Isochronous centers and foci via commutators and normal forms.

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Consider the two-dimensional autonomous systems of differential equations

$$\dot{x} = -y + \lambda x + P(x, y), \qquad \dot{y} = x + \lambda y + Q(x, y),$$

where λ is a real constant and P and Q are $\mathcal{C}^\infty$-functions of order greater or equal than two. These systems have either a center or a focus at the origin, so-called center-focus type systems. In this work, we give necessary and sufficient conditions of isochronicity using normal forms. We characterize the systems which have either an isochronous center or an isochronous focus at the origin by means of the existence of a commutator of the field. Moreover, we prove that the maximum order of a weak isochronous focus for quadratic systems is two, and for systems with cubic non linearities is three.

Centers, focus, isochronous, isochronous sections, commutator.

1. INTRODUCTION AND MAIN RESULTS.

Let us consider an autonomous differential equations system

$$\dot{x} = -y + \lambda x + P(x, y), \qquad \dot{y} = x + \lambda y + Q(x, y), \tag{1.1}$$

where $\lambda \in \mathbf{R}$ and P and Q are $\mathcal{C}^\infty$-functions of order greater or equal than two. And we denote by $\mathcal{X}$ the vector field $(-y + \lambda x + P, x + \lambda y + Q)^t$.

The origin O is a isolated singular point of the system (1.1). In general, the origin is said to be center of (1.1) if it has a punctured neighborhood filled of periodic orbits and it is focus if exists a neighborhood where all the orbits are spirals in forward or backward time.

If $\lambda \neq 0$, O is a strong focus of (1.1). In otherwise, O can be either a center or a weak focus. The problem of determining whether O is an isochronous center (all the closed orbit neighboring O has the same period) or not, has been studied by several authors. However, it is far from being completely solved, even for specific families of vector fields (see [3] and the bibliography therein). Algaba et al. [2] proved that if there exists an analytic vector field $\mathcal{W}$ with linear part $(x, y)^t$ commuting with the analytic vector field $\mathcal{X}$ for $\lambda = 0$ ($[\mathcal{X}, \mathcal{W}] \equiv 0$), then the origin of (1.1) is an isochronous center. Sabatini [15] proved the same result assuming that (1.1) has a center at the origin.

We do now the following considerations in order to understand the concept of isochronous focus. The $\mathcal{C}^\infty$-system (1.1) in polar coordinates has the form

$$\dot{r} = f(r, \theta), \quad \dot{\theta} = g(r, \theta)$$

with $g(r,\theta) = 1 + \frac{1}{r}(cos\theta Q(r\cos\theta, r\sin\theta) - sin\theta P(r\cos\theta, r\sin\theta))$, that is, $g(r,\theta)$ takes the expression $g(r,\theta) = 1 + \sum_{i\geq 1} r^i g_i(\theta) + G(r,\theta)$, where G is a $\mathcal{C}^\infty$-function in a neighborhood of $r=0$ and flat in $r=0$. Giné and Grau [10] define O as an isochronous point of (1.1) if $\mathcal{X}$ can be transformed by means of an analytic change of variables ϕ such that $\phi(O) = O$ and $D\phi(O) = I$, in a system with $g(r,\theta) = g(\theta)$. This fact implies that $g_i(\theta) = 0$, for any $i \geq 1$ and for every $\theta \in [0, 2\pi)$. In such a case, the return time T of the orbits of (1.1) on every ray of the origin is constant; concretely, T is equal to 2π. This lead us to the following definition which is less restrictive that it given in [10], since we don't require the analyticity of the change of variables. We denote as ϕ_* and ϕ^* the push-forward and pull-back defined by the $\mathcal{C}^\infty$-diffeomorphism ϕ, respectively (see [13]).

DEFINITION 1.1. *The origin of (1.1) is said to be isochronous center-focus if there exists a $\mathcal{C}^\infty$-diffeomorphism ϕ with $\phi(O) = O$ and $D\phi(O) = I$ such that $\phi_*\mathcal{X}$ takes the form $(-y,x)^t + (x,y)^t H(x,y)$ where H is a scalar $\mathcal{C}^\infty$-function, that is, $\mathcal{X}$ is $\mathcal{C}^\infty$-conjugated to $(-y,x)^t + (x,y)^t H(x,y)$.*

Let us note that if O is a focus, in general, this transformation ϕ is not convergent (see [5, 20]). There are analytic vector fields whose O is a isochronous focus, according to Definition 1.1, but these ones don't verify the definition given by Giné and Grau [10].

Giné [9] shows several families of isochronous focus of (1.1). In the analytic case, Giné and Grau [10] characterize the isochronous focus by the existence of an analytic vector field $\mathcal{Y} = (x,y)^t + \mathcal{O}(2)$ verifying $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{Y}$ with $\mu(O) = 0$ or equivalently by the existence of analytic isochronous sections, that is, a curve that meets all the orbits of (1.1) contained in a neighborhood of O, such that it is crossed at equal minimal time intervals by the orbits encircling O. Sabatini [16] studies the existence of isochronous sections of critical points of focus type, He proves that if (1.1) has a commutator $\mathcal{W}$ with linear part $(x,y)^t$ then every orbit of $(\dot{x},\dot{y})^t = \mathcal{W}(x,y)$ is an isochronous section of (1.1).

The normal form of (1.1) provides other method to the problems of center, isochronous center and isochronous focus. It is known, see [1], that there exists a $\mathcal{C}^\infty$-change of coordinates which brings $\mathcal{C}^\infty$-system (1.1) with $\lambda = 0$ to the normal form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} + \sum_{i\geq 1}(x^2+y^2)^i(\alpha_{2i+1}\begin{pmatrix} x \\ y \end{pmatrix} + \beta_{2i+1}\begin{pmatrix} -y \\ x \end{pmatrix}) + F(x,y), \quad (1.2)$$

where F is a $\mathcal{C}^\infty$-function in a neighborhood of O and flat in O. The constant α_{2k+1} is called the k-th radial coefficient and β_{2k+1} the k-th azimuthal coefficient of (1.1). It is known that if (1.1) is analytic and their radial coefficients are zero then O is a center (in fact, there exists a convergent normalizing transformation, see [5]). If α_{2r+1} is the first one non zero, that is $\alpha_3 = ... = \alpha_{2r-1} = 0, \alpha_{2r+1} \neq 0$, O is a weak focus of order r (but, in this case, it is not guaranteed the existence of a convergent normalizing transformation).

