k + \mu_{k-r} \, \mathbf{F}_r). \end{aligned}$$

Therefore, $\text{Range}(\mathcal{L}_k) \subseteq M_k(\text{Cor}(\ell_{k-r})) + \text{Range}(\mathbf{L}_k)$.

Analogously, it is possible to prove the other inclusion. □

Proposition 9. *If* $\eta_{k-r} \in \text{Ker}\,(\ell_{k-r})$ *then* $(\eta_{k-r}\mathbf{F}_r, 0) \in \text{Ker}\,(\mathcal{L}_k)$.

Proof. By (1.4.4)

$$\mathcal{L}\,((\eta_{k-r}\mathbf{F}_r, 0)) = [\eta_{k-r}\mathbf{F}_r, \mathbf{F}_r] = (\nabla \eta_{k-r} \cdot \mathbf{F}_r) \, \mathbf{F}_r + \eta_{k-r} \, [\mathbf{F}_r, \mathbf{F}_r] = 0$$

□

From Proposition 8, we deduce that, in the analysis of the simplifications provided by the reparametrizations in the time, it is enough to take $\mu_k \in \mathrm{Cor}(\ell_{k-r})$. If we include other terms in μ_k (which belong to $\mathrm{Range}(\ell_{k-r})$), they only have effect on $\mathrm{Range}(\mathbf{L}_k)$, and these terms can be simplified using $\mathcal{C}^\infty$-conjugation (i.e., we can reach these simplifications without using μ_k). This restriction on the way to select μ_k will facilitate greatly the analysis.

Moreover, if we denote by $\widehat{\mathcal{Q}}_k^{\mathbf{t}}$ a complementary subspace to $\mathrm{Ker}\,(\ell_{k-r})\,\mathbf{F}_r$ in $\mathcal{Q}_k^{\mathbf{t}}$, that is

$$\mathcal{Q}_k^{\mathbf{t}} \;=\; \mathrm{Ker}\,(\ell_{k-r})\,\mathbf{F}_r \bigoplus \widehat{\mathcal{Q}}_k^{\mathbf{t}},$$

from Proposition 9, a vector field $\mathbf{P}_k \in \mathrm{Ker}\,(\ell_{k-r})\,\mathbf{F}_r$ has no effect on $\mathrm{Range}\,(\mathbf{L}_k)$.

In this way, we modify the definition of the homological operator under $\mathcal{C}^\infty$-equivalence as

$$\begin{aligned} \mathcal{L}_k \;:\; & \widehat{\mathcal{Q}}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_{k-r}) \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \mu_k) \to \mathcal{L}_k(\mathbf{P}_k, \mu_k) = \mu_k\,\mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r]. \end{aligned} \tag{1.4.9}$$

Moreover, it is possible to annihilate in each quasi-homogeneous component of $\mathbf{F}$, the part belonging to the range of the linear operator $\mathcal{L}_k$, by selecting $(\mathbf{P}_k, \mu_k)$ adequately. Then, we have:

Theorem 10. *The system (1.4.6) is formally orbital equivalent ($\mathcal{C}^\infty$-equivalent) to (1.4.8), where $\mathbf{G}_{r+k}$ belongs to a complementary subspace to the range of the linear operator $\mathcal{L}_k$, for all $k \geq 1$.*

1.5 The Lie Transformations

In this section we present a recursive procedure, which can be organized in a Lie triangle, to compute normal forms under $\mathcal{C}^\infty$-conjugation and under $\mathcal{C}^\infty$-equivalence. Recently, many researchers have paid attention to the computation of the normal forms and efficient computation methods have been developed using computer algebra systems. This method, presented in [4], is based on Lie transforms. For a complete introduction to this form of Lie theory see [76]. The advantage of the recursive procedure based on the Lie transforms is that we have a precise management of the effect of the change of variables in terms of order greater than $k + r$.

1.5.1 The Lie Triangle under $\mathcal{C}^\infty$-conjugation

Consider a vector field $\mathbf{U} \in \mathcal{C}^\infty$ in a neighborhood of the origin such that $\mathbf{U}(0) = 0$. We apply to system (1.4.6) the change of variables $\mathbf{x} = \mathbf{u}(\mathbf{y}, \varepsilon)$, where $\mathbf{u}$ is the

unique solution of the initial value problem

$$\frac{\partial}{\partial\varepsilon}\mathbf{u}(\mathbf{y},\varepsilon)=\mathbf{U}(\mathbf{u}(\mathbf{y},\varepsilon)),\quad \mathbf{u}(\mathbf{y},0)=\mathbf{y}. \tag{1.5.1}$$

The vector field $\mathbf{U}$ is called the *generator* of the change.

The basic idea of this approach is that we can determine the transformed vector field from $\mathbf{F}$ and $\mathbf{U}$ without knowing $\mathbf{u}$, that is without knowing explicitly the change of variables.

If we use a generator, we denote $\mathbf{U}_{**}\mathbf{F} := \Phi_*\mathbf{F}$. In fact, the relation between the change of variables Φ and the generator $\mathbf{U}$ is given by

$$\Phi(\mathbf{x})=\mathbf{x}+\mathbf{U}(\mathbf{x})+\frac{1}{2!}D\mathbf{U}(\mathbf{x})\,\mathbf{U}(\mathbf{x})+\cdots. \tag{1.5.2}$$

The transformed system can be expressed in terms of nested Lie products.

$$\mathbf{G}=\mathbf{U}_{**}\mathbf{F}=\mathbf{F}+[\mathbf{F},\mathbf{U}]+\frac{1}{2!}\left[[\mathbf{F},\mathbf{U}],\mathbf{U}\right]+\frac{1}{3!}\left[[[\mathbf{F},\mathbf{U}],\mathbf{U}],\mathbf{U}\right]+\cdots.$$

Defining by induction the following operators:

$$T^{(0)}_{\mathbf{U}}(\mathbf{F}):=\mathbf{F},\quad\text{and}\quad T^{(l)}_{\mathbf{U}}(\mathbf{F}):=T^{(l-1)}_{\mathbf{U}}([\mathbf{F},\mathbf{U}]),\quad\text{for } l\geq 1,$$

we have:

$$\mathbf{U}_{**}(\mathbf{F})=\sum_{l=0}^{\infty}\frac{1}{l!}T^{(l)}_{\mathbf{U}}(\mathbf{F}).$$

Observe that $T^{(l)}_{\mathbf{U}}(\mathbf{F})=\overbrace{[\cdots[}^{l\text{ times}}\mathbf{F},\mathbf{U}],\cdots,\mathbf{U}]$.

The details of the above expressions can be found, e.g., in [4, 47] (see also references therein).

Then, the quasi-homogeneous term of order $r+k$ in the transformed vector field can be obtained as:

$$\begin{aligned}\mathbf{G}_{r+k} &= \mathbf{F}_{r+k}+[\mathbf{F},\mathbf{U}]_{r+k}+\frac{1}{2!}[[\mathbf{F},\mathbf{U}],\mathbf{U}]_{r+k} \\ &\quad+\frac{1}{3!}\left[[[\mathbf{F},\mathbf{U}],\mathbf{U}],\mathbf{U}\right]_{r+k}+\cdots,\end{aligned} \tag{1.5.3}$$

where $[\mathbf{F},\mathbf{U}]_{r+k}$ denotes the quasi-homogeneous component of degree $r+k$ in $[\mathbf{F},\mathbf{U}]$.

Assuming an expansion of the generator $\mathbf{U}=\mathbf{U}_1+\mathbf{U}_2+\cdots$ in quasi-homogeneous components, we get:

$$\begin{aligned}[\mathbf{F},\mathbf{U}]_{r+k} &= \sum_{j=1}^{k}[\mathbf{F}_{r+k-j},\mathbf{U}_j] \\ [[\mathbf{F},\mathbf{U}],\mathbf{U}]_{r+k} &= \sum_{j=1}^{k-1}[[\mathbf{F},\mathbf{U}]_{r+k-j},\mathbf{U}_j].\end{aligned}$$

To obtain $\mathbf{G}_{r+k}$ it is enough to build the sequence of vector fields $\{\mathbf{V}_{r+k,l}\}$ defined recursively by:

$$\begin{aligned}\mathbf{V}_{r+k,0} &= \mathbf{F}_{r+k} \quad k \geq 0,\\ \mathbf{V}_{r+k,l} &= \sum_{j=1}^{k-l+1} [\mathbf{V}_{r+k-j,l-1}\,, \mathbf{U}_j]\,, \quad 1 \leq l \leq k.\end{aligned}$$

In fact,

$$\mathbf{G}_{r+k} = \sum_{j=0}^{k} \frac{1}{j!}\mathbf{V}_{r+k,j} = \mathbf{F}_{r+k} + \sum_{l=1}^{k} \frac{1}{l!}\mathbf{V}_{r+k,l}.$$

The computation of the above sequence can be accomplished by means of the Lie triangle:

$\mathbf{V}_{r,0}$					
$\mathbf{V}_{r+1,0}$	$\mathbf{V}_{r+1,1}$				
$\mathbf{V}_{r+2,0}$	$\mathbf{V}_{r+2,1}$	$\mathbf{V}_{r+2,2}$			
$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$\mathbf{V}_{r+k,0}$	$\mathbf{V}_{r+k,1}$	$\mathbf{V}_{r+k,2}$	$\cdots$	$\mathbf{V}_{r+k,k}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(1.5.4)

where each row can be computed from the previous rows. Note that the first column of the triangle contains the original system, and each column indicates the number of nested Lie brackets required (in the first column the vector field appears, in the second single Lie brackets, in the third double Lie brackets, and so on).

Moreover, the jth element in the kth row depends on the elements of the $(j-1)$th column and rows $1, 2, \ldots, k-1$, and $\mathbf{G}_{r+k}$ is obtained by summing all the elements in the kth row (dividing by the corresponding factorial).

The next result, which is easy to prove, states that each row deals with a quasi-homogeneous degree:

Proposition 11. $\mathbf{V}_{r+k,l} \in \mathcal{Q}^{\mathbf{t}}_{r+k}$, *for all* $l = 0, \ldots, k$. *In particular,* $\mathbf{G}_{r+k} \in \mathcal{Q}^{\mathbf{t}}_{r+k}$.

1.5.2 The Lie Triangle under $\mathcal{C}^\infty$-equivalence

To calculate the normal form under $\mathcal{C}^\infty$-equivalence firstly we reparametrize the time in system (1.4.6) by

$$\frac{dt}{dT} = 1 - \sum_{j\geq 1} \mu_j(\mathbf{x}), \quad \text{with} \quad \mu_j \in \mathrm{Cor}\,(\ell_{j-r})\,.$$

Then, we get

$$\mathbf{x}' = \frac{d\mathbf{x}}{dT} = \sum_{k\geq 0} \left(\mathbf{F}_{r+k}(\mathbf{x}) - \sum_{j=1}^{k} \mu_j \, \mathbf{F}_{r+k-j} \right).$$

If the time-transformation is $dt = (1+\mu)dT$ with $\mu = \sum_{j\geq 1} \mu_j$, $\mu_j \in \mathcal{P}^{\mathbf{t}}_j$, then the transformed vector field is given by:

$$\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right) = \sum_{l=0}^{\infty} \frac{1}{l!} T^{(l)}_{\mathbf{U}} \left((1+\mu)\mathbf{F}\right).$$

If we separate the first three terms from the sum of the above formula, we can also write

$$\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right) = \mathbf{F} + [\mathbf{F},\mathbf{U}] + \mu\mathbf{F} + \sum_{l=1}^{\infty} \frac{1}{(l+1)!} T^{(l)}_{\mathbf{U}} \left([\mathbf{F},\mathbf{U}] + (l+1)\mu\mathbf{F}\right). \quad (1.5.5)$$

Now, consider the Lie triangle for this system, where the elements are

$$\begin{aligned}
\mathbf{V}_{r+k,0} &= \mathbf{F}_{r+k} - \sum_{j=1}^{k} \mu_j \, \mathbf{F}_{r+k-j} = \mathbf{W}_{r+k,0} - \mu_k \, \mathbf{F}_r, \\
&\quad \text{where} \quad \mathbf{W}_{r+k,0} = \mathbf{F}_{r+k} - \sum_{j=1}^{k-1} \mu_j \mathbf{F}_{r+k-j}, \\
\mathbf{V}_{r+k,1} &= \sum_{j=1}^{k} [\mathbf{V}_{r+k-j,0}, \mathbf{U}_j] = \mathbf{W}_{r+k,1} + [\mathbf{F}_r, \mathbf{U}_k], \\
&\quad \text{where} \quad \mathbf{W}_{r+k,1} = \sum_{j=1}^{k-1} [\mathbf{V}_{r+k-j,0}, \mathbf{U}_j], \\
\mathbf{V}_{r+k,l} &= \mathbf{W}_{r+k,l} = \sum_{j=1}^{k-l+1} [\mathbf{V}_{r+k-j,l-1}, \mathbf{U}_j], \quad (l \geq 2).
\end{aligned}$$

In particular, we can write

$$\mathbf{G}_{r+k} = \sum_{j=0}^{k} \frac{1}{j!} \mathbf{W}_{r+k,j} - \mu_k \, \mathbf{F}_r - [\mathbf{U}_k, \mathbf{F}_r] = \sum_{j=0}^{k} \frac{1}{j!} \mathbf{W}_{r+k,j} - \mathcal{L}_k \left(\mathbf{U}_k, \mu_k\right),$$

so that we have segregated in $\mathbf{G}_{r+k}$ the term $\mathcal{L}_k \left(\mathbf{U}_k, \mu_k\right)$, and we can select $\mathbf{U}_k, \mu_k$ in order to simplify the quasi-homogeneous terms of degree $r+k$.

The ideas above provide an algorithm well-suited for symbolic computation. This consists in building the sequence $\mathbf{W}_{r+k,l}$ in each row first of all, which provides the

homological equation under equivalence and the quasi-homogeneous normal form of degree $r+k$. Once the homological equation is solved, we can recover the sequence $\mathbf{V}_{r+k,l}$, by adding the terms segregated in $\mathbf{V}_{r+k,0}$, $\mathbf{V}_{r+k,1}$. The main steps in the algorithm are summarized below:

For $k \geq 1$:

Define

$$\begin{aligned}
\mathbf{W}_{r+k,0} &= \mathbf{F}_{r+k} - \sum_{j=1}^{k-1} \mu_j \, \mathbf{F}_{r+k-j}, \\
\mathbf{W}_{r+k,1} &= \sum_{j=1}^{k-1} \left[\mathbf{W}_{r+k-j,0}, \mathbf{U}_j\right], \\
\mathbf{W}_{r+k,l} &= \sum_{j=1}^{k-l+1} \left[\mathbf{W}_{r+k-j,l-1}, \mathbf{U}_j\right], \quad l = 2, \ldots, k.
\end{aligned}$$

Compute the quasi-homogeneous component of the normal form of degree $r+k$, by:

$$\mathbf{G}_{r+k} = \mathrm{Proy}_{\mathrm{Cor}(\mathcal{L}_k)} \left(\sum_{l=0}^{k} \frac{1}{l!} \mathbf{W}_{r+k,l} \right),$$

where $\mathrm{Cor}\left(\mathcal{L}_k\right)$ denotes a complementary subspace to the range of $\mathcal{L}_k$, selected previously.

Obtain $\mathbf{U}_k$, μ_k, from the homological equation under equivalence:

$$\mathcal{L}_k \left(\mathbf{U}_k, \mu_k\right) = \mathrm{Proy}_{\mathrm{Range}(\mathcal{L}_k)} \left(\sum_{l=0}^{k} \frac{1}{l!} \mathbf{W}_{r+k,l} \right). \tag{1.5.6}$$

Update:

$$\begin{aligned}
\mathbf{V}_{r+k,0} &= \mathbf{W}_{r+k,0} - \mu_k \, \mathbf{F}_r, \\
\mathbf{V}_{r+k,1} &= \mathbf{W}_{r+k,1} + [\mathbf{F}_r, \mathbf{U}_k], \\
\mathbf{V}_{r+k,l} &= \mathbf{W}_{r+k,l}, \quad l = 2, \ldots, k.
\end{aligned}$$

As in the classical normal form theory, the procedure sometimes allows further simplifications, by assuming adequate hypothesis in the quasi-homogeneous higher-order terms. In fact, if $\mathrm{Ker}\left(\mathcal{L}_k\right) \neq \{0\}$, then the homological equation (1.5.6) has a non-unique solution (determined up to an additive term in $\mathrm{Ker}\left(\mathcal{L}_k\right)$). So, a number of arbitrary constants will appear in the subsequent rows of the Lie triangle and can be selected in order to achieve some simplifications in the higher-order quasi-homogeneous terms.

Chapter 2

Unique Orbital Normal Form

2.1 Introduction

In this chapter we analyze the problem of determining the simplest analytical expression that can be obtained for a given autonomous system around an equilibrium point through transformations in the state and time variables, which is referred for some authors as unique orbital normal form. Our results show that such highest simplifications can always be achieved by means of linear procedures. It is important to remark that we use quasi-homogeneous expansions of the vector field to obtain the simplest normal forms, generalizing the common Taylor expansions.

This simplest orbital normal form is of primary importance in several problems of great interest in the qualitative theory of dynamical systems. Namely, it is the starting point in the study of local bifurcations in the saddle-node–Hopf and Hopf–Hopf bifurcations (see [3, 9, 64]), as well as in their nonlinear degenerate cases. Moreover, it is of primary importance in the case of the triple-zero degeneracy (see [10]). The knowledge of the unique classical normal form (and orbital normal form) allows to determine the invariants of the vector field. In fact, the unique classical normal form plays a central role in the nondegenerate center problem. Also, it is well-know that a nondegenerate planar system has a center at the origin if, and only if, it is reversible. Nevertheless, the degenerate center problem requires the use of equivalence: in [35] it is proved that a nilpotent planar system has a center at the origin if, and only if, it is reversible modulo equivalence, and we will see later that this occurs if, and only if, there exists a time-reversible unique orbital normal form.

All the results presented here must be understood in a formal sense, because we will do not address the convergence problem for the normal forms.

Along this chapter, we consider the smooth autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \text{with} \quad \mathbf{x} \in \mathbb{R}^n, \tag{2.1.1}$$

having an equilibrium point at the origin (i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$). And, as we described

in Chapter 1, the vector field of (2.1.1) can be always written as the sum of quasi-homogeneous terms of type $\mathbf{t}$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots , \tag{2.1.2}$$

where $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ for all k, and we assume that $\mathbf{F}_r \neq 0$ being $r \in \mathbb{Z}$.

2.2 Orbital Normal Form

In Chapter 1, we have shown that if system $\mathbf{y}' = \mathbf{G}(\mathbf{y})$ is obtained after the change of variables $\mathbf{x} = \Phi(\mathbf{y})$ and the time-transformation $dt = (1+\mu)dT$, $\mu \in \bigoplus_{i\geq 1} \mathcal{P}_i^{\mathbf{t}}$, then there exists a space generator $\mathbf{U} \in \bigoplus_{i\geq 1} \mathcal{Q}^{\mathbf{t}}$ such that

$$\mathbf{G} = \mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right).$$

In this section, we will prove that it is possible to obtain the vector field $\mathbf{G}$ using space and time generators belonging to subspaces of $\bigoplus_{i\geq 1} \mathcal{Q}^{\mathbf{t}}$ and $\bigoplus_{i\geq 1} \mathcal{P}_i^{\mathbf{t}}$, respectively.

We introduce the Lie derivate along the principal part $\mathbf{F}_r$:

$$\begin{array}{rcl} \ell_{k-r} & : & \mathcal{P}_{k-r}^{\mathbf{t}} \longrightarrow \mathcal{P}_k^{\mathbf{t}} \\ & & \mu_{k-r} \longrightarrow \nabla\mu_{k-r} \cdot \mathbf{F}_r. \end{array}$$

The importance of this linear operator lies in its role for selecting the time reparametrization μ. Namely, in the normal form reduction it is enough to take its quasi-homogeneous terms μ_k belonging to $\mathrm{Cor}\,(\ell_{k-r})$ (a complementary subspace to $\mathrm{Range}\,(\ell_{k-r})$).

The first lemma has been already proved in [24].

Lemma 12. *Let us consider a couple of generators:* $\mathbf{U} = \sum_{j\geq 1} \mathbf{U}_j \in \bigoplus_{j\geq 1} \mathcal{Q}_j^{\mathbf{t}}$ *and* $\mathbf{V} = \sum_{j\geq k} \mathbf{V}_j \in \bigoplus_{j\geq k} \mathcal{Q}_j^{\mathbf{t}}$ *($k \in \mathbb{N}$), corresponding to near-identity transformations* Φ *and* Ψ*, respectively. If* $\mathbf{W} = \sum_{j\geq 1} \mathbf{W}_j \in \bigoplus_{j\geq 1} \mathcal{Q}_j^{\mathbf{t}}$ *is a generator of* $\Psi \circ \Phi$*, then* $\mathbf{U}$ *and* $\mathbf{W}$ *agree up to quasi-homogeneous degree* $k-1$ *(i.e.,* $\mathcal{J}^{k-1}(\mathbf{U}) = \mathcal{J}^{k-1}(\mathbf{W})$*), and*

$$\mathbf{W}_k = \mathbf{U}_k + \mathbf{V}_k.$$

Lemma 13. *Let us consider* $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$ *($k \in \mathbb{N}$) and* $\mathbf{U} \in \bigoplus_{j\geq 1} \mathcal{Q}_j^{\mathbf{t}}$*. Then, there exists* $\lambda = \sum_{j\geq k} \lambda_j \in \bigoplus_{j\geq k} \mathcal{P}_j^{\mathbf{t}}$*, such that:*

$$(1+\mu_k)\left(\mathbf{U}_{**}\mathbf{F}\right) = \mathbf{U}_{**}\left((1+\lambda)\mathbf{F}\right),$$

where the lowest-degree quasi-homogeneous term of λ *is* $\lambda_k = \mu_k$*.*

Proof. Let us denote the successive Lie derivatives along the vector field $\mathbf{U}$ of the scalar function μ_k by $\mu_k^{[j]}$, i.e.:

$$\mu_k^{[0]} = \mu_k, \quad \text{and} \quad \mu_k^{[j]} = \nabla \mu_k^{[j-1]} \cdot \mathbf{U}, \quad \text{for} \quad j \geq 1.$$

Using induction, we can prove that

$$\mu_k T_{\mathbf{U}}^{(l)}(\mathbf{F}) = \sum_{j=0}^{l} \binom{l}{j} (-1)^j T_{\mathbf{U}}^{(l-j)} \left(\mu_k^{[j]} \mathbf{F}\right), \quad \text{for all} \quad l \geq 0.$$

Then,

$$\begin{aligned}
(1+\mu_k)(\mathbf{U} *_{**} \mathbf{F}) &= \sum_{l=0}^{\infty} \frac{1}{l!} T_{\mathbf{U}}^{(l)}(\mathbf{F}) + \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} (-1)^j T_{\mathbf{U}}^{(l-j)} \left(\mu_k^{[j]} \mathbf{F}\right) \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} T_{\mathbf{U}}^{(l)}(\mathbf{F}) + \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{1}{j!l!} (-1)^j T_{\mathbf{U}}^{(l)} \left(\mu_k^{[j]} \mathbf{F}\right) \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} T_{\mathbf{U}}^{(l)} \left(\left(1 + \sum_{j=0}^{l} \frac{1}{j!} (-1)^j \mu_k^{[j]}\right) \mathbf{F} \right) = \mathbf{U} *_{**} ((1+\lambda)\mathbf{F}),
\end{aligned}$$

as claimed, where $\lambda := \sum_{j=0}^{l} \frac{1}{j!} (-1)^j \mu_k^{[j]}$.

Moreover, it is easy to show that $\lambda_k = \mu_k$. □

Lemma 14. *Let us consider $\rho_m \in \mathcal{P}_m^{\mathbf{t}}$ ($m \in \mathbb{N}$) and $\mu \in \bigoplus_{j \geq 1} \mathcal{P}_j^{\mathbf{t}}$. Then, there exists $\eta = \sum_{j \geq 1} \eta_j \in \bigoplus_{j \geq 1} \mathcal{P}_j^{\mathbf{t}}$, such that:*

$$(\rho_m \mathbf{F}) *_{**} ((1+\mu)\mathbf{F}) = (1+\eta)\mathbf{F},$$

where η and μ agree up to quasi-homogeneous degree $m+r-1$ (i.e., $\mathcal{J}^{m+r-1}(\eta) = \mathcal{J}^{m+r-1}(\mu)$), and $\eta_{m+r} = \mu_{m+r} - \nabla \rho_m \cdot \mathbf{F}_r$.

Proof. Firstly, we will show using induction that, for each $l \in \mathbb{N}$, there exists $\eta^{(l)} \in \bigoplus_{j \geq l(m+r)} \mathcal{P}_j^{\mathbf{t}}$ satisfying:

$$T_{\rho_m \mathbf{F}}^{(l)}((1+\mu)\mathbf{F}) = \eta^{(l)} \mathbf{F}, \tag{2.2.1}$$

where $\eta_{m+r}^{(1)} = -\nabla \rho_m \cdot \mathbf{F}_r$.

Namely, for $l = 1$, from (1.4.4), i.e. $[\mu \mathbf{F}, \mathbf{G}] = (\nabla \mu \cdot \mathbf{G})\mathbf{F} + \mu [\mathbf{F}, \mathbf{G}]$, we deduce:

$$T_{\rho_m \mathbf{F}}((1+\mu)\mathbf{F}) = [(1+\mu)\mathbf{F}, \rho_m \mathbf{F}] = (\rho_m (\nabla \mu \cdot \mathbf{F}) - (1+\mu)(\nabla \rho_m \cdot \mathbf{F}))\mathbf{F},$$

and it is enough to take $\eta^{(1)} := \rho_m (\nabla \mu \cdot \mathbf{F}) - (1+\mu)(\nabla \rho_m \cdot \mathbf{F})$. Equating quasi-homogeneous terms, we get $\eta_i^{(1)} = 0$ for $i = 1, \cdots, m+r-1$, and $\eta_{m+r}^{(1)} = -\nabla \rho_m \cdot \mathbf{F}_r$.

Next, we assume that (2.2.1) holds for $l-1$. From (1.4.4), we obtain:

$$T^{(l)}_{\rho_m\mathbf{F}}\left((1+\mu)\mathbf{F}\right) = \left[T^{(l-1)}_{\rho_m\mathbf{F}}\left((1+\mu)\mathbf{F}\right), \rho_m\mathbf{F}\right] = \left[\eta^{(l-1)}\mathbf{F}, \rho_m\mathbf{F}\right] = \eta^{(l)}\mathbf{F},$$

where we have denoted $\eta^{(l)} := \rho_m\left(\nabla\eta^{(l-1)}\cdot\mathbf{F}\right) - \eta^{(l-1)}\left(\nabla\rho_m\cdot\mathbf{F}\right)$. As $\eta^{(l-1)} \in \bigoplus_{j\geq(l-1)(m+r)}\mathcal{P}^{\mathbf{t}}_j$, it is easy to show that $\eta^{(l)} \in \bigoplus_{j\geq l(m+r)}\mathcal{P}^{\mathbf{t}}_j$. This proves the equality (2.2.1).

From this equality, we obtain:

$$\begin{aligned}(\rho_m\mathbf{F}) *_{**} ((1+\mu)\mathbf{F}) &= \textstyle\sum_{l=0}^{\infty}\frac{1}{l!}T^{(l)}_{\rho_m\mathbf{F}}\left((1+\mu)\mathbf{F}\right) = (1+\mu)\mathbf{F} + \sum_{l=0}^{\infty}\frac{1}{l!}\eta^{(l)}\mathbf{F}\\ &= \left(1+\mu+\textstyle\sum_{l=1}^{\infty}\frac{1}{l!}\eta^{(l)}\right)\mathbf{F} = (1+\eta)\mathbf{F},\end{aligned}$$

where we have introduced $\eta := \mu + \displaystyle\sum_{l\geq1}\frac{1}{l!}\eta^{(l)}$. □

Next Theorem is useful in the applications, because it states that the computation of normal forms under equivalence can be achieved using elements in some subspaces of $\mathcal{Q}^{\mathbf{t}}_j$ and $\mathcal{P}^{t}_j$. In practice, this fact is of primary interest because it allows a drastic reduction of the computational effort.

Theorem 15. *Let us consider the vector field* $\mathbf{F}$ *given in (2.1.2),* $\mathbf{U} \in \bigoplus_{j\geq1}\mathcal{Q}^{\mathbf{t}}_j$ *and* $\mu \in \bigoplus_{j\geq1}\mathcal{P}^{\mathbf{t}}_j$*. Then, there exist* $\widehat{\mathbf{U}} \in \bigoplus_{j\geq1}\widehat{\mathcal{Q}}^{\mathbf{t}}_j$ *and* $\widehat{\mu} \in \bigoplus_{j\geq1}\mathrm{Cor}\left(\ell_{j-r}\right)$ *such that*

$$\mathbf{U} *_{**} ((1+\mu)\mathbf{F}) = \widehat{\mathbf{U}} *_{**} ((1+\widehat{\mu})\mathbf{F}).$$

Proof. Let us denote

$$\kappa = \min\left\{k\in\mathbb{N} : \mathrm{Proy}_{\mathrm{Ker}(\ell_{k-r})\mathbf{F}_r}\left(\mathbf{U}_k\right) \neq \mathbf{0}, \quad \text{or} \quad \mathrm{Proy}_{\mathrm{Range}(\ell_{k-r})}\left(\mu_k\right) \neq 0\right\}.$$

If $\kappa=\infty$, then $\mathbf{U} = \widehat{\mathbf{U}} \in \bigoplus_{j\geq1}\widehat{\mathcal{Q}}^{\mathbf{t}}_j$ and $\mu = \widehat{\mu} \in \bigoplus_{j\geq1}\mathrm{Cor}\left(\ell_{j-r}\right)$, and the result holds trivially.

Let us assume $\kappa<\infty$. We observe that the definition of κ implies that $\mathbf{U}_k \in \widehat{\mathcal{Q}}_k$ and $\mu_k \in \mathrm{Cor}\left(\ell_{k-r}\right)$ for $k=1,\ldots,\kappa-1$. Our goal is to show that the result is true for the κ–degree quasi-homogeneous terms. Finally, it is enough to repeat the procedure successively to get the result.

As $\mathcal{Q}^{\mathbf{t}}_\kappa = \mathrm{Ker}\left(\ell_{\kappa-r}\right)\mathbf{F}_r \bigoplus \widehat{\mathcal{Q}}^{\mathbf{t}}_\kappa$, we can write $\mathbf{U}_\kappa = \mathbf{U}^{(1)}_\kappa + \mathbf{U}^{(2)}_\kappa$, with $\mathbf{U}^{(1)}_\kappa = \rho_{\kappa-r}\mathbf{F}_r \in \mathrm{Ker}\left(\ell_{\kappa-r}\right)\mathbf{F}_r$ (for some $\rho_{\kappa-r} \in \mathrm{Ker}\left(\ell_{\kappa-r}\right)$) and $\mathbf{U}^{(2)}_\kappa \in \widehat{\mathcal{Q}}^{\mathbf{t}}_\kappa$.

Moreover, since $\mathcal{P}^{\mathbf{t}}_\kappa = \mathrm{Range}\left(\ell_{\kappa-r}\right) \bigoplus \mathrm{Cor}\left(\ell_{\kappa-r}\right)$, we can write $\mu_\kappa = \mu^{(1)}_\kappa + \mu^{(2)}_\kappa$, where $\mu^{(1)}_\kappa = \nabla\eta_{\kappa-r}\cdot\mathbf{F}_r \in \mathrm{Range}\left(\ell_{\kappa-r}\right)$ (for some $\eta_{\kappa-r} \in \mathcal{P}^{\mathbf{t}}_{\kappa-r}$) and $\mu^{(2)}_\kappa \in \mathrm{Cor}\left(\ell_{\kappa-r}\right)$. We note that $\eta_{\kappa-r} \in \mathrm{Ker}\left(\ell_{\kappa-r}\right)$ if $\mu^{(1)}_\kappa = 0$, otherwise $\eta_{\kappa-r} \in \mathcal{P}^{\mathbf{t}}_{\kappa-r} \setminus \mathrm{Ker}\left(\ell_{\kappa-r}\right)$.

Let us denote by Φ and Ψ the transformations generated by $\mathbf{U}$ and $(\rho_{\kappa-r}+\eta_{\kappa-r})\mathbf{F}$ respectively, and consider a generator $\widehat{\mathbf{U}}$ of the transformation $\Phi\circ\Psi^{-1}$. Then:

$$\begin{aligned}\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right) &= \left(\Phi\circ\Psi^{-1}\circ\Psi\right)_{*}\left((1+\mu)\mathbf{F}\right)=\left(\Phi\circ\Psi^{-1}\right)_{*}\left(\Psi_{*}\left((1+\mu)\mathbf{F}\right)\right)\\ &= \widehat{\mathbf{U}}_{**}\left(\left((\rho_{\kappa-r}+\eta_{\kappa-r})\mathbf{F}\right)_{**}\left((1+\mu)\mathbf{F}\right)\right).\end{aligned}$$

From Lemma 14, there exists $\widehat{\mu}\in\bigoplus_{j\geq 1}\mathcal{P}_j^{\mathbf{t}}$ such that

$$\widehat{\mathbf{U}}_{**}\left(\left((\rho_{\kappa-r}+\eta_{\kappa-r})\mathbf{F}\right)_{**}\left((1+\mu)\mathbf{F}\right)\right)=\widehat{\mathbf{U}}_{**}\left((1+\widehat{\mu})\mathbf{F}\right),$$

where $\widehat{\mu}$ and μ agree up to quasi-homogeneous degree $\kappa-1$ (i.e., $\mathcal{J}^{\kappa-1}(\widehat{\mu})=\mathcal{J}^{\kappa-1}(\mu)$), and

$$\widehat{\mu}_{\kappa} = \mu_{\kappa}-\nabla\rho_{\kappa-r}\cdot\mathbf{F}_r-\nabla\eta_{\kappa-r}\cdot\mathbf{F}_r=\mu_{\kappa}^{(1)}+\mu_{\kappa}^{(2)}-\nabla\eta_{\kappa-r}\cdot\mathbf{F}_r=\mu_{\kappa}^{(2)}\in\mathrm{Cor}\left(\ell_{\kappa-r}\right).$$

On the other hand, we get that $-(\rho_{\kappa-r}+\eta_{\kappa-r})\mathbf{F}_r+\cdots$ is a generator of the transformation Ψ^{-1}. Consequently, from Lemma 12, we get $\mathcal{J}^{\kappa-1}(\widehat{\mathbf{U}})=\mathcal{J}^{\kappa-1}(\mathbf{U})$, and

$$\widehat{\mathbf{U}}_{\kappa}=\mathbf{U}_{\kappa}-(\rho_{\kappa-r}+\eta_{\kappa-r})\mathbf{F}_r=\mathbf{U}_{\kappa}^{(1)}+\mathbf{U}_{\kappa}^{(2)}-\rho_{\kappa-r}\mathbf{F}_r-\eta_{\kappa-r}\mathbf{F}_r=\mathbf{U}_{\kappa}^{(2)}-\eta_{\kappa-r}\mathbf{F}_r\in\widehat{\mathcal{Q}}_{\kappa}^{\mathbf{t}}.$$

□

2.3 N-Order Orbital Normal Form

To present the procedure of obtaining further simplifications in an orbital normal form, it is convenient to write the vector field of system (2.1.2) as

$$\mathbf{F}^{(0)}:=\mathbf{F}=\mathbf{F}_r^{(0)}+\sum_{j=1}^{\infty}\mathbf{F}_{r+j}^{(0)},\tag{2.3.1}$$

where $\mathbf{F}_{r+j}^{(0)}\in\mathcal{Q}_{r+j}^{\mathbf{t}}$. In Chapter 1, we have shown that it can be reduced to orbital normal form by a sequence of transformations in the states variables and time. That orbital normal form will be referred as 1-order orbital normal form and will be denoted by $\mathbf{F}^{(1)}$.

To get further simplifications, we follow a recursive procedure that will be explained later and we will transform $\mathbf{F}^{(1)}$ into a 2-order orbital normal form $\mathbf{F}^{(2)}$, later $\mathbf{F}^{(2)}$ will be reduced to a 3-order orbital normal form $\mathbf{F}^{(3)}$ and so on.

In addition to the orbital homological operator, that will be denoted as $\mathcal{L}_k^{(1)}:=\mathcal{L}_k$, defined in (1.4.9), the further simplifications procedure involves a sequence of linear operators $\mathcal{L}_k^{(m+1)}$ which are defined inductively.

The domain of operator $\mathcal{L}_k^{(m+1)}$ is the linear subspace $\mathcal{K}_k^{(m+1)}$ of the couples $(\mathbf{U},\nu)$ where $\mathbf{U}=\mathbf{U}_k+\cdots+\mathbf{U}_{k+m}\in\bigoplus_{j=k}^{k+m}\widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\nu=\nu_k+\cdots+\nu_{k+m}\in\bigoplus_{j=k}^{k+m}\mathrm{Cor}\,(\ell_{j-r})$, and

$$\mathcal{J}^{k+m-1}(\mathbf{U},\nu)=(\mathbf{U}_k+\cdots+\mathbf{U}_{k+m-1},\nu_k+\cdots+\nu_{k+m-1})\in\mathrm{Ker}\left(\mathcal{L}_k^{(m)}\right).$$

The quoted operator is defined by

$$\begin{aligned}\mathcal{L}_k^{(m+1)}:\mathcal{K}_k^{(m+1)}&\longrightarrow\mathcal{Q}_{r+k+m}^{\mathbf{t}}\\ (\mathbf{U},\nu)\longrightarrow\mathcal{L}_k^{(m+1)}(\mathbf{U},\nu)&=-\left([\mathbf{F},\mathbf{U}]+\nu\mathbf{F}\right)_{r+k+m}\\ &=-\sum_{j=0}^{m}\left([\mathbf{F}_{r+j},\mathbf{U}_{k+m-j}]+\nu_{k+m-j}\mathbf{F}_{r+j}\right).\end{aligned}$$

Remark 1. The orbital homological operator $\mathcal{L}_k^{(1)}$ depends on the principal part $\mathbf{F}_r$. However, the operator $\mathcal{L}_k^{(m+1)}$ depends on $\mathbf{F}_r,\ldots,\mathbf{F}_{r+m}$. Sometimes, it will be important to make explicit this dependence, and then we will write it as $\mathcal{L}_k^{(m+1)}{}_{\{\mathbf{F}_r,\ldots,\mathbf{F}_{r+m}\}}$.

Remark 2. If $(\mathbf{U},\nu)\in\mathcal{K}_k^{(m+1)}$, then $(\mathbf{U}_k+\cdots+\mathbf{U}_{k+m-1},\nu_k+\cdots+\nu_{k+m-1})\in\mathrm{Ker}\left(\mathcal{L}_k^{(m)}\right)$, i.e, $\mathcal{J}^{r+m-1}\left([\mathbf{F},\mathbf{U}]+\nu\mathbf{F}\right)=0$, that is:

$$\begin{aligned}&[\mathbf{F}_r,\mathbf{U}_k]+\nu_k\mathbf{F}_r=\mathbf{0},\\ &[\mathbf{F}_r,\mathbf{U}_{k+1}]+[\mathbf{F}_{r+1},\mathbf{U}_k]+\nu_{k+1}\mathbf{F}_r+\nu_k\mathbf{F}_{r+1}=\mathbf{0},\\ &\qquad\vdots\\ &\sum_{j=0}^{m-1}\left([\mathbf{F}_{r+j},\mathbf{U}_{k+m-1-j}]+\nu_{k+m-1-j}\mathbf{F}_{r+j}\right)=\mathbf{0}.\end{aligned}$$

Remark 2 yields next result:

Lemma 16. *Let us consider $(\mathbf{U},\nu)\in\mathcal{K}_k^{(m+1)}$, with $\mathbf{U}_k=\mathbf{0}$, $\nu_k=0$ (i.e., $\mathbf{U}=\mathbf{U}_{k+1}+\cdots+\mathbf{U}_{k+m}$, $\nu=\nu_{k+1}+\cdots+\nu_{k+m}$). Then:*

$$(\mathbf{U},\nu)\in\mathrm{Ker}\left(\mathcal{L}_k^{(m+1)}\right)\quad\textit{if, and only if,}\quad(\mathbf{U},\nu)\in\mathrm{Ker}\left(\mathcal{L}_{k+1}^{(m)}\right).$$

Definition 17. *The vector field $\mathbf{F}=\sum_{j\geq0}\mathbf{F}_{r+j}$, where $\mathbf{F}_{r+j}\in\mathcal{Q}_{r+j}^{\mathbf{t}}$ for $j\geq0$, is called a N-order orbital normal form if*

$$\mathbf{F}_{r+m}\in\mathrm{Cor}\left(\mathcal{L}_1^{(m)}\right),\ \textit{for } m=1,\ldots,N-1,\quad\textit{and}\quad\mathbf{F}_{r+m}\in\mathrm{Cor}\left(\mathcal{L}_{m+1-N}^{(N)}\right),\ \textit{for } m\geq N,$$

where $\mathrm{Cor}\left(\mathcal{L}_k^{(m)}\right)$ is a complementary subspace to $\mathrm{Range}\left(\mathcal{L}_k^{(m)}\right)$ in $\mathcal{Q}_{r+k+m-1}^{\mathbf{t}}$, for each $k,m\geq1$. In this case, $\mathcal{L}_k^{(m)}=\mathcal{L}_k^{(m)}{}_{\{\mathbf{F}_r,\ldots,\mathbf{F}_{r+m-1}\}}$.

We will see that $\mathrm{Cor}\left(\mathcal{L}_k^{(m)}\right)$ can be selected as a subset of $\mathrm{Cor}\left(\mathcal{L}_k^{(1)}\right)$, so that any N-order orbital normal form is simpler than the orbital normal form.

To show the existence of the N-order orbital normal form for any given vector field, we require some previous technical results:

Lemma 18. *Let $m \in \mathbb{N}$. Consider $\mathbf{U} = \sum_{j=1}^{m} \mathbf{U}_j$, with $\mathbf{U}_j \in \mathcal{Q}_j^{\mathbf{t}}$, and $\nu = \sum_{j=1}^{m} \nu_j$, with $\nu_j \in \mathcal{P}_j^{\mathbf{t}}$ such that $\mathcal{J}^{r+m}\left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F}\right) = \mathbf{0}$.*

Then, for each $\mu = \sum_{j=1}^{m} \mu_j$, with $\mu_j \in \mathcal{P}_j^{\mathbf{t}}$, we have:

$$\left(T_{\mathbf{U}}^{(l)}\left(\mu\mathbf{F}\right)\right)_{r+k} = \sum_{j=l+1}^{k} \mu_j^{(l)} \mathbf{F}_{r+k-j},$$

for all $l \geq 0$, $k = l+1, \ldots, l+m$; where $\mu_j^{(l)}$ depend univocally on the $(j-l)$-jet of $\mathbf{U}, \nu, \mu$, and are given by:

$$\mu_j^{(0)} = \mu_j, \quad \textit{and} \quad \mu_j^{(l)} = \sum_{i=l}^{j-1} \left(\left(\nabla \mu_i^{(l-1)} \cdot \mathbf{U}_{j-i}\right) - \mu_i^{(l-1)} \nu_{j-i}\right) \in \mathcal{P}_j^{\mathbf{t}}.$$

Proof. We use induction on l.

- For $l = 0$, the result is trivial because

$$\left(T_{\mathbf{U}}^{(0)}\left(\mu\mathbf{F}\right)\right)_{r+k} = \left(\mu\mathbf{F}\right)_{r+k} = \sum_{j=1}^{k} \mu_j \mathbf{F}_{r+k-j}.$$

- Assume the result true for $l-1$, being $l > 0$. Then:

$$\begin{aligned}
&\left(T_{\mathbf{U}}^{(l)}\left(\mu\mathbf{F}\right)\right)_{r+k} = \sum_{h=1}^{k-l} \left[\left(T_{\mathbf{U}}^{(l-1)}\left(\mu\mathbf{F}\right)\right)_{r+k-h}, \mathbf{U}_h\right] \\
&= \sum_{h=1}^{k-l} \left[\sum_{j=l}^{k-h} \mu_j^{(l-1)} \mathbf{F}_{r+k-h-j}, \mathbf{U}_h\right] = \sum_{j=l}^{k-1} \sum_{h=1}^{k-j} \left[\mu_j^{(l-1)} \mathbf{F}_{r+k-j-h}, \mathbf{U}_h\right] \\
&= \sum_{j=l}^{k-1} \left(\sum_{h=1}^{k-j} \left(\nabla \mu_j^{(l-1)} \cdot \mathbf{U}_h\right) \mathbf{F}_{r+k-j-h}\right) + \sum_{j=l}^{k-1} \mu_j^{(l-1)} \sum_{h=1}^{k-j} \left[\mathbf{F}_{r+k-j-h}, \mathbf{U}_h\right] \\
&= \sum_{j=l}^{k-1} \left(\sum_{h=1}^{k-j} \left(\nabla \mu_j^{(l-1)} \cdot \mathbf{U}_h\right) \mathbf{F}_{r+k-j-h}\right) + \sum_{j=l}^{k-1} \mu_j^{(l-1)} \left[\mathbf{F}, \mathbf{U}\right]_{r+k-j}.
\end{aligned}$$

As $\mathcal{J}^{r+m}([\mathbf{F},\mathbf{U}]+\nu\mathbf{F})=\mathbf{0}$, then $[\mathbf{F},\mathbf{U}]_{r+k-j}=-(\nu\mathbf{F})_{r+k-j}=-\sum_{h=1}^{k-j}\nu_h\mathbf{F}_{r+k-j-h}$.

Hence:

$$\begin{aligned}\left(T_{\mathbf{U}}^{(l)}(\mu\mathbf{F})\right)_{r+k}&=\sum_{j=l}^{k-1}\sum_{h=1}^{k-j}\left(\left(\nabla\mu_j^{(l-1)}\cdot\mathbf{U}_h\right)-\mu_j^{(l-1)}\nu_h\right)\mathbf{F}_{r+k-j-h}\\&=\sum_{h=l+1}^{k}\left(\sum_{j=l}^{h-1}\left(\left(\nabla\mu_j^{(l-1)}\cdot\mathbf{U}_{h-j}\right)-\mu_j^{(l-1)}\nu_{h-j}\right)\right)\mathbf{F}_{r+k-h}\\&=\sum_{h=l+1}^{k}\mu_h^{(l)}\mathbf{F}_{r+k-h},\end{aligned}$$

where $\mu_h^{(l)}$ are given in the statement of the Lemma.

□

Proposition 19. *Let $m\in\mathbb{N}$. Consider $\mathbf{U}=\sum_{j\geq1}\mathbf{U}_j$, with $\mathbf{U}_j\in\widehat{\mathcal{Q}}_j^{\mathbf{t}}$, and $\nu=\sum_{j\geq1}\nu_j$, with $\nu_j\in\mathrm{Cor}(\ell_{j-r})$ such that $\mathcal{J}^{r+m}([\mathbf{F},\mathbf{U}]+\nu\mathbf{F})=\mathbf{0}$.*

Then, there exists a unique $\mu=\sum_{j=1}^{m+1}\mu_j$, with $\mu_j\in\mathcal{P}_j^{\mathbf{t}}$, such that

$$\begin{aligned}&\mathcal{J}^{r+m}(\mathbf{U}_{**}((1+\mu)\mathbf{F}))=\mathcal{J}^{r+m}(\mathbf{F}),\quad\textit{and}\\&(\mathbf{U}_{**}((1+\mu)\mathbf{F}))_{r+m+1}=\mathbf{F}_{r+m+1}+([\mathbf{F},\mathbf{U}]+\nu\mathbf{F})_{r+m+1}.\end{aligned}$$

Moreover, $\mu_1=\nu_1$ and $\mu_j=\nu_j-\sum_{l=1}^{j-1}\mu_j^{(l)}$, for $j=2,\dots,m+1$, where $\mu_j^{(l)}$ depend univocally on the $(j-l)$-jet of $\mathbf{U},\nu,\mu$.

