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Center conditions to find certain degenerate centers with characteristic directions

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Abstract

We consider the two-dimensional autonomous systems of differential equations where the origin is a monodromic degenerate singular point, i.e., with null linear part. In this work we give two heuristic procedures to obtain some center conditions (perhaps not necessary) for certain degenerate centers at the origin although they have characteristic directions.

Keywords: Nonlinear differential equations; Monodomic singular points; Center problem

1. Introduction

One of the open problems in analytic planar differential systems is to characterize when a monodromic singular point is a center when the linear part of the system is null. We recall that a monodromic singular point is a singular point for which the flow of the vector field turns around it. For an analytic system any monodromic singular point is a focus or a center. There exists different algorithms to approach the center problem when the linear part at the singular point is not degenerate, i.e., not identically zero linear part, see for instance [3,8,9,16,18,22,30] and references therein. We recall that the *center problem* consists on distinguish between a focus and a center.

When the linear part is completely null, some partial results under some generic conditions are found, see [14,15,24]. In [14] was given the first generalized Poincaré–Liapunov constant or the first focal value (see definition below) for a differential system which is the sum of two homogeneous vector fields if some nondegeneracy conditions are assumed. In [15,24] for a large class of degenerate singular points of planar differential systems the authors find a computable expression of the first generalized Poincaré–Liapunov constant, which gives the stability of the singular point. This large class corresponds to the simplest case of degenerate monodromic singular points which at the end of the desingularization process there appears a polycycle having hyperbolic saddles in the corners.

The desingularization process was studied by Dumortier [7], who proved that by means of a finite number of blow-ups, any real analytic differential system given in a neighborhood of a real isolated singular point on the plane can be transformed to an analytic field of directions given in a smooth manifold, such that the singular point is transformed to a union of projective lines having only finitely many singular points, each of them elementary

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(hyperbolic or degenerate elementary) and different from a focus or a center. This process also gives an algorithmic approach to derive the monodromy conditions, see [13]. In [4,5,27] is also developed an algorithm to determine the monodromy of a singular point using the Newton's diagram.

The principal tool to approach these problems is the return map (see definition below). The leading term of the return map for a monodromic singular point is linear, see [25]. In fact the return map is given by $\Pi(x) = V_1 x + o(x)$, where V_1 is the *first generalized Poincaré–Liapunov constant*. The procedure for the computation of this first generalized Poincaré–Liapunov constant for any monodromic singular point was given in [26]. The stability of the singular point is determined by the sign of V_1, when $V_1 \neq 1$. The determination of the stability when $V_1 = 1$ is still an open problem. However in [28], in case of two even edges of Newton diagram, was given that the return map when $V_1 = 1$ is of the form $\Pi(x) = x(1 + V_2 x^{1/n} + o(x^{1/n}))$, where $n \in \mathbb{N}$ and giving the value of V_2 under certain restrictions. The variational equations of higher order are also studied in [31,32] but in the all cases the first weight polar blow-up polycycle (see below) has not singular points. The unique case with singular points at the polycycle corresponds to the Example 3 studied here.

The aim of this paper is to study the center problem for a degenerate singular point with *characteristic directions*. Hence we consider differential systems of the form

$$\dot{x} = P(x, y) = \sum_{j \geq k} P_j(x, y), \qquad \dot{y} = Q(x, y) = \sum_{j \geq k} Q_j(x, y), \tag{1}$$

where P_j and Q_j are homogeneous polynomials of degree j and $k \geq 2$. Doing the change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ the system takes the form

$$\dot{r} = \sum_{j \geq k} F_j(\theta)\, r^{j-k+1}, \qquad \dot{\theta} = \sum_{j \geq k} G_j(\theta)\, r^{j-k}, \tag{2}$$

where

$$\begin{aligned} F_j(\theta) = &\quad \cos\theta\, P_j(\cos\theta, \sin\theta) + \sin\theta\, Q_j(\cos\theta, \sin\theta), \\ G_j(\theta) = &\quad \cos\theta\, Q_j(\cos\theta, \sin\theta) - \sin\theta\, P_j(\cos\theta, \sin\theta). \end{aligned}$$