In this work, we prove that if all the azimuthal coefficients of (1.1) are zero, then the azimuthal component of F is null, that is, O is an isochronous center-focus.

We want to point out that, as far as we know, the results obtained are not effective in order to compute families of isochronous center-focus. For this reason, one of our objectives is to give a method which allows us to detect systems with this property.

Our contribution to these problems of isochronicity are the following results.

THEOREM 1.2. *The origin is an isochronous center-focus of (1.1) if and only if there exists a $\mathcal{C}^\infty$-vector field $\mathcal{Y}$ of the form $\mathcal{Y} = (x, y)^t + \mathcal{O}(2)$ such that $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{Y}$, where μ is a scalar $\mathcal{C}^\infty$-function with $\mu(O) = 0$.*

With Theorem 1.2 we extend the one given by Giné and Grau [10] (analytic case) to the case where $\mathcal{X}$ is $\mathcal{C}^\infty$-vector field, that is we characterize the $\mathcal{C}^\infty$-system (1.1) with a isochronous center-focus at the origin by means of the existence of a $\mathcal{C}^\infty$-vector field normalized by $\mathcal{X}$. This result isn't effective computationally since it is a problem strongly nonlinear with numerous variables. But, in the particular case of isochronous center, the problem is reduced to finding a commutator of the form $(x, y)^t + \mathcal{O}(2)$.

The following theorem is a generalization of the result of Sabatini [15] and Algaba et al. [2] (case $j = 0$). Examples 2.1, 2.2 and 2.3, in the second section, show the usefulness of this result.

THEOREM 1.3. *The following properties holds:*
(i) If O is an isochronous center of (1.1) then there exists $\mathcal{W}$ commuting with $\mathcal{X}$ of the form $\mathcal{W} = (x^2 + y^2)^j (x, y)^t + \mathcal{O}(2j + 2)$ for any integer $j \geq 0$.
(ii) If there exists an integer $j \geq 0$ such that $\mathcal{W} = (x^2 + y^2)^j (x, y)^t + \mathcal{O}(2j + 2)$ commutes with $\mathcal{X}$ and $\alpha_{2j+1} = 0$, where α_{2j+1} is the radial coefficient of order j of a normal form of (1.1), then O is an isochronous center of (1.1).

The following result characterizes the weak isochronous focus of order j. Let us note that only is necessary the existence of a commutator only up to order $4j + 1$.

THEOREM 1.4. *Let $j \geq 0$ integer and we assume that O is a weak focus of order j for $\mathcal{X}$. The following statements are equivalent:*
(i) The origin is a weak isochronous focus of order j for $\mathcal{X}$,
(ii) there exists $\mathcal{W} = (x^2 + y^2)^j (x, y)^t + \mathcal{O}(2j + 2)$ such that $\mathcal{J}^{4j+1}[\mathcal{X}, \mathcal{W}] = 0$.

A strong focus is a focus of order zero. Taking $\mathcal{W} = (x, y)^t + \mathcal{O}(2)$, as $\mathcal{J}^1[\mathcal{X}, \mathcal{W}] = 0$, from Theorem 1.4, we have the following result:

COROLLARY 1.5. *Every strong focus is an isochronous focus.*

Also, from Theorems 1.3 and 1.4, it easily has

COROLLARY 1.6. *$\mathcal{X}$ has a commutator of the form $(x, y)^t + \mathcal{O}(2)$ if and only if either $\lambda \neq 0$ or the system (1.1) has an isochronous center at O.*

The following theorem is a new characterization of isochronous center-focus. It completes the results of Giné and Grau [10] and Sabatini [16].

THEOREM 1.7. *The origin is an isochronous center-focus of (1.1) if and only if there exists $j \geq 0$ integer and $\mathcal{W} = (x^2 + y^2)^j (x, y)^t + \mathcal{O}(2j + 2)$ such that $[\mathcal{X}, \mathcal{W}] = 0$.*
Moreover, if the origin is a focus for $\mathcal{X}$, the normal form of $\mathcal{X}$ determines univocally to $\mathcal{W}$.

The remainder of the paper is organized as follows. In the second section, we cite several examples and we also determine the maximum order of a weak isochronous focus for quadratic systems and systems with cubic nonlinearities (in both cases, there are higher order weak foci than weak isochronous foci). In the last section, we proof the theorems.

2. SEVERAL EXAMPLES AND APPLICATIONS

First on, as application of Theorem 1.3, we show several families which have a commutator with null linear part.

EXAMPLE 2.1. *We consider the system*

$$\begin{aligned} \dot{x} &= -y + x(ax+by)(bx-ay)(1-(x^2+y^2)^2), \\ \dot{y} &= x + y(ax+by)(bx-ay)(1-(x^2+y^2)^2), \ a,b \in \mathbf{R}, \end{aligned} \tag{2.3}$$

which is included in the family $(\dot{x},\dot{y})^t = (-y,x)^t + (x,y)^t H(x,y)$. *The system (2.3) holds* $\alpha_3 = 0$ *and it commutes with* $\mathcal{W} = (x^2+y^2)(1-(x^2+y^2)^2)(x,y)^t$. *From Theorem 1.3, the origin of (2.3) is an isochronous center.*
Let us note that this system has no polynomial commutator with non null linear part (see [3]). Therefore, the problem of finding or to be able to guarantee the existence of an analytic or $\mathcal{C}^\infty$*-commutator with non null linear part, without applying Theorem 1.3, is a problem really difficult.*