Proof. Let us write $\mathbf{U}_{**}((1+\mu)\mathbf{F})$ as in (1.5.5); that is,

$$\mathbf{U}_{**}((1+\mu)\mathbf{F})=\mathbf{F}+[\mathbf{F},\mathbf{U}]+\mu\mathbf{F}+\sum_{l=1}^{\infty}\tfrac{1}{(l+1)!}T_{\mathbf{U}}^{(l)}([\mathbf{F},\mathbf{U}]+(l+1)\mu\mathbf{F}).$$

We use induction on m. The result for $m=0$ is trivial.

Let us consider $m\geq1$. Using that $\mathcal{J}^{r+m}([\mathbf{F},\mathbf{U}])=-\mathcal{J}^{r+m}(\nu\mathbf{F})$, it can be shown that

$$\left(T_{\mathbf{U}}^{(l)}([\mathbf{F},\mathbf{U}])\right)_{r+m}=-\left(T_{\mathbf{U}}^{(l)}(\nu\mathbf{F})\right)_{r+m},$$

for any $l = 1, \dots, m$. Hence, for each $k = 1, \dots, m+1$, we can write

$$\begin{aligned}
\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)_{r+k} &= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+k} + \left((\mu-\nu)\mathbf{F}\right)_{r+k} \\
&\quad + \sum_{l=1}^{k-1} \frac{1}{(l+1)!}\left(T_{\mathbf{U}}^{(l)}\left([\mathbf{F},\mathbf{U}]\right)\right)_{r+k} + \sum_{l=1}^{k-1}\frac{1}{l!}\left(T_{\mathbf{U}}^{(l)}\left(\mu\mathbf{F}\right)\right)_{r+k} \\
&= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+k} + \left((\mu-\nu)\mathbf{F}\right)_{r+k} \\
&\quad - \sum_{l=1}^{k-1} \frac{1}{(l+1)!}\left(T_{\mathbf{U}}^{(l)}\left(\nu\mathbf{F}\right)\right)_{r+k} + \sum_{l=1}^{k-1}\frac{1}{l!}\left(T_{\mathbf{U}}^{(l)}\left(\mu\mathbf{F}\right)\right)_{r+k} \\
&= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+k} + \left((\mu-\nu)\mathbf{F}\right)_{r+k} \\
&\quad + \sum_{l=1}^{k-1} \frac{1}{(l+1)!}\left(T_{\mathbf{U}}^{(l)}\left(\left((l+1)\mu-\nu\right)\mathbf{F}\right)\right)_{r+k}.
\end{aligned}$$

From Lemma 18, we get:

$$\begin{aligned}
\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)_{r+k} &= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+k} + \left((\mu-\nu)\mathbf{F}\right)_{r+k} \\
&\quad + \sum_{l=1}^{k-1}\sum_{j=l+1}^{k} \mu_j^{(l)}\mathbf{F}_{r+k-j} \\
&= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+k} + \sum_{j=1}^{k}(\mu-\nu)_j\mathbf{F}_{r+k-j} + \sum_{j=2}^{k}\sum_{l=1}^{j-1}\mu_j^{(l)}\mathbf{F}_{r+k-j} \\
&= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+k} + (\mu-\nu)_1\mathbf{F}_{r+k-1} \\
&\quad + \sum_{j=2}^{k}\left((\mu-\nu)_j + \sum_{l=1}^{j-1}\mu_j^{(l)}\right)\mathbf{F}_{r+k-j}.
\end{aligned}$$

To complete the proof, it is enough to take μ_j as indicated in the statement. The uniqueness of μ_j follows because $\mu_j^{(l)}$ depend univocally on $\mathcal{J}^{j-l}(\mathbf{U},\nu,\mu)$. □

Next theorem shows that any vector field can be reduced to a N-order orbital normal form, for each $N \in \mathbb{N}$.

Theorem 20. *Let us consider $N \in \mathbb{N}$. Then, system (2.1.2) can be transformed, via equivalence, to a N-order orbital normal form.*

Proof. Let us write the vector field of system (2.1.2) as indicated in (2.3.1). Through the orbital homological operator $\mathcal{L}_1^{(1)} = \mathcal{L}_1^{(1)}\{\mathbf{F}_r^{(0)}\}$, we can reduce the $(r+1)$-degree quasi-homogeneous term $\mathbf{F}_{r+1}$ to orbital normal form, and then we reach a vector field:

$$\mathbf{F}^{(1)} = \mathbf{F}_r^{(1)} + \mathbf{F}_{r+1}^{(1)} + \mathbf{F}_{r+2}^{(1)} + \cdots,$$

where $\mathbf{F}_r^{(1)} = \mathbf{F}_r^{(0)}$ and $\mathbf{F}_{r+1}^{(1)} \in \mathrm{Cor}\left(\mathcal{L}_1^{(1)}\right)$. This is the 1-order orbital normal form.

Next, we consider the homological operator $\mathcal{L}_1^{(2)} = \mathcal{L}_1^{(2)}\left\{\mathbf{F}_r^{(1)}, \mathbf{F}_{r+1}^{(1)}\right\}$, and we split $\mathbf{F}_{r+2}^{(1)} = \mathbf{F}_{r+2}^{(1),r} + \mathbf{F}_{r+2}^{(1),c}$ with $\mathbf{F}_{r+2}^{(1),r} \in \mathrm{Range}\left(\mathcal{L}_1^{(2)}\right)$, $\mathbf{F}_{r+2}^{(1),c} \in \mathrm{Cor}\left(\mathcal{L}_1^{(2)}\right)$. Let us select $(\mathbf{U}^{(2)}, \nu^{(2)}) = \left(\mathbf{U}_1^{(2)} + \mathbf{U}_2^{(2)}, \nu_1^{(2)} + \nu_2^{(2)}\right) \in \mathcal{K}_1^{(2)}$ such that $\mathbf{F}_{r+2}^{(1),r} = \mathcal{L}_1^{(2)}\left(\mathbf{U}^{(2)}, \nu^{(2)}\right)$.

From Proposition 19, there exists a time transformation $dt = (1 + \mu_1^{(2)} + \mu_2^{(2)})dT$, $\mu_j^{(2)} \in \mathcal{P}_j^{\mathbf{t}}$, such that

$$\mathbf{F}^{(2)} := \mathbf{U}^{(2)}{}_{**}\left((1 + \mu_1^{(2)} + \mu_2^{(2)})\mathbf{F}^{(1)}\right) = \sum_{j\geq 0} \mathbf{F}_{r+j}^{(2)},$$

where:

$$\begin{aligned}
\mathbf{F}_{r+j}^{(2)} &= \mathbf{F}_{r+j}^{(1)}, \quad \text{for } j = 0, 1, \quad \text{and} \\
\mathbf{F}_{r+2}^{(2)} &= \mathbf{F}_{r+2}^{(1),c} + \mathbf{F}_{r+2}^{(1),r} + \left[\mathbf{F}^{(1)}, \mathbf{U}^{(2)}\right]_{r+2} + \left(\nu^{(2)}\mathbf{F}^{(1)}\right)_{r+2} \\
&= \mathbf{F}_{r+2}^{(1),c} + \mathbf{F}_{r+2}^{r} - \mathcal{L}_1^{(2)}\left(\mathbf{U}^{(2)}, \nu^{(2)}\right) = \mathbf{F}_{r+2}^{(1),c} \in \mathrm{Cor}\left(\mathcal{L}_1^{(2)}\right).
\end{aligned}$$

The procedure for the next values of m is similar to the above one. If we start from:

$$\mathbf{F}^{(m-1)} = \mathbf{F}_r^{(m-1)} + \mathbf{F}_{r+1}^{(m-1)} + \cdots + \mathbf{F}_{r+m-1}^{(m-1)} + \cdots,$$

where $\mathbf{F}_{r+j}^{(m-1)} = \mathbf{F}_{r+j}^{(j)}$, for $j = 0, 1, \ldots, m-2$ and $\mathbf{F}_{r+m-1}^{(m-1)} \in \mathrm{Cor}\left(\mathcal{L}_1^{(m-1)}\right)$, we firstly consider the homological operator $\mathcal{L}_1^{(m)} = \mathcal{L}_1^{(m)}\left\{\mathbf{F}_r^{(m-1)}, \ldots, \mathbf{F}_{r+m-1}^{(m-1)}\right\}$. Then, we split $\mathbf{F}_{r+m}^{(m-1)} = \mathbf{F}_{r+m}^{(m-1),r} + \mathbf{F}_{r+m}^{(m-1),c}$ where $\mathbf{F}_{r+m}^{(m-1),r} \in \mathrm{Range}\left(\mathcal{L}_1^{(m)}\right)$, $\mathbf{F}_{r+m}^{(m-1),c} \in \mathrm{Cor}\left(\mathcal{L}_1^{(m)}\right)$. There exist $(\mathbf{U}^{(m)}, \nu^{(m)}) = \left(\sum_{j=1}^m \mathbf{U}_j^{(m)}, \sum_{j=1}^m \nu_j^{(m)}\right) \in \mathcal{K}_1^{(m)}$ such that $\mathbf{F}_{r+m}^{(m-1),r} = \mathcal{L}_1^{(m)}\left(\mathbf{U}^{(m)}, \nu^{(m)}\right)$.

From Proposition 19, there exists a time transformation $dt = (1 + \mu_1^{(m)} + \cdots + \mu_m^{(m)})dT$, with $\mu_j^{(m)} \in \mathcal{P}_j^{\mathbf{t}}$ for $j = 1, \ldots m$, such that

$$\mathbf{F}^{(m)} := \mathbf{U}^{(m)}{}_{**}\left((1 + \mu_1^{(m)} + \cdots + \mu_m^{(m)})\mathbf{F}^{(m-1)}\right) = \sum_{j\geq 0} \mathbf{F}_{r+j}^{(m)},$$

satisfies:

$$\begin{aligned}
\mathcal{J}^{r+m-1}\left(\mathbf{F}^{(m)}\right) &= \mathcal{J}^{r+m-1}\left(\mathbf{F}^{(m-1)}\right), \quad \text{and} \\
\mathbf{F}_{r+m}^{(m)} &= \mathbf{F}_{r+m}^{(m-1),c} + \mathbf{F}_{r+m}^{(m-1),r} + \left[\mathbf{F}^{(m-1)}, \mathbf{U}^{(m)}\right]_{r+m} + \left(\nu^{(m)}\mathbf{F}^{(m-1)}\right)_{r+m} \\
&= \mathbf{F}_{r+m}^{(m-1),c} + \mathbf{F}_{r+2}^{(m-1),r} - \mathcal{L}_1^{(m)}\left(\mathbf{U}^{(m)}, \nu^{(m)}\right) = \mathbf{F}_{r+m}^{(m-1),c} \in \mathrm{Cor}\left(\mathcal{L}_1^{(m)}\right).
\end{aligned}$$

Once we have performed the above steps for $m = 1, \ldots, N$, we reach $\mathbf{F}^{(N)}$. Next, we must simplify the quasi-homogeneous terms of $\mathbf{F}^{(N)}$ of degree greater than $r+N$. This will be done using the operators $\mathcal{L}_k^{(N)}$ for successive values of k.

Firstly, we consider the homological operator $\mathcal{L}_2^{(N)} = \mathcal{L}_2^{(N)}\left\{\mathbf{F}_r^{(N)},\ldots,\mathbf{F}_{r+N-1}^{(N)}\right\}$, and denote $\mathbf{F}^{(N,1)} := \mathbf{F}^{(N)}$. Let us split $\mathbf{F}_{r+N+1}^{(N,1)} = \mathbf{F}_{r+N+1}^{(N,1),r} + \mathbf{F}_{r+N+1}^{(N,1),c}$, with $\mathbf{F}_{r+N+1}^{(N,1),r} \in \text{Range}\left(\mathcal{L}_2^{(N)}\right)$ and $\mathbf{F}_{r+N+1}^{(N,1),c} \in \text{Cor}\left(\mathcal{L}_2^{(N)}\right)$. Then there exists

$$\left(\mathbf{U}^{(N,2)}, \nu^{(N,2)}\right) = \left(\mathbf{U}_2^{(N,2)} + \cdots + \mathbf{U}_{N+1}^{(N,2)}, \nu_2^{(N,2)} + \cdots + \nu_{N+1}^{(N,2)}\right) \in \mathcal{K}_2^{(N)},$$

such that

$$\mathbf{F}_{r+N+1}^{(N,1),r} = \mathcal{L}_2^{(N)}\left(\mathbf{U}^{(N,2)}, \nu^{(N,2)}\right).$$

As $\left(\mathbf{U}^{(N,2)}, \nu^{(N,2)}\right) \in \mathcal{K}_2^{(N)}$, then

$$\left(\mathbf{U}_2^{(N,2)} + \cdots + \mathbf{U}_N^{(N,2)}, \nu_2^{(N,2)} + \cdots + \nu_N^{(N,2)}\right) \in \text{Ker}\left(\mathcal{L}_2^{(N-1)}\left\{\mathbf{F}_r^{(N-1)},\ldots,\mathbf{F}_{r+N-2}^{(N-1)}\right\}\right).$$

From Lemma 16, we get that $\left(\mathbf{U}^{(N,2)}, \nu^{(N,2)}\right) \in \mathcal{K}_1^{(N+1)}$. Moreover:

$$\mathbf{F}_{r+N+1}^{(N,1),r} = \mathcal{L}_1^{(N+1)}\left(\mathbf{U}^{(N,2)}, \nu^{(N,2)}\right).$$

From Proposition 19, there exists $\mu^{(N,2)} = \sum_{j=1}^{N+1} \mu_j^{(N,2)}$, with $\mu_j^{(N,2)} \in \mathcal{P}_j^{\mathbf{t}}$, such that the transformed vector field

$$\mathbf{F}^{(N,2)} := \mathbf{U}^{(N,2)} \ {}_{**}\left((1+\mu^{(N,2)})\mathbf{F}^{(N,1)}\right) = \sum_{j\geq 0} \mathbf{F}_{r+j}^{(N,2)},$$

satisfies:

$$\begin{aligned}
\mathcal{J}^{r+N}\left(\mathbf{F}^{(N,2)}\right) &= \mathcal{J}^{r+N}\left(\mathbf{F}^{(N,1)}\right), \text{ and} \\
\mathbf{F}_{r+N+1}^{(N,2)} &= \mathbf{F}_{r+N+1}^{(N,1),c} + \mathbf{F}_{r+N+1}^{(N,1),r} + \left(\left[\mathbf{F}^{(N,1)}, \mathbf{U}^{(N,2)}\right] + \nu^{(N,2)}\mathbf{F}^{(N,1)}\right)_{r+N+1} \\
&= \mathbf{F}_{r+N+1}^{(N,1),c} + \mathbf{F}_{r+N+1}^{(N,1),r} - \mathcal{L}_1^{(N+1)}\left(\mathbf{U}^{(N,2)}, \nu^{(N,2)}\right) \\
&= \mathbf{F}_{r+N+1}^{(N,1),c} \in \text{Cor}\left(\mathcal{L}_2^{(N)}\right).
\end{aligned}$$

The reasoning for the successive degrees is analogous.

Now, we consider $k \geq 3$ and the corresponding homological operator $\mathcal{L}_k^{(N)} = \mathcal{L}_k^{(N)}\left\{\mathbf{F}_r^{(N)},\ldots,\mathbf{F}_{r+N-1}^{(N)}\right\}$.

We split $\mathbf{F}_{r+N+k-1}^{(N,k)} = \mathbf{F}_{r+N+k-1}^{(N,k),r} + \mathbf{F}_{r+N+k-1}^{(N,k),c}$, with $\mathbf{F}_{r+N+k-1}^{(N,k),r} \in \text{Range}\left(\mathcal{L}_k^{(N)}\right)$, and $\mathbf{F}_{r+N+k-1}^{(N,k),c} \in \text{Cor}\left(\mathcal{L}_k^{(N)}\right)$.

Then, there exists

$$\left(\mathbf{U}^{(N,k)}, \nu^{(N,k)}\right) = \left(\mathbf{U}_k^{(N,k)} + \cdots + \mathbf{U}_{k+N-1}^{(N,k)}, \nu_k^{(N,k)} + \cdots + \nu_{k+N-1}^{(N,k)}\right) \in \mathcal{K}_k^{(N)},$$

such that

$$\mathbf{F}^{(N,k),r}_{r+N+k-1} = \mathcal{L}^{(N)}_k\left(\mathbf{U}^{(N,k)}, \nu^{(N,k)}\right).$$

Using that

$$\left(\mathbf{U}^{(N,k)}_k + \cdots + \mathbf{U}^{(N,k)}_{k+N-2}, \nu^{(N,k)}_k + \cdots + \nu^{(N,k)}_{k+N-2}\right) \in \mathrm{Ker}\left(\mathcal{L}^{(N-1)}_k{}_{\left\{\mathbf{F}^{(N-1)}_r,\ldots,\mathbf{F}^{(N-1)}_{r+N-2}\right\}}\right),$$

from Lemma 16, we find that $\left(\mathbf{U}^{(N,k)}, \nu^{(N,k)}\right) \in \mathcal{K}^{(k+N-1)}_1$. Moreover:

$$\mathbf{F}^{(N,k),r}_{r+N+k-1} = \mathcal{L}^{(N+k-1)}_1\left(\mathbf{U}^{(N,k)}, \nu^{(N,k)}\right).$$

From Proposition 19, there exists $\mu^{(N,k)} = \sum_{j=1}^{k+N-1} \mu^{(N,k)}_j$, with $\mu^{(N,k)}_j \in \mathcal{P}^{\mathbf{t}}_j$, such that the transformed vector field

$$\mathbf{F}^{(N,k)} := \mathbf{U}^{(N,k)}{}_{**}\left((1+\mu^{(N,k)})\mathbf{F}^{(N,k-1)}\right) = \sum_{j\geq 0} \mathbf{F}^{(N,k)}_{r+j},$$

satisfies $\mathcal{J}^{r+N+k-2}\left(\mathbf{F}^{(N,k)}\right) = \mathcal{J}^{r+N+k-2}\left(\mathbf{F}^{(N,k-1)}\right)$, and

$$\begin{aligned}
\mathbf{F}^{(N,k)}_{r+N+k-1} &= \mathbf{F}^{(N,k),c}_{r+N+k-1} + \mathbf{F}^{(N,k),r}_{r+N+k-1} + \left(\left[\mathbf{F}^{(N,k-1)}, \mathbf{U}^{(N,k)}\right] + \nu^{(N,k)}\mathbf{F}^{(N,k-1)}\right)_{r+N+k-1} \\
&= \mathbf{F}^{(N,k),c}_{r+N+k-1} + \mathbf{F}^{(N,k),r}_{r+N+k-1} - \mathcal{L}^{(N+k-1)}_1\left(\mathbf{U}^{(N,k)}, \nu^{(N,k)}\right) \\
&= \mathbf{F}^{(N,k),c}_{r+N+k-1} \in \mathrm{Cor}\left(\mathcal{L}^{(N)}_k\right).
\end{aligned}$$

The above ideas transform the vector field $\mathbf{F}^{(N)}$ into $\mathbf{F}^{(N,\infty)}$, which is a N-order orbital normal form of (2.1.2). □

2.4 Uniqueness of Orbital Normal Form

For the sake of notations, in the following, we will denote the successive vector fields $\mathbf{F}^{(N)}$ involved in the simplification procedure by $\mathbf{F}$ (i.e., we will omit the super-script). So, we can understand that the initial vector field $\mathbf{F}$ is actualized along the different steps of that procedure. In this way, we can consider that the quasi-homogeneous vector fields $\mathbf{F}_r, \mathbf{F}_{r+1}, \ldots, \mathbf{F}_{r+m-1}$, which define the homological operator $\mathcal{L}^{(m)}_k$, are fixed (really, they change in each step of the normal form procedure), and so it is not necessary to make explicit that dependence.

Definition 21. *The vector field* $\mathbf{F} = \mathbf{F}_r + \sum_{j\geq 1} \mathbf{F}_{r+j}$ *is a* ∞*-order orbital normal form if* $\mathbf{F}_{r+j} \in \mathrm{Cor}\left(\mathcal{L}^{(j)}_1\right)$ *for all* $j \in \mathbb{N}$*, where* $\mathrm{Cor}\left(\mathcal{L}^{(j)}_1\right)$ *is a complementary subspace to* $\mathrm{Range}\left(\mathcal{L}^{(j)}_1\right)$ *in* $\mathcal{Q}^{\mathbf{t}}_{r+j}$*, being* $\mathcal{L}^{(j)}_1 = \mathcal{L}^{(j)}_1{}_{\{\mathbf{F}_r,\mathbf{F}_{r+1},\ldots,\mathbf{F}_{r+j-1}\}}$ *for all* $j \geq 1$*.*

It is easy to show the existence of a ∞-order orbital normal form: it is enough to take the limit situation $N = \infty$ in the N-order orbital normal form procedure. This leads to an orbital normal form which is as simple as possible, and only linear procedures are involved. The rest of this section is devoted to show the uniqueness of the ∞-order orbital normal form.

Proposition 22. *Let $m \in \mathbb{N}$, $\mathbf{U} = \sum_{j\geq 1} \mathbf{U}_j$, and $\mu = \sum_{j\geq 1} \mu_j$, where $\mathbf{U}_j \in \widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\mu_j \in \mathrm{Cor}\,(\ell_{j-r})$. Let us assume that $\mathcal{J}^{r+m}\left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right) = \mathcal{J}^{r+m}(\mathbf{F})$. Then, there exists $\nu = \sum_{j=1}^{m+1} \nu_j$, with $\nu_j \in \mathcal{P}_j^{\mathbf{t}}$, such that*

$$\begin{aligned} &\mathcal{J}^{r+m}\left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right) = \mathbf{0}, \text{ and} \\ &\mathbf{U} *_{**} ((1+\mu)\mathbf{F})_{r+m+1} = \mathbf{F}_{r+m+1} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+m+1}. \end{aligned}$$

Proof. We use induction on m. If $m = 1$, the first part of the result is trivial because

$$\mathbf{F}_{r+1} = \mathbf{U} *_{**} ((1+\mu)\mathbf{F})_{r+1} = \mathbf{F}_{r+1} + \left([\mathbf{F},\mathbf{U}] + \mu\mathbf{F}\right)_{r+1},$$

and it is enough to take $\nu_1 = \mu_1$.

Now we need to calculate $\nu_2 \in \mathcal{P}_2^{\mathbf{t}}$. Let $\nu = \nu_1$. From Proposition 19, there exists a unique $\lambda = \lambda_1 + \lambda_2$, with $\lambda_j \in \mathcal{P}_j^{\mathbf{t}}$ such that

$$\left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right)_{r+j} = \mathbf{F}_{r+j} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+j}, \quad \text{for } j = 1, 2.$$

In particular:

$$\left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right)_{r+1} = \mathbf{F}_{r+1} = \left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right)_{r+1}.$$

Also, from the uniqueness of λ, we deduce $\lambda_1 = \mu_1$.

Moreover:

$$\begin{aligned} \left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right)_{r+2} &= \left\{\left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right)_{r+2} - \left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right)_{r+2}\right\} \\ &\quad + \left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right)_{r+2} \\ &= \left(\mathbf{U} *_{**} ((\lambda_2 - \mu_2)\mathbf{F})\right)_{r+2} + \mathbf{F}_{r+2} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+2} \\ &= (\lambda_2 - \mu_2)\,\mathbf{F}_r + \mathbf{F}_{r+2} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+2}. \end{aligned}$$

It is enough to take $\nu_2 = \lambda_2 - \mu_2$ and the second part of the statement holds.

Now, let us consider $m > 1$. From the induction hypothesis, there exist $\nu_1, \ldots, \nu_m$, such that the statement holds.

We need to calculate $\nu_{m+1} \in \mathcal{P}_{m+1}^{\mathbf{t}}$. Let us define $\nu = \sum_{j=1}^{m} \nu_j$.

Then: $\mathcal{J}^{r+m-1}\left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right) = \mathbf{0}$ and

$$\left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right)_{r+m} = \mathbf{F}_{r+m} + \left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+m} = \mathbf{F}_{r+m},$$

so: $\left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right)_{r+m} = \mathbf{0}$ and then $\mathcal{J}^{r+m}\left([\mathbf{F},\mathbf{U}] + \nu\mathbf{F}\right) = \mathbf{0}$.

On the other hand, from Proposition 19, there is a unique $\lambda = \sum_{j=1}^{m+1} \lambda_j$, $\lambda_j \in \mathcal{P}_j^{\mathbf{t}}$, satisfying

$$\left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right)_{r+j} = \mathbf{F}_{r+j} + \left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F}\right)_{r+j}, \quad \text{for } j = 1, \ldots, m+1.$$

Hence:

$$\mathcal{J}^{r+m}\left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right) = \mathcal{J}^{r+m}(\mathbf{F}) = \mathcal{J}^{r+m}\left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right).$$

The uniqueness of λ yields $\lambda_j = \mu_j$, for $j = 1, \ldots, m$.

Moreover:

$$\begin{aligned}
\left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right)_{r+m+1} &= \left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right)_{r+m+1} - \left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right)_{r+m+1} \\
&\quad + \left(\mathbf{U} *_{**} ((1+\lambda)\mathbf{F})\right)_{r+m+1} \\
&= \left(\mathbf{U} *_{**} ((\lambda_{m+1} - \mu_{m+1})\mathbf{F})\right)_{r+m+1} \\
&\quad + \mathbf{F}_{r+m+1} + \left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F}\right)_{r+m+1} \\
&= (\lambda_{m+1} - \mu_{m+1})\, \mathbf{F}_r + \mathbf{F}_{r+m+1} + \left([\mathbf{F}, \mathbf{U}] + \nu\mathbf{F}\right)_{r+m+1}.
\end{aligned}$$

It is enough to take $\nu_{m+1} = \lambda_{m+1} - \mu_{m+1}$ to get the result. □

Proposition 23. *Let $m \in \mathbb{N}$, $\mathbf{U} = \sum_{j\geq 1} \mathbf{U}_j$, and $\mu = \sum_{j\geq 1} \mu_j$, where $\mathbf{U}_j \in \widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\mu_j \in \mathrm{Cor}\,(\ell_{j-r})$. Let us assume that $\mathcal{J}^{r+m}\left(\mathbf{U} *_{**} ((1+\mu)\mathbf{F})\right) = \mathcal{J}^{r+m}(\mathbf{F})$. Then, there exists $\mathbf{V} = \sum_{j=1}^{m+1} \mathbf{V}_j$ and $\nu = \sum_{j=1}^{m+1} \nu_j$, with $\mathbf{V}_j \in \widehat{\mathcal{Q}}_j^{\mathbf{t}}$ and $\nu_j \in \mathrm{Cor}\,(\ell_{j-r})$, such that*

$$\begin{aligned}
&\mathcal{J}^{r+m}\left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right) = \mathbf{0}, \text{ and} \\
&\mathbf{U} *_{**} ((1+\mu)\mathbf{F})_{r+m+1} = \mathbf{F}_{r+m+1} + \left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right)_{r+m+1}.
\end{aligned}$$

Proof. From Proposition 22, there exists $\tilde{\nu} = \sum_{j=1}^{m+1} \tilde{\nu}_j$, with $\tilde{\nu}_j \in \mathcal{P}_j^{\mathbf{t}}$, such that

$$\begin{aligned}
&\mathcal{J}^{r+m}\left([\mathbf{F}, \mathbf{U}] + \tilde{\nu}\mathbf{F}\right) = \mathbf{0}, \text{ and} \\
&\mathbf{U} *_{**} ((1+\mu)\mathbf{F})_{r+m+1} = \mathbf{F}_{r+m+1} + \left([\mathbf{F}, \mathbf{U}] + \tilde{\nu}\mathbf{F}\right)_{r+m+1}.
\end{aligned}$$

So, it is enough to prove that there exist $\mathbf{V} = \sum_{j\geq 1} \mathbf{V}_j$ and $\nu = \sum_{j\geq 1} \nu_j$, with $\mathbf{V}_j \in \widehat{\mathcal{Q}}_j^{\mathbf{t}}$ and $\nu_j \in \mathrm{Cor}\,(\ell_{j-r})$, such that

$$\left([\mathbf{F}, \mathbf{U}] + \tilde{\nu}\mathbf{F}\right)_{r+j} = \left([\mathbf{F}, \mathbf{V}] + \nu\mathbf{F}\right)_{r+j}, \forall j \geq 0.$$

As $\mathcal{P}_i^{\mathbf{t}} = \mathrm{Range}\,(\ell_{i-r}) \bigoplus \mathrm{Cor}\,(\ell_{i-r})$ for each $i \in \mathbb{N}$, we can choose $\eta_{i-r} \in \mathcal{P}_{i-r}^{\mathbf{t}}$ and $\nu_i \in \mathrm{Cor}\,(\ell_{i-r})$ such that $\tilde{\nu}_i = \nabla\eta_{i-r} \cdot \mathbf{F}_r + \nu_i$, for $i = 1, \cdots, j-1$. So, it is

possible to choose $\eta_{i-r} = 0$ in the case $\mathrm{Proy}_{\mathrm{Range}\ell_{i-r}}(\tilde{\nu}_i) = 0$ and, in other case, $\eta_{i-r} \in \mathcal{P}^{\mathbf{t}}_{i-r} \setminus \mathrm{Ker}\,(\ell_{i-r})$. Then

$$\begin{aligned}([\mathbf{F},\mathbf{U}] + \tilde{\nu}\mathbf{F})_{r+j} &= \sum_{i=1}^{j} [\mathbf{F}_{r+j-i}, \mathbf{U}_i] + \tilde{\nu}_i \mathbf{F}_{r+j-i} \\ &= [\mathbf{F}_r, \mathbf{U}_j] + \tilde{\nu}_j \mathbf{F}_r + \sum_{i=1}^{j-1} [\mathbf{F}_{r+j-i}, \mathbf{U}_i] + (\nabla\eta_{i-r} \cdot \mathbf{F}_r + \nu_i)\,\mathbf{F}_{r+j-i}\end{aligned}$$

By (1.4.4) we get:

$$(\nabla\eta_{i-r} \cdot \mathbf{F}_r)\,\mathbf{F}_{r+j-i} = [\eta_{i-r}\mathbf{F}_{r+j-i}, \mathbf{F}_r] + \eta_{i-r}\,[\mathbf{F}_r, \mathbf{F}_{r+j-i}]$$

and

$$\eta_{i-r}\,[\mathbf{F}_r, \mathbf{F}_{r+j-i}] = [\eta_{i-r}\mathbf{F}_r, \mathbf{F}_{r+j-i}] - (\nabla\eta_{i-r} \cdot \mathbf{F}_{r+j-i})\,\mathbf{F}_r$$

So,

$$\begin{aligned}([\mathbf{F},\mathbf{U}] + \tilde{\nu}\mathbf{F})_{r+j} &= [\mathbf{F}_r, \mathbf{U}_j] + \tilde{\nu}_j \mathbf{F}_r + \sum_{i=1}^{j-1} [\mathbf{F}_{r+j-i}, \mathbf{U}_i] - [\mathbf{F}_r, \eta_{i-r}\mathbf{F}_{r+j-i}] \\ &\quad - [\mathbf{F}_{r+j-i}, \eta_{i-r}\mathbf{F}_r] - (\nabla\eta_{i-r} \cdot \mathbf{F}_{r+j-i})\,\mathbf{F}_r + \nu_i \mathbf{F}_{r+j-i} \\ &= \left[\mathbf{F}_r, \mathbf{U}_j - \sum_{i=1}^{j-1} \eta_{i-r}\mathbf{F}_{r+j-i}\right] + \left(\tilde{\nu}_j - \sum_{i=1}^{j-1} \nabla\eta_{i-r} \cdot \mathbf{F}_{r+j-i}\right)\mathbf{F}_r \\ &\quad + \sum_{i=1}^{j-1} [\mathbf{F}_{r+j-i}, \mathbf{U}_i - \eta_{i-r}\mathbf{F}_r] + \nu_i \mathbf{F}_{r+j-i}.\end{aligned}$$

If we take $\mathbf{V}_i = \mathbf{U}_i - \eta_{i-r}\mathbf{F}_r$, as $\mathbf{U}_i \in \widehat{\mathcal{Q}}^{\mathbf{t}}_i$ and $\eta_{i-r}\mathbf{F}_r \in \widehat{\mathcal{Q}}^{\mathbf{t}}_i$, we get $\mathbf{V}_i \in \widehat{\mathcal{Q}}^{\mathbf{t}}_i$ for $i = 1, \cdots, j-1$.

Now, to finish the proof, it is enough to obtain ν_j and $\mathbf{V}_j$. We take $\nu_j = \mathrm{Proy}_{\mathrm{Cor}(\ell_{j-r})}\left(\tilde{\nu}_j - \sum_{i=1}^{j-1} \nabla\eta_{i-r} \cdot \mathbf{F}_{r+j-i}\right)$, then there exists $\eta_{j-r} \in \mathcal{P}^{\mathbf{t}}_{j-r}$ such that $\nabla\eta_{j-r} \cdot \mathbf{F}_r = \mathrm{Proy}_{\mathrm{Range}(\ell_{j-r})}\left(\tilde{\nu}_j - \sum_{i=1}^{j-1} \nabla\eta_{i-r} \cdot \mathbf{F}_{r+j-i}\right)$. Then

$$\begin{aligned}([\mathbf{F},\mathbf{U}] + \tilde{\nu}\mathbf{F})_{r+j} &= \left[\mathbf{F}_r, \mathbf{U}_j - \sum_{i=1}^{j-1} \eta_{i-r}\mathbf{F}_{r+j-i}\right] + (\nabla\eta_{j-r} \cdot \mathbf{F}_r + \nu_j)\,\mathbf{F}_r \\ &\quad + \sum_{i=1}^{j-1} [\mathbf{F}_{r+j-i}, \mathbf{V}_i] + \nu_i \mathbf{F}_{r+j-i}.\end{aligned}$$

Applying again (1.4.4), we get $(\nabla\eta_{j-r}\cdot\mathbf{F}_r)\,\mathbf{F}_r = -\,[\mathbf{F}_r,\eta_{j-r}\mathbf{F}_r]$, then

$$\begin{aligned}([\mathbf{F},\mathbf{U}]+\tilde{\nu}\mathbf{F})_{r+j} &= \left[\mathbf{F}_r,\mathbf{U}_j-\eta_{j-r}\mathbf{F}_r-\sum_{i=1}^{j-1}\eta_{i-r}\mathbf{F}_{r+j-i}\right]+\nu_j\mathbf{F}_r\\ &\quad+\sum_{i=1}^{j-1}[\mathbf{F}_{r+j-i},\mathbf{V}_i]+\nu_i\mathbf{F}_{r+j-i}.\end{aligned}$$

It is enough to take $\mathbf{V}_j=\mathrm{Proy}_{\widehat{\mathcal{Q}}_j^{\mathbf{t}}}\left(\mathbf{U}_j-\sum_{i=1}^{j}\eta_{i-r}\mathbf{F}_{r+j-i}\right)$. In this case, there exists $\delta_{j-r}\in\mathrm{Ker}\,(\ell_{j-r})$ such that $\delta_{j-r}\mathbf{F}_r=\mathrm{Proy}_{\mathrm{Ker}(\ell_{j-r})\mathbf{F}_r}\left(\mathbf{U}_j-\sum_{i=1}^{j}\eta_{i-r}\mathbf{F}_{r+j-i}\right)$ and then

$$([\mathbf{F},\mathbf{U}]+\tilde{\nu}\mathbf{F})_{r+j} = [\mathbf{F}_r,\delta_{j-r}\mathbf{F}_r]+\sum_{i=1}^{j}([\mathbf{F},\mathbf{V}]+\nu\mathbf{F})_{r+j}$$

and the result follows applying (1.4.4), since $[\mathbf{F}_r,\delta_{j-r}\mathbf{F}_r]=0$. □

Next result shows the uniqueness –modulo the choosing of the complementary subspaces $\mathrm{Cor}\left(\mathcal{L}_1^{(j)}\right)$– of the ∞-order orbital normal form of a vector field.

Theorem 24. *The ∞-order orbital normal form of a vector field is unique.*

Proof. We use *reductio ad absurdum*. Let us suppose that there are two different ∞-order orbital normal forms for a vector field $\mathbf{F}=\mathbf{F}_r+\mathbf{F}_r+1+\cdots$, i.e. $\mathbf{F}$ is equivalent to:

$$\mathbf{G}=\mathbf{F}_r+\sum_{j\geq 1}\mathbf{G}_{r+j},\quad\text{and}\quad\mathbf{H}=\mathbf{F}_r+\sum_{j\geq 1}\mathbf{H}_{r+j},$$

where $\mathbf{G}_{r+j}\in\mathrm{Cor}\left(\mathcal{L}_1^{(j)}{}_{\{\mathbf{F}_r,\mathbf{G}_{r+1},\ldots,\mathbf{G}_{r+j-1}\}}\right)$, $\mathbf{H}_{r+j}\in\mathrm{Cor}\left(\mathcal{L}_1^{(j)}{}_{\{\mathbf{F}_r,\mathbf{H}_{r+1},\ldots,\mathbf{H}_{r+j-1}\}}\right)$, for all $j\geq 1$. As $\mathbf{G}\not\equiv\mathbf{H}$, we can define $k_0=\min\{j\in\mathbb{N}:\mathbf{G}_{r+j}\neq\mathbf{H}_{r+j}\}$.

Since $\mathbf{G}_{r+1}=\mathbf{H}_{r+1},\ldots,\mathbf{G}_{r+k_0-1}=\mathbf{H}_{r+k_0-1}$, then

$$\mathcal{L}_1^{(k_0)}\{\mathbf{F}_r,\mathbf{G}_{r+1},\ldots,\mathbf{G}_{r+k_0-1}\}=\mathcal{L}_1^{(k_0)}\{\mathbf{F}_r,\mathbf{H}_{r+1},\ldots,\mathbf{H}_{r+k_0-1}\}.$$

We denote this operator by $\mathcal{L}_1^{(k_0)}$.

Moreover, from Theorem 15 there exist $\mathbf{U}=\sum_{j\geq 1}\mathbf{U}_j$, $\mu=\sum_{j\geq 1}\mu_j$, ($\mathbf{U}_j\in\widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\mu_j\in\mathrm{Cor}\,(\ell_{j-r})$) such that $\mathbf{U}_{**}\,((1+\mu)\mathbf{H})=\mathbf{G}$. Then, we get:

$$\mathbf{H}=(\mathbf{H}-\mathbf{G})+\mathbf{G}=(\mathbf{H}-\mathbf{G})+\mathbf{U}_{**}\,((1+\mu)\mathbf{H})\,.$$

Hence:

$$\begin{aligned}\mathcal{J}^{r+k_0-1}(\mathbf{H}) &= \mathcal{J}^{r+k_0-1}(\mathbf{H}-\mathbf{G}) + \mathcal{J}^{r+k_0-1}(\mathbf{U}_{**}((1+\mu)\mathbf{H})) \\ &= \mathcal{J}^{r+k_0-1}(\mathbf{U}_{**}((1+\mu)\mathbf{H}))\,. \end{aligned} \tag{2.4.1}$$

From Proposition 23, there exist $\mathbf{V} = \sum_{j\geq 1}\mathbf{V}_j$ and $\nu = \sum_{j=1}^{k_0}\nu_j$ ($\mathbf{V}_j \in \widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\nu_j \in \mathrm{Cor}(\ell_{j-r})$) such that:

$$\begin{aligned}&\mathcal{J}^{r+k_0-1}([\mathbf{H},\mathbf{V}]+\nu\mathbf{H}) = \mathbf{0},\\ &\mathbf{G}_{r+k_0} = (\mathbf{U}_{**}((1+\mu)\mathbf{H}))_{r+k_0} = \mathbf{H}_{r+k_0} + ([\mathbf{H},\mathbf{V}]+\nu\mathbf{H})_{r+k_0}\,.\end{aligned}$$

Then, $\mathcal{J}^{r+k_0-1}(\mathbf{V},\nu) \in \mathrm{Ker}\left(\mathcal{L}_1^{(k_0-1)}\right)$, and so $\mathcal{J}^{r+k_0}(\mathbf{V},\nu) \in \mathcal{K}_1^{(k_0)}$. Moreover:

$$\mathbf{G}_{r+k_0} - \mathbf{H}_{r+k_0} = -\mathcal{L}_1^{(k_0)}\left(\mathcal{J}^{r+k_0}(\mathbf{V},\nu)\right) \in \mathrm{Range}\left(\mathcal{L}_1^{(k_0)}\right) \cap \mathrm{Cor}\left(\mathcal{L}_1^{(k_0)}\right) = \{\mathbf{0}\}\,.$$

In summary, $\mathbf{G}_{r+k_0} = \mathbf{H}_{r+k_0}$, which is a contradiction. □

Remark. If we take $\mu = 0$ in the proof of Theorem 24, we recover the result of [66] about the uniqueness of classical normal forms.

2.5 Towards the Unique Normal Form

Our goal in this section is to provide conditions to reach the ∞-order orbital normal form of a given vector field.

Lemma 25. $\mathrm{Range}\left(\mathcal{L}_{k+1}^{(m)}\right) \subseteq \mathrm{Range}\left(\mathcal{L}_k^{(m+1)}\right)$, *for all* $k, m \geq 1$.

Proof. The elements of $\mathrm{Range}\left(\mathcal{L}_{k+1}^{(m)}\right)$ are of the form

$$\mathcal{L}_{k+1}^{(m)}(\mathbf{U},\nu) = -\sum_{j=0}^{m-1}\left([\mathbf{F}_{r+j},\mathbf{U}_{k+m-j}] + \nu_{k+m-j}\mathbf{F}_{r+j}\right),$$

where $(\mathbf{U},\nu) = (\mathbf{U}_{k+1}+\cdots+\mathbf{U}_{k+m}, \nu_{k+1}+\cdots+\nu_{k+m}) \in \mathcal{K}_{k+1}^{(m)}$.

This means that $\mathcal{J}^{k+m-1}(\mathbf{U},\nu) \in \mathrm{Ker}\left(\mathcal{L}_{k+1}^{(m-1)}\right)$, and from Lemma 16 we get $\mathcal{J}^{k+m-1}(\mathbf{U},\nu) \in \mathrm{Ker}\left(\mathcal{L}_k^{(m)}\right)$. Hence $\mathcal{J}^{k+m-1}(\mathbf{U},\nu) \in \mathcal{K}_k^{(m+1)}$. Moreover,

$$-\sum_{j=0}^{m-1}\left([\mathbf{F}_{r+j},\mathbf{U}_{k+m-j}] + \nu_{k+m-j}\mathbf{F}_{r+j}\right) = \mathcal{L}_k^{(m+1)}\left(\mathcal{J}^{k+m-1}(\mathbf{U},\nu)\right) \in \mathrm{Range}\left(\mathcal{L}_k^{(m+1)}\right).$$

□

Corollary 26. $\dim\left(\mathrm{Cor}\left(\mathcal{L}_k^{(m+1)}\right)\right) \leq \dim\left(\mathrm{Cor}\left(\mathcal{L}_{k+1}^{(m)}\right)\right)$, *for all* $k, m \in \mathbb{N}$.

Remark. Although the complementary subspaces can be selected in different ways, it is natural (and we will do it so here) select them satisfying:

$$\mathrm{Cor}\left(\mathcal{L}_k^{(m+1)}\right) \subseteq \mathrm{Cor}\left(\mathcal{L}_{k+1}^{(m)}\right), \quad \text{for all } m, k \in \mathbb{N}.$$

In such a case, any M-order orbital normal form is a N-order orbital normal form, whenever $N \leq M$. In fact, a vector field $\mathbf{F}$ can have different N-order orbital normal forms, but it only has one ∞-order orbital normal form.

Next result provides a sufficient condition to reach a unique (or ∞-order) orbital normal form.

Theorem 27. *Let us assume that* $\mathbf{F}$ *is a* N*-order orbital normal form and*

$$\mathrm{Range}\left(\mathcal{L}_{k+1}^{(N)}\right) = \mathrm{Range}\left(\mathcal{L}_1^{(N+k)}\right), \quad \textit{for all } k \in \mathbb{N}.$$

Then, $\mathbf{F}$ *is the* ∞*-order orbital normal form.*

Proof. The considered N-order orbital normal form is

$$\mathbf{F} = \mathbf{F}_r + \cdots + \mathbf{F}_{r+N} + \sum_{j\geq 1} \mathbf{F}_{r+N+j},$$

with $\mathbf{F}_{r+k} \in \mathrm{Cor}\left(\mathcal{L}_1^{(k)}\right)$ for $k = 1, \ldots, N$, and $\mathbf{F}_{r+N+j} \in \mathrm{Cor}\left(\mathcal{L}_{j+1}^{(N)}\right)$.

As $\mathrm{Range}\left(\mathcal{L}_{1+j}^{(N)}\right) = \mathrm{Range}\left(\mathcal{L}_1^{(N+j)}\right)$ for $j \geq 1$, we get $\mathbf{F}_{r+N+j} \in \mathrm{Cor}\left(\mathcal{L}_1^{(N+j)}\right)$, which provides the result. □

Next, we present some technical results which are useful to obtain sufficient conditions to determine the unique orbital normal form of a vector field.

Lemma 28. *Let us consider* $m, k \in \mathbb{N}$ *and assume that:*

$$\mathrm{Ker}\left(\mathcal{L}_k^{(m)}\right) = (\mathbf{0}, 0) + \mathrm{Ker}\left(\mathcal{L}_{k+1}^{(m-1)}\right).$$

Then,

$$\mathrm{Range}\left(\mathcal{L}_k^{(m+1)}\right) = \mathrm{Range}\left(\mathcal{L}_{k+1}^{(m)}\right).$$

Proof. From Theorem 27, it is enough to show that Range $\left(\mathcal{L}_k^{(m+1)}\right) \subseteq$ Range $\left(\mathcal{L}_{k+1}^{(m)}\right)$ (the converse inclusion arises from Lemma 25).

The elements of Range $\left(\mathcal{L}_k^{(m+1)}\right)$ are of the form

$$\mathcal{L}_k^{(m+1)}(\mathbf{U},\nu) = -\sum_{j=0}^{m} [\mathbf{F}_{r+j}, \mathbf{U}_{k+m-j}] + \nu_{k+m-j}\mathbf{F}_{r+j},$$

where $(\mathbf{U},\nu) = (\mathbf{U}_k + \cdots + \mathbf{U}_{k+m}, \nu_k + \cdots + \nu_{k+m}) \in \mathcal{K}_k^{(m+1)}$ (i.e., $\mathcal{J}^{k+m-1}(\mathbf{U},\nu) \in$ Ker $\left(\mathcal{L}_k^{(m)}\right)$).

The hypothesis allows to write $(\mathbf{U}_k,\nu_k) = (\mathbf{0},0)$.

By Lemma 16, $(\mathbf{U},\nu) \in \mathcal{K}_k^{(m+1)}$. Finally, it is a simple matter to show that

$$\mathcal{L}_k^{(m+1)}(\mathbf{U},\nu) = \mathcal{L}_{k+1}^{(m)}(\mathbf{U},\nu),$$

which belongs to Range $\left(\mathcal{L}_{k+1}^{(m)}\right)$ □

Lemma 29. *Let us consider $N \in \mathbb{N}$ such that:*

$$\text{Ker}\left(\mathcal{L}_k^{(N)}\right) = (\mathbf{0},0) + \text{Ker}\left(\mathcal{L}_{k+1}^{(N-1)}\right),$$

for all $k \in \mathbb{N}$. Then:

$$\text{Ker}\left(\mathcal{L}_k^{(m)}\right) = (\mathbf{0},0) + \text{Ker}\left(\mathcal{L}_{k+1}^{(m-1)}\right),$$

for all $k, m \in \mathbb{N}$, $m > N$.