In fact the origin of system (1) is transformed to the solution $\{r = 0\}$. This solution is called the *first polar blow-up polycycle* or the *divisor* of the polar blow-up. Moreover each singular point $(r, \theta) = (0, \theta_*)$ is a *characteristic direction*, that is, $\theta = \theta_*$ is a *characteristic direction* for the origin of system (2) if

$$G_k(\theta_*) = \cos\theta_* Q_k(\theta_*) - \sin\theta_* P_k(\theta_*) \equiv 0.$$

The orbits either spiral tending or leaving the origin without a defined angle or the orbit tend or leave the origin with certain angle. In this last case the angle must be a characteristic direction θ_* and the orbit is called a *characteristic orbit*, see [33,34]. Consequently if the origin has no characteristic directions, it is automatically monodromic. However, instead of using the classical polar coordinates, we can use the weight polar coordinates $x = r^p\cos\theta$, $y = r^q\sin\theta$ which implies to use the type (p, q) and in these new coordinates the new $\tilde{G}_k(\theta)$ can be without any characteristic direction θ_*. In fact this happens for systems with a nilpotent singular point at the origin, taking the type $(1, n)$ where n is the Andreev number, also for some generalized nilpotent systems, see [10].

On the other hand if system (1) has a focus or a center at the origin, then the system cannot have orbits starting or ending at the origin with a defined slope. Recall that if there is an orbit defined slope then the origin is not monodromic.

When the monodromic singular point has no characteristic directions the center problem can be studied using the *Bautin method*, see the description below. The main difficulties that appear are of computational type. Moreover the problem is not algebraic solvable, that is, cannot be determined by algebraic equalities, see for instance [17,19,20,23] and references therein. The Bautin method works as follows. If we assume that $G(\theta_*) \neq 0$ for all $\theta \in [0, 2\pi]$, i.e. the origin has no characteristic directions, then the differential equation

$$\frac{dr}{d\theta} = R(r, \theta) = \frac{\sum_{j \geq k} F_j(\theta)\, r^{j-k+1}}{\sum_{j \geq k} G_j(\theta)\, r^{j-k}} = \frac{F_k(\theta)}{G_k(\theta)}\, r + O(r^2), \tag{3}$$

is well-defined at the origin. In this case we are able to compute, almost theoretically, the generalized Poincaré–Liapunov constants as follows. We write Eq. (3) as

$$\frac{dr}{d\theta} = R(r, \theta) = \sum_{i=1}^{\infty} R_i(\theta) r^i. \tag{4}$$

where $R_1(\theta) = F_k(\theta)/G_k(\theta)$. If we develop the analytic solution $r(\theta)$ of (4) with the initial condition $r(0) = r_0$, we can define

$$r(\theta; r_0) := u_1(\theta)r_0 + u_2(\theta)r_0^2 + u_3(\theta)r_0^3 + O(r_0^4). \tag{5}$$

If we take $\theta = 0$ we obtain $u_1(0) = 1$ and $u_k(0) = 0$ for all $k > 1$. Now we substitute the solution $r(\theta)$ in (4) and we obtain a recursive differential system to compute $u_k(\theta)$ whose first equations are

$$\begin{array}{rcl} u_1' &=& u_1R_1, \quad \text{with} \quad u_1(0) = 1, \\ u_2' &=& u_2R_1 + u_1^2R_2, \quad \text{with} \quad u_2(0) = 0, \\ u_3' &=& u_3R_1 + 2u_1u_2R_2 + u_1^3R_3, \quad \text{with} \quad u_3(0) = 0. \end{array} \tag{6}$$

The solutions of these linear differential equations are

$$u_1(\theta) = \exp\left\{\int_0^\theta R_1\right\}, \quad u_2(\theta) = u_1(\theta)\int_0^\theta u_1R_2, \quad u_3(\theta) = u_1(\theta)\int_0^\theta [2u_2R_2 + u_1^2R_3].$$

The return map that is equal to $\Pi(r) = V_1r_0 + o(r_0)$ is given by

$$\Pi(r) = r(2\pi) = u_1(2\pi)r_0 + u_2(2\pi)r_0^2 + u_3(2\pi)r_0^3 + O(r_0^4).$$

Consequently the first generalized Poincaré–Liapunov constant is

$$V_1 = \exp\left\{\int_0^{2\pi} \frac{F_k(\theta)}{G_k(\theta)}\right\}.$$

In fact more Poincaré–Liapunov constants can be computed using these formulas, see [6] for the detailed expressions.