EXAMPLE 2.2. *The system*

$$\dot{x} = -y - \frac{4}{9}x^4 - \frac{40}{9}x^2y^2 + \frac{4}{3}y^4, \qquad \dot{y} = x + \frac{20}{9}x^3y - \frac{28}{9}xy^3, \tag{2.4}$$

has been studied by Chavarriga et al. [6]. The authors proved that (2.4) commutes with

$$\mathcal{W} = (x^2+y^2)U(x,y)^{-\frac{1}{6}}(x(3-16y^3), y(3+12x^2y-4y^3))^t,$$

where $U(x,y) = 9 + 24x^2y + 16x^4y^2 - 24y^3 + 32x^2y^4 + 16x^6$. *As* $\alpha_3 = 0$, *from Theorem 1.3, the origin of (2.4) is an isochronous center.*

EXAMPLE 2.3. *We consider the quintic polynomial system*

$$\dot{x} = -y + y(-x^4 - 4x^2y^2 + y^4), \qquad \dot{y} = x + 2xy^2(x^2 - y^2). \tag{2.5}$$

In [7], the authors prove that

$$\mathcal{W} = (x^2+y^2)(x(1-x^2y^2-5y^4), y(1+3x^2y^2-y^4))^t$$

commutes with (2.5). As $\alpha_3 = 0$, *from Theorem 1.3 it follows that the origin of (2.5) is an isochronous center.*

Theorem 1.4 provides a effective method for obtaining the isochronous focus of a family of systems.

EXAMPLE 2.4. *Let us consider the cubic Lienard system*

$$\dot{x} = -y + a_2x^2 + a_2b_2x^3, \qquad \dot{y} = x + b_2x^2 + \frac{2}{9}(a_2^2 + 5b_2^2)x^3, \tag{2.6}$$

with a_2 *y* b_2 *different from zero. The origin of (2.6) is a weak focus of order one since* $\lambda = 0$ *and* $\alpha_3 = a_2b_2 \neq 0$. *The field commutes up to order 5 with* $(R,S)^t$ *where*

$$\begin{aligned} R = \ & (x^2+y^2)x + \tfrac{1}{3}b_2x^4 - \tfrac{4}{3}a_2x^3y - b_2x^2y^2 - \tfrac{2}{3}a_2xy^3 - \tfrac{2}{3}b_2y^4 \\ & + \tfrac{1}{9}(b_2^2 - a_2^2)x^5 - \tfrac{11}{18}a_2b_2x^4y + \tfrac{2}{9}(a_2^2 - 3b_2^2)x^3y^2 + \tfrac{2}{3}a_2b_2x^2y^3, \\ S = \ & (x^2+y^2)y + \tfrac{1}{3}a_2x^4 + \tfrac{4}{3}b_2x^3y - a_2x^2y^2 + \tfrac{2}{3}b_2xy^3 - \tfrac{2}{3}a_2y^4 \\ & + \tfrac{23}{18}a_2b_2x^5 + \tfrac{5}{9}(3b_2^2 - a_2^2)x^4y - \tfrac{4}{9}a_2b_2x^3y^2 + \tfrac{2}{3}b_2^2x^2y^3. \end{aligned}$$

From Theorem 1.4, the origin of (2.6) is a weak isochronous focus of order one.

EXAMPLE 2.5. *Let us consider the system*

$$\dot{x} = -y + 2xy - 2y^3 + (x + y^2)H(x, y), \qquad \dot{y} = x - y^2 + yH(x, y), \tag{2.7}$$

with $H(x, y) = x^2 + y^2 - 2xy^2 + y^4$, *(example 5, [9]). The field commutes with* $H(x, y)(x + y^2, y)^t$, *it which has the expresion*

$$(x^2 + y^2)(x, y)^t + (y^2(y^2 - x^2 - y^2x + y^4), (y^2 - 2x)y^3)^t.$$

From Theorem 1.7, the origin of (2.7) is an isochronous center-focus.

Last on, as application of Theorem 1.4, we obtain the maximum order of a weak isochronous foci for quadratic systems and for systems with cubic non linearities.

In relation to the first family, Bautin [4] proved that the order of a weak focus is least or equal than three, and characterized its centers. Later, Loud [12] obtained its isochronous centers.

THEOREM 2.6. *The maximum order of a weak isochronous focus for quadratic systems is two.*

Finally, for the family of cubic systems without quadratic terms, Sibirskii [17] proved that O is a center if and only if λ and the first five constants of Liapunov are zero, simultaneously. And Pleshkan [14] found the systems of the family which have got an isochronous center at the origin. We have the following result.

THEOREM 2.7. *The maximum order of a weak isochronous focus for the systems with cubic nonlinearities is three.*

3. PROOFS.

Proof Theorem 1.2. We assume that O is an isochronous center-focus of (1.1), that is there exists a $\mathcal{C}^\infty$-change of variables ϕ with $D\phi(O) = I$ such that $\phi_*\mathcal{X} = (-y, x)^t + (x, y)^t H(x, y)$ where H is a scalar $\mathcal{C}^\infty$-function.
The vector field $(x, y)^t$ verifies $[\phi_*\mathcal{X}, (x, y)^t] = (x, y)^t\mu(x, y)$ with $\mu(x, y) = xH_x + yH_y$. Therefore, the $\mathcal{C}^\infty$-vector field $\phi^*(x, y)^t$ holds $[\mathcal{X}, \phi^*(x, y)^t] = \phi^*(x, y)^t\nu(x, y)$ with $\phi^*(x, y)^t = (x, y)^t + \mathcal{O}(2)$ and $\nu(O) = 0$.