Proof. We use induction on m. Let us assume the result true for $m-1 \geq N$, and we will prove it for m.

If $(\mathbf{U},\nu) = \left(\sum_{j=0}^{m-1}\mathbf{U}_{k+j}, \sum_{j=0}^{m-1}\nu_{k+j}\right) \in \text{Ker}\left(\mathcal{L}_k^{(m)}\right)$, then $\left(\sum_{j=0}^{m-2}\mathbf{U}_{k+j}, \sum_{j=0}^{m-2}\nu_{k+j}\right) \in \text{Ker}\left(\mathcal{L}_k^{(m-1)}\right)$. From the induction hypothesis, we have

$$\text{Ker}\left(\mathcal{L}_k^{(m-1)}\right) = (\mathbf{0},0) + \text{Ker}\left(\mathcal{L}_{k+1}^{(m-2)}\right),$$

and then $(\mathbf{U}_k,\nu_k) = (\mathbf{0},0)$. So

$$\mathbf{0} = \mathcal{L}_k^{(m)}(\mathbf{U},\nu) = \mathcal{L}_{k+1}^{(m-1)}(\mathbf{U},\nu).$$

Hence, $(\mathbf{U},\nu) \in \text{Ker}\left(\mathcal{L}_{k+1}^{(m-1)}\right)$ and so $(\mathbf{U},\nu) \in (\mathbf{0},0) + \text{Ker}\left(\mathcal{L}_{k+1}^{(m-1)}\right)$. In this way:

$$\text{Ker}\left(\mathcal{L}_k^{(m)}\right) \subseteq (\mathbf{0},0) + \text{Ker}\left(\mathcal{L}_{k+1}^{(m-1)}\right).$$

The converse inclusion is trivial. □

Theorem 30. *Let us assume that* $\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots$ *is a* N*-order orbital normal form and*

$$\mathrm{Ker}\left(\mathcal{L}_k^{(N)}\right) = (\mathbf{0}, 0) + \mathrm{Ker}\left(\mathcal{L}_{k+1}^{(N-1)}\right),$$

for all $k \in \mathbb{N}$*. Then,* $\mathbf{F}$ *is the* ∞*-order orbital normal form.*

Proof. From Lemma 29, we have:

$$\mathrm{Ker}\left(\mathcal{L}_k^{(m)}\right) = (\mathbf{0}, 0) + \mathrm{Ker}\left(\mathcal{L}_{k+1}^{(m-1)}\right),$$

for all $k, m \in \mathbb{N}$, $m \geq N$. From Lemma 28, we get $\mathrm{Range}\left(\mathcal{L}_k^{(m+1)}\right) = \mathrm{Range}\left(\mathcal{L}_{k+1}^{(m)}\right)$ for all $m, k \in \mathbb{N}$, $m \geq N$; or equivalently: $\mathrm{Range}\left(\mathcal{L}_{k+1}^{(N)}\right) = \mathrm{Range}\left(\mathcal{L}_1^{(N+k)}\right)$ for all $k \in \mathbb{N}$. Applying Theorem 27, we get the result. □

Corollary 31. *Let us assume that* $\mathrm{Ker}\left(\mathcal{L}_k^{(1)}\right) = (\mathbf{0}, 0)$ *, for all* $k \in \mathbb{N}$*. Then, the 1-order orbital normal form agrees with the unique orbital normal form.*

Remark 1. This corollary is not applicable in the case that $\mathbf{F}_r$ is integrable, since if $\eta_k \in \mathcal{P}_k^{\mathbf{t}} \setminus \{0\}$ is a polynomial first integral, from (1.4.3) and (1.4.4), we have:

$$[\mathbf{F}_r, \eta_k \mathbf{D}_0] + r\eta_k \mathbf{F}_r = -\left(\nabla \eta_k \cdot \mathbf{F}_r\right) \mathbf{D}_0 - \eta_k \left[\mathbf{F}_r, \mathbf{D}_0\right] + r\eta_k \mathbf{F}_r = -r\eta_k \mathbf{F}_r + r\eta_k \mathbf{F}_r = \mathbf{0}.$$

Hence, $(\eta_k \mathbf{D}_0, r\eta_k) \in \mathrm{Ker}\left(\mathcal{L}_k^{(1)}\right) \setminus \{(\mathbf{0}, 0)\}$.

Remark 2. In [66, 97], the unique classical normal form for the Takens-Bogdanov singularity is obtained using the type $\mathbf{t} = (1, 2)$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \alpha xy + \beta x^3 \end{pmatrix} + \sum_{j \geq 2} \mathbf{F}_j, \quad \text{with } \mathbf{F}_j \in \mathcal{Q}_j^{(1,2)},$$

assuming that β/α^2 is non-algebraic. Later, in Theorem 32, we will refine this hypothesis.

2.6 Applications to Takens-Bogdanov Singularities

In this section we consider higher-order perturbations of non-integrable quasi-homogeneous Takens-Bogdanov singularities. A Takens-Bogdanov singularity can be written as:

$$\begin{aligned} \dot{x} &= y + xP(x) + y\Phi(x, y), \\ \dot{y} &= Q(x) + yR(x) + y^2\Psi(x, y), \end{aligned} \tag{2.6.1}$$

where $P(x)=\mathcal{O}(x)$, $Q(x)=\mathcal{O}(x^2)$, $R(x)=\mathcal{O}(x)$ and $\Phi(x,y)=\mathcal{O}(x,y)$.

Let us denote by m the lowest-degree in the Taylor expansion of $Q(x)$; and n is the minimum of the lowest-degrees of the Taylor expansions for $P(x)$ and $R(x)$. Hence, $m=\infty$ arises if $Q(x)\equiv 0$ and $n=\infty$ corresponds to $P(x)\equiv R(x)\equiv 0$. Then, we can write system (2.6.1) as

$$\begin{aligned}\dot{x} &= y+x^{n+1}A(x)+y\Phi(x,y),\\ \dot{y} &= x^n yB(x)+x^m C(x)+y^2\Psi(x,y),\end{aligned} \tag{2.6.2}$$

where $m,n\in\mathbb{N}\cup\{\infty\}$, $\quad A(x)=a+\mathcal{O}(x)$, $B(x)=b+\mathcal{O}(x)$, $C(x)=c+\mathcal{O}(x)$, being $c\neq 0$.

To exclude some cases corresponding to Hamiltonian lowest-degree quasi-homogeneous term since the first quasi-homogeneous component is non-integrable, we will assume $n\neq\infty$ and $m\geq 2n+1$.

By selecting the type $\mathbf{t}=(1,n+1)$, the quoted lowest-degree quasi-homogeneous term of the vector field of (2.6.1) is

$$\hat{\mathbf{F}}_n(x,y)=\left(y+ax^{n+1},bx^n y+\delta x^{2n+1}\right)^T\in\mathcal{Q}_n^{\mathbf{t}},$$

where $\delta=0$ if $m\neq 2n+1$, and $\delta=c$ if $m=2n+1$.

Following the ideas presented in [12], we can write

$$\hat{\mathbf{F}}_n=\tfrac{1}{2(n+1)}\hat{\mu}\mathbf{D}_0+\tfrac{1}{2(n+1)}\mathbf{X}_{\hat{h}},$$

where $\mathbf{D}_0=(x,(n+1)y)^T\in\mathcal{Q}_0^{\mathbf{t}}$, $\hat{\mu}=\left(\frac{b}{2(n+1)}+\frac{1}{2}a\right)x^n\in\mathcal{P}_n^{\mathbf{t}}$, and $\mathbf{X}_{\hat{h}}$ is the Hamiltonian vector field with Hamiltonian function:

$$\begin{aligned}\hat{h}(x,y) &= \tfrac{\delta}{2(n+1)}x^{2(n+1)}+\left(\tfrac{b}{2(n+1)}-\tfrac{1}{2}a\right)x^{n+1}y-\tfrac{1}{2}y^2\\ &= -\tfrac{1}{2}\left(y-\left(\tfrac{b}{2(n+1)}-\tfrac{a}{2}\right)x^{n+1}\right)^2+\left(\tfrac{\delta}{2(n+1)}+\tfrac{1}{2}\left(\tfrac{b}{2(n+1)}-\tfrac{a}{2}\right)^2\right)x^{2(n+1)}\in\mathcal{P}_n^{\mathbf{t}}.\end{aligned}$$

Now, we will assume a new hypothesis: $\frac{b}{2(n+1)}+\frac{1}{2}a\neq 0$ (excluding Hamiltonian situations with multiple factors). It is easy to show that the change of variables $u=x$, $v=y-\left(\frac{b}{2(n+1)}-\frac{a}{2}\right)x^{n+1}$, and an adequate rescaling, transform system (2.6.2) into a higher-order quasi-homogeneous perturbation of

$$\mathbf{F}_n(x,y)=\left(y+dx^{n+1},(n+1)dx^n y+\sigma(n+1)x^{2n+1}\right)^T, \tag{2.6.3}$$

where $d\neq 0$, and $\sigma\in\{-1,0,1\}$. We notice that $\mathbf{F}_n:=\mathbf{X}_h+\mu\mathbf{D}_0$, where:

$$h(x,y)=-\tfrac{1}{2}y^2+\tfrac{\sigma}{2}x^{2(n+1)}\in\mathcal{P}_n^{\mathbf{t}},\quad\text{and}\quad\mu=dx^n\in\mathcal{P}_n^{\mathbf{t}}.$$

Following the ideas presented in Sections 2.4, 2.5, we obtain the following result:

Theorem 32. *Let us consider the type* $\mathbf{t} = (1, n+1)$*, with* $n \in \mathbb{N}$*, and the vector field:*

$$\mathbf{F}(x,y) = \mathbf{F}_n(x,y) + \mathbf{F}_{n+1}(x,y) + \cdots, \tag{2.6.4}$$

where $\mathbf{F}_n$ *is given in (2.6.3), and* $\mathbf{F}_{n+k} \in \mathcal{Q}^{\mathbf{t}}_{n+k}$ *for* $k \in \mathbb{N}$*. Let us assume that* $d \neq 0$*, and*

- $\sigma \in \{-1, 0\}$*; or*
- $\sigma = 1$*,* $|d| \neq 1 + \frac{2(n+1)}{k}$*,* $|d| \neq 1 - \frac{2i(n+1)}{k}$*, for* $i = 0, \ldots, \left\lfloor \frac{k}{n+1} \right\rfloor - \left\lfloor \frac{k}{2(n+1)} \right\rfloor - 1$*, and* $k \in \mathbb{N}$ *(here,* $\lfloor \cdot \rfloor$ *denotes the* floor *function).*

Then, the unique orbital normal form is:

$$\mathbf{G} := \mathbf{F}_n + \sum_{j=n+1}^{2n} \gamma_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j \neq n}}^{2n} \gamma_j^{(l)} x^j h^l \mathbf{D}_0. \tag{2.6.5}$$

Proof. In [53], it is proved that a 1-order orbital normal form is (2.6.5), and

$$\operatorname{Ker}\left(\mathcal{L}_k^{(1)}\right) \;=\; (\mathbf{0}, 0)$$

By applying Corollary 31, (2.6.5) is the unique orbital normal form. □

Remark 1. If $n = 0$, the linearization of system (2.6.4) is non-resonant, and the above theorem agrees with the Poincaré Theorem, that states that it is analytically linearizable (see [48]).

Remark 2. The unique normal form under conjugation was obtained in [31] for the cases $m > 2n+1$ and $m < 2n+1$. The case $m = 2n+1$ was considered in [66] for $n = 1$ and in [97] for $n > 1$, assuming that d is non-algebraic. Observe that, in the case $\sigma \in \{-1, 0\}$, we do not have any restriction about d. In the case $\sigma = 1$, the hypothesis on d of Theorem 32 are weaker than those of the above-mentioned papers.

Remark 3. A normal form under equivalence is obtained in [85, 86, 87]. They assume that the initial system has been reduced to a Takens pre-normal form, and use the center-stable, center-center and center-unstable manifolds of the origin (depending on the case) in the normalization transformations. The unique normal form described in our theorem embeds the monodromic and non-monodromic cases altogether, and it is based in the original system, which allows a direct applicability in concrete cases. Also, the unique normal form is useful, as we will see later, when studying the center and reversibility problems.

Normal form (2.6.5) provides interesting dynamical information for (2.6.4). Next, we will use them to get some results characterizing the monodromy, reversibility and center problems.

The monodromy problem was considered in [25, 26]. Here we present an alternative proof:

Theorem 33. *The origin for (2.6.4) is monodromic if, and only if, $\sigma = -1$. Moreover, if n is even, the origin is an unstable focus for $d > 0$, and a stable focus for $d < 0$.*

Proof. We assume, without loss of any generality, that the vector field (2.6.4) has been reduce to the orbital normal form (2.6.5).

Let us denote by $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))^T$ is the solution of the initial value problem $\dot{\mathbf{x}} = \mathbf{X}_{\widetilde{h}}(\mathbf{x})$, $\mathbf{x}(0) = (1,0)^T$, being $\widetilde{h}(x,y) = x^{2n+2} + y^2$. It is well-known that these are periodic functions, and they satisfy $\mathrm{Cs}^{2n+2}(\theta) + \mathrm{Sn}^2(\theta) = 1$.

If we use generalized polar coordinates: $x = u\mathrm{Cs}(\theta)$, $y = u^{n+1}\mathrm{Sn}(\theta)$, and rescale the time by $dT = \frac{u^n}{2}dt$, then the unique orbital normal form $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ becomes

$$\begin{array}{rcl} \frac{du}{dT} & = & 2u\mathrm{Cs}^n(\theta)\left(d + (\sigma+1)\mathrm{Cs}^{n+1}(\theta)\mathrm{Sn}(\theta)\right) + \mathcal{O}\left(u^2\right), \\ \frac{d\theta}{dT} & = & -1 + (\sigma+1)\mathrm{Cs}^{2n+2}(\theta). \end{array} \tag{2.6.6}$$

Consequently:

- If $\sigma = -1$, there are no equilibria on the axis $u = 0$, and the origin of system (2.6.5) is monodromic. Observe that, if n is even, the stability of the focus can be easily deduced from system (2.6.6).
- If $\sigma \in \{0, 1\}$ there are some equilibria on $u = 0$. Let us denote one of such equilibria as $(u, \theta) = (0, \theta_0)$, and define $\alpha = \theta - \theta_0$. Then, system (2.6.6) becomes:

$$\begin{array}{rcl} \frac{du}{dT} & = & u\displaystyle\sum_{i,j\geq 0} a_{i,j}u^i\alpha^j + \cdots, \\ \frac{d\alpha}{dT} & = & b\alpha^k + \mathcal{O}\left(\alpha^{k+1}\right),\ b \neq 0. \end{array}$$

  Hence, the axis $\alpha = 0$ is invariant and system (2.6.5) is non-monodromic.

□

Finally, next Theorem characterizes the center problem for (2.6.4) by means of its unique orbital normal form (2.6.5).

Theorem 34. *The origin for (2.6.4) is a center if, and only if, $\sigma = -1$, and the unique orbital normal form (2.6.5) is R_x-reversible (i.e., invariant to $(x, y, t) \to (-x, y, -t)$).*

Proof. The sufficient condition is trivial, due to the monodromy (see Theorem 33) and reversibility of (2.6.5).

Let us prove the necessary condition. By Theorem 33, σ must be equal to -1.

If (2.6.5) is not R_x-reversible, let $l_0 = \min\left\{l \in \mathbb{N} \cup \{0\} : \gamma_j^{(l)} \neq 0, \text{ for some } j \text{ even}\right\}$ and $j_0 = \min\left\{0 \leq j \leq 2n, j \text{ even} : \gamma_j^{(l_0)} \neq 0\right\}$. Let

$$\mathbf{X} = \mathbf{F}_n + \sum_{\substack{j=n+1 \\ j \ odd}}^{2n} \gamma_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j \neq n, odd}}^{2n} \gamma_j^{(l)} x^j h^l \mathbf{D}_0.$$

The origin of $\dot{\mathbf{x}} = \mathbf{X}(x)$ is a center since is monodromic and $\mathbf{X}$ is R_x-reversible. Then, doing the wedge product

$$\begin{aligned}
\mathbf{G} \wedge \mathbf{X} &= \left(\sum_{\substack{j=n+1 \\ j \ even}}^{2n} \gamma_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j \neq n, even}}^{2n} \gamma_j^{(l)} x^j h^l \mathbf{D}_0 \right) \wedge \mathbf{X} \\
&= \left(\sum_{\substack{j=n+1 \\ j \ even}}^{2n} \gamma_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j \neq n, even}}^{2n} \gamma_j^{(l)} x^j h^l \mathbf{D}_0 \right) \wedge \mathbf{X}_h \\
&= 2(n+1) x^{j_0} h^{l_0+1} \left(\gamma_{j_0}^{(l_0)} + \sum_{\substack{j>j_0 \\ j \ even}}^{2n} \gamma_j^{(l_0)} x^j + \sum_{l=l_0+1}^{\infty} \sum_{\substack{j=0 \\ j \neq n, even}}^{2n} \gamma_j^{(l)} x^j h^l \right)
\end{aligned}$$

Hence, since j_0 is even, $\mathbf{G} \wedge \mathbf{X}$ does not change sign in a neighborhood of the origin, therefore the origin of system (2.6.5) is a repulsive focus if $\gamma_{j_0}^{(l_0)} > 0$ or an attractive focus if $\gamma_{j_0}^{(l_0)} < 0$. □

In [35] it is shown that the origin of system (2.6.4) is a center if, and only if, the quoted system is orbital–reversible. The above result agrees with this statement.

Remark. In the case $n = 1$, we obtain that the origin for system (2.6.4) with $\sigma = -1$ is a center if, and only if, the system is equivalent to $\dot{\mathbf{x}} = \mathbf{F}_1(\mathbf{x})$.

Chapter 3

Orbital–Reversibility via Normal Forms

3.1 Introduction

When the dynamics defined by a vector field is analyzed, it is customary to look for some features that help us to understand the skeleton of the dynamical behavior (limit cycles, global connections,...). Among these features, it is of particular interest the possibility of detecting the presence of some orbital–reversibility because then the time-reversing symmetry provides a classification of orbits: the symmetric ones (those intersecting the fixed point set of the involution defining the reversibility) and the non-symmetric ones (which come in pairs). In particular, the orbit set is symmetric with respect to the fixed point set of the involution.

In the last decades there has been an increasing interest in the study of systems with time-reversal symmetries. Reversible vector fields were first considered by Birkhoff, in the beginning of the last century. In [50] the theory was formalized by Devaney.

This chapter deals with analytic n-dimensional autonomous differential equations:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}),\ \mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n, \tag{3.1.1}$$

having an equilibrium point located at the origin. We address the *orbital-reversibility problem*, which consists in determining if it admits some reversibility (possibly non-linear) modulo $\mathcal{C}^\infty$-equivalence. In the chapter, we will consider only involutions having fixed point set with codimension one, but the ideas presented here can be of interest in a general framework.

The problem of determining if system (3.1.1) has some reversibility is considered in [24]. If the quoted system is reversible, then we know some facts about its solution set: if we have a solution, then we get another one by changing the sign

of the time variable and applying the time-reversing symmetry. A detailed study of time-reversal symmetries can be found, for instance, in [67].

In the present chapter, we plain to determine if there exists some time reparametrization such that the resulting system admits some reversibility. The presence of some orbital–reversibility is a valuable feature that helps in the understanding of the dynamical behaviour of a given system. Namely, as the time-reparametrizations do not change the orbits but only the speed in which they are traversed in time, then we know some facts about its phase portraits: if we have an orbit, then we get another one by reversing the time and applying the time-reversing symmetry. Hence, the main properties of reversible dynamics are preserved: the phase portrait is symmetric with respect to the fixed point set of the reversibility, and it is possible to classify the orbits according to the number of intersections with the quoted fixed point set.

To this respect, some results about stability of equilibria and periodic orbits for linear reversible systems can be found in [72, 73, 91].

This fact shows that, in the case of planar systems, the orbital–reversibility property has a direct application in the study of the center problem. In fact, a system with non-zero linear part has a center at the origin if, and only if, it is orbital–reversible.

Nevertheless, for more specific problems where it is not convenient to reparametrize the time variable (for instance, the isochronous-center problem, and other problems related to the period function) is more useful to deal instead with the reversibility property.

It is important to remark that our results go one step further on the reversibility problem which, as we have mentioned before, is considered in [24]. We will use some results of the quoted paper (they are included here without proof), that are related to the changes of variables and its relations with the reversibility problem. Moreover, we present here new results, mainly concerning to time-reparametrizations.

Next, we give a precise definition of the reversibility we will deal with:

Definition 35. *An involution is a local diffeomorphism $\sigma \in \mathcal{C}^\infty$, such that $\sigma \circ \sigma = Id$, $\sigma(\mathbf{0}) = \mathbf{0}$ and* $\operatorname{codim}(Fix(\sigma)) = 1$, *where $Fix(\sigma) = \{\mathbf{x} \in \mathbb{R}^n : \sigma(\mathbf{x}) = \mathbf{x}\}$ is the fixed point set of σ.*
We say that system (3.1.1) is reversible if there exists some involution σ such that $\sigma_* \mathbf{F} = -\mathbf{F}$, where $\sigma_* \mathbf{F}$ denotes the pull-back of the vector field $\mathbf{F}$ by σ.
System (3.1.1) is called orbital–reversible if there exist an involution σ and a function $\mu \in \mathcal{C}^\infty$, with $\mu(\mathbf{0}) = 1$, such that $\sigma_* (\mu \mathbf{F}) = -\mu \mathbf{F}$ (this means that $\mathbf{F}$ is reversible modulo a time-reparametrization).

The reversibility problem we are studying consists of determining if a system has some nonlinear reversibility modulo $\mathcal{C}^\infty$-equivalence, having a fixed point space with codimension one. Another kind of reversibilities, corresponding to fixed space

with codimension greater than one or more complicated groups (see, e.g., [43]) are not consider here.

In general, the involution σ is nonlinear, and we emphasize this fact by speaking of nonlinear reversibility. In the literature, the most frequent cases of reversibility that we can find corresponds to the called axis-reversibility (also called time-reversibility), where the involution is linear:

$$R_{x_p}(\mathbf{x}) = \mathsf{R}_p\mathbf{x},$$

for some $p = 1, \ldots, n$. Here, R_p is the diagonal matrix having ones on the diagonal except the p–th element which is -1. Notice that, in this case, the fixed point set is the hyperplane $x_p = 0$.

In the study of the axis-reversibility, it is possible to reduce the analysis to the first variable x_1, by reordering them adequately. In this case, we have that $\mathbf{F}$ is R_{x_1}-reversible if, and only if, $\mathbf{F}(\mathsf{R}_1\mathbf{x}) = -\mathsf{R}_1\mathbf{F}(\mathbf{x})$. For this reason, through this chapter, we will write $\mathbf{x} = (x, y)$, where $x = x_1$ and $y = (x_2, \ldots, x_n)^T \in \mathbb{R}^{n-1}$. In the same way, we will write $\mathbf{F} = (P, Q)^T$, where $P = F_1$ and $Q = (F_2, \ldots, F_n)^T$. Then, it is easy to show that $\mathbf{F}$ is R_x-reversible if P is an even function and Q is an odd vector field with respect to the variable x.

Our aim in this chapter is to adapt the ideas of the theory of normal forms under equivalence (see [7]) to characterize orbital–reversible vector fields. To this end, we introduce some definitions and terminology, related to the transformation theory in the quasi-homogeneous context. Our analysis is based in an adequate classification of system (3.1.1), obtained by using nonlinear time-reparametrizations and coordinate transformations.

Following the theory of quasi-homogeneous detailed in Chapter 1, we can write system (3.1.1) in the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \tag{3.1.2}$$

where the lowest-degree nonzero term $\mathbf{F}_r \neq \mathbf{0}$ ($r \in \mathbb{Z}$), is called the principal part with respect to the type $\mathbf{t}$ and each term $\mathbf{F}_k$ is a quasi-homogeneous vector field of type $\mathbf{t}$ and degree k.

At first glance, the type $\mathbf{t}$ may be selected arbitrarily. Nevertheless, we will choose the type $\mathbf{t}$ (and consequently the principal part) according to the following result of [13]:

Theorem 36. *Let us consider a fixed type* $\mathbf{t}$ *and assume that the vector field (3.1.2) is reversible. Then, there exists* $\Phi_0 \in \mathcal{Q}_0^{\mathbf{t}}$ *such that the diagonal elements of* $D\Phi_0$ *are zero, and* $(id + \Phi_0)_*\mathbf{F}_r$ *is* R_{x_p}*-reversible, for some* $p = 1, \cdots, n$.
Moreover, there exists $\Phi = id + \sum_{j>0} \Phi_j$, *with* $\Phi_j \in \mathcal{Q}_j^{\mathbf{t}}$, *such that* $(\Phi \circ (id + \Phi_0))_*\mathbf{F}$ *is* R_{x_p}*-reversible.*

From the above theorem, in order to determine orbital–reversibility conditions of system (3.1.2), we can assume (performing a near-identity change of variables of zero degree and reordering the state variables, if necessary), that $\mathbf{F}$ is of the form

$$\mathbf{F}(\mathbf{x}) = \widetilde{\mathbf{F}}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \quad \text{for some } \ r \in \mathbb{Z}, \tag{3.1.3}$$

where the principal part $\widetilde{\mathbf{F}}_r \neq \mathbf{0}$ is R_x–reversible, and $\mathbf{F}_{r+k} \in \mathcal{Q}^{\mathbf{t}}_{r+k}$ for all $k \in \mathbb{N}$.

Along this chapter, we will truncate quasi-homogeneous expansions of vector fields and scalar functions. So, if the vector field $\mathbf{F}$ is expanded in quasi-homogeneous terms of type $\mathbf{t}$, we define its quasi-homogeneous $(r+k)$-jet by

$$\mathcal{J}^{r+k}(\mathbf{F}) = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots + \mathbf{F}_{r+k}.$$

Sometimes, we will need to pick-up the k-degree quasi-homogeneous term of a vector field. As we have already done, we use sub-indices on vector fields to denote its projection on the space of quasi-homogeneous polynomials. In this way, $[\mathbf{F}, \mathbf{G}]_k$ will denote the k-degree quasi-homogeneous term of the Lie product.

Next section presents some definitions and properties concerning the structure of orbital–reversible vector fields. In Section 3.3 we introduce the concept of N-orbital–reversibility (orbital–reversibility truncating to quasi-homogeneous degree $r+N$) and obtain some necessary conditions: Theorem 40 states that if the system (3.1.1) is N-orbital–reversible, then there is a normal form which is R_x-reversible and this normal form can be reached by means of reversible generators and odd time-reparametrizations. Section 3.4 presents an algorithm for the orbital–reversibility, where the underlying idea is to discard cases of non N-orbital–reversibility. In next section, we characterize the orbital–reversibility via the unique normal form and the 1-order normal form. Finally, in the last section, we present some applications of the preceding results to the study of the orbital reversibility of two families of planar and tridimensional nilpotent systems. In these examples (see item (e) of Theorem 74 and item (b) of Theorem 56), we present some cases of non-reversible systems which are orbital–reversible (this means that, after reparametrizing conveniently the time variable, they become reversible).

3.2 Technical tools

If we assume that system (3.1.1) is reversible, then from Montgomery-Bochner Theorem (see [74], page 206), the vector field $\mathbf{F}$ can be expressed as $\mathbf{F}(x, y) = (f(x^2, y), xg(x^2, y))$, with $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$, in an adequate coordinate system. So, system (3.1.1) can not be reversible (respectively, orbital–reversible) provided that it cannot be expressed in the above referred form, up to conjugation (respectively, equivalence).

The above ideas suggest to introduce the following vector spaces:

- $\mathcal{O}_k^{\mathbf{t}} = \{\mu \in \mathcal{P}_k^{\mathbf{t}} : \mu(-x, y) = -\mu(x, y)\}$, the set of quasi-homogeneous scalar functions of degree k which are odd in the first variable.
- $\mathcal{E}_k^{\mathbf{t}} = \{\mu \in \mathcal{P}_k^{\mathbf{t}} : \mu(-x, y) = \mu(x, y)\}$, the set of quasi-homogeneous scalar functions of degree k which are even in the first variable.
- $\mathcal{R}_k^{\mathbf{t}} = \left\{\mathbf{F} = (P, Q)^T \in \mathcal{Q}_k^{\mathbf{t}} : P \in \mathcal{E}_{k+t_1}^{\mathbf{t}}, Q = (Q_1, \ldots, Q_{n-1})^T \text{ with } Q_i \in \mathcal{O}_{k+t_{i+1}}^{\mathbf{t}}\right\}$, the set of R_x–reversible quasi-homogeneous vector fields of degree k.
- $\mathcal{S}_k^{\mathbf{t}} := \left\{\mathbf{F} = (P, Q)^T \in \mathcal{Q}_k^{\mathbf{t}} : P \in \mathcal{O}_{k+t_1}^{\mathbf{t}}, Q = (Q_1, \ldots, Q_{n-1})^T \text{ with } Q_i \in \mathcal{E}_{k+t_{i+1}}^{\mathbf{t}}\right\}$, the set of R_x–symmetric quasi-homogeneous vector fields of degree k.

In [24], we can find the following characterization of the above sets (recall that we denoted $\mathsf{R}_1 = \operatorname{diag}(-1, 1, \ldots, 1)$):

Lemma 37. *The following properties are verified:*

(a) $\mu \in \mathcal{O}_k^{\mathbf{t}}$ *if, and only if,* $\mu(\mathsf{R}_1\mathbf{x}) = -\mu(\mathbf{x})$.

(b) $\mu \in \mathcal{E}_k^{\mathbf{t}}$ *if, and only if,* $\mu(\mathsf{R}_1\mathbf{x}) = \mu(\mathbf{x})$.

(c) $\mathbf{F} \in \mathcal{R}_k^{\mathbf{t}}$ *if, and only if,* $\mathbf{F}(\mathsf{R}_1\mathbf{x}) = -\mathsf{R}_1\mathbf{F}(\mathbf{x})$.

(d) $\mathbf{F} \in \mathcal{S}_k^{\mathbf{t}}$ *if, and only if,* $\mathbf{F}(\mathsf{R}_1\mathbf{x}) = \mathsf{R}_1\mathbf{F}(\mathbf{x})$.

The scalar monomials $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ form the canonical basis of $\mathcal{P}_k^{\mathbf{t}}$. Using that $\mathbf{x}^\alpha \in \mathcal{O}_k^{\mathbf{t}}$ if, and only if, α_1 is odd, we deduce that $\mathcal{P}_k^{\mathbf{t}} = \mathcal{O}_k^{\mathbf{t}} \bigoplus \mathcal{E}_k^{\mathbf{t}}$. Also, the canonical vector monomial $\mathbf{x}^\alpha\ \mathbf{e}_j$ (here $\mathbf{e}_j$ stands for the j–th canonical vector of $\mathbb{R}^n$) belongs to $\mathcal{R}_k^{\mathbf{t}}$ if, and only if, $j = 1$ and α_1 is even, or $j > 1$ and α_1 is odd. Hence, we have $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{R}_k^{\mathbf{t}} \bigoplus \mathcal{S}_k^{\mathbf{t}}$. This decomposition allows us to define the corresponding projection operators as follows:

$$\pi^{(\mathsf{o})}(\mu) \in \bigoplus_k \mathcal{O}_k^{\mathbf{t}}, \quad \pi^{(\mathsf{e})}(\mu) \in \bigoplus\nolimits_k \mathcal{E}_k^{\mathbf{t}}, \quad \text{for } \mu \in \bigoplus_k \mathcal{P}_k^{\mathbf{t}}, \text{ and}$$
$$\Pi^{(\mathsf{r})}(\mathbf{U}) \in \bigoplus_k \mathcal{R}_k^{\mathbf{t}}, \quad \Pi^{(\mathsf{s})}(\mathbf{U}) \in \bigoplus\nolimits_k \mathcal{S}_k^{\mathbf{t}}, \quad \text{for } \mathbf{U} \in \bigoplus_k \mathcal{Q}_k^{\mathbf{t}}.$$

Next, we present some results about the behavior of the Lie product and gradients for vector fields and scalar functions in the above-defined sets.

Before this, it is convenient to make some comments about the notation that we will use throughout the chapter: we use boldface letters to denote vector functions and variables, whereas plain letters (sometimes greek ones) are used for scalars. Usually, we will distinguish reversible vector fields by putting a tilde; and the symmetric ones by using a bar. Also, odd scalar functions are identified by a tilde and even scalar functions by a bar.

Proposition 38. *The following properties hold:*

(a) *If* $\widetilde{\mathbf{F}} \in \mathcal{R}^{\mathbf{t}}_r$, $\widetilde{\mu} \in \mathcal{O}^{\mathbf{t}}_k$, *then* $\nabla\widetilde{\mu}\cdot\widetilde{\mathbf{F}} \in \mathcal{E}^{\mathbf{t}}_{r+k}$.
If $\overline{\mathbf{F}} \in \mathcal{S}^{\mathbf{t}}_s$, $\widetilde{\mu} \in \mathcal{O}^{\mathbf{t}}_k$, *then* $\nabla\widetilde{\mu}\cdot\overline{\mathbf{F}} \in \mathcal{O}^{\mathbf{t}}_{s+k}$.
If $\widetilde{\mathbf{F}} \in \mathcal{R}^{\mathbf{t}}_r$, $\overline{\mu} \in \mathcal{E}^{\mathbf{t}}_k$, *then* $\nabla\overline{\mu}\cdot\widetilde{\mathbf{F}} \in \mathcal{O}^{\mathbf{t}}_{r+k}$.
If $\overline{\mathbf{F}} \in \mathcal{S}^{\mathbf{t}}_s$, $\overline{\mu} \in \mathcal{E}^{\mathbf{t}}_k$, *then* $\nabla\overline{\mu}\cdot\overline{\mathbf{F}} \in \mathcal{E}^{\mathbf{t}}_{s+k}$.

(b) *If* $\widetilde{\mathbf{F}} \in \mathcal{R}^{\mathbf{t}}_r$, $\widetilde{\mathbf{U}} \in \mathcal{R}^{\mathbf{t}}_k$ *then* $\left[\widetilde{\mathbf{F}},\widetilde{\mathbf{U}}\right] \in \mathcal{S}^{\mathbf{t}}_{r+k}$.
If $\widetilde{\mathbf{F}} \in \mathcal{R}^{\mathbf{t}}_r$, $\overline{\mathbf{U}} \in \mathcal{S}^{\mathbf{t}}_k$, *then* $\left[\widetilde{\mathbf{F}},\overline{\mathbf{U}}\right] \in \mathcal{R}^{\mathbf{t}}_{r+k}$.

(c) *If* $\widetilde{\mathbf{F}} \in \mathcal{R}^{\mathbf{t}}_r$, $\overline{\mu} \in \mathcal{E}^{\mathbf{t}}_k$, *then* $\overline{\mu}\,\widetilde{\mathbf{F}} \in \mathcal{R}^{\mathbf{t}}_{r+k}$.
If $\widetilde{\mathbf{F}} \in \mathcal{R}^{\mathbf{t}}_r$, $\widetilde{\mu} \in \mathcal{O}^{\mathbf{t}}_k$, *then* $\widetilde{\mu}\,\widetilde{\mathbf{F}} \in \mathcal{S}^{\mathbf{t}}_{r+k}$.
If $\overline{\mathbf{F}} \in \mathcal{S}^{\mathbf{t}}_k$, $\overline{\mu} \in \mathcal{E}^{\mathbf{t}}_k$, *then* $\overline{\mu}\,\overline{\mathbf{F}} \in \mathcal{S}^{\mathbf{t}}_{j+k}$.

Proof. Items **(a)**, **(b)** are proved in [24]. The proof of item **(c)** is a straightforward application of Lemma 37. □

The main goal of this chapter is to determine conditions for the orbital–reversibility of (3.1.3), based on the existence of a near-identity transformation Φ (more specifically, we will use its generator), and a scalar function $\mu \in \mathcal{C}^\infty$, with $\mu(\mathbf{0}) = 1$, such that $\Phi_*\,(\mu\mathbf{F})$ is R_x–reversible.

For our convenience, from now on we will write the time-reparametrization as $1+\mu$, with $\mu(\mathbf{0}) = 0$. Indeed, its quasi-homogeneous expansion is $1+\sum_{j\geq 1}\mu_j$, with $\mu_j \in \mathcal{P}^{\mathbf{t}}_j$.

3.3 N-Orbital–reversibility

Definition 39. *We say that the vector field of system (3.1.3) is* N-orbital–reversible *(*$N \in \mathbb{N}$*) if there exist a vector field* $\mathbf{U} \in \bigoplus_{j\geq 1}\mathcal{Q}^{\mathbf{t}}_j$ *and a scalar function* $\mu \in \bigoplus_{j\geq 1}\mathcal{P}^{\mathbf{t}}_j$, *such that* $\mathcal{J}^{r+N}\left(\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)\right)$ *is* R_x*-reversible.*

Trivially, if the vector field (3.1.3) is orbital–reversible then it is N-orbital–reversible for each $N \in \mathbb{N}$. Our idea is to adapt the normal form procedure in order to determine conditions under which the normalized vector field is N-orbital–reversible. As in the analysis of the integrability (see [12]), we introduce the Lie derivate along the principal part $\widetilde{\mathbf{F}}_r$:

$$\begin{aligned}\ell_{k-r} \;:\; & \mathcal{P}^{\mathbf{t}}_{k-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_k \\ & \mu_{k-r} \longrightarrow \nabla\mu_{k-r}\cdot\widetilde{\mathbf{F}}_r.\end{aligned}$$

The importance of this linear operator lies in its role for selecting the time reparametrization μ. Namely, in the normal form reduction it is enough to take its quasi-homogeneous terms μ_k belonging to $\mathrm{Cor}\,(\ell_{k-r})$ (a complementary subspace to $\mathrm{Range}\,(\ell_{k-r})$).

From now on, applying the Proposition 38 item (a), we will use the operator ℓ_{k-r} restricted to the following subspaces:

$$\ell_{k-r}|_{\mathcal{E}^{\mathbf{t}}_{k-r}} : \mathcal{E}^{\mathbf{t}}_{k-r} \longrightarrow \mathcal{O}^{\mathbf{t}}_{k} \qquad \ell_{k-r}|_{\mathcal{O}^{\mathbf{t}}_{k-r}} : \mathcal{O}^{\mathbf{t}}_{k-r} \longrightarrow \mathcal{E}^{\mathbf{t}}_{k},$$

that is, the operator ℓ_{k-r} is an anti-diagonal by blocks operator.

So, we can choose a complementary subspace to $\mathrm{Range}\,(\ell_{k-r})$ such that:

$$\mathrm{Cor}\,(\ell_{k-r}) = \mathrm{Cor}\left(\ell_{k-r}|_{\mathcal{E}^{\mathbf{t}}_{k-r}}\right) \bigoplus \mathrm{Cor}\left(\ell_{k-r}|_{\mathcal{O}^{\mathbf{t}}_{k-r}}\right). \tag{3.3.1}$$

We denote by $\widehat{\mathcal{Q}}^{\mathbf{t}}_{k}$ a complementary subspace to $\mathrm{Ker}\,(\ell_{k-r})\,\widetilde{\mathbf{F}}_r$ in $\mathcal{Q}^{\mathbf{t}}_{k}$, i.e.

$$\mathcal{Q}^{\mathbf{t}}_{k} \;=\; \mathrm{Ker}\,(\ell_{k-r})\,\widetilde{\mathbf{F}}_r \bigoplus \widehat{\mathcal{Q}}^{\mathbf{t}}_{k}.$$

On the other hand, we denote by $\widehat{\mathcal{R}}^{\mathbf{t}}_{k}$ a complementary subspace to $\mathrm{Ker}\left(\ell_{k-r}|_{\mathcal{E}^{\mathbf{t}}_{k-r}}\right)\widetilde{\mathbf{F}}_r$ in $\mathcal{R}^{\mathbf{t}}_{k}$ and by $\widehat{\mathcal{S}}^{\mathbf{t}}_{k}$ a complementary subspace to $\mathrm{Ker}\left(\ell_{k-r}|_{\mathcal{O}^{\mathbf{t}}_{k-r}}\right)\widetilde{\mathbf{F}}_r$ in $\mathcal{S}^{\mathbf{t}}_{k}$; that is:

$$\begin{aligned} \mathcal{R}^{\mathbf{t}}_{k} &= \mathrm{Ker}\left(\ell_{k-r}|_{\mathcal{E}^{\mathbf{t}}_{k-r}}\right)\widetilde{\mathbf{F}}_r \bigoplus \widehat{\mathcal{R}}^{\mathbf{t}}_{k}, \\ \mathcal{S}^{\mathbf{t}}_{k} &= \mathrm{Ker}\left(\ell_{k-r}|_{\mathcal{O}^{\mathbf{t}}_{k-r}}\right)\widetilde{\mathbf{F}}_r \bigoplus \widehat{\mathcal{S}}^{\mathbf{t}}_{k}. \end{aligned} \tag{3.3.2}$$

Since $\mathrm{Ker}\,(\ell_{k-r}) = \mathrm{Ker}\left(\ell_{k-r}|_{\mathcal{O}^{\mathbf{t}}_{k-r}}\right) \bigoplus \mathrm{Ker}\left(\ell_{k-r}|_{\mathcal{E}^{\mathbf{t}}_{k-r}}\right)$, because the operator ℓ_{k-r} is anti-diagonal by blocks, we obtain:

$$\mathcal{Q}^{\mathbf{t}}_{k} \;=\; \mathrm{Ker}\,(\ell_{k-r}) \bigoplus \widehat{\mathcal{S}}^{\mathbf{t}}_{k} \bigoplus \widehat{\mathcal{R}}^{\mathbf{t}}_{k}.$$

Thereby, it is possible to choose $\widehat{\mathcal{Q}}^{\mathbf{t}}_{k}$ such that

$$\widehat{\mathcal{Q}}^{\mathbf{t}}_{k} \;=\; \widehat{\mathcal{R}}^{\mathbf{t}}_{k} \bigoplus \widehat{\mathcal{S}}^{\mathbf{t}}_{k}. \tag{3.3.3}$$

For simplicity, we use the following notation:

$$\widehat{\mathcal{O}}^{\mathbf{t}}_{k} \;:=\; \mathrm{Cor}\left(\ell_{k-r}|_{\mathcal{E}^{\mathbf{t}}_{k-r}}\right). \tag{3.3.4}$$

The main result in this section is the following theorem, which states that the N-orbital–reversibility can be reached by means of reduced spatial and time generators.

Theorem 40. *Let us consider $N \in \mathbb{N}$ and $\mathbf{F}$ given in (3.1.3). Then, if $\mathbf{F}$ is N-orbital–reversible, there exist $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ and $\widetilde{\mu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$, such that*

$$\mathcal{J}^{r+N}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) \quad \textit{is} \quad R_x - \textit{reversible.}$$

To prove this theorem, it is enough to show that

$$\mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}.$$

This is a direct consequence of the Lemmas 12, 13, Theorem 15, stated and proved in Chapter 2, and Proposition 41 below.

The following Proposition concerns about the way of selecting the generators in the analysis of the orbital–reversibility.

Proposition 41. *Let us consider the vector field $\mathbf{F}$ given in (3.1.3), and $\mathbf{U} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{Q}}_j^{\mathbf{t}}$, $\mu \in \bigoplus_{j=1}^{N} \mathrm{Cor}\,(\ell_{j-r})$ such that $\mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right) = \mathbf{0}$, with $N \in \mathbb{N}$. Then, there exist $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ and $\widetilde{\mu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ satisfying*

$$\mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}.$$

Proof. Let us denote

$$\kappa = \min\left\{k \in \mathbb{N} : \Pi^{(\mathtt{s})}\left(\mathbf{U}_k\right) \neq \mathbf{0}, \quad \text{or} \quad \mathrm{Proy}_{\mathcal{E}_k^{\mathbf{t}}}\left(\mu_k\right) \neq 0\right\}.$$

We assume $\kappa < N$ (otherwise, taking $\widetilde{\mathbf{U}} = \mathbf{U}$, $\widetilde{\mu} = \mu$, we complete the proof).

From the definition of κ, we get that $\mathbf{U}_k \in \widehat{\mathcal{R}}_k^{\mathbf{t}}$ and $\mu_k \in \widehat{\mathcal{O}}_k^{\mathbf{t}}$ for $k = 1, \ldots, \kappa - 1$. Our goal is to show that the result is true for the κ–degree quasi-homogeneous terms and then, repeating the procedure, we get the result.

By (3.3.3), we can write $\mathbf{U}_\kappa = \mathbf{U}_\kappa^{(1)} + \mathbf{U}_\kappa^{(2)}$, where $\mathbf{U}_\kappa^{(1)} \in \widehat{\mathcal{S}}_\kappa^{\mathbf{t}}$ and $\mathbf{U}_\kappa^{(2)} \in \widehat{\mathcal{R}}_\kappa^{\mathbf{t}}$. Also, by (3.3.1), we can split $\mu_\kappa = \mu_\kappa^{(1)} + \mu_\kappa^{(2)}$, with $\mu_\kappa^{(1)} \in \mathrm{Cor}\left(\ell_{\kappa-r}|_{\mathcal{O}_\kappa^{\mathbf{t}}}\right)$ and by definition (3.3.4), $\mu_\kappa^{(2)} \in \widehat{\mathcal{O}}_\kappa^{\mathbf{t}}$.

Let us denote by $\widetilde{\mathbf{U}}$ a generator of $\Psi \circ \Phi$, where Φ and Ψ are the transformations generated by $\mathbf{U}$ and $-\mathbf{U}_\kappa^{(1)}$, respectively. From Lemma 12, we have $\mathcal{J}^{\kappa-1}(\widetilde{\mathbf{U}}) = \mathcal{J}^{\kappa-1}(\mathbf{U})$ and

$$\widetilde{\mathbf{U}}_\kappa = \mathbf{U}_\kappa - \mathbf{U}_\kappa^{(1)} = \mathbf{U}_\kappa^{(2)} \in \widehat{\mathcal{R}}_\kappa^{\mathbf{t}}.$$

Also, from Lemma 13, there exists $\lambda \in \bigoplus_{j \geq \kappa} \mathcal{P}_j^{\mathbf{t}}$ such that

$$\left(1 - \mu_\kappa^{(1)}\right)\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right) = \widetilde{\mathbf{U}}_{**}\left((1+\lambda)(1+\mu)\mathbf{F}\right) = \widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right),$$

where we have defined $\widetilde{\mu} := (1+\lambda)(1+\mu) - 1$. We observe that, as $\lambda_\kappa = -\mu_\kappa^{(1)}$, then $\mathcal{J}^{\kappa-1}(\widetilde{\mu}) = \mathcal{J}^{\kappa-1}(\mu)$, and

$$\widetilde{\mu}_\kappa = \mu_\kappa - \mu_\kappa^{(1)} = \mu_\kappa^{(2)} \in \widehat{\mathcal{O}}_\kappa^{\mathbf{t}}.$$

Moreover, from Proposition 38(b-c), we get $\Pi^{(\mathtt{s})}\left(\left[\widetilde{\mathbf{F}}_r, \mathbf{U}_N^{(1)}\right]\right) = \Pi^{(\mathtt{s})}\left(\mu_N^{(1)}\widetilde{\mathbf{F}}_r\right) = \mathbf{0}$. Hence:

$$\begin{aligned}
\mathbf{0} &= \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right) = \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(-\mathbf{U}_N^{(1)}{}_{**}\left(\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right)\right) \\
&= \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right) = \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left((1-\mu_N^{(1)})\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right)\right) \\
&= \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right).
\end{aligned}$$

□

3.4 Normal forms for N-orbital–reversible vector fields

In this section, we plain to deduce some facts about the normal form procedure for orbital–reversible vector fields. It is well known that the normal forms are not uniquely determined by the vector field, and its application to an specific problem requires to refine the normalization procedure. When dealing with the orbital–reversibility problem, it is convenient to determine normal forms modulo reversible terms, that is, to simplify as much as possible the non-reversible terms. In this way, we put in evidence the non-reversible invariant terms of the vector field.