2. Centers with characteristic directions

In order to have a degenerate center at the origin for a system (1) with characteristic directions we must to ensure that there are no orbits tending or leaving the origin through such characteristic directions. To guarantee the non-existence of such characteristic orbits we must to study the qualitative behavior of the solutions of the system near the characteristic directions. The blow-up or desingularization techniques are the known techniques to solve this problem. Moreover, these techniques are also used in several works to give necessary conditions to have a center under certain generic restrictions, see for instance [14,15,24] and in the case that the system has an inverse integrating factor, see [10,12].

In the process of desingularization the simplest case of degenerate monodromic singular points is the case when at the end of the process appears a polycycle having hyperbolic saddles in the corners. For such systems and under generic conditions in [15] was given the first generalized Poincaré Liapunov constant given by

$$V_1 = \exp\left\{GPV\int_0^{2\pi} \frac{F_k(\theta)}{G_k(\theta)}\right\}, \tag{7}$$

where GPV is an operator defined as follows: Given a function f, continuous in $[0, 2\pi] \setminus \{\theta_1, \theta_2, \dots, \theta_n\}$ where θ_i are the characteristic directions. Define $I_\delta = [0, 2\pi] \setminus \bigcup_{i=1}^n(\theta_i - \delta, \theta_i + \delta)$. Then the Cauchy Global Principal Value of $\int_0^{2\pi} f(\theta)d\theta$, is defined as the following limit if it exists

$$GPV\int_0^{2\pi} f(\theta)d\theta := \lim_{\delta\to 0}\int_{I_\delta} f(\theta)d\theta, \tag{8}$$

where f is any integrant that appears in the application of the procedure. However there are systems where at the end of the process of desingularization appear elementary degenerate singular points ((see example 2 in [14]) such that V_1 is not given by (7), because the Cauchy global principal value does not exists, and others for which at the end of the process only have hyperbolic saddles in the corners (see section 2.1 in [24]) but the previous expression of V_1 does not work. In fact, the correct expression of V_1 depend of additional terms of higher degree in the (p, q) expansion of the vector field.

We aim to give a heuristic procedure that gives some center conditions (perhaps not necessary) avoiding the desingularization process and that can be applied to certain degenerate monodromic vector fields. Our heuristic procedure is not an alternative to compute the first generalized Poincaré–Liapunov constant. In [16] was shown that any nilpotent center is limit of a linear type center also called nondegenerate centers. Moreover in such work was also studied some degenerate monodromic singular points with characteristic directions which are limit of linear type centers. The procedure is to perturb the degenerate system by $(\varepsilon y, -\varepsilon x)$, where ε is a small parameter, and then we can apply the classical Poincaré–Liapunov method to the perturbed system with $\varepsilon \neq 0$. It is clear that using this last procedure we cannot find all the degenerate centers. In fact we find degenerate centers which are algebraically solvable and limit of a linear type center, although we know that there are degenerate centers non-algebraically solvable, see [1,23] and references therein.

In the present work we aim first to study the monodromic degenerate singular points with characteristic directions that in some cases are limit of monodromic degenerate singular points without characteristic directions. The Bautin method described in the introduction does not work for such degenerate systems. The analytic Taylor expansion (3) now is only a symbolic expansion and the analytic solution $r(\theta, r_0)$ in (5) now is only a symbolic solution that will be useful if finally we obtain the sufficiency of the center conditions (perhaps not necessary) found. We clarify that, in general, $r(\theta; r_0) \neq u_1(\theta) r_0 + o(r_0)$ for all θ, i.e., when we evaluate the expression of $r(\theta; r_0)$ in 2π more linear terms coming from the queue perturb the first linear term in r_0. Therefore our center conditions can be not necessary and perhaps only some parts of the real generalized Poincaré–Liapunov constant are obtained.