Conversely. From Sternberg [18], system $(\dot{x}, \dot{y})^t = \mathcal{Y}$ is linearizable, that is there exists a change of variables $\psi(x, y) = (x, y)^t + \mathcal{O}(2)$ such that $\psi_*\mathcal{Y} = (x, y)^t$. Thus, $[\psi_*\mathcal{X}, (x, y)^t] = (x, y)^t\sigma(x, y)$. Therefore, $\psi_*\mathcal{X}$ transforms any ray of the origin $R_\xi = \{(r, \xi),\ \theta = \xi\}$ in $R_{\bar{\xi}}$, for any $\bar{\xi} \in [0, 2\pi)$, that is $\psi_*\mathcal{X}$ has constant angular speed. ∎

We now see some properties of the normal form of (1.1). We will use this results (Lemmas 3.1, 3.2 and 3.3) in order to prove the main results.
We consider the linear vector space $\mathcal{H}_i$ of the homogeneous polynomial vector field of degree i in x and y. The homological operator, see [2], which determines the normal form of (1.1) is

$$L_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i, \qquad L_i(F_i) = [F_i, (-y + \lambda x, x + \lambda y)^t].$$

If $\lambda \neq 0,\ L_i(\mathcal{H}_i) = \mathcal{H}_i$. Consequently the system (1.1) is $\mathcal{C}^\infty$-linearizable (see [18]); therefore, O is an isochronous focus.
If $\lambda = 0$, it is easy to prove that $\mathcal{C}_i = Ker\, L_i$ is a complementary subspace in $\mathcal{H}_i$ to $\mathcal{R}_i$, the range of the linear operator L_i. Also it has $\mathcal{C}_{2i} = \{0\}$ and $(x^2+y^2)^i(x,y)^t,\ (x^2+y^2)^i(-y,x)^t$ is a basis of $\mathcal{C}_{2i+1}$. Thus, by the classical normal form Theorem (see [8]), the system (1.1) can be transformed, by a $\mathcal{C}^\infty$-change of variables, into system (1.2).
From now on, $\mathcal{J}^k f$ denotes the k-jet of f at the origin.

LEMMA 3.1. *Let* $j \in \mathbf{N}$ *and* $\hat{\mathcal{X}} = (-y,x)^t + \sum_{i=1}^{j}(x^2+y^2)^i(\alpha_{2i+1}(x,y)^t + \beta_{2i+1}(-y,x)^t) + \mathcal{O}(2j+2)$ *a* $\mathcal{C}^\infty$*-vector field in a neighborhood of the origin. There exists* $\hat{\mathcal{W}} = (x^2+y^2)^j(x,y)^t + \mathcal{O}(2j+2)$ *a* $\mathcal{C}^\infty$*-vector field such that* $\mathcal{J}^{4j+1}[\hat{\mathcal{X}},\hat{\mathcal{W}}] = 0$ *if and only if* $\alpha_3 = ... = \alpha_{2j-1} = 0, \beta_3 = ... = \beta_{2j+1} = 0$.

Proof Lemma 3.1. We see the necessary condition. We impose the existence of $\hat{\mathcal{W}}$ a $\mathcal{C}^\infty$-vector field verifying $\mathcal{J}^{4j+1}[\hat{\mathcal{X}},\hat{\mathcal{W}}] = 0$. Writing $\hat{\mathcal{X}} = \hat{\mathcal{X}}_1 + \hat{\mathcal{X}}_2 + ...$ and $\hat{\mathcal{W}} = \hat{\mathcal{W}}_{2j+1} + \hat{\mathcal{W}}_{2j+2} + ...$ with $\hat{\mathcal{X}}_i$ and $\hat{\mathcal{W}}_i$ homogeneous polynomials of order $i \geq 1$, we have
1. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2j+1}] \equiv 0$,
2. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2j+2}] = 0$, which leads to $\hat{\mathcal{W}}_{2j+2} = 0$. In a similar way, for order $2i$ it has $\hat{\mathcal{W}}_{2i} = 0$, for all i.
3. $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{2j+3}] + [\hat{\mathcal{X}}_3, \hat{\mathcal{W}}_{2j+1}] = 0$, then

$$-L_{2j+3}(\hat{\mathcal{W}}_{2j+3}) + 2(1-j)\alpha_3(x^2+y^2)^{j+1}(x,y)^t + 2\beta_3(x^2+y^2)^{j+1}(-y,x)^t = 0.$$

Projecting the above equality onto the range of L_{2j+3} and onto the complement $\mathcal{C}_{2j+3}$, we deduce: $\beta_3 = 0,\ \alpha_3 = 0$ if $j \neq 1$ and $\hat{\mathcal{W}}_{2j+3} \in \mathcal{C}_{2j+3}$, that is, $\hat{\mathcal{W}}_{2j+3} = (x^2+y^2)^{j+1}(a_{2j+3}(x,y)^t + b_{2j+3}(-y,x)^t)$ with a_{2j+3} and b_{2j+3} arbitrary constants.
4. Analogously, taking into account the $(2i+1)$-th order terms of $[\hat{\mathcal{X}},\hat{\mathcal{W}}]$ with $i = j, ..., 2j-1$, we have $2(i-j)\alpha_{2i+1} = 0,\ 2i\beta_{2i+1} = 0$, for $i = 1, ..., j-1$; thus, $\alpha_{2i+1} = \beta_{2i+1} = 0$ and $\hat{\mathcal{W}}_{2i+1} \in \mathcal{C}_{2i+1}$.
5. For order $4j+1$, we get $[\hat{\mathcal{X}}_1, \hat{\mathcal{W}}_{4j+1}] + [\hat{\mathcal{X}}_{2j+1}, \hat{\mathcal{W}}_{2j+1}] = -L_{4j+1}(\hat{\mathcal{W}}_{4j+1}) + 2j\beta_{2j+1}(x^2+y^2)^{2j}(-y,x)^t = 0$ then $\beta_{2j+1} = 0$ and $\hat{\mathcal{W}}_{4j+1} \in \mathcal{C}_{4j+1}$.
Thus, $\hat{\mathcal{W}} = (x^2+y^2)^j(x,y)^t + \sum_{i=j+1}^{2j}\hat{\mathcal{W}}_{2i+1}$ verifies $\mathcal{J}^{4j+1}[\hat{\mathcal{X}},\hat{\mathcal{W}}] = 0$.
The sufficient condition is follows easily. ∎

LEMMA 3.2. *Let* $\hat{\mathcal{X}} = (-y,x)^t + \alpha_{2r+1}(x^2+y^2)^r(x,y)^t + (f(x,y), g(x,y))^t$ *where* $\alpha_{2r+1} \neq 0,\ f, g$ *are* $\mathcal{C}^\infty$*-functions in a neighborhood of the origin and* $f(x,y), g(x,y) = \mathcal{O}(|x,y|^{2r+2})$*. Then,* $\hat{\mathcal{X}}$ *is* $\mathcal{C}^\infty$*-conjugated to a* $\mathcal{C}^\infty$*-vector field of the form* $(-y,x)^t + \sum_{i \geq r}\alpha_{2i+1}(x^2+y^2)^i(x,y)^t$*, with* $\alpha_{2i+1} \in \mathbf{R}$.