To this end, we use that $\mathcal{Q}_k^{\mathbf{t}} = \mathcal{R}_k^{\mathbf{t}} \bigoplus \mathcal{S}_k^{\mathbf{t}}$, which allows to write the vector field (3.1.3) as:

$$\mathbf{F} = \widetilde{\mathbf{F}}_r + \sum_{j=1}^{\infty}\left(\widetilde{\mathbf{F}}_{r+j} + \overline{\mathbf{F}}_{r+j}\right), \tag{3.4.1}$$

where $\widetilde{\mathbf{F}}_{r+j} = \Pi^{(\mathtt{r})}\left(\mathbf{F}_{r+j}\right) \in \mathcal{R}_{r+j}^{\mathbf{t}}$ and $\overline{\mathbf{F}}_{r+j} = \Pi^{(\mathtt{s})}\left(\mathbf{F}_{r+j}\right) \in \mathcal{S}_{r+j}^{\mathbf{t}}$.

To describe a normal form procedure well adapted to the orbital-reversibility problem, let us denote the above vector field as

$$\mathbf{F}^{(0)} := \mathbf{F} = \widetilde{\mathbf{F}}_r^{(0)} + \left(\widetilde{\mathbf{F}}_{r+1}^{(0)} + \overline{\mathbf{F}}_{r+1}^{(0)}\right) + \cdots.$$

We observe that the principal part is reversible: $\widetilde{\mathbf{F}}_r^{(0)} \in \mathcal{R}_r^{\mathbf{t}}$.

The first step in the normal form procedure consists into simplifying the $(r+1)$–th degree quasi-homogeneous term $\overline{\mathbf{F}}_{r+1}^{(0)}$, which allows us to determine if it is reversible.

From Theorem 40, we consider a generator $\widetilde{\mathbf{U}}_1 \in \widehat{\mathcal{R}}_1^{\mathbf{t}} \subseteq \mathcal{Q}_1^{\mathbf{t}}$ and a time reparametrization $\widetilde{\mu}_1 \in \widehat{\mathcal{O}}_1^{\mathbf{t}} \subseteq \mathcal{P}_1^{\mathbf{t}}$. Then, the vector field $\mathbf{F}^{(0)}$ is transformed into:

$$\mathbf{F}^{(1)} = \widetilde{\mathbf{U}}_{1\,**}\left((1+\widetilde{\mu}_1)\mathbf{F}^{(0)}\right) = \widetilde{\mathbf{F}}_r^{(0)} + \widetilde{\mathbf{F}}_{r+1}^{(0)} + \overline{\mathbf{F}}_{r+1}^{(0)} + \left[\widetilde{\mathbf{F}}_r^{(0)}, \widetilde{\mathbf{U}}_1\right] + \widetilde{\mu}_1\widetilde{\mathbf{F}}_r^{(0)} + \cdots .$$

We observe that the $(r+1)$–th degree quasi-homogeneous term of $\mathbf{F}^{(1)}$ can be written as

$$\mathbf{F}_{r+1}^{(1)} = \widetilde{\mathbf{F}}_{r+1}^{(0)} + \overline{\mathbf{F}}_{r+1}^{(0)} - \overline{\mathcal{L}}^{(1)}\left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1\right),$$

where we have introduced the following homological operator:

$$\begin{array}{rcl} \overline{\mathcal{L}}^{(1)} : \widehat{\mathcal{R}}_1^{\mathbf{t}} \times \widehat{\mathcal{O}}_1^{\mathbf{t}} & \longrightarrow & \mathcal{S}_{r+1}^{\mathbf{t}} \\ \left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1\right) & \longrightarrow & -\left[\widetilde{\mathbf{F}}_r^{(0)}, \widetilde{\mathbf{U}}_1\right] - \widetilde{\mu}_1\widetilde{\mathbf{F}}_r^{(0)}. \end{array} \tag{3.4.2}$$

Observe that this linear operator depends of $\widetilde{\mathbf{F}}_r^{(0)}$, and we can make explicit this dependence by writing $\overline{\mathcal{L}}^{(1)} = \overline{\mathcal{L}}^{(1)}_{\left\{\widetilde{\mathbf{F}}_r^{(0)}\right\}}$.

Following the standard ideas of the normal form theory, we consider the decomposition $\mathcal{S}_{r+1}^{\mathbf{t}} = \text{Range}\left(\overline{\mathcal{L}}^{(1)}\right) \bigoplus \text{Cor}\left(\overline{\mathcal{L}}^{(1)}\right)$, which allows us to write $\overline{\mathbf{F}}_{r+1}^{(0)} = \overline{\mathbf{F}}_{r+1}^{r} + \overline{\mathbf{F}}_{r+1}^{c}$, where $\overline{\mathbf{F}}_{r+1}^{r} \in \text{Range}\left(\overline{\mathcal{L}}^{(1)}\right)$ and $\overline{\mathbf{F}}_{r+1}^{c} \in \text{Cor}\left(\overline{\mathcal{L}}^{(1)}\right)$. It is enough to select $\left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1\right) \in \widehat{\mathcal{R}}_1^{\mathbf{t}} \times \widehat{\mathcal{O}}_1^{\mathbf{t}}$ such that $\overline{\mathcal{L}}^{(1)}\left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1\right) = \overline{\mathbf{F}}_{r+1}^{r}$, to obtain:

$$\mathbf{F}^{(1)} = \widetilde{\mathbf{F}}_r^{(0)} + \widetilde{\mathbf{F}}_{r+1}^{(0)} + \overline{\mathbf{F}}_{r+1}^{c} + \cdots ,$$

If $\overline{\mathbf{F}}_{r+1}^{c} \neq \mathbf{0}$, we will show in Theorem 50 that the vector field (3.4.1) is not orbital–reversible.

In the case $\overline{\mathbf{F}}_{r+1}^{c} = \mathbf{0}$, the vector field is 1-orbital–reversible and we continue with the normal form procedure in order to analyze the 2-orbital–reversibility. In this situation, we have

$$\mathbf{F}^{(1)} = \widetilde{\mathbf{F}}_r^{(1)} + \widetilde{\mathbf{F}}_{r+1}^{(1)} + \widetilde{\mathbf{F}}_{r+2}^{(1)} + \overline{\mathbf{F}}_{r+2}^{(1)} + \cdots , \tag{3.4.3}$$

where $\widetilde{\mathbf{F}}_{r+j}^{(1)} := \widetilde{\mathbf{F}}_{r+j}^{(0)}$ for $j = 0, 1$.

In order to simplify the $(r+2)$–th degree quasi-homogeneous term $\overline{\mathbf{F}}_{r+2}^{(1)}$, we use a generator $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}_1 + \widetilde{\mathbf{U}}_2$, (with $\widetilde{\mathbf{U}}_j \in \widehat{\mathcal{R}}_j^{\mathbf{t}}$) and $\widetilde{\mu} = \widetilde{\mu}_1 + \widetilde{\mu}_2$ (with $\widetilde{\mu}_i \in \widehat{\mathcal{O}}_i^{\mathbf{t}}$). To leave unaltered the $(r+1)$–th degree quasi-homogeneous term, we choose $\left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1\right) \in \text{Ker}\left(\overline{\mathcal{L}}^{(1)}\right)$. In this way, the transformed vector field:

$$\mathbf{F}^{(2)} = \widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}^{(1)}\right),$$

agrees with $\mathbf{F}^{(1)}$ up to quasi-homogeneous degree $(r+1)$, and the $(r+2)$–th degree quasi-homogeneous term becomes

$$\begin{aligned}\mathbf{F}^{(2)}_{r+2} &= \widetilde{\mathbf{F}}^{(1)}_{r+2} + \overline{\mathbf{F}}^{(1)}_{r+2} + \widetilde{\mu}_1\widetilde{\mathbf{F}}^{(0)}_{r+1} + \widetilde{\mu}_2\widetilde{\mathbf{F}}^{(0)}_{r} + \left[\widetilde{\mathbf{F}}^{(0)}_{r+1}, \widetilde{\mathbf{U}}_1\right] + \left[\widetilde{\mu}_1\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_1\right] + \left[\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_2\right] \\ &\quad + \frac{1}{2!}\left[\left[\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_1\right], \widetilde{\mathbf{U}}_1\right].\end{aligned}$$

Using that $\left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1\right) \in \mathrm{Ker}\left(\overline{\mathcal{L}}^{(1)}\right)$, i.e.: $\left[\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_1\right] = -\widetilde{\mu}_1\widetilde{\mathbf{F}}^{(0)}_{r}$, we can write

$$\begin{aligned}\left[\left[\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_1\right], \widetilde{\mathbf{U}}_1\right] &= -\left[\widetilde{\mu}_1\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_1\right] = -\left(\nabla\widetilde{\mu}_1 \cdot \widetilde{\mathbf{U}}_1\right)\widetilde{\mathbf{F}}^{(0)}_{r} - \widetilde{\mu}_1\left[\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_1\right] \\ &= -\left(\nabla\widetilde{\mu}_1 \cdot \widetilde{\mathbf{U}}_1 - \widetilde{\mu}_1^2\right)\widetilde{\mathbf{F}}^{(0)}_{r}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathbf{F}^{(2)}_{r+2} &= \widetilde{\mathbf{F}}^{(1)}_{r+2} + \frac{1}{2}\left(\nabla\widetilde{\mu}_1 \cdot \widetilde{\mathbf{U}}_1 - \widetilde{\mu}_1^2\right)\widetilde{\mathbf{F}}^{(0)}_{r} \\ &\quad + \overline{\mathbf{F}}^{(1)}_{r+2} + \left[\widetilde{\mathbf{F}}^{(0)}_{r+1}, \widetilde{\mathbf{U}}_1\right] + \left[\widetilde{\mathbf{F}}^{(0)}_{r}, \widetilde{\mathbf{U}}_2\right] + \widetilde{\mu}_1\widetilde{\mathbf{F}}^{(0)}_{r+1} + \widetilde{\mu}_2\widetilde{\mathbf{F}}^{(0)}_{r}.\end{aligned}$$

We have rearranged the terms in the above expression so that the elements in the last line belong to $\mathcal{S}^{\mathbf{t}}_{r+2}$, whereas those of the first line are all reversible (see Proposition 38(a)).

Again, we define the corresponding homological operator:

$$\begin{aligned}\overline{\mathcal{L}}^{(2)} : \mathrm{Ker}\left(\overline{\mathcal{L}}^{(1)}\right) \times \left(\widehat{\mathcal{R}}^{\mathbf{t}}_2, \widehat{\mathcal{O}}^{\mathbf{t}}_2\right) &\longrightarrow \mathcal{S}^{\mathbf{t}}_{r+2} \\ \left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1; \widetilde{\mathbf{U}}_2, \widetilde{\mu}_2\right) &\longrightarrow -\left[\widetilde{\mathbf{F}}^{(1)}_{r}, \widetilde{\mathbf{U}}_2\right] - \widetilde{\mu}_2\widetilde{\mathbf{F}}^{(1)}_{r} - \left[\widetilde{\mathbf{F}}^{(1)}_{r+1}, \widetilde{\mathbf{U}}_1\right] - \widetilde{\mu}_1\widetilde{\mathbf{F}}^{(1)}_{r+1}.\end{aligned}$$

Observe that the operator $\overline{\mathcal{L}}^{(2)}$ depends on $\widetilde{\mathbf{F}}^{(1)}_{r}, \widetilde{\mathbf{F}}^{(1)}_{r+1}$, i.e. $\overline{\mathcal{L}}^{(2)} = \overline{\mathcal{L}}^{(2)}_{\left\{\widetilde{\mathbf{F}}^{(1)}_{r}, \widetilde{\mathbf{F}}^{(1)}_{r+1}\right\}}$.

Now, we proceed again as above:

We consider the splitting $\mathcal{S}^{\mathbf{t}}_{r+2} = \mathrm{Range}\left(\overline{\mathcal{L}}^{(2)}\right) \bigoplus \mathrm{Cor}\left(\overline{\mathcal{L}}^{(2)}\right)$, and write $\overline{\mathbf{F}}^{(1)}_{r+2} = \overline{\mathbf{F}}^{r}_{r+2} + \overline{\mathbf{F}}^{c}_{r+2}$, with $\overline{\mathbf{F}}^{r}_{r+2} \in \mathrm{Range}\left(\overline{\mathcal{L}}^{(2)}\right)$ and $\overline{\mathbf{F}}^{c}_{r+2} \in \mathrm{Cor}\left(\overline{\mathcal{L}}^{(2)}\right)$. Selecting $\left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1; \widetilde{\mathbf{U}}_2, \widetilde{\mu}_2\right) \in \mathrm{Ker}\left(\overline{\mathcal{L}}^{(1)}\right) \times \left(\widehat{\mathcal{R}}^{\mathbf{t}}_2, \widehat{\mathcal{O}}^{\mathbf{t}}_2\right)$ such that $\overline{\mathcal{L}}^{(2)}\left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1; \widetilde{\mathbf{U}}_2, \widetilde{\mu}_2\right) = \overline{\mathbf{F}}^{r}_{r+2}$, we get:

$$\mathbf{F}^{(2)} = \widetilde{\mathbf{F}}^{(2)}_{r} + \widetilde{\mathbf{F}}^{(2)}_{r+1} + \left(\widetilde{\mathbf{F}}^{(2)}_{r+2} + \overline{\mathbf{F}}^{c}_{r+2}\right) + \cdots,$$

where $\widetilde{\mathbf{F}}^{(2)}_{r+j} = \widetilde{\mathbf{F}}^{(1)}_{r+j}$ for $j = 0, 1, 2$.

If $\overline{\mathbf{F}}^{c}_{r+2} \neq \mathbf{0}$, we will show in Theorem 50 that the vector field (3.4.1) is not orbital–reversible.

If $\overline{\mathbf{F}}^{c}_{r+2} = \mathbf{0}$, the quoted vector field is 2-orbital–reversible, and we could continue analyzing the 3-orbital–reversibility.

Next, we describe the normal form procedure at an arbitrary quasi-homogeneous degree $r+m$, $(m \in \mathbb{N})$. We assume that we deal with a $(r+m-1)$-orbital–reversible vector field:

$$\mathbf{F}^{(m-1)} = \widetilde{\mathbf{F}}^{(m-1)}_{r} + \cdots + \widetilde{\mathbf{F}}^{(m-1)}_{r+m-1} + \sum_{j \geq m} \left(\widetilde{\mathbf{F}}^{(m-1)}_{r+j} + \overline{\mathbf{F}}^{(m-1)}_{r+j} \right),$$

where $\widetilde{\mathbf{F}}^{(m-1)}_{r+j} = \widetilde{\mathbf{F}}^{(m-2)}_{r+j}$ for $j = 0, 1, \ldots, m-1$ and $\widetilde{\mathbf{F}}^{(m-1)}_{r+j} \in \mathcal{R}^{\mathbf{t}}_{r+j}$, $\overline{\mathbf{F}}^{(m-1)}_{r+j} \in \mathcal{S}^{\mathbf{t}}_{r+j}$ for $j \geq m$. The analysis of the simplification of the $(r+m)$th-degree quasi-homogeneous term (without changing the lower degree terms) involves the corresponding homological operator $\overline{\mathcal{L}}^{(m)}$, which is defined by:

$$\begin{aligned} \overline{\mathcal{L}}^{(m)} : \operatorname{Ker}\left(\overline{\mathcal{L}}^{(m-1)}\right) \times \left(\widehat{\mathcal{R}}^{\mathbf{t}}_{m}, \widehat{\mathcal{O}}^{\mathbf{t}}_{m}\right) &\longrightarrow \mathcal{S}^{\mathbf{t}}_{r+m} \\ \left(\widetilde{\mathbf{U}}_1, \widetilde{\mu}_1, \cdots, \widetilde{\mathbf{U}}_{m-1}, \widetilde{\mu}_{m-1}; \widetilde{\mathbf{U}}_m, \widetilde{\mu}_m\right) &\longrightarrow -\sum_{j=0}^{m-1} \left(\left[\widetilde{\mathbf{F}}^{(m-1)}_{r+j}, \widetilde{\mathbf{U}}_{m-j} \right] + \widetilde{\mu}_{m-j} \widetilde{\mathbf{F}}^{(m-1)}_{r+j} \right). \end{aligned}$$

Proposition 38(b-c) yields that $\overline{\mathcal{L}}^{(m)}$ is well defined. Also, it is evident that the operator $\overline{\mathcal{L}}^{(m)}$ depends on $\widetilde{\mathbf{F}}^{(m-1)}_{r}, \cdots, \widetilde{\mathbf{F}}^{(m-1)}_{r+m-1}$: $\overline{\mathcal{L}}^{(m)} = \overline{\mathcal{L}}^{(m)}_{\left\{\widetilde{\mathbf{F}}^{(m-1)}_{r}, \ldots, \widetilde{\mathbf{F}}^{(m-1)}_{r+m-1}\right\}}$.

The normal form procedure at order $r+m$ is presented in Proposition 45. Before the statement of this proposition, we require some technical results.

The following two lemmas deal with vector fields having a Lie symmetry up to order $r+m$. The first one has been proved in Chapter 2, Lemma 18; however, we include here its statement to completeness.

Lemma 42. *Let us consider the vector field* $\mathbf{F}$ *given in (3.4.1), and assume that exists* $\mathbf{U} \in \bigoplus_{j=1}^{m} \mathcal{Q}^{\mathbf{t}}_{j}$ *such that*

$$\mathcal{J}^{r+m}\left([\mathbf{F}, \mathbf{U}]\right) = \mathcal{J}^{r+m}\left(\nu \mathbf{F}\right), \quad \textit{for some } \nu \in \bigoplus_{j=1}^{m} \mathcal{P}^{\mathbf{t}}_{j},$$

where $m \in \mathbb{N}$. *Then, for any* $\mu \in \bigoplus_{j=1}^{m} \mathcal{P}^{\mathbf{t}}_{j}$, $l \geq 0$, *and* $k = l+1, \ldots, l+m$, *we have:*

$$\left(T^{(l)}_{\mathbf{U}}\left(\mu \mathbf{F}\right)\right)_{r+k} = \sum_{j=l+1}^{k} \mu^{(l)}_{j} \mathbf{F}_{r+k-j},$$

for some $\mu^{(l)}_{j} \in \mathcal{P}^{\mathbf{t}}_{j}$.

Lemma 43. *Let us consider the vector field* $\mathbf{F}$ *given in (3.4.1), and assume that there exist* $\mathbf{U} \in \bigoplus_{j=1}^{m+1} \mathcal{Q}_j^{\mathbf{t}}$ *such that*

$$\mathcal{J}^{r+m}\left([\mathbf{F},\mathbf{U}]\right) = \mathcal{J}^{r+m}\left(\nu\mathbf{F}\right), \quad \textit{for some } \nu \in \bigoplus_{j=1}^{m} \mathcal{P}_j^{\mathbf{t}},$$

where $m \in \mathbb{N}$. *Then, for any* $\mu \in \bigoplus_{j=1}^{m+1} \mathcal{P}_j^{\mathbf{t}}$, *there exists* $\lambda \in \bigoplus_{j=1}^{m+1} \mathcal{P}_j^{\mathbf{t}}$ *such that*

$$\mathcal{J}^{r+m+1}\left(\mathbf{U} *_{**} \left((1+\mu)\mathbf{F}\right)\right) = \mathcal{J}^{r+m+1}\left([\mathbf{F},\mathbf{U}] + (1+\lambda)\mathbf{F}\right).$$

Proof. As $[\mathbf{F},\mathbf{U}]_{r+k} = (\nu\mathbf{F})_{r+k}$ for $k = 1,\dots,m$, it can be shown that

$$\left(T_{\mathbf{U}}^{(l)}\left([\mathbf{F},\mathbf{U}]\right)\right)_{r+k} = \left(T_{\mathbf{U}}^{(l)}\left(\nu\mathbf{F}\right)\right)_{r+k},$$

for all $l \geq 0$, $k = l+1,\dots,l+m$. Using that

$$\mathbf{U} *_{**} \left((1+\mu)\mathbf{F}\right) = \mathbf{F} + [\mathbf{F},\mathbf{U}] + \mu\mathbf{F} + \sum_{l=1}^{\infty} \tfrac{1}{(l+1)!} T_{\mathbf{U}}^{(l)}\left([\mathbf{F},\mathbf{U}] + (l+1)\mu\mathbf{F}\right),$$

we have

$$\begin{aligned}
\left(\mathbf{U} *_{**} \left((1+\mu)\mathbf{F}\right)\right)_{r+k} &= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \mu\mathbf{F}\right)_{r+k} \\
&\quad + \sum_{l=1}^{k-1} \frac{1}{(l+1)!} \left(T_{\mathbf{U}}^{(l)}\left([\mathbf{F},\mathbf{U}] + (l+1)\mu\mathbf{F}\right)\right)_{r+k} \\
&= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \mu\mathbf{F}\right)_{r+k} \\
&\quad + \sum_{l=1}^{k-1} \frac{1}{(l+1)!} \left(T_{\mathbf{U}}^{(l)}\left((\nu + (l+1)\mu)\mathbf{F}\right)\right)_{r+k}.
\end{aligned}$$

From Lemma 42, we get:

$$\begin{aligned}
\left(\mathbf{U} *_{**} \left((1+\mu)\mathbf{F}\right)\right)_{r+k} &= \mathbf{F}_{r+k} + \left([\mathbf{F},\mathbf{U}] + \mu\mathbf{F}\right)_{r+k} + \sum_{l=1}^{k-1} \sum_{j=l+1}^{k} \mu_j^{(l)} \mathbf{F}_{r+k-j} \\
&= \mathbf{F}_{r+k} + [\mathbf{F},\mathbf{U}]_{r+k} + \sum_{j=1}^{k} \mu_j \mathbf{F}_{r+k-j} + \sum_{j=2}^{k} \sum_{l=1}^{j-1} \mu_j^{(l)} \mathbf{F}_{r+k-j} \\
&= \mathbf{F}_{r+k} + [\mathbf{F},\mathbf{U}]_{r+k} + \sum_{j=1}^{k} \left(\mu_j + \sum_{l=1}^{j-1} \mu_j^{(l)}\right) \mathbf{F}_{r+k-j},
\end{aligned}$$

which provides the result by introducing $\lambda_j = \mu_j + \sum_{l=1}^{j-1} \mu_j^{(l)}$, for $j = 1,\dots,m+1$. We notice that $\mu_j^{(l)}$ depend univocally on $\mathcal{J}^{j-l}(\mathbf{U},\nu,\mu)$. □

Lemma 44. *Let us consider the vector field* $\mathbf{F}$ *given in (3.4.1). Then, for each* $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{m} \mathcal{R}_j^{\mathbf{t}}$ *and* $\widetilde{\lambda} \in \bigoplus_{j=1}^{m} \mathcal{O}_j^{\mathbf{t}}$ *($m \in \mathbb{N}$), there exist* $\widetilde{\mathbf{V}} \in \bigoplus_{j=1}^{m} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ *and* $\widetilde{\nu} \in \bigoplus_{j=1}^{m} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ *such that*

$$\mathcal{J}^{r+m}\left(\Pi^{(s)}\left(\left[\mathbf{F}, \widetilde{\mathbf{U}}\right] + \widetilde{\lambda}\mathbf{F}\right)\right) = \mathcal{J}^{r+m}\left(\Pi^{(s)}\left(\left[\mathbf{F}, \widetilde{\mathbf{V}}\right] + \widetilde{\nu}\mathbf{F}\right)\right).$$

Proof. Let us denote

$$\kappa := \min\left\{j \in \mathbb{N} : \mathrm{Proy}_{\mathrm{Range}(\ell_{j-r})}\left(\widetilde{\lambda}_j\right) \neq 0 \ \text{ or } \ \mathrm{Proy}_{\mathrm{Ker}(\ell_{j-r})\widetilde{\mathbf{F}}_r}\left(\widetilde{\mathbf{U}}_j\right) \neq \mathbf{0}\right\}.$$

Observe that $\kappa \leq m$, otherwise $\widetilde{\mathbf{V}} = \widetilde{\mathbf{U}}$ and $\widetilde{\nu} = \widetilde{\lambda}$.

As $\mathcal{O}_\kappa^{\mathbf{t}} = \mathrm{Range}\left(\ell_{\kappa-r}|_{\mathcal{E}_k^{\mathbf{t}}}\right) \bigoplus \widehat{\mathcal{O}}_\kappa^{\mathbf{t}}$, we can write $\widetilde{\lambda}_\kappa = \widetilde{\lambda}_\kappa^{(1)} + \widetilde{\lambda}_\kappa^{(2)}$, where $\widetilde{\lambda}_\kappa^{(1)} = \nabla\overline{\rho} \cdot \widetilde{\mathbf{F}}_r \in \mathrm{Range}\left(\ell_{\kappa-r}|_{\mathcal{E}_k^{\mathbf{t}}}\right)$ $(\overline{\rho} \in \mathcal{E}_{\kappa-r}^{\mathbf{t}})$ and $\widetilde{\lambda}_\kappa^{(2)} \in \widehat{\mathcal{O}}_\kappa^{\mathbf{t}}$. We observe that $\overline{\rho} = 0$ if $\widetilde{\lambda}_\kappa^{(1)} = 0$, and $\overline{\rho} \in \mathcal{E}_{\kappa-r}^{\mathbf{t}} \setminus \mathrm{Ker}\left(\ell_{\kappa-r}\right)$ otherwise.

Moreover, by (3.3.2), we can write $\widetilde{\mathbf{U}}_\kappa = \widetilde{\mathbf{U}}_\kappa^{(1)} + \widetilde{\mathbf{U}}_\kappa^{(2)}$, where $\widetilde{\mathbf{U}}_\kappa^{(1)} = \overline{\zeta}\widetilde{\mathbf{F}}_r \in \mathrm{Ker}\left(\ell_{\kappa-r}|_{\mathcal{E}_{\kappa-r}^{\mathbf{t}}}\right)\widetilde{\mathbf{F}}_r$ $(\overline{\zeta} \in \mathrm{Ker}\left(\ell_{\kappa-r}|_{\mathcal{E}_{\kappa-r}^{\mathbf{t}}}\right))$ and $\widetilde{\mathbf{U}}_\kappa^{(2)} \in \widehat{\mathcal{R}}_\kappa^{\mathbf{t}}$.

Let us denote $\widetilde{\mathbf{F}} = \Pi^{(\mathtt{r})}(\mathbf{F})$ and define $\widetilde{\mathbf{V}} := \widetilde{\mathbf{U}} - (\overline{\rho} + \overline{\zeta})\widetilde{\mathbf{F}}$, where $\widetilde{\nu} := \widetilde{\lambda} - \nabla\overline{\rho} \cdot \widetilde{\mathbf{F}} - \nabla\overline{\zeta} \cdot \widetilde{\mathbf{F}}$. From Proposition 38, we obtain:

$$\begin{aligned}
\left(\Pi^{(\mathtt{s})}\left(\left[\mathbf{F}, \widetilde{\mathbf{U}}\right] + \widetilde{\lambda}\mathbf{F}\right)\right)_{r+\kappa} &= \left(\left[\widetilde{\mathbf{F}}, \widetilde{\mathbf{U}}\right] + \widetilde{\lambda}\widetilde{\mathbf{F}}\right)_{r+\kappa} = \left(\left[\widetilde{\mathbf{F}}, \widetilde{\mathbf{V}} + (\overline{\rho} + \overline{\zeta})\widetilde{\mathbf{F}}\right] + \widetilde{\lambda}\widetilde{\mathbf{F}}\right)_{r+\kappa} \\
&= \left(\left[\widetilde{\mathbf{F}}, \widetilde{\mathbf{V}}\right] + \left[\widetilde{\mathbf{F}}, (\overline{\rho} + \overline{\lambda})\widetilde{\mathbf{F}}\right] + \widetilde{\lambda}\widetilde{\mathbf{F}}\right)_{r+\kappa} \\
&= \left(\left[\widetilde{\mathbf{F}}, \widetilde{\mathbf{V}}\right] + \left(\widetilde{\lambda} - \nabla\overline{\rho} \cdot \widetilde{\mathbf{F}} - \nabla\overline{\zeta} \cdot \widetilde{\mathbf{F}}\right)\widetilde{\mathbf{F}}\right)_{r+\kappa} \\
&= \left(\left[\widetilde{\mathbf{F}}, \widetilde{\mathbf{V}}\right] + \widetilde{\nu}\widetilde{\mathbf{F}}\right)_{r+\kappa} \\
&= \left(\Pi^{(\mathtt{s})}\left(\left[\mathbf{F}, \widetilde{\mathbf{V}}\right] + \widetilde{\nu}\mathbf{F}\right)\right)_{r+\kappa}.
\end{aligned}$$

To complete the proof, it is enough to observe that:

$$\begin{aligned}
\widetilde{\mathbf{V}}_\kappa &= \widetilde{\mathbf{U}}_\kappa^{(1)} + \widetilde{\mathbf{U}}_\kappa^{(2)} - (\overline{\rho} + \overline{\zeta})\widetilde{\mathbf{F}}_r = \widetilde{\mathbf{U}}_\kappa^{(2)} - \overline{\rho}\widetilde{\mathbf{F}}_r \in \widehat{\mathcal{R}}_\kappa^{\mathbf{t}} \\
\widetilde{\nu}_\kappa &= \widetilde{\lambda}_\kappa^{(1)} + \widetilde{\lambda}_\kappa^{(2)} - \nabla\overline{\rho} \cdot \widetilde{\mathbf{F}}_r - \nabla\overline{\zeta} \cdot \widetilde{\mathbf{F}}_r = \widetilde{\lambda}_\kappa^{(2)} \in \widehat{\mathcal{O}}_{\kappa-r}^{\mathbf{t}}.
\end{aligned}$$

□

Proposition 45. *Let us assume that the vector field* $\mathbf{F}$ *given in (3.4.1) is* R_x*-reversible up to order* $r + N - 1$*, with* $N \in \mathbb{N}$*. Then, for all* $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ *and* $\widetilde{\mu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ *such that* $\mathcal{J}^{r+N-1}\left(\Pi^{(s)}\left(\widetilde{\mathbf{U}}_{**}\left((1 + \widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}$*, there exist* $\overline{\lambda} \in \bigoplus_{j=1}^{N} \mathcal{E}_j^{\mathbf{t}}$*,* $\widetilde{\mathbf{V}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ *and* $\widetilde{\nu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ *such that:*

(a) $\mathcal{J}^{r+N-1}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) = \mathcal{J}^{r+N-1}\left((1+\overline{\lambda})\mathbf{F}\right)$.

(b) $\mathrm{Proy}_{\mathcal{S}^{\mathbf{t}}_{r+N}}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) = \overline{\mathbf{F}}_{r+N} - \overline{\mathcal{L}}^{(N)}_{\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\}}\left(\widetilde{\mathbf{V}},\widetilde{\nu}\right)$.

(c) $\mathrm{Proy}_{\mathcal{R}^{\mathbf{t}}_{r+N}}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) = \left((1+\overline{\lambda})\widetilde{\mathbf{F}}\right)_{r+N}$.

Proof. Firstly, we show that there exists $\lambda \in \bigoplus_{j=1}^{N}\mathcal{P}^{\mathbf{t}}_j$ such that

$$\mathcal{J}^{r+N}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) = \mathcal{J}^{r+N}\left(\left[\mathbf{F},\widetilde{\mathbf{U}}\right] + (1+\lambda)\mathbf{F}\right).$$

We use induction on N. If $N=1$, the result is trivial. Moreover

$$\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)_{r+1} = \mathbf{F}_{r+1} + \left([\mathbf{F},\widetilde{\mathbf{U}}] + \widetilde{\mu}\mathbf{F}\right)_{r+1},$$

and it is enough to take $\lambda_1 = \widetilde{\mu}_1$.

Now, let us consider $N>1$ and assume that there exists $\lambda = \sum_{j=1}^{N-1}\lambda_j$, such that

$$\mathcal{J}^{r+N-1}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) = \mathcal{J}^{r+N-1}\left(\left[\mathbf{F},\widetilde{\mathbf{U}}\right] + (1+\lambda)\mathbf{F}\right).$$

The hypothesis $\mathcal{J}^{r+N-1}\left(\Pi^{(\mathsf{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}$ yields

$$\mathcal{J}^{r+N-1}\left(\Pi^{(\mathsf{s})}\left(\left[\mathbf{F},\widetilde{\mathbf{U}}\right] + (1+\lambda)\mathbf{F}\right)\right) = \mathcal{J}^{r+N-1}\left(\left[\mathbf{F},\widetilde{\mathbf{U}}\right] + (1+\widetilde{\lambda})\mathbf{F}\right) = \mathbf{0},$$

where $\widetilde{\lambda} = \pi^{(\mathsf{o})}(\lambda)$ and $\overline{\lambda} = \pi^{(\mathsf{e})}(\lambda)$. Hence $\mathcal{J}^{r+N-1}\left(\left[\mathbf{F},\widetilde{\mathbf{U}}\right] + \widetilde{\lambda}\mathbf{F}\right) = \mathbf{0}$ and from Lemma 43, the statement holds for the value N. Consequently, we have:

$$\begin{aligned}\mathcal{J}^{r+N}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) &= \mathcal{J}^{r+N}\left(\left[\mathbf{F},\widetilde{\mathbf{U}}\right] + (1+\lambda)\mathbf{F}\right)\\ &= \mathcal{J}^{r+N}\left(\left[\mathbf{F},\widetilde{\mathbf{U}}\right] + \widetilde{\lambda}\mathbf{F}\right) + \mathcal{J}^{r+N}\left((1+\overline{\lambda})\mathbf{F}\right).\end{aligned}$$

Let us define $\widetilde{\mathbf{V}}$, $\widetilde{\nu}$ as indicated in Lemma 44. We observe that:

$$\mathcal{J}^{r+N}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) = \mathcal{J}^{r+N}\left(\left[\mathbf{F},\widetilde{\mathbf{V}}\right] + \widetilde{\nu}\mathbf{F}\right) + \mathcal{J}^{r+N}\left((1+\overline{\lambda})\mathbf{F}\right).$$

As $\mathbf{F}$ is R_x-reversible up to order $r+N-1$ and $\mathcal{J}^{r+N-1}\left(\Pi^{(\mathsf{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}$, we prove the item **(a)**. Moreover, we obtain that $(\widetilde{\mathbf{V}},\widetilde{\nu})$ belongs to the domain of $\overline{\mathcal{L}}^{(N)}_{\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\}}$, and then items **(b)** and **(c)** hold. □

Lemma 46. *Let us assume that the vector field* $\mathbf{F}$ *given in (3.4.1) is* R_x*-reversible up to order* $r+N-1$*, with* $N \in \mathbb{N}$*, and consider* $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ *and* $\mu \in \widehat{\mathcal{O}}_1^{\mathbf{t}} \oplus \bigoplus_{j=2}^{N} \mathcal{P}_j^{\mathbf{t}}$ *such that* $\mathcal{J}^{r+N-1}\left(\Pi^{(s)}\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right) = \mathbf{0}$*. Then, there exists* $\widetilde{\mu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ *such that*

(a) $\mathcal{J}^{r+N-1}\left(\Pi^{(s)}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}$,

(b) $\Pi^{(s)}\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right)_{r+N} = \Pi^{(s)}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)_{r+N}$.

Proof. Let us denote

$$\kappa = \min\left\{k \in \mathbb{N} : \mathrm{Proy}_{\mathcal{E}_k^{\mathbf{t}}}(\mu_k) \neq 0\right\}.$$

We assume $2 \leq \kappa \leq N$ (otherwise, taking $\widetilde{\mu} = \mu$, we complete the proof).

From the definition of κ, we get that $\mu_k \in \widehat{\mathcal{O}}_k^{\mathbf{t}}$ for $k = 1, \ldots, \kappa - 1$. Our goal is to show that the result is true for the κ–degree quasi-homogeneous terms and then, repeating the procedure, we get the result.

By (3.3.1) and (3.3.4), we can split $\mu_\kappa = \mu_\kappa^{(1)} + \mu_\kappa^{(2)}$, with $\mu_\kappa^{(1)} \in \mathrm{Cor}\left(\ell_{\kappa-r}|_{\mathcal{O}_\kappa^{\mathbf{t}}}\right)$ and $\mu_\kappa^{(2)} \in \widehat{\mathcal{O}}_\kappa^{\mathbf{t}}$.

From Lemma 13, there exists $\lambda \in \bigoplus_{j\geq\kappa} \mathcal{P}_j^{\mathbf{t}}$ such that

$$\left(1-\mu_\kappa^{(1)}\right)\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right) = \widetilde{\mathbf{U}}_{**}\left((1+\lambda)(1+\mu)\mathbf{F}\right) = \widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right),$$

where we have defined $\widetilde{\mu} := (1+\lambda)(1+\mu) - 1$. We observe that, as $\lambda_\kappa = -\mu_\kappa^{(1)}$, then $\mathcal{J}^{\kappa-1}(\widetilde{\mu}) = \mathcal{J}^{\kappa-1}(\mu)$, and

$$\widetilde{\mu}_\kappa = \mu_\kappa - \mu_\kappa^{(1)} = \mu_\kappa^{(2)} \in \widehat{\mathcal{O}}_\kappa^{\mathbf{t}}.$$

Then:

$$\begin{aligned}
&\mathcal{J}^{r+N-1}\left(\Pi^{(\mathtt{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right) = \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left((1-\mu_\kappa^{(1)})\left(\widetilde{\mathbf{U}}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right)\right) \\
&= \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\lambda)(1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathcal{J}^{r+N}\left(\Pi^{(\mathtt{s})}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}.
\end{aligned}$$

To complete the proof, it is enough to repeat the reasoning on $\widetilde{\mu}$ until we get $\kappa > N$. □

Proposition 47. *Let us assume that the vector field* $\mathbf{F}$ *given in (3.4.1) is* R_x*-reversible up to order* $r+N-1$*, with* $N \in \mathbb{N}$*. Then, for all* $\widetilde{\mathbf{V}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ *and*

$\widetilde{\nu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ *such that* $\left(\widetilde{\mathbf{V}}, \widetilde{\nu}\right)$ *belongs to the domain of* $\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\right\}}$, *there exists* $\widetilde{\mu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ *such that:*

$$\mathcal{J}^{r+N-1}\left(\Pi^{(s)}\left(\widetilde{\mathbf{V}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) = \mathbf{0}, \quad \textit{and}$$
$$\Pi^{(s)}\left(\widetilde{\mathbf{V}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)_{r+N} = \overline{\mathbf{F}}_{r+N} - \overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\right\}}\left(\widetilde{\mathbf{V}}, \widetilde{\nu}\right).$$

Proof. As $\left(\widetilde{\mathbf{V}}, \widetilde{\nu}\right)$ belongs to the domain of the operator $\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\right\}}$, then we obtain $\mathcal{J}^{r+N-1}\left(\Pi^{(\mathsf{s})}\left([\mathbf{F}, \widetilde{\mathbf{V}}] + \widetilde{\nu}\mathbf{F}\right)\right) = \mathbf{0}$. Also, as $\mathbf{F}$ is R_x-reversible up to order $r+N-1$, from Proposition 38(b-c), we get $\mathcal{J}^{r+N-1}\left([\mathbf{F}, \widetilde{\mathbf{V}}] + \widetilde{\nu}\mathbf{F}\right) = \mathbf{0}$.

From Lemma 43, for each $\mu = \sum_{j=1}^{N} \mu_j \in \bigoplus_{j=1}^{N} \mathcal{P}_j^{\mathbf{t}}$, there exists $\lambda = \sum_{j=1}^{N} \lambda_j \in \bigoplus_{j=1}^{N} \mathcal{P}_j^{\mathbf{t}}$, such that

$$\mathcal{J}^{r+N}\left(\widetilde{\mathbf{V}}_{**}\left((1+\mu)\mathbf{F}\right)\right) = \mathcal{J}^{r+N}\left(\left[\mathbf{F}, \widetilde{\mathbf{V}}\right] + (1+\lambda)\mathbf{F}\right),$$

with $\lambda_j = \mu_j + \sum_{l=1}^{j-1} \mu_j^{(l)}$, for $k = 1, \ldots, m$, where $\mu_j^{(l)}$ depend univocally on $\mathcal{J}^{j-l}(\mathbf{U}, \widetilde{\nu}, \mu)$.

Let us select μ such that $\lambda = \widetilde{\nu}$ (this is possible by solving, degree by degree, the equations $\widetilde{\nu}_j = \mu_j + \sum_{l=1}^{j-1} \mu_j^{(l)}$). Observe that:

$$\mathcal{J}^{r+N}\left(\widetilde{\mathbf{V}}_{**}\left((1+\mu)\mathbf{F}\right)\right) = \mathcal{J}^{r+N}\left(\left[\mathbf{F}, \widetilde{\mathbf{V}}\right] + (1+\widetilde{\nu})\mathbf{F}\right).$$

In particular, as $\mathbf{F}$ is $(N-1)$-R_x-reversible, we obtain:

$$\begin{aligned}\mathbf{0} &= \mathcal{J}^{r+N-1}\left(\Pi^{(\mathsf{s})}\left(\left[\mathbf{F}, \widetilde{\mathbf{V}}\right] + \widetilde{\nu}\mathbf{F}\right)\right) + \mathcal{J}^{r+N-1}\left(\overline{\mathbf{F}}\right) \\ &= \mathcal{J}^{r+N-1}\left(\Pi^{(\mathsf{s})}\left(\left[\mathbf{F}, \widetilde{\mathbf{V}}\right] + (1+\widetilde{\nu})\mathbf{F}\right)\right) = \mathcal{J}^{r+N-1}\left(\Pi^{(\mathsf{s})}\left(\widetilde{\mathbf{V}}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right),\end{aligned}$$

where we have denoted $\overline{\mathbf{F}} = \Pi^{(\mathsf{s})}(\mathbf{F})$. To complete the proof, we use Lemma 46, which provides the existence of $\widetilde{\mu}$ such that

$$\begin{aligned}\mathcal{J}^{r+N}\left(\Pi^{(\mathsf{s})}\left(\widetilde{\mathbf{V}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right)\right) &= \mathcal{J}^{r+N}\left(\Pi^{(\mathsf{s})}\left(\widetilde{\mathbf{V}}_{**}\left((1+\mu)\mathbf{F}\right)\right)\right) \\ &= \mathcal{J}^{r+N}\left(\Pi^{(\mathsf{s})}\left(\left[\mathbf{F}, \widetilde{\mathbf{V}}\right] + (1+\widetilde{\nu})\mathbf{F}\right)\right).\end{aligned}$$

We notice that $\widetilde{\mu}_1 = \widetilde{\nu}_1$, and $\widetilde{\mu}_j = \widetilde{\nu}_j - \sum_{l=1}^{j-1} \widetilde{\mu}_j^{(l)}$, for $j = 2, \cdots, N$; where $\widetilde{\mu}_j^{(l)}$ depends univocally on $\mathcal{J}^{j-l}(\widetilde{\mathbf{V}}, \widetilde{\nu}, \widetilde{\mu})$. □

The following result proves that the non-reversible $(r+N)$–jet of any two N-orbital–reversible normal form is unique, modulus reversible generators.

Proposition 48. *Let us consider two vector fields* $\mathbf{F}$, $\mathbf{G}$, *which are* R_x*-reversible up to order* $r+N-1$, *(*$N \in \mathbb{N}$*):*

$$\begin{aligned}\mathbf{F} &= \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{F}}_{r+1} + \cdots + \widetilde{\mathbf{F}}_{r+N-1} + \left(\widetilde{\mathbf{F}}_{r+N} + \overline{\mathbf{F}}_{r+N}\right) + \cdots,\\ \mathbf{G} &= \widetilde{\mathbf{G}}_r + \widetilde{\mathbf{G}}_{r+1} + \cdots + \widetilde{\mathbf{G}}_{r+N-1} + \left(\widetilde{\mathbf{G}}_{r+N} + \overline{\mathbf{G}}_{r+N}\right) + \cdots.\end{aligned}$$

Then, for all $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ *and* $\widetilde{\mu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ *such that* $\mathcal{J}^{r+N}\left(\widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right)\right) = \mathcal{J}^{r+N}(\mathbf{G})$, *we have:*

(a) $\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}_{\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\}}\right) = \mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}_{\{\widetilde{\mathbf{G}}_r,\widetilde{\mathbf{G}}_{r+1},\ldots,\widetilde{\mathbf{G}}_{r+N-1}\}}\right)$ *(we will denote both by* $\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}\right)$*).*

(b) *Moreover, if* $\mathrm{Proy}_{\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}\right)}\left(\overline{\mathbf{F}}_{r+N}\right) = \mathrm{Proy}_{\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}\right)}\left(\overline{\mathbf{G}}_{r+N}\right) = \mathbf{0}$, *then:*

$$\overline{\mathbf{G}}_{r+N} = \overline{\mathbf{F}}_{r+N}.$$

Proof. **(a)** From Proposition 45, there exists $\overline{\lambda} \in \bigoplus_{j=1}^{N} \mathcal{E}_j^{\mathbf{t}}$ such that

$$\mathcal{J}^{r+N-1}(\mathbf{G}) = \mathcal{J}^{r+N-1}\left((1+\overline{\lambda})\mathbf{F}\right).$$

Let us take $\left(\widetilde{\mathbf{V}},\widetilde{\nu}\right)$ in the domain of $\overline{\mathcal{L}}^{(N)}_{\{\widetilde{\mathbf{G}}_r,\widetilde{\mathbf{G}}_{r+1},\ldots,\widetilde{\mathbf{G}}_{r+N-1}\}}$, $\widetilde{\mathbf{V}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$, $\widetilde{\nu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ and define $\widetilde{\zeta} = \frac{\nabla\overline{\lambda}\cdot\widetilde{\mathbf{V}}}{1+\overline{\lambda}} + \widetilde{\nu}$. Then:

$$\begin{aligned}&\mathcal{J}^{r+N}\left(\left[\mathbf{G},\widetilde{\mathbf{V}}\right] + \widetilde{\nu}\mathbf{G}\right) = \mathcal{J}^{r+N}\left(\left[(1+\overline{\lambda})\mathbf{F},\widetilde{\mathbf{V}}\right] + \widetilde{\nu}(1+\overline{\lambda})\mathbf{F}\right)\\ =\ &\mathcal{J}^{r+N}\left((1+\overline{\lambda})\left[\mathbf{F},\widetilde{\mathbf{V}}\right] + \left(\nabla(1+\overline{\lambda})\cdot\widetilde{\mathbf{V}}\right)\mathbf{F} + \widetilde{\nu}(1+\overline{\lambda})\mathbf{F}\right)\\ =\ &\mathcal{J}^{r+N}\left((1+\overline{\lambda})\left(\left[\mathbf{F},\widetilde{\mathbf{V}}\right] + \left(\tfrac{\nabla\overline{\lambda}\cdot\widetilde{\mathbf{V}}}{1+\overline{\lambda}} + \widetilde{\nu}\right)\mathbf{F}\right)\right)\\ =\ &\mathcal{J}^{r+N}\left((1+\overline{\lambda})\left(\left[\mathbf{F},\widetilde{\mathbf{V}}\right] + \widetilde{\zeta}\mathbf{F}\right)\right).\end{aligned}$$

As we have taken $\left(\widetilde{\mathbf{V}},\widetilde{\nu}\right)$ in the domain of $\overline{\mathcal{L}}^{(N)}_{\{\widetilde{\mathbf{G}}_r,\widetilde{\mathbf{G}}_{r+1},\ldots,\widetilde{\mathbf{G}}_{r+N-1}\}}$, then:

$$\mathcal{J}^{r+N-1}\left(\left[\mathbf{G},\widetilde{\mathbf{V}}\right] + \widetilde{\nu}\mathbf{G}\right) = \mathbf{0}, \quad \text{and so} \quad \mathcal{J}^{r+N-1}\left(\left[\mathbf{F},\widetilde{\mathbf{V}}\right] + \widetilde{\zeta}\mathbf{F}\right) = \mathbf{0}.$$

Consequently,

$$\left(\left[\mathbf{G},\widetilde{\mathbf{V}}\right]+\widetilde{\nu}\mathbf{G}\right)_{r+N}=\left(\left[\mathbf{F},\widetilde{\mathbf{V}}\right]+\widetilde{\zeta}\mathbf{F}\right)_{r+N}.$$

Finally, from Lemma 44, we get

$$\text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\right\}}\right)\supseteq\text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{G}}_r,\widetilde{\mathbf{G}}_{r+1},\ldots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)$$

The converse inclusion can be proven analogously.