A first procedure is based in assuming that condition (A) is satisfied. The condition (A) is defined by the existence of the Cauchy global principal value (8) at any step of the heuristic procedure and also the existence of any $u_i(\theta)$ of (5) up to certain value of i. Moreover the existence of the Cauchy global principal value means that the limit exists and has a finite value.

In second procedure we introduce a parameter ε that tending to zero we recover the original monodromic degenerate singular point with characteristic directions. But for values of $\varepsilon \neq 0$ either the system has not characteristic directions because the Hamiltonian part of the first component is monodromic or the system has a non-zero linear part following the method described in [16]. In the first case we can study the center problem via the classical Bautin method because the monodromic singular point has no characteristic directions and later impose the existence of limit $\varepsilon \to 0$ to recover center conditions for the original one under the restriction that the limit $\varepsilon \to 0$ exist at each step of the procedure. In the second case the monodromic singularity can have characteristic directions and in such case we use the Cauchy global principal value at each step of the procedure.

Indeed the condition (B) is the existence of the limit

$$\lim_{\varepsilon \to 0} GPV \int_0^{2\pi} f(\theta) d\theta = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{I_\delta} f(\theta) d\theta, \tag{9}$$

and also the existence of any $u_i(\theta)$(5) up to certain value of i. From the results obtained when the procedure is applied to several examples we can establish the following conjecture.

Conjecture 1. *The conditions obtained for monodromic degenerate singular points at the origin of system (1) computed by the heuristic procedure described above satisfying at any step either condition (A) or (B) always define subsets of the center variety.*

We clarify that we only obtain, in general, subsets of the center variety and not all the center variety. The verification that the heuristic procedure using that either condition (A) or condition (B) are fulfilled ends seeing that the sufficiency is satisfied for any component of the subset of the variety defined by the center conditions found. In order to compute $u_i(\theta)$ that appear in (5) we have to compute a Cauchy principal value of the form

$$u_i(\theta) = u_1(\theta)\, GPV \int_0^{\theta} F(u_1, \ldots, u_{i-1}, R_2, \ldots, R_i),$$

which is a continuous function in $[0, 2\pi] \setminus \{\theta_1, \theta_2, \ldots, \theta_n\}$ and which is not the solution of the initial value problem established in (6). In fact the functions $u_i(\theta)$ are solutions of the corresponding differential equation except in the points $\{\theta_1, \theta_2, \ldots, \theta_n\}$ where the differential equation is not defined and therefore the functions $u_i(\theta)$ are not unique.

3. Examples

We exemplify the heuristic procedure with some examples. In the first example we apply the first procedure based on the existence of the Cauchy global principal value at each step.

Example 1.

Consider the differential system

$$\dot{x} = x^2 y + a x^3 + y^5, \qquad \dot{y} = -x y^2 + b x^2 y - x^3. \tag{10}$$

The origin of system (10) is monodromic if and only if, $(a-b)2 - 8 < 0$, see [2]. System (10) has two edges in the Newton polygon associate to it. One edge is of type (1, 1) and writing the system as sum of quasihomogeneous systems of type (1, 1) we get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2 y + a x^3 \\ -x y^2 + b x^2 y - x^3 \end{pmatrix} + \begin{pmatrix} y^5 \\ 0 \end{pmatrix}. \tag{11}$$

The Hamilton function of the first quasihomogeneous component is $h = -x^2(x^2 + 2y^2 + (a-b)xy)/4$. Moreover the second edge is of type (2, 1) and the system can be written as sum of quasihomogeneous systems of type (2, 1) as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2 y + y^5 \\ -x y^2 \end{pmatrix} + \begin{pmatrix} a x^3 \\ b x^2 y - x^3 \end{pmatrix}. \tag{12}$$

Taking into account that the Hamiltonian part in type (1, 1) has parameters and the Hamiltonian part in type (2, 1) has no parameters, the better is to use the second type. The type (2, 1) suggest to take the generalized polar coordinates $x = r^2 \cos\theta$ and $y = r \sin\theta$ and Eq. (3) takes the form (4) where

$$R_1(\theta) = -\frac{\cos^5\theta}{\sin\theta(3\cos^2\theta + \sin^4\theta)}.$$