Proof Lemma 3.2. It is well-known that if we perform the change of variables $\phi_U(x,y) = u(x,y,1)$ where u is the unique solution of the initial values problem

$$\frac{\partial}{\partial \epsilon}u(x,y,\varepsilon) = U(u(x,y,\varepsilon)), \quad u(x,y,0) = (x,y),$$

with $U \in \mathcal{C}^\infty$ in a neighborhood of the origin and $U(O) = O$, the vector field $\mathcal{X}$ is transformed into $\phi_{U^*}\mathcal{X} = \mathcal{X} + [\mathcal{X},U] + \frac{1}{2!}[[\mathcal{X},U],U] + \frac{1}{3!}[[[\mathcal{X},U],U],U] + \cdots$, (see [1]).

Using $U = U_{2k+1} = B(x^2+y^2)^k(-y,x)^t \in \mathcal{C}_{2k+1},\ k \geq 1$, the vector field $\phi_{U^*}\hat{\mathcal{X}}$ is also in normal form, since $[\hat{\mathcal{X}}_{2i+1}, U_{2k+1}] \in \mathcal{C}_{2k+2i+1}$.
As $U_{2k+1} \in \mathcal{C}_{2k+1} = Ker\, L_{2k+1}$, we have that the order of $[\hat{\mathcal{X}}, U]$ is greater or equal than $2k+2r+1$, therefore the transformed vector field remains unaltered up to order $2k+2r-1$. And the term of order $(2k+2r+1)$ is $\hat{\mathcal{X}}_{2k+2r+1} + [\hat{\mathcal{X}}_{2r+1}, U_{2k+1}]$, that is $(x^2+y^2)^{k+r}(\alpha_{2k+2r+1}(x,y)^t + (\beta_{2k+2r+1} - 2kB\alpha_{2r+1})(-y,x)^t)$. Thus, taking $B = \frac{\beta_{2k+2r+1}}{2k\alpha_{2r+1}}$ it is annihilated the $(k+r)$-th azimuthal constant of $\phi_{U^*}\hat{\mathcal{X}}$.
Making successive change of variables over $\hat{\mathcal{X}}$ and by Borel's Theorem (see [11]), we arrive at that there is a ϕ $\mathcal{C}^\infty$-diffeomorphism such that $\phi_*\hat{\mathcal{X}} = (-y,x)^t + \sum_{i\geq r}\alpha_{2i+1}(x^2+y^2)^i(x,y)^t + (\hat{f}(x,y), \hat{g}(x,y))^t$ where $\hat{f}, \hat{g}$ are $\mathcal{C}^\infty$-functions in a neighborhood of the origin and flat in O.
By Tokarev [20], there exists ψ $\mathcal{C}^\infty$-diffeomorphism such that $\psi_*\phi_*\hat{\mathcal{X}} = (-y,x)^t + \sum_{i\geq r}\alpha_{2i+1}(x^2+y^2)^i(x,y)^t + \bar{f}(x^2+y^2)(x,y)^t + \bar{g}(x^2+y^2)(-y,x)^t$ where $\bar{f}, \bar{g}$ are $\mathcal{C}^\infty$-functions in a neighborhood of 0 and flat in 0. By Takens [19], there exists a φ $\mathcal{C}^\infty$-change of variables of the form $\varphi(x,y) = (x+\varphi_1(x^2+y^2)x, y+\varphi_2(x^2+y^2)y)^t$ such that $\varphi_*\psi_*\phi_*\hat{\mathcal{X}} = (-y,x)^t + \sum_{i\geq r}\alpha_{2i+1}(x^2+y^2)^i(x,y)^t + h(x^2+y^2)(-y,x)^t$ where h is a $\mathcal{C}^\infty$-function in a neighborhood of 0 and flat in 0.
Finally, we complete the proof if we make the $\mathcal{C}^\infty$-change $(x - F(x^2+y^2)y, y + F(x^2+y^2)x)$ where $F(z) = -\int_0^z \frac{h(\bar{z})}{\sum_{i\geq r}\alpha_{2i+1}\bar{z}^{2i}} d\bar{z}$ is a $\mathcal{C}^\infty$-function in a neighborhood of 0 and flat in 0. ∎
The following lemma gives a normal form of the systems with constant angular speed.

LEMMA 3.3. *Let $\mathcal{X} = (-y + xH(x,y), x + yH(x,y))^t$ with H a scalar $\mathcal{C}^\infty$-function in a neighborhood of the origin. $\mathcal{X}$ is $\mathcal{C}^\infty$-conjugated either to $(-y,x)^t$ or to a $\mathcal{C}^\infty$-vector field of the form $(-y,x)^t + \sum_{i\geq r}\alpha_{2i+1}(x^2+y^2)^i(x,y)^t$ with $\alpha_{2i+1} \in \mathbf{R}$ and $\alpha_{2r+1} \neq 0$.*