(b) From Proposition 45 **(b)**, there exists $\left(\widetilde{\mathbf{V}},\widetilde{\nu}\right)$ in the domain of $\overline{\mathcal{L}}^{(N)}$ (with $\widetilde{\mathbf{V}}\in\bigoplus_{j=1}^{N}\widehat{\mathcal{R}}^{\mathbf{t}}_j$, $\widetilde{\nu}\in\bigoplus_{j=1}^{N}\widehat{\mathcal{O}}^{\mathbf{t}}_j$), such that:

$$\overline{\mathbf{G}}_{r+N}=\overline{\mathbf{F}}_{r+N}-\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\widetilde{\mathbf{F}}_{r+1},\ldots,\widetilde{\mathbf{F}}_{r+N-1}\right\}}\left(\widetilde{\mathbf{V}},\widetilde{\nu}\right),$$

Thus, $\overline{\mathbf{G}}_{r+N}-\overline{\mathbf{F}}_{r+N}\in\text{Range}\left(\overline{\mathcal{L}}^{(N)}\right)$. As we assume $\text{Proy}_{\text{Range}\left(\overline{\mathcal{L}}^{(N)}\right)}\left(\overline{\mathbf{G}}_{r+N}-\overline{\mathbf{F}}_{r+N}\right)=\mathbf{0}$, we get $\overline{\mathbf{G}}_{r+N}=\overline{\mathbf{F}}_{r+N}$.

□

The following Proposition defines a procedure which provides us the necessary conditions of orbital–reversibility up to a fixed order.

Proposition 49. *Let us consider a $(N-1)$-orbital–reversible vector field $\mathbf{F}$, ($N\in\mathbb{N}$). Then, there exist $\mathbf{U}\in\bigoplus_{j=1}^{N}\widehat{\mathcal{R}}^{\mathbf{t}}_j$ and $\mu\in\bigoplus_{j=1}^{N}\widehat{\mathcal{O}}^{\mathbf{t}}_j$ such that $\mathbf{G}=\mathbf{U}_{**}\left((1+\mu)\mathbf{F}\right)$ is R_x–reversible up to order $r+N-1$. Moreover:*

$$\mathbf{G}:=\widetilde{\mathbf{F}}_r+\widetilde{\mathbf{G}}_{r+1}+\cdots+\widetilde{\mathbf{G}}_{r+N-1}+\left(\widetilde{\mathbf{G}}_{r+N}+\overline{\mathbf{G}}_{r+N}\right)+\cdots.$$

where $\text{Proy}_{\text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\widetilde{\mathbf{G}}_{r+1},\ldots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)}\left(\overline{\mathbf{G}}_{r+N}\right)=\mathbf{0}$.

Proof. From Theorem 40, there exist $\widetilde{\mathbf{U}}^{(N-1)}\in\bigoplus_{j=1}^{N-1}\widehat{\mathcal{R}}^{\mathbf{t}}_j$ and $\widetilde{\mu}^{(N-1)}\in\bigoplus_{j=1}^{N-1}\widehat{\mathcal{O}}^{\mathbf{t}}_j$ such that $\mathbf{F}^{(N-1)}:=\widetilde{\mathbf{U}}^{(N-1)}{}_{**}\left((1+\widetilde{\mu}^{(N-1)})\mathbf{F}\right)$ is R_x-reversible up to order $r+N-1$, i.e.

$$\mathbf{F}^{(N-1)}=\widetilde{\mathbf{F}}_r+\widetilde{\mathbf{F}}^{(N-1)}_{r+1}+\cdots+\widetilde{\mathbf{F}}^{(N-1)}_{r+N-1}+\left(\widetilde{\mathbf{F}}^{(N-1)}_{r+N}+\overline{\mathbf{F}}^{(N-1)}_{r+N}\right)+\cdots.$$

We consider the decomposition $\overline{\mathbf{F}}^{(N-1)}_{r+N}=\overline{\mathbf{F}}^{r}_{r+N}+\overline{\mathbf{F}}^{c}_{r+N}$, with $\overline{\mathbf{F}}^{r}_{r+N}\in\text{Range}\left(\overline{\mathcal{L}}^{(N)}\right)$, and $\overline{\mathbf{F}}^{c}_{r+N}\in\text{Cor}\left(\overline{\mathcal{L}}^{(N)}\right)$, (here: $\overline{\mathcal{L}}^{(N)}=\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\widetilde{\mathbf{F}}^{(N-1)}_{r+1},\ldots,\widetilde{\mathbf{F}}^{(N-1)}_{r+N-1}\right\}}$).

Let us take $\left(\widetilde{\mathbf{U}}^{(N)}, \widetilde{\nu}^{(N)}\right)$ in the domain of $\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r^{(N-1)},\ldots,\widetilde{\mathbf{F}}_{r+N-1}^{(N-1)}\right\}}$, with $\widetilde{\mathbf{U}}^{(N)} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$, and $\widetilde{\nu}^{(N)} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$, such that

$$\overline{\mathbf{F}}^{r}_{r+N} = \overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{F}}_{r+1}^{(N-1)},\ldots,\widetilde{\mathbf{F}}_{r+N-1}^{(N-1)}\right\}} \left(\widetilde{\mathbf{U}}^{(N)}, \widetilde{\nu}^{(N)}\right).$$

From Proposition 47, there exists $\widetilde{\mu}^{(N)} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ such that the transformed vector field $\mathbf{G} := \widetilde{\mathbf{U}}^{(N)}{}_{**}\left((1+\widetilde{\mu}^{(N)})\mathbf{F}^{(N-1)}\right)$ satisfies

$$\begin{aligned}
&\mathcal{J}^{r+N-1}\left(\Pi^{(\mathtt{s})}\left(\mathbf{G}\right)\right) = \mathbf{0},\\
&\overline{\mathbf{G}}_{r+N} = \overline{\mathbf{F}}^{(N-1)}_{r+N} - \overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{F}}_{r+1}^{(N-1)},\ldots,\widetilde{\mathbf{F}}_{r+N-1}^{(k-1)}\right\}} \left(\widetilde{\mathbf{U}}^{(N)}, \widetilde{\nu}^{(N)}\right) = \overline{\mathbf{F}}^{c}_{r+N}.
\end{aligned}$$

Using Proposition 48 **(a)**, we get

$$\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{F}}_{r+1}^{(N-1)},\ldots,\widetilde{\mathbf{F}}_{r+N-1}^{(N-1)}\right\}}\right) = \mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{G}}_{r+1},\ldots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right).$$

Hence,

$$\mathrm{Proy}_{\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{G}}_{r+1},\ldots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)}\left(\overline{\mathbf{G}}_{r+N}\right) = \mathrm{Proy}_{\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r, \widetilde{\mathbf{F}}_{r+1}^{(N-1)},\ldots,\widetilde{\mathbf{F}}_{r+N-1}^{(N-1)}\right\}}\right)}\left(\mathbf{F}^{c}_{r+N}\right) = \mathbf{0}.$$

From Lemma 13, there exists μ such that

$$(1+\widetilde{\mu}^{(N)})\widetilde{\mathbf{U}}^{(N-1)}{}_{**}\left((1+\widetilde{\mu}^{(N-1)})\mathbf{F}\right) = \widetilde{\mathbf{U}}^{(N-1)}{}_{**}\left((1+\mu)\mathbf{F}\right).$$

Let us take a generator $\mathbf{U}$ of the transformation $\Phi^{(N)} \circ \Phi^{(N-1)}$, where $\Phi^{(N)}$, $\Phi^{(N-1)}$ are the transformations corresponding to the generators $\widetilde{\mathbf{U}}^{(N)}$, $\widetilde{\mathbf{U}}^{(N-1)}$, respectively. Finally:

$$\mathbf{G} = \widetilde{\mathbf{U}}^{(N)}{}_{**}\left((1+\widetilde{\mu}^{(N)})\mathbf{F}^{(N-1)}\right) = \Phi^{(N)}{}_{*}\left(\Phi^{(N-1)}{}_{*}\left((1+\mu)\mathbf{F}\right)\right) = \mathbf{U}{}_{**}\left((1+\mu)\mathbf{F}\right).$$

□

The following result characterizes the $(N+1)$-orbital–reversibility of a N-orbital–reversible vector field. Proceeding degree by degree, we will obtain an algorithm to determine the orbital–reversibility.

Theorem 50. *Let us consider a vector field* $\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots + \widetilde{\mathbf{F}}_{r+N-1} + \left(\widetilde{\mathbf{F}}_{r+N} + \overline{\mathbf{F}}_{r+N}\right) + \cdots$, *satisfying* $\overline{\mathbf{F}}_{r+N} \neq 0$ *and* $\mathrm{Proy}_{\mathrm{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\ldots,\widetilde{\mathbf{F}}_{r+N-1}\right\}}\right)}\left(\overline{\mathbf{F}}_{r+N}\right) = \mathbf{0}$, *for some* $N \in \mathbb{N}$. *Then,* $\mathbf{F}$ *is not orbital–reversible.*

Proof. Let us assume, on the contrary, that $\mathbf{F}$ is orbital–reversible. Then, $\mathbf{F}$ is N-orbital–reversible and from Proposition 49, there exist $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{N+1} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ and $\widetilde{\mu} \in \bigoplus_{j=1}^{N+1} \widehat{\mathcal{O}}_j^{\mathbf{t}}$ such that

$$\begin{aligned}\mathbf{G} &:= \widetilde{\mathbf{U}}_{**}((1+\widetilde{\mu})\mathbf{F}) \\ &= \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{G}}_{r+1} + \cdots + \widetilde{\mathbf{G}}_{r+N-1} + \left(\widetilde{\mathbf{G}}_{r+N} + \overline{\mathbf{G}}_{r+N}\right) + \left(\widetilde{\mathbf{G}}_{r+N+1} + \overline{\mathbf{G}}_{r+N+1}\right) + \cdots,\end{aligned}$$

where $\overline{\mathbf{G}}_{r+N} = \mathbf{0}$. From Proposition 48, we deduce $\overline{\mathbf{F}}_{r+N} = \mathbf{0}$, which is a contradiction. □

Taking into account the Hilbert Basis Theorem and the above Theorem, we obtain the following result.

Corollary 51. *Let* $\mathcal{F} = \left\{\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots : \mathbf{F} \text{ polynomial vector field}\right\}$ *a family of polynomial vector fields. Then, there exists* $M \in \mathbb{N}$ *such that:*

$$\mathbf{F} \in \mathcal{F} \text{ is orbital–reversible if, and only if, } \mathbf{F} \text{ is } M\text{-orbital–reversible.}$$

This number $M \in \mathbb{N}$ *is called orbital–reversibility order of* $\mathcal{F}$.

At this moment, it is convenient to summarize the previous ideas, by introducing non-linear operators:

By Montgomery-Bochner theorem, system (3.1.3) is orbital–reversible if, and only if, there exist a diffeomorphism Φ and a scalar function μ (with $\mu(\mathbf{0}) = 0$) such that the transformed system $\Phi_*((1+\mu)\mathbf{F})$ is R_x-reversible. Or, equivalently $\mathbf{U}_{**}((1+\mu)\mathbf{F})$ is R_x-reversible, being $\mathbf{U}$ a generator of the transformation Φ. So, our objective is to determine conditions for the orbital–reversibility of (3.1.3) based on the existence of such $\mathbf{U}$ and μ.

As we have seen in Chapter 1, the transformed vector field is given by:

$$\mathbf{G} = \mathbf{U}_{**}((1+\mu)\mathbf{F}) = \mathbf{F} + \mu\mathbf{F} + [\mathbf{F},\mathbf{U}] + [\mu\mathbf{F},\mathbf{U}] + \frac{1}{2}[[\mathbf{F},\mathbf{U}],\mathbf{U}] + \cdots$$

We introduce the following non-linear operators by induction:

$$\begin{aligned}\overline{\mathcal{NL}}^{(1)} : \mathcal{Q}_1^{\mathbf{t}} \times \mathcal{P}_1^{\mathbf{t}} &\longrightarrow \mathcal{S}_{r+1}^{\mathbf{t}} \\ (\mathbf{U},\mu) &\longrightarrow -\left(\overline{\mathbf{U}_{**}((1+\mu)\mathbf{F})}\right)_{r+1} + \overline{\mathbf{F}}_{r+1}.\end{aligned}$$

$$\begin{aligned}\overline{\mathcal{NL}}^{(N)} : \mathrm{Ker}(\overline{\mathcal{NL}}^{(N-1)}) \times \mathcal{Q}_N^{\mathbf{t}} \times \mathcal{P}_N^{\mathbf{t}} &\longrightarrow \mathcal{S}_{r+N}^{\mathbf{t}} \\ (\mathbf{U},\mu) &\longrightarrow -\left(\overline{\mathbf{U}_{**}((1+\mu)\mathbf{F})}\right)_{r+N} + \overline{\mathbf{F}}_{r+N}.\end{aligned}$$

From Theorem 40, it is possible to choose $\widetilde{\mathbf{U}} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ and $\widetilde{\mu} \in \bigoplus_{j=1}^{N} \widehat{\mathcal{O}}_j^{\mathbf{t}}$, such that $\widetilde{\mathbf{U}}_{**}((1+\widetilde{\mu})\mathbf{F})$ is also N-orbital–reversible. So, we define the reduced non-linear operators as follows:

$$\begin{aligned} \overline{\mathcal{NL}}^{(1)} : \widehat{\mathcal{R}}_1^{\mathbf{t}} \times \widehat{\mathcal{O}}_1^{\mathbf{t}} &\longrightarrow \mathcal{S}_{r+1}^{\mathbf{t}} \\ \left(\widetilde{\mathbf{U}}, \widetilde{\mu}\right) &\longrightarrow -\left(\overline{\widetilde{\mathbf{U}}_{**}((1+\widetilde{\mu})\mathbf{F})}\right)_{r+1} + \overline{\mathbf{F}}_{r+1}. \end{aligned}$$

$$\begin{aligned} \overline{\mathcal{NL}}^{(N)} : \mathrm{Ker}(\overline{\mathcal{NL}}^{(N-1)}) \times \widehat{\mathcal{R}}_N^{\mathbf{t}} \times \widehat{\mathcal{O}}_N^{\mathbf{t}} &\longrightarrow \mathcal{S}_{r+N}^{\mathbf{t}} \\ \left(\widetilde{\mathbf{U}}, \widetilde{\mu}\right) &\longrightarrow -\left(\overline{\widetilde{\mathbf{U}}_{**}((1+\widetilde{\mu})\mathbf{F})}\right)_{r+N} + \overline{\mathbf{F}}_{r+N}. \end{aligned}$$

Taking into account the definition of N-orbital–reversibility and the previous non-linear operators, it is obvious that we are interested in the study of $\mathrm{Range}(\overline{\mathcal{NL}}^{(N)})$ because the fact that $\mathcal{J}^{r+N}(\mathbf{U}_{**}((1+\mu)\mathbf{F}))$ is R_x-reversible, is equivalent to $\mathcal{J}^{r+N}\left(\overline{\widetilde{\mathbf{U}}_{**}((1+\widetilde{\mu})\mathbf{F})}\right) = \mathbf{0}$.

However, although the transformations are non-linear, along this section, we have described a normal form procedure to study the N-orbital–reversibility using linear operators. This is possible because

$$\mathrm{Range}(\overline{\mathcal{NL}}^{(N)}) = \mathrm{Range}(\overline{\mathcal{L}}^{(N)}).$$

To prove this fact, it is enough to take into account Propositions 45 and 47.

Therefore, the study of the N-orbital–reversibility using linear operators is equivalent to the study using non-linear operators. But, in the first situation we are able to solve the problem.

3.5 Orbital–Reversibility versus Normal Forms

In this section, we prove two results. The first one characterizes the orbital–reversibility through the unique normal form and in the second one via the 1-order normal form. We apply this last result to a nilpotent system.

Theorem 52. *Let $\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots$ be a vector field and let $N \in \mathbb{N}$ such that $\mathbf{G} = \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{G}}_{r+1} + \cdots + \widetilde{\mathbf{G}}_{r+N-1} + \overline{\mathbf{G}}_{r+N} + \cdots$ with $\widetilde{\mathbf{G}}_{r+j} \in \mathcal{R}_{r+j}^{\mathbf{t}}$, $1 \leq j < N$ and $\overline{\mathbf{G}}_{r+N} \in \mathcal{S}_{r+N}^{\mathbf{t}} \setminus \{\mathbf{0}\}$ is the unique normal form of $\mathbf{F}$ then $\mathbf{F}$ is not orbital–reversible.*

Proof. As $\mathbf{G}$ is the unique normal form of $\mathbf{F}$, it is verified that $\widetilde{\mathbf{G}}_{r+j} \in \mathrm{Cor}\left(\mathcal{L}_1^{(j)}\right)$ for $1 \leq j \leq N-1$ and $\overline{\mathbf{G}}_{r+N} \in \mathrm{Cor}\left(\mathcal{L}_1^{(N)}\right)$, i.e., $\mathrm{Proy}_{\mathrm{Range}\left(\mathcal{L}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r, \cdots, \widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)}\left(\overline{\mathbf{G}}_{r+N}\right) = \mathbf{0}$

Moreover, as $\text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\cdots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right) \subset \text{Range}\left(\mathcal{L}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\cdots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)$, we get

$$\text{Proy}_{\text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\cdots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)}\left(\overline{\mathbf{G}}_{r+N}\right) = \mathbf{0}.$$

By Theorem 50, $\mathbf{G}$ is not orbital–reversible and then $\mathbf{F}$ is not orbital–reversible □

Theorem 53. *Let* $\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots$ *be a vector field and let* $N \in \mathbb{N}$ *such that* $\mathbf{G} = \widetilde{\mathbf{F}}_r + \widetilde{\mathbf{G}}_{r+1} + \cdots + \widetilde{\mathbf{G}}_{r+N-1} + \overline{\mathbf{G}}_{r+N} + \cdots$ *with* $\widetilde{\mathbf{G}}_{r+j} \in \mathcal{R}^{\mathbf{t}}_{r+j}$, $1 \leq j < N$ *and* $\overline{\mathbf{G}}_{r+N} \in \mathcal{S}^{\mathbf{t}}_{r+N} \setminus \{\mathbf{0}\}$ *is a 1-order orbital normal form of* $\mathbf{F}$. *If* $\text{Ker}\left(\mathcal{L}^{(1)}_k\right) \subset \mathcal{S}^{\mathbf{t}}_k \times \mathcal{E}^{\mathbf{t}}_k$ *for all* $k \in \mathbb{N}$, *then* $\mathbf{F}$ *is not orbital–reversible.*

Proof. As $\mathbf{G}$ is a 1-order orbital normal form of $\mathbf{F}$, it has that $\widetilde{\mathbf{G}}_{r+j} \in \text{Cor}\left(\mathcal{L}^{(1)}_j\right)$ for $1 \leq j \leq N-1$ and $\overline{\mathbf{G}}_{r+N} \in \text{Cor}\left(\mathcal{L}^{(1)}_N\right)$, i.e., $\text{Proy}_{\text{Range}\left(\mathcal{L}^{(1)}_{\left\{\widetilde{\mathbf{F}}_r\right\}}\right)}\left(\overline{\mathbf{G}}_{r+N}\right) = \mathbf{0}$

As $\text{Range}\left(\overline{\mathcal{L}}^{(1)}_{\left\{\widetilde{\mathbf{F}}_r\right\}}\right) \subset \text{Range}\left(\mathcal{L}^{(1)}_{\left\{\widetilde{\mathbf{F}}_r\right\}}\right)$, we get

$$\text{Proy}_{\text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r\right\}}\right)}\left(\overline{\mathbf{G}}_{r+N}\right) = \mathbf{0}.$$

Moreover, as $\text{Ker}\left(\mathcal{L}^{(1)}_k\right) \subset \mathcal{S}^{\mathbf{t}}_k \times \mathcal{E}^{\mathbf{t}}_k$ for all $k \in \mathbb{N}$ then $\text{Ker}\left(\overline{\mathcal{L}}^{(N)}_k\right) = \{(\mathbf{0},0)\} + \text{Ker}\left(\overline{\mathcal{L}}^{(N-1)}_{k+1}\right)$. Following the ideas of the proof of Theorem 30 for linear operator $\overline{\mathcal{L}}$, we obtain that $\text{Range}\left(\overline{\mathcal{L}}^{(m+1)}_k\right) = \text{Range}\left(\overline{\mathcal{L}}^{(m)}_{k+1}\right)$ for all $k, m \in \mathbb{N}$, $m > N$, or equivalently $\text{Range}\left(\overline{\mathcal{L}}^{(N)}_k\right) = \text{Range}\left(\overline{\mathcal{L}}^{(N)}_1\right)$ for all $k \in \mathbb{N}$. Then $\text{Range}\left(\overline{\mathcal{L}}^{(1)}_{\left\{\widetilde{\mathbf{F}}_r\right\}}\right) = \text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\cdots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)$ and we get

$$\text{Proy}_{\text{Range}\left(\overline{\mathcal{L}}^{(N)}_{\left\{\widetilde{\mathbf{F}}_r,\cdots,\widetilde{\mathbf{G}}_{r+N-1}\right\}}\right)}\left(\overline{\mathbf{G}}_{r+N}\right) = \mathbf{0}.$$

By Theorem 50, $\mathbf{G}$ is not orbital–reversible and then $\mathbf{F}$ is not orbital–reversible □

Theorem 54. *Let* $\mathbf{F} = \widetilde{\mathbf{F}}_r + \cdots$ *a nilpotent vector field with* $\widetilde{\mathbf{F}}_r = (y, \sigma x^{2n-1})^T \in \mathcal{R}^{(1,n)}_{n-1}$, $\sigma = \pm 1$, $n \in \mathbb{N}$. $\mathbf{F}$ *is orbital–reversible if, and only if,* $\mathbf{F}$ *is orbital equivalent to*

$$\mathbf{G} = (y, \sigma x^{2n-1})^T + \sum_{\substack{i=n+1 \\ i \ odd}}^{2n-2} \alpha^{(0)}_i x^i \mathbf{D}_0 + \sum_{j\geq 1} \sum_{\substack{i=0 \\ i \ odd}}^{2n-2} \alpha^{(j)}_i x^i h^j \mathbf{D}_0,$$

where $\mathbf{D}_0 = (x, ny)^T \in \mathcal{S}_0^{(1,n)}$ *and* $h = \sigma x^{2n} - ny^2 \in \mathcal{E}_{2n}^{(1,n)}$.

Proof. The sufficiency is trivial since $\mathbf{G}$ is R_x-reversible.

We will prove the necessary condition. By [21, Theorem 16] a 1-order orbital normal form of vector field $\mathbf{F}$ is

$$\mathbf{G} = (y, \sigma x^{2n-1})^T + \sum_{i=n+1}^{2n-2} \alpha_i^{(0)} x^i \mathbf{D}_0 + \sum_{j\geq 1} \sum_{i=0}^{2n-2} \alpha_i^{(j)} x^i h^j \mathbf{D}_0.$$

If $\alpha_i^{(j)} = 0$ for all j and i even, then the result is proved. Otherwise let

$$\begin{aligned} j_0 &= \min\left\{ j \in \mathbb{N} \cup \{0\} : \alpha_i^{(j)} \neq 0 \text{ for some i even} \right\} \\ i_0 &= \min\left\{ i \in \left\{ \begin{array}{ll} n \leq i \leq 2n-2 & \text{if} \quad j_0 = 0 \\ 0 \leq i \leq 2n-2 & \text{if} \quad j_0 > 0 \end{array} \right\}, i \text{ even } : \alpha_i^{(j_0)} \neq 0 \right\}. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{G} \;=\; & (y, \sigma x^{2n-1})^T + \sum_{\substack{i=n \\ i \text{ odd}}}^{i_0-1} \widetilde{\mathbf{G}}_{n-1+2nj_0+i} + \sum_{j=1}^{j_0-1} \sum_{\substack{i=0 \\ i \text{ odd}}}^{2n-2} \widetilde{\mathbf{G}}_{n-1+2nj_0+i} + \sum_{\substack{i=0 \\ i \text{ odd}}}^{i_0-1} \widetilde{\mathbf{G}}_{n-1+2nj_0+i} \\ & + \overline{\mathbf{G}}_{n-1+2nj_0+i_0} + \cdots \end{aligned}$$

where $\widetilde{\mathbf{G}}_{n-1+2nj+i} = \alpha_i^{(j)} x^i h^j \mathbf{D}_0 \in \mathcal{R}_{n-1+2nj+i}^{(1,n)}$ if i is odd, and $\overline{\mathbf{G}}_{n-1+2nj_0+i_0} = \alpha_{i_0}^{(j_0)} x^{i_0} h^{j_0} \mathbf{D}_0 \in \mathcal{S}_{n-1+2nj_0+i_0}^{(1,n)}$.

It easy to prove that

$$\operatorname{Ker}\left(\mathcal{L}_k^{(1)}\right) = \left\{ \begin{array}{cc} (h^j \mathbf{D}_0, -(n-1)h^j) & \text{if} \quad k = 2nj \\ (\mathbf{0}, 0) & \text{otherwise} \end{array} \right.$$

Since

$$\begin{aligned} \mathcal{L}_{2nj}\left(h^j \mathbf{D}_0, -(n-1)h^j\right) &= \left[\widetilde{\mathbf{F}}_r, h^j \mathbf{D}_0\right] + (n-1)h^j \widetilde{\mathbf{F}}_r \\ &= -\overbrace{\left(\nabla h^j \cdot \widetilde{\mathbf{F}}_r\right)}^{=0} \mathbf{D}_0 - h^j \left[\widetilde{\mathbf{F}}_r, \mathbf{D}_0\right] + (n-1)h^j \widetilde{\mathbf{F}}_r \\ &= -(n-1)\widetilde{\mathbf{F}}_r + (n-1)\widetilde{\mathbf{F}}_r = \mathbf{0} \end{aligned}$$

Then $\operatorname{Ker}\left(\mathcal{L}_k^{(1)}\right) \in \mathcal{S}_k^{(1,n)} \times \mathcal{E}_k^{(1,n)}$ and by Theorem 53, $\mathbf{F}$ is not orbital–reversible. □

3.6 Applications

This last section is devoted to apply the above results to some families of nilpotent vector fields. We have selected two cases to show the applicability of the method: the first one corresponds to a planar situation and the other one to a tridimensional family.

3.6.1 A case study of Takens-Bogdanov singularity

Let us consider the following family of planar vector fields:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x^{4q+1} \end{pmatrix} + \begin{pmatrix} a_1 xy + a_2 x^{2q+2} \\ b_1 y^2 + b_2 x^{2q+1} y + b_3 x^{4q+2} \end{pmatrix}, \tag{3.6.1}$$

where $q \in \mathbb{N}$. The analytic integrability and the center problem for this family has been studied in [44] and [55], respectively. The reversibility problem for this family is completely solved in [24]. With respect to the orbital–reversibility problem, we have the following result:

Theorem 55. *System (3.6.1) with $q \geq 2$ is orbital–reversible if, and only if, one of the following conditions is satisfied:*

(a) $b_1 = -a_1$, $a_2 = 0$, $b_3 = a_1$.

(b) $a_1 = b_1 = b_3 = 0$.

(c) $a_2 = b_2 = 0$.

(d) $a_1 + 2b_1 = 0$, $b_2 = -2(q+1)a_2$.

(e) $a_2 = 0$, $b_3 = -\frac{1}{2}a_1$, $b_1 = \frac{2q+1}{2}a_1$, $a_1 b_2 \neq 0$.

(f) $a_2 = 0$, $b_3 = -2a_1$, $b_1 = 2(q+1)a_1$, $a_1 b_2 \neq 0$.

(g) $b_3 = 0$, $b_2 = (2q+1)a_2$, $b_1 = (2q+1)a_1$.

Remark. Items (a)–(d) agree with the cases of reversibility, considered in [24]. Items (e)–(g) correspond to situations of non-reversible systems that are orbital–reversible. Moreover, this result extends the one obtained in [55]. In fact, items (a)–(f) agree with the cases (ii), (vi), (i), (v), (iv), (iii) of the above reference. It is remarkable that item (g) provides a new case in the family corresponding to a center, which is not covered in the above referenced papers.

On the other hand, items (a)–(g) for $q = 1$ are cases where the origin of system (3.6.1) is a center. Moreover for $q = 1$, we have found the case $b_2 = 35a_2$, $b_1 = \frac{5}{3}a_1$, $b_3 = -\frac{4}{9}a_1$, $405a_2^2 = a_1^2$, $a_1 \neq 0$ that could be a new center but it is a open problem. Moreover, these should be all cases of centers for $q = 1$.

Proof. The vector field of the statement can be written as $\mathbf{F} = \widetilde{\mathbf{F}}_r + \mathbf{F}_{r+1}$, where

$$\widetilde{\mathbf{F}}_r := (y, -x^{4q+1})^T \in \mathcal{Q}^{\mathbf{t}}_{2q}, \text{ and } \mathbf{F}_{r+1} \in \mathcal{Q}^{\mathbf{t}}_{2q+1},$$

being $r = 2q$ and $\mathbf{t} = (1, 2q+1)$. We observe that $\widetilde{\mathbf{F}}_{2q}$ is R_x- and R_y- reversible, and these are the unique zero degree involutions which carry $\mathbf{F}_{2q}$ to R_x-reversible.

We start seeking the R_x-reversibility. To obtain reversible conditions we take the generator $\widetilde{\mathbf{U}} \in \bigoplus_{j\geq 1} \mathcal{R}^{\mathbf{t}}_j$, where:

$$\begin{aligned}
\widetilde{\mathbf{U}}_1 &= (\alpha_1 x^2, \alpha_2 xy)^T, \\
\widetilde{\mathbf{U}}_2 &= (0, \alpha_3 x^{2q+3})^T, \\
\widetilde{\mathbf{U}}_3 &= (\alpha_4 x^4, \alpha_5 x^3 y)^T, \\
\widetilde{\mathbf{U}}_4 &= (\chi_{\{q=1\}}\alpha_6 x^2 y, \alpha_7 x^{2q+5} + \chi_{\{q=1\}}\alpha_8 xy^2)^T,
\end{aligned}$$

where $\chi_{\{q=n\}}$ is 1 when $q=n$ and 0 otherwise, and $\widetilde{\mu}_j = \gamma_j x^j$ $j = 1, \cdots, 4$. Choosing adequately α_i, $i = 1, \ldots, 8$, γ_i, $i = 1, \ldots, 4$ as indicated in the previous Section (see Proposition 49), and using *Maple* in the computations, we have obtained the following normal form:

$$\begin{aligned}
\mathbf{F}^{(1)}(\mathbf{x}) &= \widetilde{\mathbf{U}}_{**}\left((1+\widetilde{\mu})\mathbf{F}\right) = \widetilde{\mathbf{F}}_{2q} + \widetilde{\mathbf{F}}_{2q+1} + \left(\widetilde{\mathbf{F}}_{2q+2} + \frac{1}{4q+3}\lambda^{(2)}\begin{pmatrix} 0 \\ x^{2q+2}y \end{pmatrix}\right) \\
&+\widetilde{\mathbf{F}}_{2q+3} + \left(\widetilde{\mathbf{F}}_{2q+4} + \frac{1}{3(4q+5)(4q+3)^3}\lambda^{(4)}\begin{pmatrix} 0 \\ x^{2q+4}y \end{pmatrix}\right) \\
&+\widetilde{\mathbf{F}}_{2q+5} + \left(\widetilde{\mathbf{F}}_{2q+6} + \frac{1}{30(4q+7)(4q+5)(4q+3)^5}\lambda^{(6)}\begin{pmatrix} 0 \\ x^{2q+6}y \end{pmatrix}\right) \\
&+\widetilde{\mathbf{F}}_{2q+7} + \left(\widetilde{\mathbf{F}}_{2q+8} + \frac{1}{3(4q+5)(4q+3)^3}\lambda^{(8)}\begin{pmatrix} 0 \\ x^{2q+8}y \end{pmatrix}\right) + \cdots .
\end{aligned}$$

So, applying Theorem 50, if $\mathbf{F}$ is orbital–reversible then $\lambda^{(2j)} = 0$ for all j.

The first coefficient is:

$$\lambda^{(2)} = \det\begin{pmatrix} -\left((2q+3)(2q+1)a_2 + 2qb_2\right) & b_2 + 2(q+1)a_2 \\ -(4q+3)b_1 - (2q+3)b_3 & a_1 + 2b_1 \end{pmatrix}. \quad (3.6.2)$$

The vanishing of this coefficient leads to the following cases:

(1) $-(4q+3)b_1 - (2q+3)b_3 = a_1 + 2b_1 = 0$,

(2) $\left((4q+3)b_1 + (2q+3)b_3\right)^2 + (a_1+2b_1)^2 > 0$.

We analyze both cases:

(1) In the first case, we have $a_1 = -2b_1$, $b_3 = -\frac{(4q+3)}{2q+3}b_1$. For these values, we have obtained:

$$\begin{aligned}
\lambda^{(4)} &= b_1\left(2(q+1)a_2 + b_2\right) \times \\
&\left(-16q(4q+5)(q+1)b_1^2 + 3(2q+5)(2q+3)^2 a_2\left((2q+3)a_2 - b_2\right)\right).
\end{aligned}$$

There are some possibilities:

(1a) $b_1 = 0$, which leads to $a_1 = b_1 = b_3 = 0$. In this case the system is R_x-reversible. This is the situation described in item **(b)**.

(1b) $b_2 = -2(q+1)a_2$, $b_1 \neq 0$. In this case the change of variables $u = x - 2b_1xy + a_2y^{2q+2}$, $v = y$ transforms the system (3.6.1) in

$$\begin{aligned}\dot{u} &= u\left(b_1u^2 + v^{4q+1} - (4b_1+b_3)\,v^{4q+2} + 2\left(2b_3b_1 + 2b_1^2 - (q+1)a_2^2\right)v^{4q+3} + \right.\\ &\qquad \left.\left(3b_1a_2^2 - 4b_3b_1^2 + 4qb_1a_2^2\right)v^{4q+4}\right)/\left(2b_1v-1\right),\\ \dot{v} &= u,\end{aligned}$$

that is R_u-reversible. This situation is described in **(d)**.

(1c) If $b_1^2 = \frac{3(2q+5)(2q+3)^2a_2((2q+3)a_2-b_2)}{16q(4q+5)(q+1)}$, $b_1\left(2(q+1)a_2+b_2\right) \neq 0$, then we obtain:

$$\lambda^{(6)} = a_2(2q+5)(2q+3)(16q^2+150q+161) - 2(16q^3+130q^2+251q+140)b_2.$$

The vanishing of this coefficient leads to $b_2 = \frac{(2q+5)(2q+3)(16q^2+150q+161)}{2(16q^3+130q^2+251q+140)}a_2$, and then

$$b_1^2 = -\frac{45(2q+7)(2q+5)(2q+3)^3}{32(q+1)q(16q^3+130q^2+251q+140)}a_2^2.$$

So we cannot obtain $\lambda^{(4)} = \lambda^{(6)} = 0$.

(2) In the second case, the two rows of the matrix in (3.6.2) are proportional:

$$\begin{aligned}a_2 &= \beta\left(-\tfrac{2q}{4q+3}a_1 + \tfrac{3}{4q+3}b_1 + \tfrac{2q+3}{(4q+3)}b_3\right),\\ b_2 &= \beta\left(\tfrac{(2q+3)(2q+1)}{4q+3}a_1 + \tfrac{2q}{4q+3}b_1 - \tfrac{2(2q+3)(q+1)}{(4q+3)}b_3\right).\end{aligned}$$

In this situation, we have obtained:

$$\begin{aligned}\lambda^{(4)} &= \beta(a_1+2b_1)\times\\ &\quad \{3(2q+5)\left[(2q+3)(4q+5)b_3 - (2q+3)(4q+1)a_1\right.\\ &\quad \left.+(4q+9)b_1\right]\times\left[(2q+3)b_3 + (3b_1-2qa_1)\right]\times\\ &\quad \left[2(q+1)b_3 + (b_1-(2q+1)a_1)\right]\beta^2 - p(a_1,b_1,b_3)\}\\ p(a_1,b_1,b_3) &= -2b_3^3q(2q+5)(4q+5)(6q+7)\\ &\quad +3b_3^2(2q+5)(4q+5)((12q^2+26q+9)a_1 + (2q+9)b_1)\\ &\quad +3b_3\left[(288q^4+1032q^3+1192q^2+460q+45)a_1^2\right.\\ &\quad +2(84q^2+233q+180)b_1^2\\ &\quad \left.-(480q^3+1576q^2+1592q+405)a_1b_1\right] -\\ &\quad ((2q+1)a_1-b_1)\left[5(52q+81)b_1^2 + 2q(120q^2+202q+49)a_1^2\right.\\ &\quad \left.-(512q^2+844q+135)a_1b_1\right]\end{aligned}$$

Next, we will study some cases: the first three ones correspond to the vanishing of one factor of $\lambda^{(4)}$. A particular situation, where the expression for $p(a_1, b_1, b_3)$ is simple, is also included in item (2d).

(2a) $\beta = 0$, which provides $a_2 = b_2 = 0$. In this case, covered in item **(c)**, the system is R_y-reversible.

(2b) $a_1 + 2b_1 = 0$, $\beta \neq 0$. Then, $b_2 + 2(q+1)a_2 = 0$, $a_2 \neq 0$. Now, the system (3.6.1) is Hamiltonian, with Hamiltonian function given by

$$h(x,y) = -\tfrac{1}{2}y^2 - \tfrac{1}{2(2q+1)}x^{4q+2} + b_1xy^2 - a_2x^{2q+2}y + \tfrac{b_3}{4q+3}x^{4q+3}.$$

If we denote $u = x$, $v = y - 2b_1xy + a_2x^{2q+2}$, then system (3.6.1) becomes:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u^{4q+4} + (b_3 + 2(q+1)a_2^2 + 2b_1)u^{4q+2} - 2b_1b_3u^{4q+3} + \tfrac{a_2u^{4q+4} - b_1v^2}{1-2b_1u}, \end{aligned}$$

which is R_v-reversible (item **(d)**).

(2c) $(2q+3)b_3 + (3b_1 - 2qa_1) = 0$, $\beta(a_1 + 2b_1) \neq 0$. Once we have $\lambda^{(2)} = \lambda^{(4)} = 0$, we will compute the next normal form coefficient, which is

$$\lambda^{(6)} = (a_1 + b_1)((2q+1)a_1 - 2b_1)(2(q+1)a_1 - b_1).$$

This coefficient vanishes in three cases:

(2ci) $a_1 + b_1 = 0$, which leads to $a_2 = 0$, $b_3 = a_1$. In this case (and the following ones), it is more convenient to work with system (3.6.1) with the transformation $x = u$, $y = v(1 + a_1u)^{\frac{b_1}{a_1}}$, i.e.:

$$\begin{aligned} \dot{u} &= v(1 + a_1u)^{\frac{b_1}{a_1}+1} + a_2u^{2q+2}, \\ \dot{v} &= \frac{-u^{4q+1} + b_3u^{4q+2}}{(1 + a_1u)^{\frac{b_1}{a_1}}} + \frac{b_2 + (b_2a_1 - b_1a_2)u}{1 + a_1u}u^{2q+1}v. \end{aligned} \tag{3.6.3}$$

The assumption of the present item reduces (3.6.3) to:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u^{4q+1} + b_2u^{2q+1}v + a_1^2u^{4q+3}, q \in \mathbb{N}, \end{aligned}$$

which is R_u-reversible (item **(a)**).

(2cii) $(2q+1)a_1 - 2b_1 = 0$, which implies $a_2 = 0$, $b_3 = -\frac{1}{2}a_1$, $b_1 = \frac{2q+1}{2}a_1$, $a_1 \neq 0$. Now, system (3.6.3) becomes:

$$\begin{aligned} \dot{u} &= v(1+a_1u)^{\frac{2q+3}{2}}, \\ \dot{v} &= \frac{-u^{4q+1}(1+\frac{a_1}{2}u)}{(1+a_1u)^{\frac{2q+1}{2}}} + b_2u^{2q+1}v. \end{aligned}$$

The transformation $Y = v$, $X = \frac{u}{\sqrt{1+a_1u}}$, $dt = \frac{dT}{(1+\frac{a_1}{2}u)(1-a_1u)^q}$ leads to

$$\begin{aligned} X' &= Y, \\ Y' &= -X^{4q+1} + b_2\frac{X^{2q+1}}{\sqrt{1+\frac{a_1^2}{4}X^2}}Y, \end{aligned}$$

which is R_X-reversible (item **(e)**).

(2ciii) $2(q+1)a_1 - b_1 = 0$, which provides: $a_2 = 0$, $b_3 = -2a_1$, $b_1 = 2(q+1)a_1$, $a_1 \neq 0$. System (3.6.3) reduces to:

$$\begin{aligned} \dot{u} &= v(1+a_1u)^{2q+3}, \\ \dot{v} &= -\frac{1+2a_1u}{(1+a_1u)^{2q+2}}u^{4q+1} + b_2u^{2q+1}v, \end{aligned}$$

If we denote $Y = v$, $X = \frac{u}{1+a_1u}$, $dt = (1-a_1X)^{2q+1}dT$, then we get:

$$\begin{aligned} X' &= Y, \\ Y' &= -X^{4q+1} + 2(q+1)a_1X^{2q+1}Y + a_1^2X^{4q+3}, \end{aligned}$$

which is R_X-reversible (item **(f)**).

(2d) We will assume in this item that $2(q+1)b_3+(b_1-(2q+1)a_1) = 0$, $\beta(a_1+2b_1) \neq 0$. Under this hypothesis, the coefficient $\lambda^{(4)}$ reduces to:

$$\lambda^{(4)} = (a_1+b_1)(b_1+(2q+3)a_1)(b_1-(2q+1)a_1),$$

and its vanishing leads to the following possibilities:

(2di) $a_1 + b_1 = 0$, which corresponds to $a_2 = 0$, $b_3 = a_1$. This case is covered in item **(a)**.

(2dii) $b_1 = -(2q+3)a_1$. Now, we are able to compute the next normal form coefficient $\lambda^{(6)}$, and we can check that it does not vanish.

(2diii) $b_1 = (2q+1)a_1$, which corresponds to $b_3 = 0$, $b_2 = (2q+1)a_2$, $b_1 = (2q+1)a_1$. System (3.6.3) reduces to:

$$\begin{aligned} \dot{u} &= v(1+a_1u)^{2q+2} + a_2u^{2q+2}, \\ \dot{v} &= \frac{-u^{4q+1}}{(1+a_1u)^{2q+1}} + \frac{(2q+1)a_2}{1+a_1u}u^{2q+1}v. \end{aligned}$$

The transformation $Y = v$, $X = \frac{u}{1+a_1u}$, $dT = (1+a_1X)^{2q}dt$ yields

$$\begin{aligned} X' &= Y + a_2X^{2q+2}, \\ Y' &= -X^{4q+1} + (2q+1)a_2X^{2q+1}Y, \end{aligned}$$

which is R_X-reversible (item **(g)**).

(2e) In the case $\beta(a_2+2b_1)\left[2(q+1)b_3 + b_1 - (2q+1)a_1\right] \neq 0$, using a computer algebra system SINGULAR (see [63]), we have not found any new conditions.

Now we deal with the R_y-reversibility. By interchanging the variables: $x \leftrightarrow y$, $y \leftrightarrow x$, the system (3.6.1) becomes:

$$\begin{aligned} \dot{x} &= -y^{4q+1} + b_1x^2 + b_2xy^{2q+1} + b_3y^{4q+2}, \\ \dot{y} &= x + a_1xy + a_2y^{2q+2}. \end{aligned} \tag{3.6.4}$$

The vector field $\mathbf{F}$ of (3.6.4) is a sum of two quasi-homogeneous vector fields of type $\mathbf{t} = (2q+1, 1)$ and degree $2q$ and $2q+1$, respectively. The first quasi-homogeneous term, $\widetilde{\mathbf{F}}_{2q} := (-y^{4q+1}, x)^T$ is already in a desired simplified form (it is R_x-reversible). Choosing adequately $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}_1 + \widetilde{\mathbf{U}}_2$, as specified in the Proposition 49, we obtain:

$$\begin{aligned} \widetilde{\mathbf{U}}_{**}\mathbf{F}(\mathbf{x}) &= \begin{pmatrix} -y^{4q+1} \\ x \end{pmatrix} + \left(\widetilde{\mathbf{F}}_{2q+1} + \lambda^{(1)}\begin{pmatrix} xy^{2q+1} \\ 0 \end{pmatrix}\right) \\ &\quad + \left(\widetilde{\mathbf{F}}_{2q+2} - \lambda^{(2)}\begin{pmatrix} xy^{2q+2} \\ 0 \end{pmatrix}\right) + \cdots . \end{aligned}$$

From Theorem 50, if $\mathbf{F}$ is orbital–reversible then $\lambda^{(1)} = \lambda^{(2)} = 0$. This yields

$$b_2 = -2(q+1)a_2, \quad a_2(2b_1 + a_1) = 0.$$

We find the following possibilities:

(1) $a_2 = 0$. In this case, the system (3.6.1) is R_y-reversible and so it is reversible. This situation is described in **(c)**.

(2) $a_1 = -2b_1$. This case agrees with the one described in item (1b).

This completes the proof. □

3.6.2 A case study of the triple-zero singularity

Finally, we study the reversibility of the following three-dimensional system:

$$\begin{aligned} \dot{x} &= y + a_1 x^3 + a_2 xy, \\ \dot{y} &= xz + b_1 x^2 y + b_2 x^2 z, \\ \dot{z} &= x^5 + c_1 x^2 z + c_2 y^2 + c_3 x^6. \end{aligned} \tag{3.6.5}$$

Observe that the equilibrium at the origin has a triple-zero eigenvalue with two Jordan blocks, so that the linear part corresponds to a coupling of a Takens-Bogdanov and a simple zero degeneracies.

Theorem 56. *System (3.6.5) is orbital–reversible if, and only if, one of the following conditions is satisfied:*

(a) $a_2 = b_2 = 0$, $c_1 = c_2 = -(b_1 + 3a_1)$, $c_3 = a_1(b_1 + 3a_1)(b_1 + a_1)$.

(b) $a_1 = b_1 = c_1 = c_2 = 0$, $b_2 = c_3 = a_2$.

Proof. The vector field $\mathbf{F}$ in (3.6.5) is a sum of three quasi-homogeneous vector fields of type $\mathbf{t} = (3, 8, 10)$ and degrees 5, 6, 8, respectively. The principal part $\mathbf{F}_5 := (y, xz, x^5)^T$ is already in a desired simplified form (it is R_x–reversible). Moreover, it can be shown that the unique near-identity change of variables of zero degree which carries $\mathbf{F}_5$ to axis-reversible is the identity (see Theorem 59). Hence, it is enough to study the R_x-orbital–reversibility of the vector field $\mathbf{F}$.