The characteristic directions are $\{0, \pi\}$. So we compute $u_1(2\pi)$ through the expression

$$u_1(2\pi) := \exp\left\{GPV \int_0^{2\pi} R_1(\theta) d\theta\right\} = \exp\left\{\lim_{\delta\to 0} \int_{I_\delta} R_1(\theta) d\theta\right\}$$

and by the periodicity of $R_1(\theta)$ we have that $u_1(2\pi) = 1$ and condition (A) is satisfied. In order to compute the next center condition we first compute

$$u_1(\theta) := \exp\left\{GPV \int_0^{\theta} R_1(\theta) d\theta\right\} = \frac{c_1}{2^{1/3} \sin^{1/3}\theta (15 - 8\cos 2\theta + \cos^4\theta)^{1/6}},$$

which is a continuous solution in $(0, 2\pi) \setminus \{\pi\}$ and c_1 is an arbitrary constant. Now we use the formula

$$u_2(\theta) = u_1(\theta) GPV \int_0^{\theta} u_1(\theta) R_2(\theta) d\theta.$$

and the computation of $u_2(2\pi)$ is given by the expression

$$u_2(2\pi) = GPV \int_0^{2\pi} u_1(\theta) R_2(\theta) d\theta := \lim_{\delta\to 0} \int_{I_\delta} u_1(\theta) R_2(\theta) d\theta,$$

and in this case $u_2(2\pi)$ takes the form

$$GPV \int_0^{2\pi} \frac{-2c_1 \times 2^{2/3} \cos^2\theta (3 + \cos(2\theta))(4a + 7b + 4a\cos(2\theta) + b\cos(4\theta))}{(15 + 8\cos(2\theta) + \cos(4\theta))^{13/6} (\sin(\theta))^{7/3}} d\theta,$$

which can be written as

$$u_2(2\pi) = a\left(GPV \int_0^{2\pi} A_1(\theta) d\theta\right) + b\left(GPV \int_0^{2\pi} A_2(\theta) d\theta\right).$$

By the periodicity of the integrants, both Cauchy global principal values are zero and consequently zero $u_2(2\pi) \equiv 0$. Using the definition of Cauchy global principal value (8) we can get also the same result therefore condition (A) is satisfied. Next we compute the $u_2(\theta)$ adding an arbitrary constant c_2 and substituting into the definition of $u_3(2\pi)$ we have

$$u_3(2\pi) = GPV \int_0^{2\pi} [2u_2 R_2 + u_1^2 R_3]\, d\theta := \lim_{\delta\to 0} \int_{I_\delta} [2u_2 R_2 + u_1^2 R_3]\, d\theta,$$

This expression can be written as

$$u_3(2\pi) = (3a+b)\left(GPV\int_0^{2\pi} B_1(\theta)d\theta\right) + \left(GPV\int_0^{2\pi} B_2(\theta)d\theta\right).$$

Using the periodicity of the integrants we can see that the first Cauchy global principal value is nonzero and the second is zero independent of the values of c_1 and c_2. Hence the condition (A) is satisfied and the center condition is $3a+b=0$. Moreover in the case $3a+b=0$ system (13) is a Hamiltonian system and therefore it has a degenerate center at the origin.

In the second example we present a system where the quasi-homogeneous leading term is Hamiltonian and adding a term with a parameter ε we obtain a monodromic Hamiltonian leading term. However we will see that we cannot finish the computations using this approach. Nevertheless doing a perturbation of lower order we can solve the problem.

Example 2.