Proof Lemma 3.3. Let $\mathcal{H}_i^r = \{P_i \in \mathcal{H}_i, P_i = (x,y)^t F_{i-1}$ with $F_{i-1} \in \mathcal{H}_{i-1},\ i \geq 1\}$. It is easy to prove that the following properties hold:
(a) If $[(-y,x)^t, U_{i+2}] \in \mathcal{H}_{i+1}^r$ then $U_{i+2} \in \mathcal{H}_{i+2}^r$,
(b) if $P_i \in \mathcal{H}_i^r,\ Q_j \in \mathcal{H}_j^r$ then $[P_i, Q_j] \in \mathcal{H}_{i+j-1}^r$.
Therefore, the generator U which transforms $\mathcal{X}$ into $\hat{\mathcal{X}} = \phi_{U^*}\mathcal{X}$ can be chosen such that $U_i \in \mathcal{H}_i^r$ and $\hat{\mathcal{X}}_i \in \mathcal{H}_i^r$ also.
So, if all $\hat{\mathcal{X}}_i$ are zero, $\mathcal{X}$ is $\mathcal{C}^\infty$-conjugated to $(-y,x)^t$. Otherwise, $\mathcal{X}$ is $\mathcal{C}^\infty$-conjugated to $(-y,x)^t + \sum_{i\geq r}\alpha_{2i+1}(x^2+y^2)^i(x,y)^t$ with $\alpha_{2r+1} \neq 0$. ∎

Remark. By Bruno [5], if O is a center of (1.1) there exists a convergent normalizing transformation since the normal form of (1.1) verifies the conditions "A" and "ω", whereas if O is a weak focus, in general, we cannot guarantee the existence of a convergent transformation.

Proof Theorem 1.3. (i) By definition, if (1.1) has an isochronous center at the origin, there exists a change ϕ bringing $\mathcal{X}$ to the form $\hat{\mathcal{X}} = (-y,x)^t$, it which commutes with $\hat{\mathcal{W}} = (x^2+y^2)^j(x,y)^t$, for every $j \geq 0$. The vector field $\phi^*\hat{\mathcal{W}}$ has the form $(x^2+y^2)^j(x,y)^t + \mathcal{O}(2j+2)$ and commutes with $\mathcal{X}$ since $[\phi^*\hat{\mathcal{X}}, \phi^*\hat{\mathcal{W}}] = \phi^*[\hat{\mathcal{X}}, \hat{\mathcal{W}}] = 0$.
(ii) Let a change ϕ be with $\phi(O) = O$ and $D\phi(O) = I$ bringing $\mathcal{X}$ to its normal form (1.2) where $\lambda = \alpha_3 = \alpha_5 = ... = \alpha_{2j+1} = 0$ and let $\hat{\mathcal{X}}$ be its vector field associ-

ated. The vector field $\hat{\mathcal{W}}=\phi_*\mathcal{W}$ also has the form $(x^2+y^2)^j(x,y)^t+\mathcal{O}(2j+2)$. By Lemma 3.1, $\beta_{2j+1}=0$, that is $\hat{\mathcal{X}}=(-y,x)^t+\mathcal{O}(2j+3)$ and $\hat{\mathcal{W}}=(x^2+y^2)^j(x,y)^t+\sum_{i=j+1}^{2j}\hat{\mathcal{W}}_{2i+1}+\mathcal{O}(4j+3)$ with $\hat{\mathcal{W}}_{2i+1}=(x^2+y^2)^i(a_{2i+1}(x,y)^t+b_{2i+1}(-y,x)^t)$ where $a_{2i+1},b_{2i+1},i=j+1,...,2j$ are arbitrary constants.
For order $4j+3$, $[\hat{\mathcal{X}}_1,\hat{\mathcal{W}}_{4j+3}]+[\hat{\mathcal{X}}_{2j+3},\hat{\mathcal{W}}_{2j+1}]=0$, that is, $\hat{\mathcal{W}}_{4j+3}\in\mathcal{C}_{4j+3}$ and $2\alpha_{2j+3}=0,\quad 2(j+1)\beta_{2j+3}=0$. Thus, we have $\alpha_{2j+3}=\beta_{2j+3}=0$. And so on it arrives to $\alpha_{2i+1}=\beta_{2i+1}=0$, for any $i\geq 1$. Therefore, $\hat{\mathcal{X}}=(-y,x)^t$, i.e. $\mathcal{X}$ has an isochronous center at O. ∎

Proof Theorem 1.4. (i)⇒(ii) If O is a weak isochronous focus of order j for $\mathcal{X}$, there exists a change of variables ϕ such that $\phi_*\mathcal{X}$ takes the form $(-y,x)^t+(x,y)^tH(x,y)$ where H is a scalar $\mathcal{C}^\infty$-function. From Lemma 3.3 it has $\lambda=0,\alpha_3=...=\alpha_{2j-1}=0,\alpha_{2j+1}\neq 0$ and $\beta_{2i+1}=0$, for all i. From Lemma 3.1, there exists $\hat{\mathcal{W}}=(x^2+y^2)^j(x,y)^t+\mathcal{O}(2j+2)$ such that $\mathcal{J}^{4j+1}[\hat{\mathcal{X}},\hat{\mathcal{W}}]=0$. Thus, $\mathcal{W}=\phi^*\hat{\mathcal{W}}$ verifies $\mathcal{J}^{4j+1}[\mathcal{X},\mathcal{W}]=\mathcal{J}^{4j+1}\phi^*[\hat{\mathcal{X}},\hat{\mathcal{W}}]=0$.
(ii)⇒(i) From Lemma 3.1, it has that $\beta_3=...=\beta_{2j+1}=0$, and by Lemma 3.2, $\beta_{2i+1}=0$, for all $i\geq 1$. That is, O is an isochronous focus for $\mathcal{X}$. ∎