As we will see later, it is sufficient to reach the $N = 11$-orbital–reversibility to solve the orbital–reversibility problem in this case. To reduce the vector field $\mathbf{F}$ to the normal form $\mathbf{F}^{(11)}$, we take a generator $\widetilde{\mathbf{U}} = \sum_{j=1}^{11} \widetilde{\mathbf{U}}_j \in \bigoplus_{j=1}^{11} \widehat{\mathcal{R}}_j^{\mathbf{t}}$ and $\widetilde{\mu} = \sum_{j=1}^{11} \widetilde{\mu}_j \in \bigoplus_{j=1}^{11} \widehat{\mathcal{O}}_j^{\mathbf{t}}$, as specified in Proposition 49. In this way, we obtain the following expression of the normal form:

$$\begin{aligned} \widetilde{\mathbf{U}}_{**}((1+\widetilde{\mu})\mathbf{F}) = \widetilde{\mathbf{F}}_5 &+ \left(\widetilde{\mathbf{F}}_6 + \begin{pmatrix} 0 \\ 0 \\ \lambda_1^{(1)} x^2 z + \lambda_2^{(1)} y^2 \end{pmatrix}\right) + \widetilde{\mathbf{F}}_7 + \left(\widetilde{\mathbf{F}}_8 + \begin{pmatrix} 0 \\ 0 \\ \lambda_1^{(3)} yz \end{pmatrix}\right) \\ &+ \widetilde{\mathbf{F}}_9 + \left(\widetilde{\mathbf{F}}_{10} + \begin{pmatrix} \lambda_1^{(5)} xz \\ 0 \\ \lambda_2^{(5)} z^2 + \lambda_3^{(5)} x^4 y \end{pmatrix}\right) + \widetilde{\mathbf{F}}_{11} + \left(\widetilde{\mathbf{F}}_{12} + \begin{pmatrix} \lambda_1^{(7)} x^5 \\ 0 \\ \lambda_2^{(7)} x^4 z \end{pmatrix}\right) \\ &+ \widetilde{\mathbf{F}}_{13} + \left(\widetilde{\mathbf{F}}_{14} + \begin{pmatrix} 0 \\ 0 \\ \lambda_1^{(9)} x^8 \end{pmatrix}\right) + \widetilde{\mathbf{F}}_{15} + \left(\widetilde{\mathbf{F}}_{16} + \begin{pmatrix} 0 \\ \lambda_1^{(11)} yz^2 \\ \lambda_2^{(11)} z^3 + \lambda_3^{(11)} x^{10} \end{pmatrix}\right) + \cdots \end{aligned}$$

From Theorem 50, if $\mathbf{F}$ is orbital–reversible, then all the coefficients $\lambda_i^{(2j-1)}$ must vanish.

We have obtained that $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$ if, and only if,

$$c_1 = c_2 = -(b_1 + 3a_1).$$

For these parameter values, we have $\lambda_1^{(3)} = 0$ if, and only if,

$$c_3 = a_1(b_1 + 3a_1)(b_1 + a_1) + \tfrac{1}{3}(7b_2 - 4a_2).$$

Moreover, for this value, we have $\lambda_1^{(5)} = \lambda_2^{(5)} = 0$ and

$$\lambda_3^{(5)} = \frac{a_2\left[2b_1^2+2a_1b_1+3a_1^2\right]-b_2\left[(8b_1^2+23a_1b_1+102a_1^2)\right]}{3}.$$

The vanishing of this coefficient leads to the following possibilities:

(1) $2b_1^2 + 2a_1b_1 + 3a_1^2 \neq 0$. In this case we obtain:

$$a_2 = \frac{b_2(8b_1^2+23a_1b_1+102a_1^2)}{2b_1^2+2a_1b_1+3a_1^2}.$$

For this value, we have obtained

$$\begin{aligned}\lambda_1^{(7)} &= -\frac{b_2^2(28b_1^5+86b_1^4a_1+502b_1^3a_1^2-546a_1^3b_1^2-2907b_1a_1^4-9315a_1^5)}{15(2b_1^2+2a_1b_1+3a_1^2)^2}\\ \lambda_2^{(7)} &= \frac{b_2^2(b_1+3a_1)(291a_1^2+59a_1b_1+14b_1^2)}{3(2b_1^2+2a_1b_1+3a_1^2)}\end{aligned}$$

Both coefficients vanish if, and only if, $b_2 = 0$. In this case, the change of variables $x = u$, $y = v - a_1u^3$, $z = w + a_1b_1u^4 - (b_1 + 3a_1)uv$, transforms the system (3.6.5) into a R_u-reversible system.

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= uw\\ \dot{w} &= u^5 + (8a_1b_1 + 15a_1^2 + b_1^2 - 4b_1)u^3v.\end{aligned}$$

This situation is described in **(a)**. In fact, in this case the system is reversible (it is not necessary to reparametrize the time to reach reversibility), and returning to the original variables, it is possible to get the analytical expression for the involution:

$$\sigma(x, y, z) = (-x, y + 2a_1x^3, z + 2(b_1 + 3a_1)x(y + a_1x^3)).$$

(2) $2b_1^2+2a_1b_1+3a_1^2 = 0$. This condition is equivalent to $a_1 = b_1 = 0$ and we obtain $\lambda_1^{(7)} = \lambda_2^{(7)} = 0$ and

$$\lambda_1^{(9)} = \tfrac{1210}{9}(b_2 - a_2)\left(b_2 - \tfrac{65+\sqrt{705}}{110}a_2\right)\left(b_2 - \tfrac{65-\sqrt{705}}{110}a_2\right)$$

This coefficient vanishes in the following subcases:

- $b_2 - a_2 = 0$. In this case, the change of variables in the time $dt = (1+a_2x)dT$ transforms the system (3.6.5) into the following R_x-reversible system:

$$\begin{aligned} x' &= y, \\ y' &= xz \\ z' &= x^5. \end{aligned}$$

  This situation is described in **(b)**. In this case, the system is not reversible, but it is orbital–reversible. As we commented before, the time reparametrization is given by $dt = (1 + a_2x)dT$, and the involution for the time-reparametrized system is

$$\sigma(x, y, z) = (-x, y, z).$$

- $b_2 = \frac{65\pm\sqrt{705}}{110}a_2$. In this case: $\lambda_1^{(11)} = \lambda_2^{(11)} = 0$, $\lambda_3^{(11)} = \frac{32}{69178725}(-22236 \pm 1721\sqrt{705})a_2^5$. The vanishing of this coefficient leads to the case **(a)**

□

From the above proof, we get that system (3.6.5) is orbital–reversible if, and only if, it is 11-orbital–reversible. This means that the orbital–reversibility order of the family (3.6.5) is 11.

Chapter 4

Orbital–Reversibility of Planar Dynamical Systems

4.1 Introduction

The presence of involutory reversing symmetries (*reversibilities*) is very common in systems that occur in many disciplines of science and engineering (see e.g., [67, 81]). The most frequent cases of reversibility that we can find in the literature are associated to linear involutions. In fact, they usually correspond to the called axis–reversibility (some authors call it time–reversibility), where the involution consists in the change of sign in some variable. In such cases, the presence of some reversibility is a valuable fact in the study, because each solution has a twin, obtained by changing the sign of the time variable and applying the time–reversing symmetry. This property is of utility in the understanding of the phase portrait as well as in the long time behavior of the system, being an important tool to detect periodic orbits.

In particular, for planar systems, there is a strong connection between the center problem and the reversibility property. In fact, the vector field $(-y, x)^T + \cdots$ has a center at the origin if, and only if, it is reversible (see [78]). The reversibility of some families of vector fields are known; for instance, the reversibility of $(-y, x)^T + (P_m(x, y), Q_m(x, y))^T$ is known for $m = 2$ and $m = 3$ (see [32, 83, 100, 101] and references therein). However, this does not happen for the nilpotent case: the vector field $(-y, x^{2n+1})^T + \cdots$ has a center at the origin if, and only if, it is orbital–reversible (see [35]). Therefore, the monodromic and reversible planar systems of type nilpotent are centers. However, the reverse claim is not true.

The connections between centers, analytic integrability and reversibility of planar vector fields has been investigated by many authors (see [15, 24, 34, 35, 46, 70, 86, 90, 99, 102] and references therein).

In this chapter, we are concerned with the *orbital–reversibility* problem: a system

is called *orbital–reversible* if there exists some time-reparametrization such that the resulting system admits some (nonlinear) reversibility; and our goal is to determine, in the planar case, conditions on the system to be orbital–reversible. As with the reversibility, the presence of some orbital–reversibility is useful in the understanding of the dynamical behavior of the system, because the time-reparametrizations do not change the orbits but only the speed in which they are traversed in time.

Sometimes, the concept of orbital–reversibility is more helpful than the one of reversibility. For instance, in the study of the characterization of centers, the orbital–reversibility property is a wider concept than the reversibility; in fact, there exist nilpotent vector fields having a center that are orbital–reversible but not reversible. An example is presented below:

Let us consider the system

$$\begin{aligned} \dot{x} &= y + xy + x^4, \\ \dot{y} &= -x^5 + 3y^2 + 3x^3 y. \end{aligned}$$

In [24], it is shown that it is not reversible. Nevertheless, if we reparametrize the time by $\frac{dt}{dT} = \frac{1}{(1+x)^2}$, we get a system which is reversible with respect to the involution

$$\sigma(x, y) = \left(\frac{-x}{1+2x}, \frac{y}{(1+2x)^3} \right),$$

defined for $x < 1/2$. It is a simple task to show that $\mathrm{Fix}\,(\sigma)$ is the y-axis and, as the system is monodromic, we can conclude that the origin is a center, which is orbital–reversible but not reversible.

Along this chapter, we consider a planar analytic autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}),\ \mathbf{x} = (x, y)^T \in \mathbb{R}^2, \tag{4.1.1}$$

having an equilibrium point located at the origin.

In this chapter, the underlying idea in our analysis is the reduction of the system to a pre-normal form and later to analyze the axis–reversibility modulo orbital–equivalence (see Proposition 66). In this way, we get necessary conditions for orbital–reversibility that provide an algorithm, discarding non-orbital–reversible cases. Hence, it is not necessary to pay attention to the convergence aspects in the normalization procedure. In other words, our analysis does not require addressing issues of convergence, so that the vector fields and scalar functions along this chapter should be understood in the formal sense.

The algorithm we will use to obtain conditions for orbital–reversibility is based on the following definition of orbital equivalence:

Definition 57. *Let us consider two vector fields* $\mathbf{F}, \mathbf{G}$. *We say that* $\mathbf{F}$ *is orbital equivalent to* $\mathbf{G}$ *if there exist a diffeomorphism* Φ *and a function* μ, *with* $\mu(\mathbf{0}) = 1$ *such that* $\mathbf{G} = \Phi_* (\mu \mathbf{F})$.

This chapter is organized as follows. In Section 4.2, we present necessary conditions for orbital–reversibility based on invariants of the vector field, which are used as a negative criterion for a class of planar vector fields (see Proposition 66). These results are applied in Section 4.3 to the study of the orbital–reversibility for nilpotent vector fields and, in Section 4.4, to some vector fields having null linear part.

4.2 Necessary conditions for orbital–reversibility of planar vector fields

In our analysis, we use some definitions and terminology, related to the quasi-homogeneous expansions for planar vector fields. We have described them in Chapter 1. However, we include here a summary for the planar case.

A scalar function f is quasi-homogeneous of type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$ and degree k if $f(\varepsilon^{t_1} x, \varepsilon^{t_2} y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous scalar functions of type $\mathbf{t}$ and degree k is denoted by $\mathcal{P}_k^{\mathbf{t}}$.

A vector field $\mathbf{F} = (P, Q)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$, and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of quasi-homogeneous vector fields of type $\mathbf{t}$ and degree k is denoted by $\mathcal{Q}_k^{\mathbf{t}}$.

Given a vector field $\mathbf{F}$, we can formally write it as a quasi-homogeneous expansion corresponding to a fixed type $\mathbf{t}$:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots , \tag{4.2.1}$$

where each term $\mathbf{F}_k$ is a quasi-homogeneous vector field of type $\mathbf{t}$ and degree k.

Given a scalar function μ, to obtain its quasi-homogeneous expansion $\mu = \sum_k \mu_k$, with $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$, it is enough to collect the terms of its Taylor expansion whose exponents lie in the straightline $t_1 x + t_2 y = k$, for each k. The expansion in quasi-homogeneous terms of a vector field $\mathbf{F} = (P, Q)^T$ is obtained easily from the ones of the components P, Q.

The lowest non–zero quasi-homogeneous term $\mathbf{F}_r \neq \mathbf{0}$ in (4.2.1) (for some $r \in \mathbb{Z}$) is called the principal part of $\mathbf{F}$ with respect to the type $\mathbf{t}$. A priori, the type $\mathbf{t}$ may be selected arbitrarily. Nevertheless, we choose the type $\mathbf{t}$ so that the corresponding principal part $\mathbf{F}_r$ is topologically determined (see [39, 51]). In this way, we facilitate the procedure to reduce the expression of the vector field to an adequate form (see Proposition 66), and also the invariants of (4.2.1) are easier to detect (see Lemma 65).

At this moment, it is convenient to recall the definition of reversibility we will deal with:

Definition 58. *An involution is a diffeomorphism σ such that $\sigma \circ \sigma = Id$, $\sigma(\mathbf{0}) = \mathbf{0}$ and $\dim(Fix(\sigma)) = 1$ where $Fix(\sigma) = \{\mathbf{x} \in \mathbb{R}^2 : \sigma(\mathbf{x}) = \mathbf{x}\}$ is the fixed point set of*

σ. It is important to remark here that the involution could be nonlinear.
We say that system (4.1.1) (or the vector field $\mathbf{F}$) is reversible if there exists some involution σ such that $\sigma_* \mathbf{F} = -\mathbf{F}$ (we denote pull-back of a vector field of $\mathbf{F}$ by a transformation Φ as $\Phi_* \mathbf{F}$). Sometimes, we make explicit the involution by saying that $\mathbf{F}$ is σ–reversible.
We say that system (4.1.1) (or the vector field $\mathbf{F}$) is orbital–reversible if there exist an involution σ and a scalar function μ, with $\mu(\mathbf{0}) = 1$ such that $\sigma_* (\mu\mathbf{F}) = -\mu\mathbf{F}$, (this means that reparametrizing the time adequately, the system becomes reversible).

Next result, which is deduced from [13, Theorem 2.2], shows that it is possible to linearize the involution (in fact, it can be reduced to the axis–reversibility).

Theorem 59. *Let us assume that the vector field (4.2.1) is orbital–reversible. Then, for any fixed type $\mathbf{t}$, there exists $\Phi_0 \in \mathcal{Q}_0^{\mathbf{t}}$ such that the diagonal elements of $D\Phi_0$ are zero, and $(Id + \Phi_0)_* \mathbf{F}_r$ is axis–reversible (i.e., R_x or R_y–reversible).*
Also, there is a near-identity transformation $\Phi = Id + \sum_{j>0} \Phi_j$, with $\Phi_j \in \mathcal{Q}_j^{\mathbf{t}}$, and a scalar function μ, with $\mu(0) = 1$, such that $(\Phi \circ (Id + \Phi_0))_* (\mu\mathbf{F})$ is axis–reversible ($\mathbf{F}$ is orbital R_x or R_y–reversible modulo a change of variables that includes zero-degree terms).

The above theorem gives a first necessary condition for orbital–reversibility of the vector field based on its principal part $\mathbf{F}_r$ with respect to the type considered.

Two situations can occur: if $\mathbf{F}_r$ is not R_x neither R_y–reversible modulo a zero-degree change of variables, then the full vector field $\mathbf{F}$ is not longer orbital–reversible. Next, we present an example that illustrates this situation.

Example 60. *Any perturbation with higher-order quasi-homogeneous terms of type $\mathbf{t} = (3, 5)$ for the system $\dot{x} = y^2$, $\dot{y} = x^4$, cannot be orbital–reversible.*

Proof. The vector field of the quoted system $\mathbf{F}_7 = (y^2, x^4)^T \in \mathcal{Q}_7^{\mathbf{t}}$ is not R_x neither R_y–reversible. Moreover, the unique zero-degree change of variables of type $\mathbf{t}$ is a scaling of variables. Therefore, there is not $\Phi_0 \in \mathcal{Q}_0^{\mathbf{t}}$ with the diagonal elements of $D\Phi_0$ equal to zero, such that $(\mathrm{Id} + \Phi_0)_* \mathbf{F}_7$ is axis–reversible. From Theorem 59, we obtain the result. □

Next, we analyze the second situation, that arises when $\mathbf{F}_r$ is R_x or R_y–reversible (after a near-identity change of variables of zero degree, if necessary). Hence, we assume that $\mathbf{F}$ is of the form

$$\mathbf{F}(\mathbf{x}) = \widetilde{\mathbf{F}}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \quad \text{for some } \ r \in \mathbb{Z}, \tag{4.2.2}$$

where the principal part $\widetilde{\mathbf{F}}_r \neq \mathbf{0}$ is axis–reversible, and $\mathbf{F}_{r+k} \in \mathcal{Q}_{r+k}^{\mathbf{t}}$ for all $k \in \mathbb{N}$.

Note 61. *In order to study the orbital–reversibility respect to an arbitrary involution for the original system (4.1.1), we must reduce it to the form (4.2.2) by means of a zero degree near-identity transformation, and later analyze the axis–reversibility modulo orbital equivalence, or in other words, the orbital–reversibility with respect to R_x or R_y modulo a near-identity transformation. For instance, let us consider the system $\dot{x} = y^2 + \mathcal{O}(|x,y|^3)$, $\dot{y} = x^2 + \mathcal{O}(|x,y|^3)$. We select the type $\mathbf{t} = (1,1)$, because in this case the principal part $\mathbf{F}_1 = (y^2, x^2)$ is topologically determined. We note that the quoted principal part is reversible with respect to the involution $\sigma(x,y) = (y,x)$. In this case, the zero-degree change of variables $(x,y) \longrightarrow (x+y, -x+y)$ leads to $\dot{x} = (x^2+y^2)/2 + \mathcal{O}(|x,y|^3)$, $\dot{y} = -xy + \mathcal{O}(|x,y|^3)$, and now the principal part is R_x–reversible.*

We will require some properties about operations involving quasi-homogeneous functions and/or vector fields (for more details, see [12]); the first four have been stated in Chapter 1, Lemma 4, although we include them again to facilitate reading.

- If $\mu \in \mathcal{P}_k^{\mathbf{t}}$ and $\mathbf{F} \in \mathcal{Q}_l^{\mathbf{t}}$, then $\mu\,\mathbf{F} \in \mathcal{Q}_{k+l}^{\mathbf{t}}$.
- If $\mu \in \mathcal{P}_k^{\mathbf{t}}$ and $\mathbf{F} \in \mathcal{Q}_l^{\mathbf{t}}$, then $\nabla\mu \cdot \mathbf{F} \in \mathcal{P}_{k+l}^{\mathbf{t}}$.
- The divergence of a vector field $\mathbf{F} = (P,Q)^T$ is $\mathrm{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.
  For all $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$, we have $\mathrm{div}\,(\mathbf{F}) \in \mathcal{P}_k^{\mathbf{t}}$.
- The Lie product of two vector fields $\mathbf{F}$ and $\mathbf{G}$ is $[\mathbf{F},\mathbf{G}] = D\mathbf{F}\,\mathbf{G} - D\mathbf{G}\,\mathbf{F}$.
  If $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$, $\mathbf{G} \in \mathcal{Q}_l^{\mathbf{t}}$, then $[\mathbf{F},\mathbf{G}] \in \mathcal{Q}_{k+l}^{\mathbf{t}}$.
- We denote $\mathbf{D}_0 = (t_1 x, t_2 y)^T \in \mathcal{Q}_0^{\mathbf{t}}$. Observe that it is a radial vector field.
  For any $p \in \mathcal{P}_k^{\mathbf{t}}$ we have $\nabla p \cdot \mathbf{D}_0 = k\,p$ (Euler's Theorem for quasi-homogeneous vector fields).
- We denote by $\mathbf{X}_h$ the Hamiltonian vector field associated to Hamiltonian h, that is, $\mathbf{X}_h = \left(-\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x}\right)^T$.
  It is easy to check that $\mathbf{X}_h \in \mathcal{Q}_k^{\mathbf{t}}$ if, and only if, $h \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$, (here, $|\mathbf{t}| := t_1 + t_2$ denotes the modulus of $\mathbf{t}$).
- The wedge product (see [64]) of two vector fields $\mathbf{F} = (P,Q)^T$ and $\mathbf{G} = (R,S)^T$ is the scalar function $\mathbf{F} \wedge \mathbf{G} = PS - QR$.
  If we take $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$, $\mathbf{G} \in \mathcal{Q}_l^{\mathbf{t}}$, then $\mathbf{F} \wedge \mathbf{G} \in \mathcal{P}_{k+l+|\mathbf{t}|}^{\mathbf{t}}$.

A key fact, which can be found in [12], is the following. Any planar quasi-homogenous vector field can be decomposed uniquely as the sum of two quasi-homogeneous vector fields, one of them having zero divergence (*conservative part*) and the other one with divergence equal to the original vector field (*dissipative part*).

Lemma 62. *Let us consider a quasi-homogeneous planar vector field* $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$. *Then, there exists an unique couple of quasi-homogeneous polynomials:* $h \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$, $\mu \in \mathcal{P}_k^{\mathbf{t}}$, *such that*

$$\mathbf{F}_k = \mathbf{X}_h + \mu \mathbf{D}_0, \ \ \textit{with} \ \ h = \frac{1}{k+|\mathbf{t}|}\mathbf{D}_0 \wedge \mathbf{F}_k, \ \ \mu = \frac{1}{k+|\mathbf{t}|}\mathrm{div}(\mathbf{F}_k).$$

Now we introduce the concept and some properties of invariant curves of vector fields, in order to study the orbital–reversibility property.

Definition 63. *We denote by* $\mathbb{R}[[x,y]]$ *the ring of formal power series with coefficients in* $\mathbb{R}$. *We say that* $V \in \mathbb{R}[[x,y]]$, $V(\mathbf{0}) = 0$, *is an invariant curve of system (4.1.1) with cofactor* $\kappa \in \mathbb{R}[[x,y]]$ *if* $\nabla V(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) = \kappa(\mathbf{x}) V(\mathbf{x})$, *for all* $\mathbf{x} \in \mathbb{R}^2$.

Next lemma shows that if we know an invariant curve for system (4.1.1), then we also know it for any system obtained by transforming the time and the state variables.

Lemma 64. *Let us consider the vector field* $\mathbf{G} = \Phi_* (\mu \mathbf{F})$ *(*$\mathbf{G}$ *is orbital equivalent to* $\mathbf{F}$*), where* Φ *is a formal diffeomorphism and* μ *a formal scalar function such that* $\mu(\mathbf{0}) \neq 0$. *If* V *is an invariant curve of system (4.1.1) with cofactor* κ, *then* $V(\Phi(\mathbf{y}))$ *is an invariant curve of* $\mathbf{y}' = \frac{d\mathbf{y}}{dT} = \mathbf{G}(\mathbf{y})$ *with cofactor* $\mu(\Phi(\mathbf{y}))\,\kappa(\Phi(\mathbf{y}))$.

Proof. Let us denote $\widetilde{\mathbf{V}} = V(\Phi(\mathbf{y}))$. Then:

$$\begin{aligned} \nabla\widetilde{\mathbf{V}} \cdot \mathbf{G} &= \nabla V(\Phi(\mathbf{y})) \cdot \mathbf{G} = \nabla V(\Phi(\mathbf{y})) D\Phi(\mathbf{y})\, D\Phi(\mathbf{y})^{-1}\, \mu(\Phi(\mathbf{y}))\, \mathbf{F}(\Phi(\mathbf{y})) \\ &= \nabla V(\mathbf{x}) \cdot (\mu(\mathbf{x})\mathbf{F}(\mathbf{x})) = \mu(\mathbf{x})\kappa(\mathbf{x})V(\mathbf{x}) = \mu(\Phi(\mathbf{y}))\kappa(\Phi(\mathbf{y}))\widetilde{\mathbf{V}}. \end{aligned}$$

□

Lemma 65. *Let us assume that the system (4.1.1) is* σ*–reversible. If* $V(x,y)$ *is an invariant curve for system (4.1.1) with cofactor* $\kappa(x,y)$, *then* $V(\sigma(x,y))$ *is also an invariant curve for system (4.1.1) with cofactor* $-\kappa(\sigma(x,y))$.

Proof. We consider the time-reparametrization $dt = \mu(\mathbf{x})dT$, with $\mu(\mathbf{x}) \equiv -1$ (in fact, we are reversing the time), and the change of variables $(x,y) = \sigma(u,v)$. It is immediate to check that $\sigma_* (\mu\mathbf{F}) = \mathbf{F}$. Applying Lemma 64, we obtain that $V(\sigma(x,y))$ is an invariant curve of $\mathbf{F}$ with cofactor $-\kappa(\sigma(x,y))$. □

From now on, we will consider a class of vector fields which is specified in the next proposition. Specifically, we are interested in dissipative perturbations of a conservative quasi-homogeneous vector field modulo orbital–equivalence.

A wide range of interesting cases, after a normal form reduction, fall into this class, which include systems having an equilibrium with non–zero linearization:

hyperbolic saddles, linear centers and nilpotent singularities (see Section 4.3), as well as some cases of zero linear part (some examples are included in Section 4.4). Another interesting case included in this class corresponds to the systems $\dot{x} = -y^3 + \mathcal{O}(|x,y|^4)$, $\dot{y} = x^3 + \mathcal{O}(|x,y|^4)$ (see [15]). These are monodromic systems, and the analysis of the orbital–reversibility could allow to find nonlinear centers.

We recall the definition of the Lie-derivative operator associated to the principal part of the vector field (4.2.1):

$$\begin{array}{rcl} \ell_{k-r} : \mathcal{P}^{\mathbf{t}}_{k-r} & \to & \mathcal{P}^{\mathbf{t}}_{k} \\ p & \to & \ell_{k-r}(p) = \nabla p \cdot \mathbf{F}_r. \end{array}$$

As in the previous chapters, we will denote by $\mathrm{Cor}(\ell_{k-r})$ to a complementary subspace to the range of this linear operator.

The following result gives necessary conditions for the orbital–reversibility for systems that have been reduced to a suitable prenormal form. In [68], similar ideas about normal forms for reversible systems with non–zero linear part are presented.

Proposition 66. *Let us assume that* $\mathbf{F} = \mathbf{X}_h + \sum_{j>r} \mu_j \mathbf{D}_0$, *where:*

- $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, $h(-x,y) = h(x,y)$, *and the factorization of* h *on* $\mathbb{R}[x,y]$ *only has simple factors; and*
- *there exists* $N > r$ *such that* $\mu_j \in \mathrm{Cor}(\ell_{j-r})$, $\mu_j(-x,y) = -\mu_j(x,y)$, *for* $r < j < N$, *and* $\mu_N \in \mathrm{Cor}(\ell_{N-r})$, $\mu_N(-x,y) + \mu_N(x,y) \in \mathrm{Cor}(\ell_{N-r}) \setminus \{0\}$, *and* $\mathcal{P}^{\mathbf{t}}_{N-r} \neq \{0\}$.

Then, system (4.1.1) is not orbital R_x*–reversible modulo near-identity transformations.*

Proof. Applying Euler's Theorem, we get $\nabla h \cdot (\mathbf{X}_h + \mu \mathbf{D}_0) = (r + |\mathbf{t}|) h \sum_{j>r} \mu_j$. Then, h is an invariant curve of $\mathbf{F}$ with cofactor $(r + |\mathbf{t}|) \sum_{j>r} \mu_j$.

We use *reductio ad absurdum*: let us assume that there exist a time-reparametrization $dt = \left(1 + \sum_{j \geq N-r} \lambda_j\right) dT$, $\lambda_j \in \mathcal{P}^{\mathbf{t}}_j$, and a near-identity transformation $(x,y) = \Phi(u,v) = (u,v) + \sum_{j \geq N-r} \Phi_j(u,v)$, $\Phi_j \in \mathcal{Q}^{\mathbf{t}}_j$, such that the transformed vector field $\mathbf{G} = \Phi_* \left((1 + \sum_{j \geq N-r} \lambda_j) \mathbf{F} \right)$ is R_u–reversible.

From Lemma 64, we get that

$$V(u,v) := h(\Phi(u,v)) = h(u,v) + h(\Phi_{N-r}(u,v)) + \cdots$$

is an invariant curve of $\mathbf{G}$ with cofactor

$$\begin{array}{rcl} \kappa(u,v) & = & \left(1 + \sum_{j \geq N-r} \lambda_j\right) (r + |\mathbf{t}|) \mu(\Phi(u,v)) \\ & = & (r + |\mathbf{t}|) \left(\displaystyle\sum_{j>r}^{N-1} \mu_j(u,v) + \mu_N(u,v) + \cdots \right). \end{array}$$

On the other hand, applying Lemma 65 with $\sigma = R_x$, we obtain that

$$\widetilde{\mathbf{V}}(u,v) = V(-u,v) = h(u,v) + \sum_{j \geq N+|\mathbf{t}|} \widetilde{\mathbf{V}}_j$$

is another invariant curve of $\mathbf{G}$ with cofactor

$$\tilde{\kappa}(u,v) = -\kappa(-u,v) = (r+|\mathbf{t}|)\left(\sum_{j>r}^{N-1} \mu_j(u,v) - \mu_N(-u,v) + \cdots\right).$$

Let W be the invariant curve of $\mathbf{F}$ which comes from $\widetilde{\mathbf{V}}$, undoing the change of variables. From Lemma 64, we have that $W(x,y) = h(x,y) + \sum_{j \geq N+|\mathbf{t}|} W_j$ with cofactor

$$\hat{\kappa}(x,y) \;=\; (r+|\mathbf{t}|)\left(\sum_{j>r}^{N-1} \mu_j(x,y) - \mu_N(-x,y) + \cdots\right).$$

Hence $\nabla W \cdot \mathbf{F} = \hat{\kappa} W$, which implies

$$\nabla\left(h + \sum_{j \geq N+|\mathbf{t}|} W_j\right) \cdot \left(\mathbf{X}_h + \sum_{j>r}^{N-1} \mu_j \mathbf{D}_0 + \mu_N(x,y)\mathbf{D}_0 + \cdots\right) =$$
$$= (r+|\mathbf{t}|)\left(\sum_{j>r}^{N-1} \mu_j(x,y) - \mu_N(-x,y) + \cdots\right)\left(h + \sum_{j \geq N+|\mathbf{t}|} W_j\right),$$

and then

$$\nabla W_{N+|\mathbf{t}|} \cdot \mathbf{X}_h + (r+|\mathbf{t}|)\left(\mu_N(x,y) + \mu_N(-x,y)\right) h + \cdots = 0.$$

By hypothesis, $\mu_N(x,y) + \mu_N(-x,y) \in \mathrm{Cor}\left(\ell_{N-r}\right) \setminus \{0\}$ and applying [12, Prop. 3.18], since h has simple factors on $\mathbb{R}[x,y]$ and $\mathcal{P}^{\mathbf{t}}_{N-r} \neq \{0\}$, we get

$$\nabla W_{N+|\mathbf{t}|} \cdot \mathbf{X}_h = -(r+|\mathbf{t}|)\left(\mu_N(x,y) + \mu_N(-x,y)\right) h \in \mathrm{Cor}\left(\ell_{N+|\mathbf{t}|}\right) \setminus \{0\},$$

which is a contradiction since $\nabla W_{N+|\mathbf{t}|} \cdot \mathbf{X}_h \in \mathrm{Range}\left(\ell_{N+|\mathbf{t}|}\right)$. □

Analogously we have the following result:

Proposition 67. *Let us assume that* $\mathbf{F} = \mathbf{X}_h + \sum_{j>r} \mu_j \mathbf{D}_0$, *where:*

- $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, $h(x,-y) = h(x,y)$, *and the factorization of* h *on* $\mathbb{R}[x,y]$ *only has simple factors; and*

- *there exists* $N > r$ *such that* $\mu_j \in \mathrm{Cor}\,(\ell_{j-r})$, $\mu_j(x,-y) = -\mu_j(x,y)$, *for* $r < j < N$, *and* $\mu_N \in \mathrm{Cor}\,(\ell_{N-r})$, $\mu_N(x,-y) + \mu_N(x,y) \in \mathrm{Cor}\,(\ell_{N-r}) \setminus \{0\}$, *and* $\mathcal{P}^{\mathbf{t}}_{N-r} \neq \{0\}$.

Then, system (4.1.1) is not orbital R_y*–reversible modulo near-identity transformations.*

As a consequence of Propositions 66 and 67, we get the following results:

Theorem 68. *Let* $\mathbf{F} = \mathbf{X}_h + \sum_{j>r} \mu_j \mathbf{D}_0$, *where* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, $h(-x,y) = h(x,y)$, h *only has simple factors in its decomposition on* $\mathbb{R}[x,y]$*; and* $\mu_j \in \mathrm{Cor}\,(\ell_{j-r})$. *Then, system (4.1.1) is orbital* R_x*–reversible modulo near-identity transformations if, and only if, it is* R_x*–reversible.*

Theorem 69. *Let* $\mathbf{F} = \mathbf{X}_h + \sum_{j>r} \mu_j \mathbf{D}_0$, *where* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, $h(x,-y) = h(x,y)$, h *only has simple factors in its decomposition on* $\mathbb{R}[x,y]$*; and* $\mu_j \in \mathrm{Cor}\,(\ell_{j-r})$. *Then, system (4.1.1) is orbital* R_y*–reversible modulo near-identity transformations if, and only if, it is* R_y*–reversible.*

4.3 Orbital–reversibility of nilpotent vector fields

In this section, we consider perturbations of the following nilpotent vector field:

$$\mathbf{F}_r = (y, \sigma x^m)^T, \tag{4.3.1}$$

being $\sigma = \pm 1$, $m \in \mathbb{N}$.

It is easy to check that, if m is even then this vector field is quasi-homogeneous of type $\mathbf{t} = (2, m+1)$ and degree $r = m-1$; whereas if $m = 2n-1$ is odd then it is quasi-homogeneous of type $\mathbf{t} = (1,n)$ and degree $r = n-1$.

Our goal is to analyze the orbital–reversibility of higher-order quasi-homogeneous perturbations of (4.3.1):

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots . \tag{4.3.2}$$

From Note 61, it is enough to study the orbital R_x and R_y–reversibilities modulo near-identity transformations of (4.3.2), because for this type, the unique zero-degree term are $\Phi_0(x,y) = (ax, by)^T \in \mathcal{Q}^{\mathbf{t}}_0$, and then the unique reversibilities with degree zero that (4.3.1) admits are the axis–reversibilities.

The following result characterizes the orbital R_y–reversibility:

Proposition 70. *The vector field (4.3.2) is orbital* R_y*–reversible modulo near-identity transformations if, and only if, it is orbital equivalent to (4.3.1).*

Proof. The sufficient condition is trivial since the vector field (4.3.1) is R_y–reversible.

We prove the necessary condition. By [21, Theorems 14, 16], the vector field (4.3.2) is orbital–equivalent to the following normal form:

$$\mathbf{F}_r + \sum_{j=\lfloor \frac{m+1}{2} \rfloor}^{m-1} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{m-1} \alpha_j^{(l)} x^j h^l \mathbf{D}_0,$$

where $\lfloor \cdot \rfloor$ denotes the *floor* function, and:

- if m is even, we have $\mathbf{D}_0 = (2x, (m+1)y)^T$ and $h = 2\sigma x^{m+1} - (m+1)y^2$;
- if $m = 2n-1$ is odd, then $\mathbf{D}_0 = (x, ny)^T$ and $h = \sigma x^{2n} - ny^2$.

Also, by [21, Propositions 13, 15], we have that $\alpha_j^{(0)} x^j \in \mathrm{Cor}\,(\ell_{j-r})$, if $\lfloor \frac{m+1}{2} \rfloor \leq j \leq m-1$; and $\alpha_j^{(l)} x^j h^l \in \mathrm{Cor}\,(\ell_{j-r})$ for $l \geq 1$, $0 \leq j \leq m-1$.

Hence, from Theorem 69, we obtain that $\alpha_j^{(l)} = 0$ for all l, j, and the result follows. □

Corollary 71. *Let us assume that the vector field (4.3.2), with $m = 2n-1$ odd, is orbital R_y–reversible modulo near-identity transformations. Then, it is orbital R_x–reversible modulo near-identity transformations.*

Proof. From Proposition 70, the vector field (4.3.2) is orbital–equivalent to (4.3.1), which is R_x–reversible. □

Now we deal with the orbital R_x–reversibility:

Proposition 72. *The vector field (4.3.2) is orbital R_x–reversible modulo near-identity transformations if, and only if, $m = 2n-1$ is odd and it is orbital equivalent to the following normal form:*

$$\mathbf{F}_r + \sum_{\substack{j=n \\ j \ odd}}^{2n-2} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j \ odd}}^{2n-2} \alpha_j^{(l)} x^j h^l \mathbf{D}_0,$$

where $\mathbf{D}_0 = (x, ny)^T \in \mathcal{Q}_0^{\mathbf{t}}$, and $h = \sigma x^{2n} - ny^2 \in \mathcal{P}_{2n}^{\mathbf{t}}$.

Proof. If m is even, the principal part (4.3.1) is not R_x–reversible. Moreover, the zero-degree changes of variables for this type are just the scaling of variables. From Theorem 59, we obtain that the vector field (4.3.2) is not orbital R_x–reversible modulo near-identity transformations.

If $m = 2n-1$ is odd, from [21, Theorem 16], system (4.3.2) is orbital equivalent to the following normal form:

$$\mathbf{F}_r + \sum_{j=n}^{2n-2} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{2n-2} \alpha_j^{(l)} x^j h^l \mathbf{D}_0, \tag{4.3.3}$$

where $\mathbf{D}_0 = (x, ny)^T$ and $h = \sigma x^{2n} - ny^2$.

Moreover, by [21, Propositions 15], we have that $\alpha_j^{(0)} x^j \in \mathrm{Cor}\,(\ell_{j-r})$ for $n \leq j \leq 2n-2$, and $\alpha_j^{(l)} x^j h^l \in \mathrm{Cor}\,(\ell_{j-r})$ for $l \geq 1$, $0 \leq j \leq 2n-2$.

Hence, applying Theorem 68, we get $\alpha_j^{(l)} = 0$ for j even and for all l, and the result holds. □

Using Note 61, Propositions 70, 72 and Corollary 71, we get the main result in this section:

Theorem 73. *System (4.3.2) is orbital–reversible if, and only if,*

- *m is even and it is orbital equivalent to $\mathbf{F}_r$; or*
- *$m = 2n-1$ is odd and it is orbital equivalent to*

$$\mathbf{F}_r + \sum_{\substack{j=n \\ j \ odd}}^{2n-2} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j \ odd}}^{2n-2} \alpha_j^{(l)} x^j h^l \mathbf{D}_0, \tag{4.3.4}$$

where $\mathbf{D}_0 = (x, ny)^T \in \mathcal{Q}_0^{\mathbf{t}}$ and $h = \sigma x^{2n} - ny^2 \in \mathcal{P}_{2n}^{\mathbf{t}}$.

4.3.1 A particular family

Now, let us consider the following family of nilpotent planar vector fields:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \sigma x^{4q+1} \end{pmatrix} + \begin{pmatrix} a_1 xy + a_2 x^{2q+2} \\ b_1 y^2 + b_2 x^{2q+1} y \end{pmatrix}, \tag{4.3.5}$$

where $\sigma = \pm 1$, $q \in \mathbb{N}$.

This family has been studied by several authors. Namely, the analytic integrability has been studied in [44]; the center problem for $\sigma = -1$ (which corresponds to the monodromic situation) has been partially studied in [55]; and the reversibility problem is completely solved in [24]. Next result solves the orbital–reversibility problem:

Theorem 74. *System (4.3.5) is orbital–reversible if, and only if, one of the following conditions holds:*

(a) $a_2 = b_2 = 0$.

(b) $a_2 = a_1 = b_1 = 0$, $b_2 \neq 0$.

(c) $a_1 = b_1 = 0$, $a_2 \neq 0$.

(d) $a_1 + 2b_1 = b_2 + 2(q+1)a_2 = 0$, $a_2 b_1 \neq 0$.

(e) $b_2 = (2q+1)a_2$, $b_1 = (2q+1)a_1$, $a_2(a_1 + 2b_1) \neq 0$.

Note 75. *Items* **(a)**-**(d)** *agree with the cases of reversibility, considered in [24]. Item* **(e)** *corresponds to a situation of a non–reversible system that is orbital–reversible. This item for* $\sigma = -1$ *presents a new case of center at the origin, not included in [55], that has been detected by the orbital–reversible algorithm.*

Proof. The vector field of the statement can be written as $\mathbf{F} = \widetilde{\mathbf{F}}_r + \mathbf{F}_{r+1}$, where:

$$\widetilde{\mathbf{F}}_r = (y, \sigma x^{4q+1})^T \in \mathcal{Q}^{\mathbf{t}}_r,\ \mathbf{F}_{r+1} = \left(a_1 xy + a_2 x^{2q+2}, b_1 y^2 + b_2 x^{2q+1} y\right)^T \in \mathcal{Q}^{\mathbf{t}}_{r+1},$$

with $r = 2q$ and $\mathbf{t} = (1, 2q+1)$. We notice that it is a particular case of the vector field (4.3.2), with $m = 2n - 1$, being $n = 2q + 1$.

From Theorem 73, the characterization of the orbital–reversibility for system (4.3.5) depends on the normal form (4.3.3). More specifically, system (4.3.5) is orbital–reversible if, and only if, the normal form (4.3.3) reduces to (4.3.4). Therefore, the following coefficients of the quoted normal form must vanish:

- $\alpha_j^{(0)} = 0$, if $2q - 1 \leq j \leq 4q$, j even, and
- $\alpha_j^{(l)} = 0$, for all $l \in \mathbb{N}$, $0 \leq j \leq 4q$, j even.

The effective computation of the normal form in this example is performed following a procedure based on Lie transforms. The details of the procedure can be found in [7], and the main tool is the Lie product of vector fields. Using *Maple* in the computations, we have obtained the following expressions for the coefficients of the normal form (4.3.3).

The first coefficient is:

$$\alpha^{(0)}_{2q+2} = a_2((2q+3)(2q+1)a_1 + 2qb_1) + b_2(2qa_1 - 3b_1). \tag{4.3.6}$$

To study the vanishing of this coefficient, we consider the following two possibilities:

(1) $2qa_1 - 3b_1 = 0$, and then $\alpha^{(0)}_{2q+2}$ vanishes in a couple of cases:

(1a) $a_2 = 0$. In this case, the next normal form coefficient is

$$\alpha^{(0)}_{2q+4} = b_2 a_1^3,$$

which vanishes if

- $b_2 = 0$. In this case, covered in item **(a)**, the system is R_y–reversible; or
- $a_1 = 0$. The system now is R_x–reversible, this situation appears in item **(b)**.

(1b) $a_2 \neq 0$, $(2q+3)(2q+1)a_1 + 2qb_1 = 0$. In this case, a trivial computation provides $a_1 = b_1 = 0$. For these values, the system is R_x–reversible. This is the situation described in item **(c)**.

(2) $2qa_1 - 3b_1 \neq 0$, and then the normal form coefficient $\alpha^{(0)}_{2q+2}$ vanishes if, and only if,

$$b_2 = -\frac{(2q+3)(2q+1)a_1 + 2qb_1}{2qa_1 - 3b_1} a_2. \tag{4.3.7}$$

For this value of b_2, we are able to compute the next normal form coefficient:

$$\alpha^{(0)}_{2q+4} = a_2(a_1 + 2b_1)(b_1 - (2q+1)a_1)\, C_4,$$

where:

$$\begin{aligned} C_4 \;=\; & 3(2q+5)(4q+3)^2[(2q+3)(4q+1)a_1 - (4q+9)b_1]a_2^2 \\ & +\sigma(2qa_1 - 3b_1)[2q(120q^2 + 202q + 49)a_1^2 \\ & \qquad -(512q^2 + 844q + 135)a_1b_1 + 5(52q+81)b_1^2], \end{aligned}$$

depends on a_2, a_1, b_1, q, σ. To discuss the vanishing of $\alpha^{(0)}_{2q+4}$, we consider some subcases:

(2a) $a_2 = 0$. This implies $b_2 = 0$, and we get item **(a)** again.

(2b) $a_2 \neq 0$, $a_1 + 2b_1 = 0$. This hypothesis implies that $b_1 \neq 0$ (otherwise, $a_1 = b_1 = 0$, and so $2qa_1 - 3b_1 = 0$)). Moreover, from (4.3.7) we get $b_2 = -2(q+1)a_2$. Now, the system (4.3.5) is Hamiltonian, with Hamiltonian function:

$$h(x,y) = -\frac{1}{2}y^2 + \frac{\sigma}{2(2q+1)}x^{4q+2} + b_1xy^2 - a_2x^{2q+2}y.$$

The transformation $u = x$, $v = y - 2b_1xy + a_2x^{2q+2}$, leads the system (4.3.5) to:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \sigma u^{4q+4} + (2(q+1)a_2^2 - 2b_1\sigma)u^{4q+2} + \frac{a_2u^{4q+4} - b_1v^2}{1 - 2b_1u}, \end{aligned}$$

which is R_v–reversible. This case is covered in item **(d)**.

(2c) $a_2(a_1+2b_1)\neq 0$, $b_1=(2q+1)a_1$. Now, from (4.3.7) we obtain $b_2=(2q+1)a_2$. If we reparametrize the time by $dT=(1+a_1x)^{2q}dt$, and change the variables by $X=\frac{x}{1+a_1x}$, $Y=y(1+a_1x)^{-2q-1}$, then the system (4.3.5) becomes:

$$\begin{aligned} X' &= Y+a_2X^{2q+2},\\ Y' &= \sigma X^{4q+1}+(2q+1)a_2X^{2q+1}Y, \end{aligned}$$

which is R_X–reversible. This case is included in item **(e)**.

(2d) $a_2(a_1+2b_1)(b_1-(2q+1)a_1)\neq 0$, $C_4=0$. In this case, a deeper study shows that the next normal form coefficients $\alpha^{(0)}_{2q+6}$ and $\alpha^{(0)}_{2q+8}$ cannot vanish simultaneously. Hence, from Theorem 73, the system is not orbital–reversible.

□

4.4 Orbital–reversibility for some degenerate vector fields

In this last section, we take the type $\mathbf{t}=(1,1)$ (i.e., we will use Taylor expansions), and consider the degenerate quadratic vector field

$$\mathbf{F}_1=(x^2+y^2,-2xy)^T\in\mathcal{Q}_1^{\mathbf{t}}. \tag{4.4.1}$$

Observe that $\mathbf{F}_1$ is Hamiltonian: $\mathbf{F}_1=\mathbf{X}_h$, with $h=-\frac{y^3+3x^2y}{3}$. We analyze the orbital–reversibility for higher-order perturbations of (4.4.1):

$$\mathbf{F}=(x^2+y^2,-2xy)^T+\mathbf{F}_2+\mathbf{F}_3+\cdots, \tag{4.4.2}$$

where $\mathbf{F}_2\in\mathcal{Q}_2^{\mathbf{t}}$ are cubic terms, etc.