Consider the differential system

$$\dot{x} = x^2y + ax^5 + y^5, \qquad \dot{y} = -xy^2 - x^5 + bx^4y, \tag{13}$$

where $a, b \in \mathbb{R}$. In [2] it is proved that the origin of system (13) is always monodromic. System (13) has characteristic directions because the value of $G_3(\theta) = -2\cos^2\theta\sin^2\theta$. Moreover system (13) has two edges in the Newton polygonal with $(p,q) = (1,2)$ and $(p,q) = (2,1)$. Therefore we write system (13) as a sum of quasihomogeneous systems of type $(1,2)$ because the leading term has no parameters and system (13) becomes

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}x^2y\\ -xy^2-x^5\end{pmatrix} + \begin{pmatrix}ax^5\\ bx^4y\end{pmatrix} + \begin{pmatrix}y^5\\ 0\end{pmatrix}. \tag{14}$$

The Hamilton function of first quasihomogeneous component is $h = -x^2(x^4+3y^2)/6$. We construct from this Hamiltonian function a monodromic Hamiltonian adding a term with a small parameter ε. So we have $h = -(x^6 + 3x^2y^2 + \varepsilon^2y^2)/6$. Then we consider the new system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}x^2y+\varepsilon^2y\\ -xy^2-x^5\end{pmatrix} + \begin{pmatrix}ax^5\\ bx^4y\end{pmatrix} + \begin{pmatrix}y^5\\ 0\end{pmatrix}, \tag{15}$$

Taking into account the type $(1,2)$ of the system we take $\varepsilon^2 = r^2\mu^2$. The type $(1,2)$ suggest to take the generalized polar coordinates $x = r\cos\theta$ and $y = r^2\sin\theta$ and Eq. (3) takes the form

$$\frac{dr}{d\theta} = R_1(\theta)r + R_2(\theta)r^2 + O(r^3). \tag{16}$$

where

$$R_1(\theta) = \frac{\cos\theta(-\mu^2\sin\theta - \cos^2\theta\sin\theta + \cos^4\theta\sin\theta + \sin^3\theta)}{2(\cos^6\theta + 2\mu^2\sin^2\theta + 3\cos^2\theta\sin^2\theta)}.$$

However we cannot compute not even the $u_1(2\pi)$ in a explicit way. Then although the system has not characteristic directions for $\mu \neq 0$ and we can apply the classical Bautin method we cannot go further in the computations.

Then we do not apply the substitution $\varepsilon^2 = r^2\mu^2$ and we work directly with system (15) with ε arbitrary small. Now we substitute the symbolic expansion of the form (5) into the expansion of similar form as (4) and we get an equation for $u_1(\theta)$ given by $u_1(\theta)\cot\theta + 2u_1'(\theta) = 0$. This differential equation does not define an initial value problem. First we assume that we can compute $u_1(2\pi)$ through the expression

$$u_1(2\pi) = \exp\left\{GPV\int_0^{2\pi} -\frac{\cot\theta}{2}\,d\theta\right\} = \exp\left\{\lim_{\delta\to 0}\int_{I_\delta} -\frac{\cot\theta}{2}\,d\theta\right\} = 1. \tag{17}$$

In order to continue the computations we first calculate

$$u_1(\theta) = \exp\left\{\int_0^{\theta} -\frac{\cot\theta}{2}d\theta\right\} = \frac{c_1}{\sqrt{\sin\theta}},$$

which is a continuous function in $[0,2\pi)\setminus\{0,\pi\}$ and c_1 is an arbitrary constant of integration different from zero. The equation for $u_2(\theta)$ is $u_2(\theta)\cot\theta + 2u_2'(\theta) = 0$, and we take $u_2(\theta) = c_2/\sqrt{\sin\theta}$ where c_2 is an arbitrary constant.

Next we made the change $u_3(\theta) = u_1(\theta)v_3(\theta)$ and the equation for $v_3(\theta)$ is

$$v_3'(\theta) = \frac{c_1^2(\cos\theta - \cos(3\theta) + 4\cot\theta\csc^3\theta)}{16\varepsilon^2}.$$

Therefore we have

$$\begin{aligned} u_3(2\pi) = {} & u_1(2\pi)\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} \frac{c_1^2(\cos\theta - \cos(3\theta) + 4\cot\theta\csc^3\theta)}{16\varepsilon^2} d\theta \\ & = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{I_\delta} \frac{c_1^2(\cos\theta - \cos(3\theta) + 4\cot\theta\csc^3\theta)}{16\varepsilon^2} d\theta = \lim_{\varepsilon\to 0}\frac{0}{16\varepsilon^2} = 0. \end{aligned}$$