Proof Theorem 1.7. If O is either an isochronous center or an isochronous strong focus ($\lambda\neq 0$), by Corollary 1.6, the vector field $\mathcal{X}$ has a commutator with linear part $(x,y)^t$.
If (1.1) has a weak isochronous focus of order j at the origin, then there exists a change ϕ with $\phi(O)=O$ and $D\phi(O)=I$ transforming $\mathcal{X}$ into $\hat{\mathcal{X}}=(-y,x)^t+\sum_{i\geq j}(x^2+y^2)^i\alpha_{2i+1}(x,y)^t$ with $\alpha_{2j+1}\neq 0$. It is easy to check that this field commutes with $\hat{\mathcal{W}}=(x^2+y^2)^j(x,y)^t+\frac{1}{\alpha_{2j+1}}\sum_{i\geq j+1}(x^2+y^2)^i\alpha_{2i+1}(x,y)^t$. Thus, the vector field $\mathcal{W}=\phi^*\hat{\mathcal{W}}$ has the form $(x^2+y^2)^j(x,y)^t+\mathcal{O}(2j+2)$ and also $[\mathcal{X},\mathcal{W}]=0$.
Conversely, we assume that there exists a change of variables ϕ which transforms $\mathcal{X}$ into $\hat{\mathcal{X}}$, the vector field associated to (1.2). In such case, $\phi_*\mathcal{W}=\hat{\mathcal{W}}$ has the form $(x^2+y^2)^j(x,y)^t+\mathcal{O}(2j+2)$. By Lemma 3.1, $\hat{\mathcal{X}}=(-y,x)^t+\alpha_{2j+1}(x^2+y^2)^j(x,y)^t+\mathcal{O}(2j+3)$ and by Lemma 3.2, O is an isochronous focus of (1.1).
We see that the commutator is unique. We can assume that $\hat{\mathcal{X}}=(-y,x)^t+\alpha_{2j+1}(x^2+y^2)^j(x,y)^t+\sum_{i\geq j+1}\alpha_{2i+1}(x^2+y^2)^i(x,y)^t$. From Lemma 3.1 we have $\hat{\mathcal{W}}=(x^2+y^2)^j(x,y)^t+\sum_{i=j+1}^{2j}\hat{\mathcal{W}}_{2i+1}+\mathcal{O}(4j+2)$ with $\hat{\mathcal{W}}_{2i}=0$ and $\hat{\mathcal{W}}_{2i+1}=(x^2+y^2)^i(a_{2i+1}(x,y)^t+b_{2i+1}(-y,x)^t)$, where a_{2i+1} and b_{2i+1} are arbitrary constants.
For order $4j+3$, $[\hat{\mathcal{X}}_1,\hat{\mathcal{W}}_{4j+3}]+[\hat{\mathcal{X}}_{2j+1},\hat{\mathcal{W}}_{2j+3}]+[\hat{\mathcal{X}}_{2j+3},\hat{\mathcal{W}}_{2j+1}]=0$, that is, $\hat{\mathcal{W}}_{4j+3}\in\mathcal{C}_{4j+3}$ and $-2\alpha_{2j+1}a_{2j+3}+2\alpha_{2j+3}=0,\quad -2(j+1)\alpha_{2j+1}b_{2j+3}=0$. Thus, $a_{2j+3}=\frac{\alpha_{2j+3}}{\alpha_{2j+1}}$ and $b_{2j+3}=0$.
In a similar way, for order $4j+2i+1$ we have $\hat{\mathcal{W}}_{4j+2i+1}\in\mathcal{C}_{4j+2i+1}$, and

$$-2i\alpha_{2j+1}a_{2j+2i+1}+2i\alpha_{2j+2i+1}=0,\quad -2(j+i)\alpha_{2j+1}b_{2j+2i+1}=0.$$

Thus, $a_{2j+2i+1}=\frac{\alpha_{2j+2i+1}}{\alpha_{2j+1}}$, $b_{2j+2i+1}=0$, for $i\geq 1$. Therefore,

$$\hat{\mathcal{W}}=(x^2+y^2)^j(x,y)^t+\frac{1}{\alpha_{2j+1}}\sum_{i\geq j+1}(x^2+y^2)^i\alpha_{2i+1}(x,y)^t$$

commutes with $\hat{\mathcal{X}}$. ∎

Proof Theorem 2.6. The quadratic systems, by means of a rotation of axes, we can carry them to the form given by Bautin [4],

$$\begin{aligned} \dot{x} &= -y + \lambda x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= \ \ x + \lambda y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{aligned} \tag{3.8}$$

He proved that the origin of (3.8) is a focus if:

$$\lambda \neq 0 \text{ (order 0)} \tag{3.9}$$
$$\lambda = 0,\ \lambda_3 \neq \lambda_6, \lambda_5 \neq 0 \text{ (order 1)} \tag{3.10}$$
$$\lambda = \lambda_5 = 0,\ \lambda_2 \neq 0, 0 \neq \lambda_4 \neq 5(\lambda_6 - \lambda_3) \neq 0 \text{ (order 2)} \tag{3.11}$$
$$\lambda = \lambda_5 = 0, \lambda_4 = 5(\lambda_6 - \lambda_3) \neq 0, \lambda_2^2 \neq \lambda_6(\lambda_3 - 2\lambda_6) \text{ (order 3).} \tag{3.12}$$

Otherwise, it is a center.
If $\lambda \neq 0$, (3.8) is a strong focus which is, by Corollary 1.5, an isochronous focus.
The systems (3.8) whose origin is a weak isochronous focus of order one must hold (3.10). Under these conditions, by imposing the existence of a vector field of the form $\mathcal{W} = (x^2+y^2)(x,y)^t + \mathcal{O}(4)$ satisfying $\mathcal{J}^5[\mathcal{X}, \mathcal{W}] = 0$, we obtain the condition of compatibility $r = (\lambda_5 + 4\lambda_2)^2$, with

$$r := r(\lambda_3, \lambda_4, \lambda_6) = -\lambda_4^2 - 18\lambda_3^2 - 10\lambda_6^2 - 9\lambda_3\lambda_4 + 12\lambda_3\lambda_6 + \lambda_4\lambda_6.$$

We now obtain systems (3.8) whose origin is a weak isochronous focus of order two. That is, from Theorem 1.4, such systems must verify (3.11) and by imposing the existence of a vector field of the form $\mathcal{W} = (x^2 + y^2)^2(x, y)^t + \mathcal{O}(6)$ such that $\mathcal{J}^9[\mathcal{X}, \mathcal{W}] = 0$, we derive the conditions $r = 16\lambda_2^2 > 0$ and

$$(\lambda_3 - \lambda_6)(\lambda_4 + 6\lambda_3 - 6\lambda_6)\Sigma(\lambda_3, \lambda_4, \lambda_6) = 0,$$

where $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 70\lambda_6^2 - 220\lambda_6\lambda_3 - 55\lambda_6\lambda_4 + 150\lambda_3^2 + 71\lambda_3\lambda_4 + 7\lambda_4^2$.
If λ_6 were zero, it has

$$r = -(3\lambda_3 + \lambda_4)(6\lambda_3 + \lambda_4), \quad \lambda_3(50\lambda_3 + 7\lambda_4)r = 0.$$