Proposition 76. *The vector field (4.4.2) is not orbital R_y–reversible modulo near-identity transformations.*

Proof. We observe that $\mathbf{F}_1$ is not R_y–reversible. Moreover, it is easy to show that if $\Phi_0\in\mathcal{Q}_0^{\mathbf{t}}$ satisfies that $D\Phi_0$ has zero diagonal elements, then $\Phi_0(u,v)=(av,bu)^T$. Hence, the change of variables $(x,y)^T=(\mathrm{Id}+\Phi_0)(u,v)$, with $ab\neq 1$, leads to

$$\mathbf{G}_1 = (\mathrm{id}+\Phi_0)_*\mathbf{F}_1=\begin{pmatrix} -\frac{b^2+1+2ab}{ab-1}u^2-\frac{2(2a+b+a^2b)}{ab-1}uv-\frac{3a^2+1}{ab-1}v^2\\ \frac{b(3+b^2)}{ab-1}u^2+\frac{2(b^2+1+2ab}{ab-1}uv+\frac{2a+b+a^2b}{ab-1}v^2\end{pmatrix}.$$

Since $3a^2+1\neq 0$, we can assure that $\mathbf{G}_1$ is not R_v–reversible. To complete the proof, it is enough to use Theorem 59. □

Next result yields a normal form for system (4.4.2):

Proposition 77. *The vector field (4.4.2) is orbital–equivalent to the following normal form:*

$$\mathbf{G} = \mathbf{X}_h + \alpha_0^{(4)} y^2 \mathbf{D}_0 + \sum_{l=1}^{\infty} \left(\alpha_l^{(1)} + \alpha_l^{(2)} x + \alpha_l^{(3)} y + \alpha_l^{(4)} y^2 \right) h^l \mathbf{D}_0, \tag{4.4.3}$$

where $h = -\frac{y^3+3x^2y}{3} \in \mathcal{P}_3^{\mathbf{t}}$ *and* $\mathbf{D}_0 = (x, y)^T \in \mathcal{Q}_0^{\mathbf{t}}$.

Proof. Recall that $\mathbf{F}_1 = \mathbf{X}_h$, with $h = -\frac{y^3+3x^2y}{3}$. It is easy to obtain complementary subspaces $\mathrm{Cor}\,(\ell_k)$ to the range of the Lie derivative operators ℓ_k for $k = 0, 1, 2$:

- $k = 0$. We have $\mathcal{P}_0^{\mathbf{t}} = \mathrm{span}\{1\}$ and $\mathcal{P}_1^{\mathbf{t}} = \mathrm{span}\{x, y\}$. Moreover, taking $\mu_0 = \alpha_0 \in \mathcal{P}_0^{\mathbf{t}}$, we find $\ell_0(\mu_0) = 0$. Therefore, $\mathrm{Range}(\ell_0) = \mathrm{span}\{0\}$ and $\mathrm{Cor}(\ell_0) = \mathrm{span}\{x, y\}$.

- $k = 1$. We have $\mathcal{P}_2^{\mathbf{t}} = \mathrm{span}\{x^2, xy, y^2\}$. If we take $\mu_1 = \alpha_0 x + \alpha_1 y \in \mathcal{P}_1^{\mathbf{t}}$ then a simple computation gives $\ell_1(\mu_1) = \alpha_0 x^2 - 2\alpha_1 xy + \alpha_0 y^2$. Therefore, $\mathrm{Range}(\ell_1) = \mathrm{span}\{x^2 + y^2, xy\}$ and $\mathrm{Cor}(\ell_1) = \mathrm{span}\{y^2\}$.

- $k = 2$. We take the following basis of $\mathcal{P}_2^{\mathbf{t}} = \mathrm{span}\{x^3, x^2y, xy^2, h\}$. If we take $\mu_2 = \alpha_0 x^2 + \alpha_1 xy + \alpha_2 y^2 \in \mathcal{P}_2^{\mathbf{t}}$ then $\ell_2(\mu_2) = 2\alpha_0 x^3 - 4\alpha_1 x^2 y + 2(\alpha_0 - 2\alpha_2)xy^2 - 3\alpha_1 h$. Hence, $\mathrm{Range}(\ell_k) = \mathrm{span}\{x^3 + xy^2, -4x^2y - 3h, xy^2\}$ and $\mathrm{Cor}(\ell_2) = \mathrm{span}\{h\}$.

A normal form under equivalence is given by [21, Proposition 12]:

$$\mathbf{G} = \mathbf{X}_h + \sum_{j \geq 2} \left(\mathbf{X}_{h_j} + \mu_j \mathbf{D}_0 \right), \tag{4.4.4}$$

where h_j belongs to a complementary subspace to $h\mathrm{Cor}\,(\ell_{j-2})$ in $\mathrm{Cor}\,(\ell_{j+1})$ and $\mu_j \in \mathrm{Cor}\,(\ell_{j-1})$.

The normal form of the statement arises by taking into account that, since $\mathcal{P}_{j-2}^{\mathbf{t}} \neq \{0\}$ for all $j \geq 2$, from [6, Proposition 3.18] we get $\mathrm{Cor}\,(\ell_{j+1}) = h\mathrm{Cor}\,(\ell_{j-2})$. This means that $h_j = 0$ for $j \geq 2$, and $\mu_j \in \mathrm{Cor}\,(\ell_{j-1}) = h\mathrm{Cor}\,(\ell_{j-4})\,\mathbf{D}_0 = \cdots = h^l \mathrm{Cor}\,(\ell_{i-1})\,\mathbf{D}_0$ if $j = 3l + i$, $l \in \mathbb{N}$, $1 \leq i \leq 3$. Using these bases for the complementary subspaces in (4.4.4), we get (4.4.3). □

Next theorem characterizes the orbital–reversibility for (4.4.2) in terms of the coefficients of the above normal form:

Theorem 78. *The vector field (4.4.2) is orbital–reversible if, and only if, the coefficients of the normal form (4.4.3) verify:* $\alpha_l^{(1)} = \alpha_l^{(3)} = 0$ *for all* $l \in \mathbb{N}$, *and* $\alpha_l^{(4)} = 0$ *for all* $l \in \mathbb{N} \cup \{0\}$.

Proof. From Note 61 and Proposition 76, the vector field (4.4.2) is orbital–reversible if, and only if, it is orbital R_x–reversible modulo near-identity transformations. On the other hand, by Proposition 77, it is orbital R_x–reversible modulo near-identity transformations if, and only if, system (4.4.3) is orbital R_x–reversible modulo near-identity transformations.

We can apply Theorem 68 to the normal form (4.4.3) because $h(-x,y)=h(x,y)$, h only has simple factors in its decomposition on $\mathbb{R}[x,y]$ and the dissipative terms belong to $\mathrm{Cor}\,(\ell_j)$ for some appropriate j (see the proof of Proposition 77).

In summary, the vector field system (4.4.2) is orbital reversible if, and only if, the normal form (4.4.3) is R_x–reversible, which gives the result. □

4.4.1 A case study

We will apply Theorem 78 to a specific example, namely, we consider the family:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ -2xy \end{pmatrix} + \begin{pmatrix} A_0x^3+A_1x^2y+A_2xy^2+A_3y^3 \\ A_0x^2y+B_2xy^2+B_3y^3 \end{pmatrix}. \tag{4.4.5}$$

Theorem 79. *System (4.4.5) is orbital–reversible if, and only if, one of the following conditions is satisfied:*

(a) $A_0=A_2=B_3=0$.

(b) $A_0=0,\ A_1=-B_2,\ A_2=-3B_3$.

(c) $B_3=A_2=A_0,\ B_2=A_1,\ A_3=0$.

Proof. We obtain necessary conditions by using Theorem 78. We compute the coefficients of the normal form (4.4.3), and we impose that $\alpha_l^{(1)}=\alpha_l^{(3)}=0$ for all $l\in\mathbb{N}$ and $\alpha_l^{(4)}=0$ for all $l\in\mathbb{N}\cup\{0\}$.

The conservative/dissipative splitting given in Lemma 62 allows us to write the vector field of the above system as

$$\mathbf{F}_1+\mathbf{X}_{h_4}+\mu_2\mathbf{D}_0, \tag{4.4.6}$$

where

$$h_4=c_{22}x^2y^2+c_{13}xy^3+c_{04}y^4\in\mathcal{P}_4^{\mathbf{t}},\quad \mu_2=(d_{20}x^2+d_{11}xy+d_{02}y^2)\in\mathcal{P}_2^{\mathbf{t}},$$

with

$$c_{22}=\frac{1}{4}(B_2-A_1),\ \ c_{13}=\frac{1}{4}(B_3-A_2),\ \ c_{04}=-\frac{1}{4}A_3,$$
$$d_{20}=A_0,\ \ d_{11}=\frac{1}{2}(A_1+B_2),\ \ d_{02}=\frac{1}{4}(A_2+3B_3).$$

The computation of the normal form coefficients has been carried out by using *Maple*. The expression for the first normal form coefficient is

$$\alpha_0^{(4)} = d_{02} - d_{20},$$

which vanishes if, and only if, $d_{02} = d_{20}$.

For this value, we have computed the second normal form coefficient:

$$\alpha_1^{(1)} = -\frac{3}{10} d_{11} c_{31} - \frac{6}{5} d_{20}(c_{22} + 2c_{04}).$$

At this point, we distinguish two cases: $d_{20} = 0$ and $d_{20} \neq 0$:

- Case $d_{20} = 0$. Here, the third coefficient is

  $$\alpha_1^{(3)} = -\frac{3}{10} d_{11} c_{31},$$

  which vanishes in two situations:

  - Case $c_{31} = 0$. In this case, the system is R_x–reversible. (Item **(a)**).
  - Case $d_{11} = 0$. In this case, the change of variables $x = \frac{B_3 v^2 + 2u}{2 - B_2 v}$, $y = v$, transforms (4.4.5) into

    $$\begin{aligned} \dot{u} &= \frac{2u^2}{2 - B_2 v} + \frac{4v^2 + 4(A_3 - B_2)v^3 + (B_2^2 - 5B_3^2 - 4A_3 B_2)v^4 + B_2(A_3 B_2 + 2B_3^2)v^5}{4 - 2B_2 v}, \\ \dot{v} &= -2uv, \end{aligned}$$

    which is R_u–reversible. (Item **(b)**)

- Case $d_{20} \neq 0$. The vanishing of $\alpha_1^{(1)}$ provides $c_{04} = -\frac{4c_{22}d_{20} + d_{11}c_{13}}{8d_{20}}$. The next normal form coefficient is now

  $$\alpha_1^{(3)} = -\frac{1}{2}(2c_{13}d_{20} + 8c_{22}^2 + 3c_{13}^2) - \frac{1}{2} c_{22}(2d_{20} + 3c_{13}) d_{11}.$$

  To study its vanishing, we consider the following three possibilities: $c_{22} = 0$, $2d_{20} + 3c_{13} = 0$ and $c_{22}(2d_{20} + 3c_{13}) \neq 0$:

  - Case $c_{22} = 0$. In this case, $\alpha_1^{(3)} = -\frac{1}{2} d_{20} c_{13}(2d_{20} + 3c_{13})$ and it is zero in two cases:
    - Case $c_{13} = 0$. In this case, the change of variable and the time-reparametrization $x = \frac{u}{1 - A_0 u}$, $y = -\frac{v}{1 - A_0 u}$, $dt = (1 - A_0 u)dT$ transforms (4.4.5) into

      $$\begin{aligned} \frac{du}{dT} &= u^2 + v^2 - A_1 u^2 v, \\ \frac{dv}{dT} &= -2uv - A_1 u v^2, \end{aligned}$$

      which is R_u–reversible (Item **(c)**). Therefore, (4.4.5) is orbital–reversible but not reversible.

  - Case $c_{13} = -\frac{2}{3}d_{20}$. In this case, $\alpha_1^{(4)} = \frac{1}{21}d_{20}^3 d_{11} = 0$. Therefore $d_{11} = 0$ and the next coefficient is $\alpha_2^{(1)} = -\frac{1}{3}d_{20}^5 \neq 0$. So in this case the system is not orbital–reversible.

- Case $c_{13} = -\frac{2}{3}d_{20}$, $c_{22} \neq 0$. In this case, $\alpha_1^{(4)} = -4c_{22}^2 d_{20} \neq 0$ and the system is not orbital–reversible.
- Case $c_{22}(2d_{20} + 3c_{13}) \neq 0$. From $\alpha_1^{(3)} = 0$ we obtain the expression of $d_{11} = -\frac{8c_{22}^2 + c_{13}(2d_{20}+3c_{13})}{c_{22}(2d_{20}+3c_{13})}$. In this case, the next coefficients $\alpha_1^{(4)}$ and $\alpha_2^{(1)}$ do not vanish simultaneously. Hence the system is not orbital–reversible.

□

Chapter 5

Local Integrability for some degenerate nilpotent vector fields

In this chapter we study the analytic integrability around the origin inside a family of degenerate nilpotent vector fields. The integrability problem for planar vector fields with first Hamiltonian component having simple factors in its factorization on $\mathbb{C}[x, y]$ is solved in [12]. Nevertheless, when the Hamiltonian function has multiple factors on $\mathbb{C}[x, y]$ is an open problem. In this second case our problem is framed. More concretely, we consider degenerate systems of the form

$$\dot{x} = -y(x^{2n} + ny^2) + \cdots, \quad \dot{y} = x^{2n-1}(x^{2n} + ny^2) + \cdots,$$

with $n \in \mathbb{N}$, whose Hamiltonian function, of its first quasi-homogeneous component, is given by $(x^{2n} + ny^2)^2/(2n)$. The analytic integrability of the above system is not completely solved and only partial results are obtained. The results are applied to some particular families of degenerate vector fields for which the integrability problem is completely solved.

5.1 Introduction

The integrability problem consists in proving the existence of first integrals and finding the functional class which this first integral of a given differential system must belong to. This is one of the main open problems in the qualitative theory of differential systems in $\mathbb{R}^2$ (see [12, 20, 22, 45, 82] and references therein).

A classical method for determining the local qualitative picture of the phase plane of a given planar system around an equilibrium point consists in obtaining a first integral; that is, a nonconstant function, defined in some nonempty open subset of $\mathbb{R}^2$, which is constant along the solution curves of the system.

Another open problem is the center problem, which consists in the distinction between a center and a focus. This problem is closely related to the integrability

and the reversibility problem (see [8, 11, 13, 14, 24, 44, 45, 77, 86] and references therein).

Throughout this chapter, we will write the system as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{5.1.1}$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$ and $\mathbf{F} = (P, Q)^T$ with P and Q analytic functions that vanish at the origin.

If there exists a formal (analytical) first integral of the above system, then it is said to be formally (analytically) integrable. Clearly, H is a $\mathcal{C}^1$ first integral if, and only if, it satisfies $\nabla H \cdot \mathbf{F} = 0$ (integrability equation).

With regard to the center problem, it is well-known that a system has a center at a singular point only if it is monodromic and it has either linear part of center type, i.e. with imaginary eigenvalues (nondegenerate point), or nilpotent linear part (nilpotent point) or null linear part (degenerate point). Any nondegenerate center has always a local analytic first integral in a neighborhood of its singular point (see [14, 45, 69, 78, 79, 80, 82]), that is, the center problem and the integrability problem are equivalent to a nondegenerate singular point. However, there are nilpotent and degenerate centers without a local analytic first integral (see [46, 58, 59, 60, 61, 75, 77, 82, 90] and references therein). There are methods to detect nondegenerate and nilpotent centers of a given family of polynomial vector field (see [2, 17, 19, 27, 56, 57, 61, 75, 77]). However there is not any method to detect centers for a general degenerate singular point.

The characterization of a nilpotent center in terms of its normal form is due to Moussu [75] and Berthier and Moussu [35]. Strózyna and Zoładek [86] have obtained the orbital normal form of nilpotent centers with analytic first integral. Any nilpotent center has a local analytic first integral if, and only if, it is analytically equivalent to the Hamiltonian system $\dot{x} = y$, $\dot{y} = -x^{2k-1}$ where $k > 1$. Chavarriga et al. [46] study the analytical integrability for reversible nilpotent centers. The integrability problem has been studied for a few families of degenerate singular points (see [1, 12, 15, 16, 56, 57] and references therein).

In [12], the integrability problem for a class of planar systems is studied and the following results are established.

Theorem 80. *Let us assume that the lowest-degree quasi-homogeneous term of the degenerate system* $\mathbf{F}$ *is* $\mathbf{F}_r = \mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$, *where* h *has only simple factors in its factorization on* $\mathbb{C}[x, y]$. *Then, the quoted system is formally integrable if, and only if, it is formally conjugated, via dissipative transformations, to a divergence-free system.*

Theorem 81. *Let us assume that the lowest-degree quasi-homogeneous term of the degenerate system* $\mathbf{F}$ *is* $\mathbf{F}_r = \mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$, *where* h *has only simple*

factors in its factorization on $\mathbb{C}[x,y]$. *Let us assume that there exist analytic (or formal) first integrals for the quoted system. Then, one of them is of the form* $I = h + \cdots$, *where the dots denote higher-order quasi-homogeneous terms.*

Using these results, the integrability problem for the next families can be solved.

In [15] the analytic integrability problem for degenerate systems of the form

$$\dot{x} = y^3 + 3\mu x^2 y + o(|x,y|^3), \qquad \dot{y} = -x^3 - 3\mu x y^2 + o(|x,y|^3), \qquad \mu \in \mathbb{R}, \quad (5.1.2)$$

was analyzed and the following result was established.

Theorem 82. *System* (5.1.2) *is analytically integrable if, and only if, it is formally orbital equivalent to* $\dot{x} = y^3 + 3\mu x^2 y$, $\dot{y} = -x^3 - 3\mu x y^2$.

In [16] the analytic integrability problem for degenerate systems of the form

$$\dot{x} = y^3 + 2ax^3 y + \cdots, \qquad \dot{y} = -x^5 - 3ax^2 y^2 + \cdots, \qquad a \in \mathbb{R}, \qquad (5.1.3)$$

was also studied where here the dots mean terms of higher order than the first component in the quasi-homogeneous expansion (see definition of quasi-homogeneous expansion below). The next result was established in [16].

Theorem 83. *System* (5.1.3) *is analytically integrable if, and only if, it is formally orbital equivalent to* $\dot{x} = y^3 + 2ax^3 y - 2\beta_9 x^4 y$, $\dot{y} = -x^5 - 3ax^2 y^2 + 4\beta_9 x^3 y^2$, *where* β_9 *is a function of the parameters of the first quasi-homogeneous components of system* (5.1.3).

Theorem 80 solves the integrability problem for $\mathbf{F} = \mathbf{F}_r + \cdots$, in the case that $\mathbf{F}_r = \mathbf{X}_h$ where all the irreducible factors of h on $\mathbb{C}[x,y]$ are simple. The case $\operatorname{div}(\mathbf{F}_r) \neq 0$ or $\mathbf{F}_r = \mathbf{X}_h$ where h has multiple factors on $\mathbb{C}[x,y]$ is not solved and it is an open problem.

However a necessary condition so that $\mathbf{F}$ be integrable is that $\mathbf{F}_r$ be also integrable. The integrability problem for quasi-homogeneous vector fields, $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$ with $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, $\mu \neq 0$, is solved in [20].

It is known, see [1], that the integrability problem of a quasi-homogeneous vector field with non null divergence is equivalent to the integrability problem of a quasi-homogeneous Hamiltonian vector field where its Hamilton function has multiple factors on $\mathbb{C}[x,y]$.

In [1], we studied the analytic integrability around the origin inside a family of degenerate centers and we showed how difficult is the integrability problem even inside a simple family of degenerate centers. Concretely, there we considered degenerate systems of the form

$$\dot{x} = -y(x^2 + y^2) + \cdots, \quad \dot{y} = x(x^2 + y^2) + \cdots, \qquad (5.1.4)$$

where here the dots mean terms of higher order than the first component in the homogeneous expansion and which corresponds to $\mathbf{X}_h + \cdots$ whose $h = (x^2+y^2)^2/4$. This type of systems was studied in [95] where it was proved that there exist centers inside this family without an analytic first integral.

Recall that if h has not simple factors on $\mathbb{C}[x,y]$ Theorem 80 can not be applied. In [1], we obtain an useful result to the integrability problem for this kind of systems, because it gives us some necessary conditions to have formal integrability and also the value of $n \in \mathbb{N}$ to have first integrals of the form $I = (x^2+y^2)^n + \cdots$. Nevertheless, there is a particular case which remains open.

Summarizing, the nodes and saddle-nodes are not analytically integrable. A singular point with linear part of center type is analytically integrable if, and only if, it is a center and if, and only if, it is orbitally linearizable. System (5.1.1) is not analytically integrable around a non-resonant saddle singular point, but a resonant saddle has an analytic first integral around the singular point if, and only if, it is orbitally linearizable. A nilpotent vector field, with first quasi-homogeneous component non-reducible, is analytically integrable if, and only if, it is orbitally equivalent to its first quasi-homogeneous component, see [12, 18]. The remaining global open case is the case when we have a degenerate singular point. However there are some partial results. The analytic integrability problem when $\mathbf{F}_r = \mathbf{X}_h$, with h having only simple factors on $\mathbb{C}[x,y]$ is completely solved in [12]. The case $\mathbf{F}_r = \mathbf{X}_h$, with h having multiple factors on $\mathbb{C}[x,y]$ is still open, although in [1] there are some partial results.

We have structured this chapter as follows. In the next section, we summarize some preliminary definitions and some technical results. In the third section, we present our system and we give a formal orbital equivalent normal form for it. We transform this system into another one whose first quasi-homogeneous component be irreducible and we analyze the different Newton diagrams. In the fourth and fifth sections we provide integrability conditions for our system. First, we study when this system is reducible and then, we suppose it irreducible, give necessary conditions for the integrability. Finally, we finish the chapter with two applications to particular families of degenerate vector fields.

5.2 Technical results and preliminary definitions

As we have already mentioned throughout the memory, any vector field $\mathbf{F}$ can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. So, we write the vector field $\mathbf{F}$ as

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \cdots,$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \not\equiv \mathbf{0}$. Such expansions will be expressed as $\mathbf{F} = \mathbf{F}_r + q - h.h.o.t.$

We will denote by $\mathbf{D}_0 = (t_1 x, t_2 y)^T \in \mathcal{Q}_0^{\mathbf{t}}$ (a dissipative quasi-homogeneous vector field) and by $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (the Hamiltonian vector field associated to the polynomial h). If $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, then $\mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$, where $|\mathbf{t}| = t_1 + t_2$. Moreover, it is proved that every $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_k = \mathbf{X}_h + \mu \mathbf{D}_0 \tag{5.2.1}$$

with $h = (\mathbf{D}_0 \wedge \mathbf{F}_k)/(k+|\mathbf{t}|)$ and $\mu = \mathrm{div}(\mathbf{F}_k)/(k+|\mathbf{t}|)$, where $\mathbf{D}_0 \wedge \mathbf{F}_k \in \mathcal{P}^{\mathbf{t}}_{k+|\mathbf{t}|}$ is the wedge product of both vector fields and $\mathrm{div}(\mathbf{F}_k) \in \mathcal{P}_k^{\mathbf{t}}$ is the divergence of $\mathbf{F}_k$, see [12]. This sum is known as the conservative-dissipative splitting of a quasi-homogeneous vector field.

In the procedure to obtain a normal form under orbital equivalence for the system (5.1.1), described in Chapter 1, we have introducedthe homological operator under formal orbital equivalence:

$$\begin{aligned} \mathcal{L}_k \;:\; & \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \mu_k) \to \mathcal{L}_k(\mathbf{P}_k, \mu_k) = [\mathbf{P}_k, \mathbf{F}_r] - \mu_k \mathbf{F}_r. \end{aligned} \tag{5.2.2}$$

To study the homological operator in the case $\mathbf{F}_r = \mathbf{X}_H$ with $H = h^2$, $h \in \mathcal{P}^{\mathbf{t}}_{\frac{r+|\mathbf{t}|}{2}}$, we define the following linear operators

$$\begin{array}{rcl} \ell_{k-r} : \mathcal{P}^{\mathbf{t}}_{k-r} & \longrightarrow & \mathcal{P}^{\mathbf{t}}_{k} \\ \mu_{k-r} & \to & \nabla \mu_{k-r} \cdot \mathbf{X}_H \end{array} \qquad \begin{array}{rcl} \tilde{\ell}_{k-r} : \mathcal{P}^{\mathbf{t}}_{k-r} & \longrightarrow & \mathcal{P}^{\mathbf{t}}_{k-\frac{r+|\mathbf{t}|}{2}} \\ \mu_{k-r} & \to & \nabla \mu_{k-r} \cdot \mathbf{X}_h, \end{array}$$

i.e. the Lie derivative of the lowest degree quasi-homogeneous term of $\mathbf{X}_H$ and $\mathbf{X}_h$, respectively.

Following the developing made in [21], we have the following lemma adapted to our purposes.

Lemma 84. *Let us assume that the lowest-degree quasi-homogeneous term of system* (5.1.1) *is* $\mathbf{F}_r = \mathbf{X}_H \in \mathcal{Q}_r^{\mathbf{t}}$, *where* $H \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$. *A complementary subspace to the range (co-range) of* $\mathcal{L}_k$, *with* $k \geq 1$, *can be written as*

$$\mathrm{Cor}(\mathcal{L}_k) = \mathbf{X}_{S_{r+|\mathbf{t}|+k}} \oplus \mathrm{Cor}(\ell_k)\mathbf{D}_0,$$

being $S_{r+|\mathbf{t}|+k}$ *a subspace verifying*

$$\mathrm{Cor}(\ell_{k+|\mathbf{t}|}) = S_{r+|\mathbf{t}|+k} \oplus \left(H\mathrm{Cor}(\ell_{k-r}) \cap \mathrm{Cor}(\ell_{k+|\mathbf{t}|})\right),$$

where $\mathrm{Cor}(\ell_{k-r})$ *is a complementary subspace to the* $\mathrm{Range}(\ell_{k-r})$ *in* $\mathcal{P}_k^{\mathbf{t}}$, *such that* $H\mathrm{Cor}(\ell_{k-r}) \cap \mathrm{Cor}(\ell_{k+|\mathbf{t}|})$ *has maximal dimension.*

Proposition 85. *Let us assume that the lowest-degree quasi-homogeneous term of system* (5.1.1) *is* $\mathbf{F}_r = \mathbf{X}_H \in \mathcal{Q}_r^{\mathbf{t}}$, *with* $H = h^2$, $h \in \mathcal{P}^{\mathbf{t}}_{\frac{r+|\mathbf{t}|}{2}}$. *A complementary subspace of the range of* ℓ_{k-r} *is*

$$\mathrm{Cor}(\ell_{k-r}) = h\mathrm{Cor}(\tilde{\ell}_{k-r}) \oplus \Delta_k,$$

where Δ_k *is such that* $\mathcal{P}_k^{\mathbf{t}} = \Delta_k \oplus h \cdot \mathcal{P}^{\mathbf{t}}_{k-\frac{r+|\mathbf{t}|}{2}}$.

Proof. For all $\mu \in \mathcal{P}^{\mathbf{t}}_{k-r}$, we have $\ell_{k-r}(\mu) = \nabla\mu \cdot \mathbf{X}_{h^2} = 2h\nabla\mu \cdot \mathbf{X}_h = 2h\tilde{\ell}_{k-r}(\mu)$, from where we deduce $Range(\ell_{k-r}) \subset h \cdot Range(\tilde{\ell}_{k-r})$.

On the other hand, $h \cdot Range(\tilde{\ell}_{k-r}) \subset Range(\ell_{k-r})$ because $h \cdot \tilde{\ell}_{k-r}(\mu) = h \cdot \nabla\mu \cdot \mathbf{X}_h = \nabla\frac{\mu}{2} \cdot \mathbf{X}_{h^2} = \ell_{k-r}(\frac{\mu}{2})$.

Hence we have the equality $Range(\ell_{k-r}) = h \cdot Range(\tilde{\ell}_{k-r})$. The result follows from here. □

5.3 A family of perturbed degenerate centers

The objective of this chapter is to study the formal integrability around the origin of an analytic system of the form

$$(\dot{x}, \dot{y})^T = (x^{2n} + ny^2)(-y, x^{2n-1})^T + q - h.h.o.t. \tag{5.3.1}$$

First we highlight that in [71] it is proved that if an analytic vector field is formally integrable around an isolated singular point then it is analytically integrable. Hence, the formal integrability is equivalent to the analytic integrability.

For the case $n = 1$, in [98], the blow up

$$x = u(u^2 + v^2)^2, \quad y = v(u^2 + v^2)^2, \tag{5.3.2}$$

and the scaling of time $dT = (u^2+v^2)^5 dt$ transform system (5.3.1) in a nondegenerate analytic system

$$(u', v')^T = (-v, u)^T + q - h.h.o.t. \tag{5.3.3}$$

Note that this transformation is an homeomorphism but not a diffeomorphism. Therefore this transformation preserves the topological properties but not the differentiable ones. So, if system (5.3.1) is formal integrable then system (5.3.3) is also formal integrable. But the converse is not true. If system (5.3.3) is $\mathcal{C}^\omega$-integrable with a formal first integral H, undoing the transformation (5.3.2) we obtain a first integral of system (5.3.1) which is not always formal. Therefore, the fact that system (5.3.3) be $\mathcal{C}^\omega$-integrable does not implies that system (5.3.1) be also $\mathcal{C}^\omega$-integrable.

Analogously, it is possible to extend the above comments for $n > 1$.

In order to study the formal integrability of system (5.3.1), we first compute a formal orbital equivalent normal form for such system.

Theorem 86. *A formally orbital equivalent normal form for system (5.3.1) is*

$$(\dot{x},\dot{y})^T = (x^{2n}+ny^2)(-y,x^{2n-1})^T + \sum_{j=3n}^{\infty} \mathbf{X}_{a_j x^{j+n+1}+b_j x^{j+1}y} + (c_j x^j + d_j x^{j-n}y)\mathbf{D}_0 + \sum_{i=n}^{2n-2} e_i^{(1)} x^i h \mathbf{D}_0 + \sum_{j=2}^{\infty}\sum_{i=0}^{2n-2} e_i^{(j)} x^i h^j \mathbf{D}_0, \tag{5.3.4}$$

where $h = \frac{1}{2n}(x^{2n}+ny^2)$.

Proof. In this case, $r = 3n-1$ and $|\mathbf{t}| = n+1$.

In order to calculate a co-range of $\mathcal{L}_k$, that is, a complementary subspace of the range of the homological operator $\mathcal{L}_k$, we apply Lemma 84, i.e. we have that $\mathrm{Cor}(\mathcal{L}_k) = \mathbf{X}_{S_{4n+k}} \oplus \mathrm{Cor}(\ell_k)\mathbf{D}_0$ with $k \geq 1$. Therefore, we must compute $\mathrm{Cor}(\ell_k)$ and the subspace S_{4n+k}.

From Proposition 85, we obtain

$$\mathrm{Cor}(\ell_k) = h\mathrm{Cor}(\tilde{\ell}_{k-r}) \oplus \Delta_{k+3n-1},$$

$k \geq 1$, where we can choose $\Delta_j =< x^j, x^{j-n}y >$, $j \geq n$.

From [21], $\mathrm{Cor}(\tilde{\ell}_{k-r})$ is

$$\begin{cases} < x^{k+n-1} > & \text{if } 1 \leq k \leq n-1, \\ < x^j h^l > & \text{if } k+n-1 = 2nl+j,\ 0 \leq j \leq 2n-2,\ l \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

So, $\mathrm{Cor}(\ell_k)$ is:

- If $1 \leq k \leq n-1$,
$$\mathrm{Cor}(\ell_k) =< x^{k+3n-1}, x^{k+2n-1}y, x^{k+n-1}h > .$$
- If $k+3n-1 = 2nl+j$, $0 \leq j \leq 2n-2$, $l \in \mathbb{N}$,
$$\mathrm{Cor}(\ell_k) =< x^{k+3n-1}, x^{k+2n-1}y, x^j h^{l+1} > .$$
- If $k+3n-1 = 2nl+2n-1$, $l \in \mathbb{N} \cup \{0\}$,
$$\mathrm{Cor}(\ell_k) =< x^{k+3n-1}, x^{k+2n-1}y > .$$

The subspace S_{4n+k} is such that

$$\mathrm{Cor}(\ell_{k+n+1}) = S_{4n+k} \oplus \left(H\mathrm{Cor}(\ell_{k-3n+1}) \cap \mathrm{Cor}(\ell_{k+n+1})\right).$$

Taking into account the previous calculations, we can deduce $S_{4n+k} =< x^{4n+k}, x^{3n+k}y >$, with $k \geq 1$.

Therefore, a co-range of the homological operator is:

- If $1 \leq k \leq n-1$,

$$\mathrm{Cor}(\mathcal{L}_k) =< \mathbf{X}_{x^{k+4n}}, \mathbf{X}_{x^{k+3n}y}, x^{k+3n-1}\mathbf{D}_0, x^{k+2n-1}y\mathbf{D}_0, x^{k+n-1}h\mathbf{D}_0 > .$$

- If $k+3n-1 = 2nl+j$, $0 \leq j \leq 2n-2$, $l \in \mathbb{N}$,

$$\mathrm{Cor}(\mathcal{L}_k) =< \mathbf{X}_{x^{k+4n}}, \mathbf{X}_{x^{k+3n}y}, x^{k+3n-1}\mathbf{D}_0, x^{k+2n-1}y\mathbf{D}_0, x^j h^{l+1}\mathbf{D}_0 > .$$

- If $k+3n-1 = 2nl+2n-1$, $l \in \mathbb{N} \cup \{0\}$,

$$\mathrm{Cor}(\mathcal{L}_k) =< \mathbf{X}_{x^{k+4n}}, \mathbf{X}_{x^{k+3n}y}, x^{k+3n-1}\mathbf{D}_0, x^{k+2n-1}y\mathbf{D}_0 > .$$

□

As a consequence, we obtain the following result.

Theorem 87. *System* (5.3.1) *is formally integrable if, and only if, system* (5.3.4) *is formally integrable.*

From now on we will study the formal integrability of system (5.3.4). In the following lemma we apply the blow-up technique to transform system (5.3.4) into another whose first quasi-homogeneous component be irreducible and thus learn more about necessary conditions of integrability and existence of invariant curves of system (5.3.4).

Lemma 88. *The blow up $x = u$, $y = u^n v$, the scaling of time $dT = u^{3n-1}dt$ and then the translation $x_2 = \sqrt{n}v + i\sigma$, where $\sigma = \pm 1$ with $i = \sqrt{-1}$ transforms system (5.3.4) into*

$$\begin{aligned}
u' &= \frac{2}{n}ux_2 + \frac{3i\sigma}{n}ux_2^2 - \frac{1}{n}ux_2^3 - \frac{1}{\sqrt{n}}\sum_{j\geq 3n} b_j u^{j-3n+2} + \frac{1}{\sqrt{n}}\sum_{j\geq 3n}(c_j - \frac{i\sigma}{\sqrt{n}}d_j)u^{j-3n+2} \\
&\quad + \frac{1}{n}\sum_{j\geq 3n} d_j u^{j-3n+2}x_2 + \frac{1}{2n\sqrt{n}}\sum_{i=n}^{2n-2} e_i^{(1)}u^{i-n+2}x_2(x_2 - 2i\sigma) \\
&\quad + \frac{1}{\sqrt{n}}\sum_{j\geq 2}\sum_{i=0}^{2n-2}\frac{1}{(2n)^j}e_i^{(j)}u^{i+2nj-3n+2}x_2^j(x_2-2i\sigma)^j, \qquad (5.3.5) \\
x_2' &= -4x_2^2 - 4i\sigma x_2^3 + x_2^4 + \sum_{j\geq 3n}(j+n+1)(a_j - \frac{i\sigma}{\sqrt{n}}b_j)u^{j-3n+1} \\
&\quad + \frac{1}{\sqrt{n}}\sum_{j\geq 3n}(j+n+1)b_j u^{j-3n+1}x_2.
\end{aligned}$$

In order to study the integrability conditions for system (5.3.4), we analyze the different Newton diagrams in function of the parameters of system (5.3.5) (see Figure 5.1), and we make some definitions in terms of the coefficients of the normal form that we will use later.

The Newton diagram associated to system (5.3.5) (see [39, 51]), has always a non compact edge leaving from the inner vertex $V_0 = (1, 2)$ associated to the vector $(\frac{2}{n}ux_2, -4x_2^2)^T$.

We define the following non-negative integers

$$\begin{aligned} m &:= \min\left\{ j \in \mathbb{N}, j \geq 3n : a_j^2 + \frac{1}{n} b_j^2 \neq 0 \right\}, \\ l &:= \min\left\{ j \in \mathbb{N}, j \geq 3n : c_j^2 + \frac{1}{n} d_j^2 \neq 0 \right\}, \\ k &:= \min\left\{ 2nj + i : e_i^{(j)} \neq 0, j \in \mathbb{N} \text{ and } \left\{ \begin{array}{c} i = 0, \ldots, 2n-2 \ \ if \ \ j \geq 2, \\ \text{or} \\ i = n, \ldots, 2n-2 \ \ if \ \ j = 1 \end{array} \right\} \right\}, \end{aligned} \tag{5.3.6}$$

where $\min(\emptyset) := +\infty$.

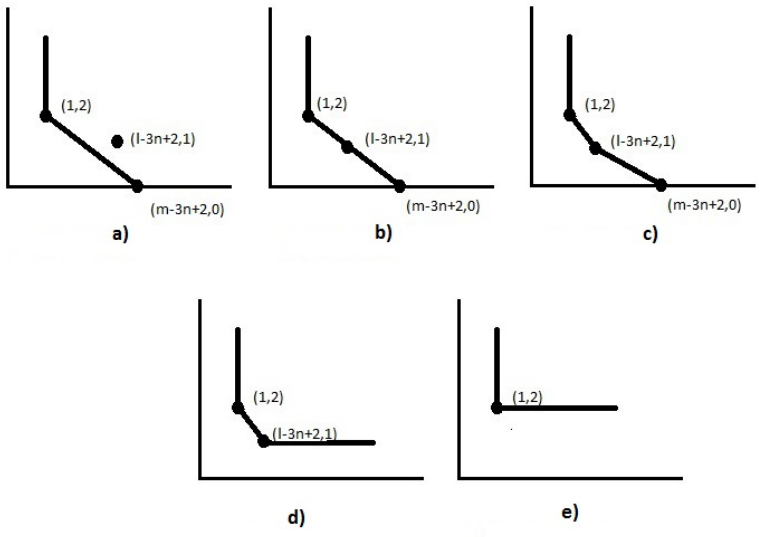


Figure 5.1: **a)** case $m < 2l-3n+1$, **b)** case $m = 2l-3n+1$, **c)** case $m > 2l-3n+1$, **d)** case $m < +\infty$, $l = +\infty$, **e)** case $m = l = +\infty$

- In the case $m < 2l-3n+1$ with $m < +\infty$ the Newton diagram has a exterior vertex $V_2 = (m-3n+2, 0)$ associated to the vector field $(0, (m+n+1)(a_m - \frac{i\sigma}{\sqrt{n}} b_m) u^{m-3n+1})^T$.

- The case $2l-3n+1 < m < +\infty$ has also the inner vertex $V_1 = (l-3n+2, 1)$ associated to the vector field $(\frac{1}{\sqrt{n}}(c_l - \frac{i\sigma}{\sqrt{n}} d_l) u^{l-3n+2}, 0)^T$ and two compact edges.

- In the case $m = 2l-3n+1 < +\infty$, the vector field $(\frac{1}{\sqrt{n}}(c_l - \frac{i\sigma}{\sqrt{n}} d_l) u^{l-3n+2}, 0)^T$ is not associated to any vertex, the points in the Newton diagram in this case are located on the unique compact edge of the Newton diagram.

- If $m = +\infty$, we will see by Theorem 92 that system (5.3.1) is integrable if $l = k = +\infty$.

  In this case the system is formally orbital equivalent to system $(x^{2n} + ny^2)(-y, x^{2n-1})^T$.

  Hence the system has a first integral of the form $I = (x^{2n} + ny^2) + \cdots$.

- If $m < +\infty$, with $l = k = +\infty$, system (5.3.1) is formally integrable because system (5.3.4) is Hamiltonian of the form

$$\begin{aligned}(\dot{x}, \dot{y})^T &= (x^{2n} + ny^2)(-y, x^{2n-1})^T + \sum_{j=3n}^{\infty} \mathbf{X}_{a_j x^{j+n+1} + b_j x^{j+1} y} \\ &= n\mathbf{X}_{h^2} + \sum_{j=3n}^{\infty} \mathbf{X}_{a_j x^{j+n+1} + b_j x^{j+1} y}\end{aligned}$$

  where $h = (x^{2n} + ny^2)/(2n)$. Hence system (5.3.1) has a first integral of the form $I = (x^{2n} + ny^2)^2 + \cdots$.

Consequently, from now on, we assume $l < +\infty$ or $k < +\infty$; i.e., we assume that the vector field is not formally equivalent to a Hamiltonian one.

5.4 The main results

First we study the case when system (5.3.1) is reducible.

Definition 89. *We say that the vector field* $\mathbf{F}$ *is reducible if there exists a scalar function* f*, with* $f(0) = 0$*, such that* $\mathbf{F} = f \cdot \mathbf{G}$*. In such case,* f *is the reducibility factor of* $\mathbf{F}$*.*

Theorem 90. *Assume that system (5.3.1) is formally integrable and suppose that its associated vector field is* $\mathbf{F} = (x^{2n} + ny^2)(-y, x^{2n-1})^T + \cdots$*. System (5.3.1) has a first integral of the form* $I = (x^{2n} + ny^2) + \cdots$ *if, and only if,* $\mathbf{F}$ *is reducible.*

Proof. We assume that the vector field $\mathbf{F}$ is reducible. Hence it can be written

$$\mathbf{F} = (x^{2n} + ny^2 + \cdots)[(-y, x^{2n-1})^T + \cdots] = f \cdot \mathbf{G}.$$

The vector field $\mathbf{G}$ is formally integrable because $\mathbf{F}$ is formally integrable. Moreover, by Theorem 81, a first integral of $\mathbf{G}$ is of the form $I = (x^{2n} + ny^2) + \cdots$. Therefore, $I = (x^{2n} + ny^2) + \cdots$ is a first integral of system (5.3.1).

Now, we assume that $I = (x^{2n} + ny^2) + \cdots$ is a first integral of $\mathbf{F}$. By the Arnold Theorem (see [28]), there exists a change of variables $(x, y) = \phi(u, v)$ such that $I(\phi(u, v)) = h(u, v)$, where we denote $h = x^{2n} + ny^2$, being $I(\phi(u, v)) = h(u, v)$ a first integral of the transformed vector field $\mathbf{G} = \phi_*\mathbf{F}$.
We write $\mathbf{G}$ as $\mathbf{G} = (x^{2n} + ny^2)(-y, x^{2n-1})^T + \sum_{j>3n} \mathbf{G}_j$.
We claim that $\mathbf{G}_j = f_j \cdot (-y, x^{2n-1})^T$, $\forall j > 3n$.
If h is a first integral of $\mathbf{G}$, then it is a first integral for each $\mathbf{G}_j$. So, $\nabla h \cdot \mathbf{G} = 0$ and then $\nabla h \cdot (x^{2n} + ny^2)(-y, x^{2n-1})^T + \sum_{j>3n} \nabla h \cdot \mathbf{G}_j = 0$ which implies $\nabla h \cdot \mathbf{G}_j = 0$, for all $j > 3n$.

Now, we write each $\mathbf{G}_j$ as $\mathbf{G}_j = \mathbf{X}_{g_j} + \mu_j \mathbf{D}_0 + \lambda_j(-y, x^{2n-1})^T$, with $g_j \in \Delta_j$, for all $j > 3n$, (see definition in Proposition 85). Then, we have $\nabla h \cdot \mathbf{G}_j = \nabla h \cdot \mathbf{X}_{g_j} + 2\mu_j h = 0$; therefore, h is an invariant curve of $\mathbf{X}_{g_j}$ and we can write $g_j = f_j \cdot h$, $\forall j > 3n$. But this is in contradiction with $g_j \in \Delta_j$; then, we have $g_j = 0$, $\forall j > 3n$. Consequently, $\nabla h \cdot \mathbf{G}_j = 2\mu_j h = 0$, hence $\mu_j = 0$, for all $j > 3n$.
From here $\mathbf{G}_j = \lambda_j(-y, x^{2n-1})^T$, $\forall j > 3n$.

Thus we have

$$\begin{aligned} \mathbf{G} &= (x^{2n} + ny^2)(-y, x^{2n-1})^T + \sum_{j>3n} \lambda_j(-y, x^{2n-1})^T = \\ & (x^{2n} + ny^2 + \sum_{j>3n} \lambda_j)(-y, x^{2n-1})^T. \end{aligned}$$

In short, we can write the vector field $\mathbf{G} = f \cdot (-y, x^{2n-1})^T$.
Undoing the change ϕ, we obtain $\mathbf{F} = \tilde{f} \cdot [(-y, x^{2n-1})^T + \cdots]$, where $\tilde{f} = \phi^{-1}(f) = x^{2n} + ny^2 + \cdots$.
Finally, we can write $\mathbf{F}$ as

$$\mathbf{F} = (x^{2n} + ny^2 + \cdots) \cdot [(-y, x^{2n-1})^T + \cdots],$$

thus the vector field $\mathbf{F}$ is reducible. □

Next, we assume that the vector field $\mathbf{F}$ associated to system (5.3.4) is irreducible.

Now we recall the following result proved in [20] that we have adapted to our purpose.

Theorem 91. *The quasi-homogeneous system* $\dot{\mathbf{x}} = \mathbf{F}_r = (P, Q)^T$, *with* $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$, P, Q *coprimes,* $PQ \not\equiv 0$ *and* $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$, *is polynomially integrable if, and only if,*

$$(\mathbf{D}_0 \wedge \mathbf{F}_r)(x, y) = cx^{\delta_x} y^{\delta_y} \prod_{i=1}^{m} (y^{t_1} - \lambda_i x^{t_2}), \tag{5.4.1}$$

with $r + |\mathbf{t}| = t_1\delta_x + t_2\delta_y + t_1t_2m$, $c \neq 0$, $\delta_x, \delta_y \in \{0, 1\}$, $\delta_x + \delta_y + m \geq 2$ *and* $\lambda_1, ..., \lambda_m$ *distinct complex numbers not zero and exist* n_x, n_y, n_i, $i = 1, \cdots, m$ *non-negative integers, not all zero, verifying*

$$\begin{cases} Res[\eta(x, 1), 0] = -\frac{1}{t_2} + \frac{(n_x+1)(r+|\mathbf{t}|)}{t_2\widetilde{M}} & \text{if } \delta_x = 1, \\ Res[\eta(1, y), 0] = \frac{1}{t_1} - \frac{(n_y+1)(r+|\mathbf{t}|)}{t_1\widetilde{M}} & \text{if } \delta_y = 1, \\ Res[\eta(1, y), \lambda_i^{1/t_1}] = \frac{1}{t_1} - \frac{(n_i+1)(r+|\mathbf{t}|)}{t_1\widetilde{M}} & i = 1, \cdots, m, \end{cases} \tag{5.4.2}$$

where $\widetilde{M} = t_1(n_x + 1)\delta_x + t_2(n_y + 1)\delta_y + t_1t_2 \sum_{j=1}^{m}(n_j + 1)$ *and* $\eta = \dfrac{\operatorname{div}(\mathbf{F}_r)}{\mathbf{D}_0 \wedge \mathbf{F}_r}$.

Moreover a first integral is of the form

$$I = x^{\delta_x(n_x+1)} y^{\delta_y(n_y+1)} \prod_{i=1}^{m} (y^{t_1} - \lambda_i x^{t_2})^{n_i+1}. \tag{5.4.3}$$

We denote by $\mathbf{G}$ the vector field associated to system (5.3.5). Recall that if system (5.3.4) is formally integrable then system (5.3.5) is also formally integrable.