Due to the periodicity of the integrant we see that the Cauchy global principal value is zero and then the limit when ε tends to zero is zero and condition (B) is satisfied. Next we compute $u_3(\theta)$ and we obtain

$$u_3(\theta) = \frac{c_1^3(\sin^3\theta^3 - \csc^3\theta)}{12e^2\sqrt{\sin\theta}} + \frac{c_1c_3}{\sqrt{\sin\theta}},$$

which is a continuous function in $[0, 2\pi)\backslash\{0, \pi\}$ and c_3 is an arbitrary constant. Doing the change $u_4(\theta) = u_1(\theta)v_4(\theta)$ and the equation for $v_4(\theta)$ is

$$v_4'(\theta) = \frac{c_1\cot\theta\csc^3\theta(\cos^2\theta + 2\sin^2\theta)(3c_2\cos^4\theta - bc_1^2\cos^3\theta\sqrt{\sin\theta} + 3c_2\sin^2\theta)}{\varepsilon^2}.$$

From here we get

$$u_4(2\pi) = u_1(2\pi)\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} v_4'(\theta)d\theta = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{I_\delta} v_4'(\theta)d\theta = 0,$$

also by the periodicity of the integrant. Now doing the change $u_5(\theta) = u_1(\theta)v_5(\theta)$ the equation for $v_5(\theta)$ is

$$\begin{aligned} v_5'(\theta) = {} & \frac{1}{4096\varepsilon^4}(\cos(2\theta) - 3)\cot\theta\csc^6\theta \Big[3c_1^4\cos^2\theta \Big(211 - 120\cos(2\theta) + 44\cos(4\theta) \\ & - 8\cos(6\theta) + \cos(8\theta) \Big) + 384bc_1^2c_2e^2\cos\theta\sin^{3/2}\theta \\ & + 64e^2\sqrt{\sin\theta} \Big(-3(c_2^2 + c_1^2c_3)(7 + \cos(4\theta))\sin^{5/2}\theta - bc_1^2c_2\sin(6\theta) \Big) \Big]. \end{aligned}$$

Therefore we have

$$u_5(2\pi) = u_1(2\pi)\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} v_5'(\theta)d\theta = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{I_\delta} v_5'(\theta)d\theta = 0,$$

also by the periodicity of the integrant. The next step is to take $u_6(\theta) = u_1(\theta)v_6(\theta)$ and the equation for $v_6(\theta)$ is

$$v_6'(\theta) = \frac{1}{61440c_1\varepsilon^4}\,\mathcal{R}_6(\theta, a, b),$$

where $\mathcal{R}_6$ is a rational function of trigonometric functions that we do not write here due to its extension but the readers can compute it easily. Now we compute $u_6(2\pi)$ using the expression

$$u_6(2\pi) = u_1(2\pi)\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} v_6'(\theta)d\theta \qquad = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{I_\delta} v_6'(\theta)d\theta,$$

which can be written as

$$u_6(2\pi) = a\left(\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} C_1(\theta)d\theta\right) + b\left(\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} C_2(\theta)d\theta\right).$$

It is easy to see, by the periodicity of the integrants, that both Cauchy global principal values are zero and consequently $u_6(2\pi) \equiv 0$. Using the definition of Cauchy global principal value (8) for each integral we can get the same result. Next we compute the $u_7(\theta)$ doing the change $u_7(\theta) = u_1(\theta)v_7(\theta)$ and the equation for $v_7(\theta)$ is

$$v_7'(\theta) = \frac{1}{503316480\varepsilon^6}\,\mathcal{R}_7(\theta, a, b),$$

where $\mathcal{R}_7$ is also a very big is a rational function of trigonometric functions. In order to compute $u_7(2\pi)$ we have to write

$$u_7(2\pi) = u_1(2\pi) \lim_{\varepsilon\to 0} GPV \int_0^{2\pi} v_7'(\theta) d\theta = \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \int_{I_\delta} v_7'(\theta) d\theta.$$

Nevertheless this expression last expression can be written as

$$u_7(2\pi) = (5a+b)\left(\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} D_1(\theta) d\theta\right) + \left(\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} D_2(\theta) d\theta\right).$$

Moreover by the periodicity of the integrants we can see that the first Cauchy global principal value is nonzero and the second is zero. Hence the center condition is $5a+b=0$. Moreover in this case system (13) is a Hamiltonian system and therefore it has a degenerate center at the origin.