As $\lambda_3 \neq 0$ and $r > 0$, it has $\lambda_4 = -50/7\lambda_3$; in such case, r is negative. Therefore, without loss of generality, we assume that $\lambda_6 \neq 0$.
As $\lambda_3 \neq \lambda_6$, by imposing the second condition it has that either $\lambda_4 = 6(\lambda_6 - \lambda_3)$ or $\Sigma(\lambda_3, \lambda_4, \lambda_6) = 0$.
In both cases, it is easy to show that there are systems satisfying such conditions.
We compute the systems (3.8) whose origin is a weak isochronous focus of order 3. So, under the conditions (3.12), and by applying Theorem 1.4 with $j = 3$, we have the condition $\lambda_3 - \lambda_6 = 0$, it which is nonzero. Therefore, O cannot be weak isochronous focus of order three. ■

Proof Theorem 2.7. We consider the cubic systems center-focus type without quadratic terms. By means of a rotation of axes, these ones can be written of the following form, given by Sibirskii [17],

$$\begin{aligned} \dot{x} &= -y + \lambda x + (\mu_3 - \mu_1 - \mu_2)x^3 + (3\mu_5 - \mu_4)x^2y \\ &\qquad + (3\mu_2 - 3\mu_1 - 2\mu_3 + \mu_6)xy^2 + (\mu_7 - \mu_5)y^3, \\ \dot{y} &= \ \ x + \lambda y + (\mu_5 + \mu_7)x^3 + (3\mu_1 + 3\mu_2 + 2\mu_3)x^2y \\ &\qquad + (\mu_4 - 3\mu_5)xy^2 + (\mu_1 - \mu_2 - \mu_3)y^3. \end{aligned} \tag{3.13}$$

If $\lambda \neq 0$ the system (3.13) is a strong focus which is, by Corollary 1.5, an isochronous focus.
In it that follows, we assume $\lambda = 0$. Up to a positive factor, we have that:

$$\begin{aligned}
&\alpha_3 = \mu_6,\\
&\alpha_5 = -\mu_3\mu_7, \text{ if } \alpha_3 = 0,\\
&\alpha_7 = \mu_3\mu_2\mu_1, \text{ if } \alpha_3 = \alpha_5 = 0,\\
&\alpha_9 = \mu_3^2\mu_2\mu_4, \text{ if } \alpha_3 = \alpha_5 = \alpha_7 = 0,\\
&\alpha_{11} = -\mu_3^2\mu_2[4(\mu_2^2 + \mu_5^2) - \mu_3^2], \text{ if } \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = 0,\\
&\alpha_{2k+1} = 0, \ k \geq 6, \text{ if } \alpha_{2i+1} = 0, \ i = 1, 2, 3, 4, 5.
\end{aligned}$$

Imposing the existence of a vector field of the form $\mathcal{W} = (x^2 + y^2)(x, y)^t + \mathcal{O}(4)$ satisfying $\mathcal{J}^5[\mathcal{X}, \mathcal{W}] = 0$, we have that μ_4 musts be zero. Therefore, from Theorem 1.4, O is a weak isochronous focus of order one if and only if $\mu_4 = 0, \ \mu_6 \neq 0$.
The system (3.13) has a focus of order two at the origin if $\alpha_3 = 0$ and $\alpha_5 \neq 0$, i.e. $\mu_6 = 0, \mu_7 \neq 0, \ \mu_3 \neq 0$, and it will be isochronous if $\mu_4 = 0$ and by imposing the existence of the commutator up order 9 of the form $\mathcal{W} = (x^2 + y^2)^2(x, y)^t + \mathcal{O}(6)$, we have the condition

$$6(\mu_2^2 + \mu_5^2 + \mu_7^2) = (4\mu_1 + \mu_3)(\mu_3 - 6\mu_1). \tag{3.14}$$

From the expression of α_5 and α_7 it has that O is focus of order three if and only if $\mu_6 = \mu_7 = 0, \ \mu_1 \neq 0, \ \mu_2 \neq 0, \ \mu_3 \neq 0$. In this case, O is a weak isochronous focus of order three if $\mu_4 = 0$, it satisfies (3.14) and, by imposing the existence of a vector field of the form $\mathcal{W} = (x^2 + y^2)^3(x, y)^t + \mathcal{O}(8)$ satisfying $\mathcal{J}^{13}[\mathcal{X}, \mathcal{W}] = 0$, we have $\mu_5(\mu_3 + 24\mu_1) = 0$. If $\mu_5 = 0$, it musts hold $(4\mu_1 + \mu_3)(6\mu_1 - \mu_3) = -6\mu_2^2$, and if $\mu_3 = -24\mu_1$ it has $\mu_2^2 + \mu_5^2 = 100\mu_1^2$.
The systems (3.13) which O is a weak focus of order four verify, among others conditions, $\mu_4 \neq 0$. So, O can not be isochronous one, since μ_4 musts be zero.
Last on, O of (3.13) is a weak focus of order five if verifies $\mu_6 = \mu_7 = \mu_1 = \mu_4 = 0, \ \mu_2 \neq 0, \ \mu_3 \neq 0, \ \mu_3^2 \neq 4(\mu_2^2 + \mu_5^2)$. Imposing the existence of the commutator up order 17 of the form $\mathcal{W} = (x^2 + y^2)^4(x, y)^t + \mathcal{O}(10)$, we arrive to $\mu_2^2 + \mu_5^2 = 0$. Therefore, there existn't a weak isochronous focus of order five for (3.13). ■

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