In the following result we give integrability necessary conditions of system (5.3.4), (a formal normal form of system (5.3.1)). In its proof we will use Theorem 91.

Theorem 92. *Let* $\mathbf{F}$ *be the associated vector field of system (5.3.4) and consider* m, l, k *defined in (5.3.6). We assume* $\mathbf{F}$ *irreducible. If* $\mathbf{F}$ *is formally integrable then the following conditions are satisfied.*

- $\min\{l, m\} < k$, $k < +\infty$,
- $m \leq 2l - 3n + 1$, $l < +\infty$,
- *If* $m = 2l - 3n + 1$, $m < +\infty$, *then*

$$(a_m - \frac{i\sigma}{\sqrt{n}} b_m) + \frac{(l - 3n + 1)^2}{16(l - n + 1)^2} (c_l - \frac{i\sigma}{\sqrt{n}} d_l)^2 \neq 0.$$

Proof. We will prove the converse result. If any of the following conditions is verified, then system (5.3.4) is not formally integrable.

- $\min\{l, m\} \geq k$, $k < +\infty$,

- $m > 2l - 3n + 1$, $l < +\infty$,
- $m = 2l-3n+1$, $m < +\infty$ and $\frac{(l-3n+1)^2}{16(l-n+1)^2}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)^2 + (a_m - \frac{i\sigma}{\sqrt{n}}b_m) = 0$.

We assume that $\min\{l, m\} \geq k$. If $\mathbf{F}$ is the associated vector field of system (5.3.4), then we have $\mathbf{F} = \mathbf{F}_{3n-1} + \mathbf{F}_k + \cdots$ with $\mathbf{F}_{3n-1} = \frac{1}{2n}h\mathbf{X}_h$, $h = x^{2n} + ny^2$ and $\mathbf{F}_k = (e_{i_0}^{(j_0)}x^{i_0}h^{j_0} + \mu_l)\mathbf{D}_0 + \mathbf{X}_{g_m}$, where

- $e_{i_0}^{(j_0)} \neq 0$ where $j_0 \geq 2$ and $i_0 = 0, \ldots, 2n-2$ or $j_0 = 1$ and $i_0 = n, \ldots, 2n-2$,
- $\mu_l = c_l x^l + d_l x^{l-n}y$ if $k = l$, otherwise $\mu_l = 0$,
- $g_m = a_m x^{m+n+1} + b_m x^{m+1}y$ if $k = m$, otherwise $g_m = 0$.

If the system were integrable, a first integral would be of the form $I = h^p + \sum_{j>2np} I_j$ with $I_j \in \mathcal{P}_j^{(1,n)}$, by Theorem 90 we have that $p \geq 2$. The integrability condition ($\nabla I \cdot \mathbf{F} = 0$) is satisfied and at the lower degree $2np + k$ is

$$\begin{aligned} 0 &= \nabla h^p \cdot \mathbf{F}_k + \nabla I_{2np+k-3n+1} \cdot \mathbf{F}_{3n-1} \\ &= 2npe_{i_0}^{(j_0)}x^{i_0}h^{p+j_0} + 2np\mu_l h^p + ph^{p-1}\nabla h \cdot \mathbf{X}_{g_m} + h\nabla I_{2np+k-3n+1} \cdot \mathbf{X}_{\frac{h}{2n}} \\ &= h[2npe_{i_0}^{(j_0)}x^{i_0}h^{p+j_0-1} + 2np\mu_l h^{p-1} - 2nph^{p-2}\nabla g_m \cdot \mathbf{X}_{\frac{h}{2n}} + \nabla I_{2np+k-3n+1} \cdot \mathbf{X}_{\frac{h}{2n}}] \end{aligned}$$

which is compatible only if $e_{i_0}^{(j_0)} = 0$; this is a contradiction.

Now, we assume that $m > 2l - 3n + 1$. In this case the Newton diagram of (5.3.5), after applying the blow-up to system (5.3.4), would have two compact edges (if $m < +\infty$) or one (if $m = +\infty$). In any case, we focus in the edge of type $t = (1, l - 3n + 1)$, common to both situations. This edge has associated the vector field

$$\mathbf{G}_r = (\frac{2}{n}ux_2 + \frac{1}{\sqrt{n}}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+2}, -4x_2^2)^T,$$

where $r = l - 3n + 1$.

If $\mathbf{F}$ is formally integrable then $\mathbf{G}$ is formally integrable and $\mathbf{G}_r$ is also formally integrable. To apply Theorem 91, first we compute the Hamiltonian function of $\mathbf{G}_r$

$$h = \frac{1}{r + |\mathbf{t}|}(u, (l - 3n + 1)x_2)^T \wedge \mathbf{G}_r = -\frac{2(l-n+1)}{n(2l-6n+3)}ux_2[x_2 - \lambda u^{l-3n+1}],$$

where $\lambda = -\frac{\sqrt{n}(l-3n+1)}{2(l-n+1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)$.

On the other hand, the dissipative part of $\mathbf{G}_r$ is

$$\mu = \frac{1}{r + |\mathbf{t}|}div(\mathbf{G}_r) = -\frac{2(4n-1)}{n(2l-6n+3)}[x_2 - \frac{\sqrt{n}(l-3n+2)}{2(4n-1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+1}].$$

Applying Theorem 91 and computing the residue expressions, we have

$$\begin{aligned} Res[\eta(u,1),0] &= \frac{4n-1}{l-n+1} = \frac{-\widetilde{M}+(2l-6n+3)(n_x+1)}{(l-3n+1)\widetilde{M}}, \\ Res[\eta(1,x_2),0] &= -\frac{l-3n+2}{l-3n+1} = \frac{\widetilde{M}-(2l-6n+3)(n_y+1)}{\widetilde{M}}, \\ Res[\eta(1,x_2),\lambda] &= \frac{l^2+2l-9n^2+2n+1}{(l-n+1)(l-3n+1)} = \frac{\widetilde{M}-(2l-6n+3)(n_1+1)}{\widetilde{M}}, \end{aligned}$$

where $\widetilde{M} = (n_x+1)+(l-3n+1)(n_y+1)+(l-3n+1)(n_1+1)$.

From the second equation we get $n_x+1 = -(l-3n+1)(n_1+1)$ and taking into account that $l \geq 3n$, we would have that n_x should take negative values which implies a contradiction. Hence $\mathbf{G}_r$ is not formally integrable.

Finally we assume that $m = 2l-3n+1$ and

$$\frac{(l-3n+1)^2}{16(l-n+1)^2}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)^2 + (a_m - \frac{i\sigma}{\sqrt{n}}b_m) = 0.$$

In this case, $\mathbf{G}_r = (P,Q)^T$ is

$$\mathbf{G}_r = \begin{pmatrix} \frac{2}{n}ux_2 + \frac{1}{\sqrt{n}}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+2} \\ -4x_2^2 + 2(l-n+1)(a_m - \frac{i\sigma}{\sqrt{n}}b_m)u^{2(l-3n+1)} \end{pmatrix}.$$

Moreover we have the following dissipative-conservative decomposition:

$$\begin{aligned} h &= \frac{-2(l-n+1)}{n(2l-6n+3)}u(x_2 + \frac{\sqrt{n}(l-3n+1)}{4(l-n+1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+1})^2 := \alpha\, u\cdot f^2 \\ \mu &= -\frac{2(4n-1)}{n(2l-6n+3)}(x_2 - \frac{\sqrt{n}(l-3n+2)}{2(4n-1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+1}) \neq 0. \end{aligned}$$

We will see that $\mathbf{G}_r$ is irreducible. Otherwise $\mathbf{G}_r = g\cdot\widehat{\mathbf{G}}_s$, with $\widehat{\mathbf{G}}_s \in \mathcal{Q}_s^{\mathbf{t}}$ and $g \in \mathcal{P}_{r-s}^{\mathbf{t}}$.

Thus we have $\mathbf{D}_0\wedge\mathbf{G}_r = (r+|\mathbf{t}|)\,h = g\mathbf{D}_0\wedge\widehat{\mathbf{G}}_s$; therefore, the irreducible factors of g are irreducible factors of h. That is, the possible reducible factors of $\mathbf{G}_r$ are u or f. But u is not a factor of Q and f is not a factor of $P = \frac{2}{n}u(x_2 + \frac{\sqrt{n}}{2}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+1})$; otherwise $(x_2 + \frac{\sqrt{n}}{2}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+1}) = f$, from where $l = -(n+1)$, which is a contradiction.

By Theorem 91, as $\mathbf{G}_r$ is irreducible and $\mathbf{D}_0 \wedge \mathbf{G}_r$ has not simple factors on $\mathbb{C}[x,y]$, $\mathbf{G}_r$ is not polynomially integrable and therefore system (5.3.5) is not formally integrable. □

In the following result we give necessary conditions of integrability for system (5.3.1).

Theorem 93. *Let* $\mathbf{F}$ *be the associated vector field of system (5.3.4) irreducible. Consider* m *and* l *defined in (5.3.6). If* $\mathbf{F}$ *is formally integrable, then it is verified*

(a) *If* $m < 2l - 3n + 1$, *with* $m - 3n + 1$ *odd, then a first integral of system (5.3.4) is of the form* $I = (x^{2n} + ny^2)^2 + \cdots$.

(b) *If* $m < 2l - 3n + 1$, *with* $m - 3n + 1$ *even, then a first integral of system (5.3.4) is of the form* $I = f_1 f_2$ *where* $f_1 \not\equiv f_2$ *y* $f_i = (x^{2n} + ny^2) + \cdots$ *for* $i = 1, 2$.

(c) *If* $m = 2l - 3n + 1$ *and* $(a_m - \frac{i\sigma}{\sqrt{n}} b_m) \neq \frac{n}{2(l-n+1)}(c_l - \frac{i\sigma}{\sqrt{n}} d_l)^2$ *then exist* $M \in \mathbb{N}$ *and* $N \in \mathbb{N}$, *with* $M \neq N$, *such that*

$$\begin{aligned} a_m &= \frac{1}{16(l-n+1)^2}\left((l+n+1)^2\frac{(M+N)^2}{(M-N)^2} - (l-3n+1)^2\right)(c_l^2 - \frac{1}{n}d_l^2), \\ b_m &= \frac{1}{8(l-n+1)^2}\left((l+n+1)^2\frac{(M+N)^2}{(M-N)^2} - (l-3n+1)^2\right)c_l d_l. \end{aligned}$$

In this case, a first integral of system (5.3.4) is of the form $I = f_1^M f_2^N$ *where* $f_1 \not\equiv f_2$ *and* $f_i = (x^{2n} + ny^2) + \cdots$ *for* $i = 1, 2$.

The proof of Theorem 93 is given in the next section.

Remark. The case $m = 2l - 3n + 1$ with $(a_m - \frac{i\sigma}{\sqrt{n}} b_m) = \frac{n}{2(l-n+1)}(c_l - \frac{i\sigma}{\sqrt{n}} d_l)^2$, that corresponds to the case where the principal part of system (5.3.5) is reducible, remains open.

5.5 Proof of Theorem 93

i) If $m < 2l - 3n + 1$, the Newton diagram of system (5.3.5) has a unique compact face of type $(2, m - 3n + 1)$ associated to the vector field

$$\mathbf{G}_{m-3n+1} = (\frac{2}{n}ux_2, -4x_2^2 + (m+n+1)(a_m - \frac{i\sigma}{\sqrt{n}}b_m)u^{m-3n+1})^T$$

with the Hamiltonian function h and the dissipative function μ (conservative and dissipative part of $\mathbf{G}_{m-3n+1}$ respectively) given by

$$\begin{aligned} h &= \frac{1}{r+|\mathbf{t}|}\mathbf{D}_0 \wedge \mathbf{G}_{m-3n+1} = -\frac{m+n+1}{n(m-3n+2)}u\left[x_2^2 - n(a_m - \frac{i\sigma}{\sqrt{n}}b_m)u^{m-3n+1}\right], \\ \mu &= \frac{1}{r+|\mathbf{t}|}div(\mathbf{G}_{m-3n+1}) = -\frac{4n-1}{n(m-3n+2)}x_2. \end{aligned} \tag{5.5.1}$$

Here we distinguish if $m - 3n + 1$ is odd or even.

- If $m-3n+1$ is odd, the Hamiltonian function has the irreducible factor $[x_2^2-\lambda u^{m-3n+1}]$, where $\lambda = n(a_m - \frac{i\sigma}{\sqrt{n}}b_m)$. Now, in order to apply Theorem 91, we compute the residues

$$\begin{aligned} Res[\eta(u,1),0] &= \frac{4n-1}{m+n+1} = \frac{-\widetilde{M}+2(m-3n+2)(n_x+1)}{(m-3n+1)\widetilde{M}}, \\ Res[\eta(1,x_2),\lambda^{1/2}] &= \frac{4n-1}{2(m+n+1)} = \frac{\widetilde{M}-2(m-3n+2)(n_1+1)}{2\widetilde{M}}, \end{aligned}$$

where $\widetilde{M} = 2(n_x+1)+2(m-3n+1)(n_1+1)$.
Equating the expressions of the residues and taking into account the value of $\widetilde{M}$, we obtain

$$(n_x+1) = 4n(n_1+1).$$

Hence, by Theorem 91, a first integral of $\mathbf{G}_{m-3n+1}$ is of the form

$$\begin{aligned} I &= u^{n_x+1}(x_2^2-\lambda u^{m-3n+1})^{n_1+1} = u^{4n(n_1+1)}(x_2^2-\lambda u^{m-3n+1})^{n_1+1} = \\ &\quad (u^{4n}(x_2^2-\lambda u^{m-3n+1}))^{n_1+1} = (u^{4n}(x_2^2-\lambda u^{m-3n+1}))^N, \end{aligned}$$

where $N = n_1+1$. Then, $I = (u^{4n}(x_2^2-\lambda u^{m-3n+1}))^N + \cdots$ is a first integral of system (5.3.5). By abuse of notation, we use the same letter I to refer to the first integral of the different vector fields.
Since the first component of the vector field is integrable, we can write $I = (u^{4n}(x_2^2-\lambda u^{m-3n+1})+\cdots)^N \cdot U$, where U is a unity in the ring of the formal series of the variables u, x_2. Consequently a first integral of system (5.3.5) is of the form

$$I = u^{4n}(x_2^2-\lambda u^{m-3n+1})+\cdots. \tag{5.5.2}$$

On the other hand, if system (5.3.1) is formally integrable, a first integral is of the form $I = (x^{2n}+ny^2)^N+\cdots$, with $N\in\mathbb{N}$. If we apply the change of variables described in Lemma 88, this first integral becomes $I = u^{2nN}x_2^N(x_2-2\sigma i)^N + O(u^{2nN+1})$, that we can write as

$$I = u^{2nN}x_2^N + O(u^{2nN+1}). \tag{5.5.3}$$

Considering the type $\mathbf{t}=(3,1)$ in system (5.3.5), the first quasi-homogeneous component of the vector field is $\mathbf{G}_1 = (\frac{2}{n}ux_2, -4x_2^2)^T$. Ordering respect to this type the first integral (5.5.2), we get $I = u^{4n}x_2^2+\cdots$..
Now we consider (5.5.3) with respect to the type $\mathbf{t}=(3,1)$ and we have $I = u^{2nN}x_2^N+\cdots$.
Equating both expressions we deduce that $N=2$.
Therefore, for system (5.3.1) a first integral is of the form $I = (x^{2n}+ny^2)^2+\cdots$ and this corresponds to case **(a)**.

- If $m-3n+1$ is even, then we can factorize h. From the expression of (5.5.1) we have

$$h = -\frac{(m+n+1)}{n(m-3n+2)}u\left[x_2-\sqrt{n(a_m-\frac{i\sigma}{\sqrt{n}}b_m)}u^{\frac{m-3n+1}{2}}\right]\times \left[x_2+\sqrt{n(a_m-\frac{i\sigma}{\sqrt{n}}b_m)}u^{\frac{m-3n+1}{2}}\right].$$

Consider $R^2=\sqrt{n^2a_m^2+nb_m^2}$ and α such that

$$R^2\cos(2\alpha)=na_m,$$

$$R^2\sin(2\alpha)=-\sqrt{n}\sigma b_m.$$

Thereby $\sqrt{n(a_m-\frac{i\sigma}{\sqrt{n}}b_m)}=R(\cos(\alpha)+i\sin(\alpha))$.
Then, we have

$$\begin{aligned} h &= -\frac{m+n+1}{n(m-3n+2)}u[x_2-\lambda_1u^{\frac{m-3n+1}{2}}][x_2-\lambda_2u^{\frac{m-3n+1}{2}}],\\ \mu &= -\frac{4n-1}{n(m-3n+2)}x_2, \end{aligned}$$

where $\lambda_1=R\cos(\alpha)+iR\sin(\alpha)$ and $\lambda_2=-R\cos(\alpha)-iR\sin(\alpha)$.
Considering the type $\mathbf{t}=(1,\frac{m-3n+1}{2})$ and applying Theorem 91 we obtain

$$\begin{aligned} Res[\eta(u,1),0]=\frac{2(4n-1)}{m+n+1} &= \frac{-2\widetilde{M}+2(m-3n+2)(n_x+1)}{(m-3n+1)\widetilde{M}},\\ Res[\eta(1,x_2),\lambda_i]=\frac{4n-1}{m+n+1} &= \frac{\widetilde{M}-(m-3n+2)(n_i+1)}{\widetilde{M}},\ i=1,2, \end{aligned}$$

where $\widetilde{M}=(n_x+1)+\frac{m-3n+1}{2}(n_1+1)+\frac{m-3n+1}{2}(n_2+1)$.
Defining $M:=n_1+1$, $N:=n_2+1$ and solving the equations we get

$$\begin{aligned} M &= N=\frac{1}{4n}(n_x+1),\\ \widetilde{M} &= \frac{1}{4n}(m+n+1)(n_x+1). \end{aligned}$$

Hence by Theorem 91, a first integral of $\mathbf{G}_{m-3n+1}$ is of the form

$$I=u^{n_x+1}(x_2-\lambda_1u^{\frac{m-3n+1}{2}})^{n_1+1}(x_2-\lambda_2u^{\frac{m-3n+1}{2}})^{n_2+1},$$

that we can write as

$$\begin{aligned} I &= (u^{4n}(x_2 - \lambda_1 u^{\frac{m-3n+1}{2}})(x_2 - \lambda_2 u^{\frac{m-3n+1}{2}}))^N = \\ &\left[(u^{2n}x_2 - \lambda_1 u^{m-3n+1})(u^{2n}x_2 - \lambda_2 u^{m-3n+1})\right]^N. \end{aligned}$$

Therefore, reasoning as the previous case, as the first quasi-homogeneous component is polynomially integrable, we can prove that a first integral of system (5.3.5) is of the form

$$I = (u^{2n}x_2 - \lambda_1 u^{m-3n+1} + \cdots)(u^{2n}x_2 - \lambda_2 u^{m-3n+1} + \cdots).$$

As in the previous case, we can consider the type $\mathbf{t} = (3,1)$ in system (5.3.5), and we deduce that a first integral of system (5.3.1) is of the form

$$I = (x^{2n} + ny^2 + \cdots)(x^{2n} + ny^2 + \cdots).$$

This corresponds to case **(b)**.

ii) If $m = 2l - 3n + 1$, the Newton diagram of system (5.3.5) has a unique compact edge of type $(1, l-3n+1)$ associated to the vector field

$$\mathbf{G}_{l-3n+1} = \begin{pmatrix} \frac{2}{n}ux_2 + \frac{1}{\sqrt{n}}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+2} \\ -4x_2^2 + 2(l-n+1)(a_m - \frac{i\sigma}{\sqrt{n}}b_m)u^{2(l-3n+1)} \end{pmatrix}.$$

As $(a_m - \frac{i\sigma}{\sqrt{n}}b_m) \neq \frac{n}{2(l-n+1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)^2$, the vector field associated to system (5.3.5) is irreducible since its first quasi-homogeneous component is irreducible.

Therefore we can apply Theorem 91, where the Hamiltonian function and the dissipative term of $\mathbf{G}_{l-3n+1}$ are respectively

$$\begin{aligned} h &= -\frac{2(l-n+1)}{n(2l-6n+3)}u\Big[\left(x_2 + \frac{\sqrt{n}}{4}\frac{(l-3n+1)}{(l-n+1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+1}\right)^2 \\ &- \left(\frac{n(l-3n+1)^2}{16(l-n+1)^2}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)^2 + n(a_m - \frac{i\sigma}{\sqrt{n}}b_m)\right)u^{2(l-3n+1)}\Big], \\ \mu &= -\frac{2(4n-1)}{n(2l-6n+3)}\left[x_2 - \frac{\sqrt{n}(l-3n+2)}{2(4n-1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)u^{l-3n+1}\right]. \end{aligned}$$

By Theorem 92, we have that $\frac{(l-3n+1)^2}{16(l-n+1)^2}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)^2 + (a_m - \frac{i\sigma}{\sqrt{n}}b_m) \neq 0$, hence we can write h as

$$h = \frac{-2(l-n+1)}{n(2l-6n+3)}u\left[x_2 - \lambda_1 u^{l-3n+1}\right]\left[x_2 - \lambda_2 u^{l-3n+1}\right],$$

where

$$\lambda_1 = -\frac{\sqrt{n}(l-3n+1)}{4(l-n+1)}c_l - R\cos(\alpha) + \left(\frac{\sqrt{n}(l-3n+1)}{4(l-n+1)}\sigma d_l - R\sin(\alpha)\right)i,$$
$$\lambda_2 = -\frac{\sqrt{n}(l-3n+1)}{4(l-n+1)}c_l + R\cos(\alpha) + \left(\frac{\sqrt{n}(l-3n+1)}{4(l-n+1)}\sigma d_l + R\sin(\alpha)\right)i,$$

and α is an angle such that

$$R^2\cos(2\alpha) = \left(na_m + \tfrac{n(l-3n+1)^2}{16(l-n+1)^2}(c_l^2 - \frac{1}{n}d_l^2)\right),$$
$$R^2\sin(2\alpha) = -\left(b_m + \tfrac{(l-3n+1)^2}{8(l-n+1)^2}c_l d_l\right)\sqrt{n}\sigma,$$
$$R^2 = \sqrt{\left(na_m + \tfrac{n(l-3n+1)^2}{16(l-n+1)^2}(c_l^2 - \frac{1}{n}d_l^2)\right)^2 + n\left(b_m + \tfrac{(l-3n+1)^2}{8(l-n+1)^2}c_l d_l\right)^2}.$$

Now, applying Theorem 91, there exist n_1, n_2, n_x non–negative integers, not all zero, verifying

$$Res[\eta(u,1),0] = \frac{4n-1}{l-n+1} = \frac{-\widetilde{M} + (2l-6n+3)(n_x+1)}{(l-3n+1)\widetilde{M}},$$
$$Res[\eta(1,x_2),\lambda_1] = \frac{(4n-1)[\lambda_1 - \frac{\sqrt{n}(l-3n+2)}{2(4n-1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)]}{(l-n+1)(\lambda_1-\lambda_2)} = 1 - \frac{(2l-6n+3)(n_1+1)}{\widetilde{M}},$$
$$Res[\eta(1,x_2),\lambda_2] = \frac{(4n-1)[\lambda_2 - \frac{\sqrt{n}(l-3n+2)}{2(4n-1)}(c_l - \frac{i\sigma}{\sqrt{n}}d_l)]}{(l-n+1)(\lambda_2-\lambda_1)} = 1 - \frac{(2l-6n+3)(n_2+1)}{\widetilde{M}},$$

where $\widetilde{M} = (n_x+1) + (l-3n+1)[(n_1+1)+(n_2+1)]$.

From these equations we get

$$c_l\sin(\alpha) + \frac{\sigma}{\sqrt{n}}d_l\cos(\alpha) = 0, \tag{5.5.4}$$
$$\frac{1}{4R}\frac{l+n+1}{l-n+1}\sqrt{n}(c_l\cos(\alpha) - \frac{\sigma}{\sqrt{n}}d_l\sin(\alpha)) = -\frac{M-N}{M+N}, \tag{5.5.5}$$

where $M = n_1+1$, $N = n_2+1$ and $n_x+1 = 2n(M+N)$

We distinguish four cases:

1) Case $c_l \neq 0$, $d_l = 0$.
   From (5.5.4) we have $\sin\alpha = 0$ and $|\cos(\alpha)| = 1$, hence $\cos(2\alpha) = 1$ and $\sin(2\alpha) = 0$.

From the expression $R^2\sin(2\alpha) = -\left(b_m + \frac{(l-3n+1)^2}{8(l-n+1)^2}c_l d_l\right)\sqrt{n}\sigma$, we deduce $b_m = 0$.
And from the expression $R^2\cos(2\alpha) = \left(na_m + \frac{n(l-3n+1)^2}{16(l-n+1)^2}(c_l^2 - \frac{1}{n}d_l^2)\right)$, we get $na_m = R^2 - \frac{n(l-3n+1)^2}{16(l-n+1)^2}c_l^2$.
On the other hand, condition (5.5.5) can be written as

$$\sqrt{n}(l+n+1)c_l\cos(\alpha)(M+N) = -4R(l-n+1)(M-N).$$

We multiply by $\cos\alpha$ and we isolate $R\cos(\alpha) = -\frac{\sqrt{n}(l+n+1)}{4(l-n+1)}\frac{M+N}{M-N}c_l$ $(M \neq N)$.
We have $na_m = R^2 - \frac{n(l-3n+1)^2}{16(l-n+1)^2}c_l^2 = R^2\cos^2(\alpha) - \frac{n(l-3n+1)^2}{16(l-n+1)^2}c_l^2$, and replacing $R\cos\alpha$, we obtain the expression given in the statement.

2) Case $c_l = 0$, $d_l \neq 0$.
Analogously to the previous case. From (5.5.4) we have $\cos\alpha = 0$ and $|\sin(\alpha)| = 1$, hence $\cos(2\alpha) = -1$ and $\sin(2\alpha) = 0$.
From the expression $R^2\sin(2\alpha) - \left(b_m + \frac{(l-3n+1)^2}{8(l-n+1)^2}c_l d_l\right)\sqrt{n}\sigma$, we deduce $b_m = 0$.
And from the expression $R^2\cos(2\alpha) = \left(na_m + \frac{n(l-3n+1)^2}{16(l-n+1)^2}(c_l^2 - \frac{1}{n}d_l^2)\right)$, we get $na_m = \frac{(l-3n+1)^2}{16(l-n+1)^2}d_l^2 - R^2$.
Condition (5.5.5) can be written as

$$(l+n+1)\sigma d_l\sin(\alpha)(M+N) = 4R(l-n+1)(M-N).$$

We multiply by $\sin\alpha$ and we isolate $R\sin(\alpha) = \frac{l+n+1}{4(l-n+1)}\frac{M+N}{M-N}\sigma d_l$ $(M \neq N)$.
We have $na_m = \frac{(l-3n+1)^2}{8(l-n+11)^2}d_l^2 - R^2 = \frac{(l-3n+1)^2}{8(l-n+1)^2}d_l^2 - R^2\sin^2(\alpha)$ and replacing $R\sin\alpha$, we obtain the expression given in the statement.

3) Case $c_l^2 + d_l^2 \neq 0$, $c_l^2 - \frac{1}{n}d_l^2 = 0$.
From (5.5.4) we have $\cos^2\alpha = \sin^2\alpha$.
From the expression $R^2\cos(2\alpha) = \left(na_m + \frac{n(l-3n+1)^2}{16(l-n+1)^2}(c_l^2 - \frac{1}{n}d_l^2)\right)$, we deduce $a_m = 0$.
In (5.5.4) we can isolate $c_l = -\frac{\sigma d_l\cos(\alpha)}{\sqrt{n}\sin(\alpha)}$, and replacing in (5.5.5) we get

$$\frac{(l+n+1)\sigma d_l}{4(l-n+11)R\sin(\alpha)} = \frac{M-N}{M+N}.$$

As $M \neq N$, we can isolate $R\sin(\alpha) = \frac{(l+n+1)\sigma d_l}{4(l-n+1)}\frac{M+N}{M-N}$.

On the other hand, if we multiply (5.5.4) by $\sin\alpha$, we can isolate $\sin^2\alpha = -\frac{\sigma d_l \cos(\alpha)\sin(\alpha)}{\sqrt{n}c_l}$, and substituting into the last equality we obtain

$$-R^2\frac{\sigma d_l\cos(\alpha)\sin(\alpha)}{\sqrt{n}c_l} = \frac{(l+n+1)^2 d_l^2}{16(l-n+1)^2}\frac{(M+N)^2}{(M-N)^2}.$$

From here we can isolate $R^2\cos(\alpha)\sin(\alpha) = -\frac{(l+n+1)^2}{16(l-n+1)^2}\frac{(M+N)^2}{(M-N)^2}\sqrt{n}\sigma c_l d_l$. Substituting this last equality in the expression $-\left(b_m + \frac{(l-3n+1)^2}{8(l-n+1)^2}c_l d_l\right)\sqrt{n}\sigma = R^2\sin(2\alpha) = 2R^2\cos(\alpha)\sin(\alpha)$, we can isolate b_m and we obtain the expression given in the statement.

4) Case $c_l^2 + d_l^2 \neq 0$, $c_l^2 - \frac{1}{n}d_l^2 \neq 0$.
From (5.5.4) we have $\tan\alpha = -\frac{\sigma}{n}\frac{d_l}{c_l}$. Moreover

$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)} = \frac{-2\sigma c_l d_l}{\sqrt{n}(c_l^2 - \frac{1}{n}d_l^2)}.$$

On the other hand,

$$\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{-(b_m + \frac{(l-3n+1)^2}{8(l-n+1)^2}c_l d_l)\sqrt{n}\sigma}{na_m + \frac{n(l-3n+1)^2}{16(l-n+1)^2}(c_l^2 - \frac{1}{n}d_l^2)}.$$

Equating both expressions and isolating a_m, we deduce that

$$a_m = \frac{c_l^2 - \frac{1}{n}d_l^2}{2c_l d_l}b_m.$$

Reasoning as in the previous case, we obtain the expression of b_m given in the statement, that substituted in the last equality provides the expression of a_m given in the statement.

Hence in this case, we have that a first integral of system (5.3.1) is of the form

$$I = u^{2n(M+N)}(x_2 - \lambda_1 u^{l-3n+1})^M(x_2 - \lambda_2 u^{l-3n+1})^N.$$

Reasoning as in previous cases, since the first quasi-homogeneous component of the vector field is polynomially integrable, we can write

$$I = u^{2n(M+N)}(x_2 - \lambda_1 u^{l-3n+1} + \cdots)^M(x_2 - \lambda_2 u^{l-3n+1} + \cdots)^N,$$

i.e.,

$$I = (u^{2n}x_2 - \lambda_1 u^{l-n+1} + \cdots)^M(u^{2n}x_2 - \lambda_2 u^{l-n+1} + \cdots)^N.$$

Ordering respect to the type $\mathbf{t} = (3, 1)$, this first integral becomes

$$I = (u^{2n}x_2 + \cdots)^M (u^{2n}x_2 + \cdots)^N = (u^{2n}x_2)^{M+N} + \cdots .$$

On the other hand, we know that a first integral of system (5.3.1) is of the form $I = (x^{2n} + ny^2)^{\widetilde{N}} + \cdots$, $\widetilde{N} \in \mathbb{N}$.
Applying the change of variables described in Lemma 88, it can be written as $I = u^{2n\widetilde{N}}x_2^{\widetilde{N}} + \cdots = (u^{2n}x_2)^{\widetilde{N}} + \cdots$.
Equating both expressions, we deduce that $\widetilde{N} = M + N$.

Therefore, a first integral of system (5.3.1) is of the form

$$I = (x^{2n} + ny^2 + \cdots)^M (x^{2n} + ny^2 + \cdots)^N.$$

This corresponds to the case **(c)**.

5.6 Applications

In this section, we are going to apply our results to two particular families of degenerate vector fields; the first one is for $n = 1$ and the second one is the case $n = 2$.

Firstly, we are going to study the integrability problem for the following family:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2 + y^2) \begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^3y \\ bx^4 + cx^2y^2 \end{pmatrix}, \tag{5.6.1}$$

where $a, b, c \in \mathbb{R}$.

Theorem 94. *System (5.6.1) has a first integral of the form $I = x^2 + y^2 + \cdots$ if, and only if, one of the following conditions is satisfied:*

(1) $a = 0$, $c = b$.

(2) $b = -a$, $c = 0$.

Proof. First, we assume that $I = x^2 + y^2 + \cdots$ is a first integral of system (5.6.1). Imposing the condition of integrability $\nabla I \cdot \mathbf{F} = 0$ and making an analysis for each degree, we obtain the conditions of the statement.

Reciprocally, we first assume that $a = 0$ and $c = b$. In this case system (5.6.1) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2 + y^2) \left[\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ bx^2 \end{pmatrix} \right], \tag{5.6.2}$$

which is integrable with a first integral of the form $I = x^2 + y^2 + \cdots$.

We assume now that $b = -a$ and $c = 0$. With this values system (5.6.1) yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2 + y^2 - ax^3) \begin{pmatrix} -y \\ x \end{pmatrix}, \tag{5.6.3}$$

which is integrable because it is a non-degenerate center doing a scaling of time. Moreover, as it is reducible, applying Theorem 90, a first integral is of the form $I = x^2 + y^2 + \cdots$. □

Theorem 95. *System (5.6.1) with $a(a+b) \neq 0$, is analytically integrable if, and only if, one of these following conditions is verified:*

(1) $c = -\frac{3}{2}a$, $5a + 2b \neq 0$.

(2) $c = a + b$, $b \neq 0$, $5a - 4b \neq 0$, $5a + 2b \neq 0$ *and exist* $M, N \in \mathbb{N}$, *with* $M \neq N$, *such that* $b = -\frac{2M+3N}{2M}a$.

(3) $a = -\frac{2}{5}b$, $c = \frac{3}{5}b$, $b \neq 0$.

Moreover in the Cases 1 *and* 3 *a first integral is of the form* $I = (x^2 + y^2)^2 + \cdots$ *and in the Case* 2 *is of the form* $I = (x^2 + y^2)^{M+N} + \cdots$.

Proof. The first coefficients of the normal form of system (5.6.1) are

$$\begin{aligned} a_3 = \frac{1}{5}(a + b - c), \;\; b_3 = 0, \\ c_3 = 0, \;\; d_3 = \frac{1}{5}(3a + 2c). \end{aligned} \tag{5.6.4}$$

First, we consider the case $a_3 \neq 0$; that is, $a + b - c \neq 0$, and we apply Theorem 93 statement **(a)**, since $m = 3$ and $m < 2(l-1)$ for all $l > 3$. In this case, we have that system (5.6.1) is integrable and it has a first integral of the form $I = (x^2 + y^2)^2 + \cdots$. Now we assume that the system is integrable and we impose the integrability condition $\nabla I \cdot \mathbf{F} = 0$ obtaining that $c = -\frac{3}{2}a$. Reciprocally, if it is verified such relation of statement (1), system (5.6.1) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2 + y^2) \begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} ax^3 y \\ bx^4 - \frac{3}{2}ax^2 y^2 \end{pmatrix}, \tag{5.6.5}$$

which is Hamiltonian and therefore integrable.

We now assume that $a_3 = 0$, that is, $c = a + b$. With this condition the following coefficients of the normal form for system (5.6.1) are

$$\begin{aligned} a_3 &= 0, \quad b_3 = 0, \\ c_3 &= 0, \quad d_3 = \frac{1}{5}(5a + 2b), \\ a_4 &= \frac{1}{150}b(5a - 4b), \;\; b_4 = 0. \end{aligned}$$

We first consider the case $a_4 \neq 0$ and $d_3 \neq 0$. In this case, we are under the conditions of Theorem 93 statement **(c)**, that is, $m = 2(l-1)$ with $l = 3$ y $m = 4$. Therefore, we have that

$$a_m = \frac{1}{16l^2}\left((l+2)^2\frac{(M+N)^2}{(M-N)^2} - (l-2)^2\right)(c_l^2 - d_l^2).$$

In our case this implies

$$a_4 = \frac{1}{144}\left(1 - 25\frac{(M+N)^2}{(M-N)^2}\right)d_3^2.$$

If we substitute the values of a_4 and d_3, we can isolate $b = -\frac{2M+3N}{2M}a$. Hence the relations given in statement (2) are satisfied.

On the other hand, doing the change $\{u = x, v = x^2 + y^2\}$ and scaling the time, if statement (2) is satisfied, system (5.6.1) is transformed to the quasi-homogeneous system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -v + au^3 \\ 2(a+b)u^2 v \end{pmatrix}. \tag{5.6.6}$$

Hence we can apply Theorem 91 and system (5.6.1) is integrable with a first integral of the form $I = (x^2+y^2)^M(x^2+y^2+\frac{2b-a}{3}x^3)^N$. This completes the proof of statement (2).

We now assume that $a_3 = 0$, $d_3 = 0$ and $a_4 \neq 0$, that is $a = -\frac{2}{5}b$, $c = \frac{3}{5}b$ and $b \neq 0$. We apply Theorem 93 statement **(b)** and we deduce that system (5.6.1) is integrable with a first integral of the form $I = (x^2 + y^2 + \cdots)(x^2 + y^2 + \cdots)$. Reciprocally, if it is satisfied statement (3), system (5.6.1) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} -\frac{2}{5}bx^3y \\ bx^4 + \frac{3}{5}bx^2y^2 \end{pmatrix}, \tag{5.6.7}$$

which is Hamiltonian with a first integral of the form $I = (x^2+y^2)(x^2+y^2+\frac{4}{5}x^3)$. □

Lemma 96. *System (5.6.1) under the conditions of statement* (2) *of Theorem 95 is not orbitally equivalent to a Hamiltonian vector field.*

Proof. The result is proved by noting that in this case the coefficient of the normal form a_3 is not null. □

Remark. Notice that in the case $a(a+b) = 0$ the first component of the associated vector field (5.3.5) to system (5.3.1) is reducible and this case remains open.

Finally, we study the integrability problem for the family:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^4+2y^2)\begin{pmatrix} -y \\ x^3 \end{pmatrix} + \begin{pmatrix} a_{51}x^5y + a_{13}xy^3 \\ b_{80}x^8 + b_{42}x^4y^2 + b_{04}y^4 \end{pmatrix}. \tag{5.6.8}$$

Theorem 97. *System (5.6.8) has a first integral of the form $I = x^4 + 2y^2 + \cdots$ if, and only if, one of the following conditions is satisfied*

(1) $a_{13} = 2a_{51}$, $b_{80} = 0$, $b_{04} = 2b_{42}$.

(2) $b_{80} = -a_{51}$, $b_{42} = -a_{13}$, $b_{04} = 0$.

Proof. To obtain the necessary conditions we apply Theorem 90, that is, **F** must be reducible. As the first component is reducible, the second component must be reducible. Thus we obtain:

- $a_{13} = 2a_{51}$, $b_{80} = 0$, $b_{04} = 2b_{42}$; and in this case, $(x^4 + 2y^2)$ is the factor of reducibility,
- or $b_{80} = -a_{51}$, $b_{42} = -a_{13}$ and $b_{04} = 0$; in this case, $(x^4 + 2y^2 - a_{51}x^5 - a_{13}xy^2)$ is the common factor of both components.

Reciprocally, we first assume that $a_{13} = 2a_{51}$, $b_{80} = 0$ and $b_{04} = 2b_{42}$. In this case system (5.6.8) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^4 + 2y^2)\left[\begin{pmatrix} -y \\ x^3 \end{pmatrix} + \begin{pmatrix} a_{51}xy \\ b_{42}y^2 \end{pmatrix}\right], \tag{5.6.9}$$

which is integrable with a first integral of the form $I = x^4 + 2y^2 + \cdots$.

We assume now that $b_{80} = -a_{51}$, $b_{42} = -a_{13}$ and $b_{04} = 0$. With these values system (5.6.8) yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (x^4 + 2y^2 - a_{51}x^5 - a_{13}xy^2)\begin{pmatrix} -y \\ x^3 \end{pmatrix}, \tag{5.6.10}$$

which is integrable. Moreover, as it is reducible, applying Theorem 90, a first integral is of the form $I = x^4 + 2y^2 + \cdots$. □

Theorem 98. *System (5.6.8), in generic conditions ($A \neq 0$), is analytically integrable if, and only if, one of the following conditions is verified.*

(1) $4a_{51} - 2a_{13} + 4b_{80} - 2b_{42} + b_{04} \neq 0$.

(2) $a_{13} = 2a_{51} + 2b_{80} - b_{42} + \frac{1}{2}b_{04}$, $16a_{51} - 4b_{80} + 10b_{42} - 9b_{04} \neq 0$, $2a_{51} + 4b_{80} - b_{42} \neq 0$, $64a_{51} + 164b_{80} - 50b_{42} + 9b_{04} \neq 0$ *and exist* $M, N \in \mathbb{N}$, *with* $M \neq N$, *such that* $4(-4a_{51} - 4b_{80} + b_{04})N + 5(4b_{80} - 2b_{42} + b_{04})M = 0$.

(3) $a_{13} = 10a_{51} + 4b_{42} - 4b_{04}$, $b_{80} = 4a_{51} + \frac{5}{2}b_{42} - \frac{9}{4}b_{04}$, $2a_{51} + b_{42} - b_{04} \neq 0$,

where $A = -81b_{04}^2 + 1296b_{80}^2 + 324a_{51}b_{04} - 648a_{51}b_{42} + 1296a_{51}b_{80} + 162b_{04}b_{42} - 648b_{42}b_{80}$.

Moreover in the Cases 1 *and* 3 *a first integral of system (5.6.8) is of the form* $I = (x^4 + 2y^2)^2 + \cdots$ *and in the Case* 2 *is of the form* $I = (x^4 + 2y^2)^{M+N} + \cdots$.

Proof. The first coefficients of the normal form of system (5.6.8) are

$$a_6 = \frac{1}{36}(4a_{51} - 2a_{13} + 4b_{80} - 2b_{42} + b_{04}), \;\; b_6 = 0,$$
$$c_6 = 0, \;\; d_6 = \frac{1}{18}(10a_{51} - a_{13} + 4b_{42} - 4b_{04}). \tag{5.6.11}$$

First we consider the case $a_6 \neq 0$, that is, $4a_{51} - 2a_{13} + 4b_{80} - 2b_{42} + b_{04} \neq 0$, and we apply Theorem 93 statement **(a)**, since $m = 6$ and $m < 2l - 5$ for all $l \geq 6$. In this case we have that system (5.6.8) is integrable and it has a first integral of the form $I = (x^4 + 2y^2)^2 + \cdots$.

We now assume that $a_6 = 0$, that is, $a_{13} = 2a_{51} + 2b_{80} - b_{42} + \frac{1}{2}b_{04}$. With this condition the coefficients of the normal form for system (5.6.8) are

$$\begin{aligned} a_6 &= 0, \quad b_6 = 0, \\ c_6 &= 0, \quad d_6 = \frac{1}{36}(16a_{51} - 4b_{80} + 10b_{42} - 9b_{04}), \\ a_7 &= -\frac{1}{6480}(2a_{51} + 4b_{80} - b_{42})(64a_{51} + 164b_{80} - 50b_{42} + 9b_{04}), \;\; b_7 = 0. \end{aligned}$$

We first consider the case $a_7 \neq 0$ and $d_6 \neq 0$. In this case we are under the conditions of Theorem 93 statement **(c)**, that is, $m = 2l - 5$ with $l = 6$ y $m = 7$. Therefore we have that

$$a_m = \frac{1}{16(l-1)^2}\left((l+3)^2\frac{(M+N)^2}{(M-N)^2} - (l-5)^2\right)(c_l^2 - \frac{1}{2}d_l^2).$$

In our case this implies

$$a_7 = \frac{1}{400}\left[1 - 81\frac{(M+N)^2}{(M-N)^2}\right]\frac{1}{2}d_6^2.$$

If we substitute the values of a_7 and d_6, we can write it as $4(-4a_{51} - 4b_{80} + b_{04})N + 5(4b_{80} - 2b_{42} + b_{04})M = 0$. Hence the relations given in statement (2) are satisfied.

We now assume that $a_6 = 0$, $d_6 = 0$ and $a_7 \neq 0$. We apply Theorem 93 statement **(b)** and we deduce that system (5.6.8) is formally integrable with a first integral of the form $I = (x^4 + 2y^2 + \cdots)(x^4 + 2y^2 + \cdots)$. □

Conclusiones

A continuación exponemos, de manera breve, las conclusiones más relevantes de esta memoria.

En primer lugar, en la forma normal clásica, hemos reducido el dominio del operador homológico bajo $\mathcal{C}^\infty$-equivalencia; esto es, los generadores temporales se pueden elegir en $\mathrm{Cor}(\ell_{k-r})$. Inicialmente, estos generadores temporales se elegían en $\mathcal{P}_k^{\mathbf{t}}$, espacio vectorial cuya dimensión tiende a infinito cuando el grado k tiende a infinito. Sin embargo, la dimensión de $\mathrm{Cor}(\ell_{k-r})$ se mantiene acotada. Esto nos permite llegar a una forma normal en la que los generadores temporales no son redundantes, es decir, usa el mínimo número de parámetros posibles; por tanto, se hace operativa a grado más alto. Por otro lado, también hemos conseguido reducir el dominio del citado operador para los generadores espaciales.

En el segundo capítulo, hemos definido el concepto de forma hipernormal bajo orbital equivalencia (aquella que obtiene el número mínimo de invariantes del campo vectorial). Además, hemos proporcionado un algoritmo basado en transformaciones de Lie para su cálculo. De igual forma, hemos demostrado la existencia y unicidad de la forma hipernormal para orbital equivalencia (en el futuro, forma normal única bajo orbital equivalencia). Además, aunque su construcción requiere un procedimiento fuertemente no lineal basado en el triángulo de Lie, hemos demostrado que es posible conseguirla mediante procedimientos lineales. Usando la forma normal única, hemos podido caracterizar el problema de centro en una familia nilpotente cuya primera componente cuasi-homogénea es no integrable.

En cuanto a la reversibilidad orbital de campos vectoriales, la hemos caracterizado mediante formas normales usando operadores homológicos reducidos. Esto nos ha llevado a la construcción de un algoritmo para reconocer la reversibilidad orbital de un campo vectorial. Los resultados obtenidos han sido aplicados a un sistema tridimensional con una singularidad en el origen del tipo triple-zero y a una familia de campos nilpotentes, obteniendo una nueva demostración del famoso teorema de Berthier & Moussu que afirma que un sistema nilpotente es centro si, y sólo si, es orbital–reversible.

De conformidad con el artículo 34 del Reglamento de los estudios de doctorado de la Universidad de Huelva, toda tesis redactada en otro idioma distinto del español, deberá incluir unas conclusiones en castellano.

Usando propiedades de las curvas invariantes, caracterizamos la reversibilidad orbital de campos vectoriales planos. Esto nos permite construir un algoritmo para caracterizar centros. Caracterizamos una familia semi cuasi-homogénea de centros nilpotentes. Hemos completado el problema de centro para familias de sistemas nilpotentes monodrómicos que estaban parcialmente estudiados en la literatura por el método de Cherkas, pero ahora gracias a la reversibilidad orbital.

Por último, en cuanto al problema de la integrabilidad analítica de un campo vectorial, hemos conseguido condiciones necesarias de integrabilidad. Se aborda el problema de la integrabilidad analítica de campos vectoriales planos cuya primera componente cuasi-homogénea es degenerada (no es irreducible), es decir, tiene una curva invariante de puntos de equilibrio en un entorno del origen. Presentamos dos aplicaciones en las que la integrabilidad analítica está completamente resuelta: una para el caso en el que la primera componente cuasi-homogénea es de la forma $(x^2+y^2)(-y,x)^T$ y la otra para el caso en el que es $(x^4+2y^2)(-y,x^3)^T$.

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