Example 3.

Consider the differential system

$$\dot{x} = y(\alpha x^2 + bxy + cy^2), \qquad \dot{y} = y^2(\alpha x + by) + x^5, \tag{18}$$

where α, b and $c \in \mathbb{R}$. In fact, in [14, page 726] the stability of the monodromic singular point at origin this system when $\alpha < 0$ and $c < 0$ was studied. Under such conditions system (18) is monodromic, see [14]. The authors show that doing easy computations, taking classical polar coordinates, that the generalized Cauchy principal value (7) diverge when δ tends to zero. More specifically

$$GPV \int_0^{2\pi} \frac{F_k(\theta)}{G_k(\theta)} = -\frac{4b}{c} \lim_{\delta\to 0} \frac{\cos\delta}{\sin\delta} = \operatorname{sign}(b)\infty,$$

taking into account that $c < 0$. Consequently condition (A) is not satisfied. To satisfy condition (A) we can to impose $b = 0$. However in [14] using the desingularization process it is proved that the Poincaré return map is given by $\Pi(x_0) = x_0 + o(x_0)$.

Now we can use the second procedure to find a center condition. System (18) has characteristic directions because the value of $G_3(\theta) = c\sin^4\theta$. Moreover system (18) has two edges in the Newton polygonal with $(p,q) = (1,1)$ and $(p,q) = (1,2)$. Hence we write system (18) as a sum of quasihomogeneous systems of type $(1,1)$ and we get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha x^2 y + bxy^2 + cy^3 \\ \alpha xy^2 + by^3 \end{pmatrix} + \begin{pmatrix} 0 \\ x^5 \end{pmatrix}, \tag{19}$$

Next we consider the new system the perturbed system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha x^2 y + \varepsilon^2 y + bxy^2 + cy^3 \\ \alpha xy^2 + by^3 \end{pmatrix} + \begin{pmatrix} 0 \\ x^5 \end{pmatrix}, \tag{20}$$

Now taking polar coordinates and substituting the analytic solution (5) into Eq. (4) we get an equation for $u_1(\theta)$ given by $u_1(\theta)\cot\theta + u_1'(\theta) = 0$. So we can compute $u_1(2\pi)$ doing the same that in (17) and we get the expression $u_1(2\pi) = 1$. This is not in contradiction attending to remark of [11] that shows that the symbolic expansion $r(\theta, r_0)$ depends on the chosen coordinate system. What is invariant is the Poincaré return map and the variety defined by the true necessary center conditions.

In order to continue the computations we first compute

$$u_1(\theta) = \exp\left\{\int_0^\theta -\cot\theta d\theta\right\} = \frac{c_1}{\sin\theta},$$

which is a continuous solution in $[0, 2\pi) \setminus \{0, \pi\}$ where c_1 is an arbitrary constant. The next equation is for $u_2(\theta)$ and reads as $u_2(\theta)\cot\theta + 2u_2'(\theta) = 0$. Hence we can take $u_2(\theta) = c_2/\sin\theta$. In the next equation we made the change $u_3(\theta) = u_1(\theta)v_3(\theta)$ and the equation for $v_3(\theta)$ is

$$v_3'(\theta) = \frac{c_1^2}{\varepsilon^2}(-b\csc^2\theta - \alpha\cot\theta\csc^2\theta).$$

Consequently the expression for $u_3(2\pi)$ can be written as

$$u_3(2\pi) = -b\left(\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} \frac{c_1^2\csc^2\theta}{\varepsilon^2} d\theta\right) - \alpha\left(\lim_{\varepsilon\to 0} GPV \int_0^{2\pi} \frac{c_1^2\cot\theta\csc^2\theta}{\varepsilon^2} d\theta\right).$$

Moreover by the periodicity of the integrants we can see that the first Cauchy global principal value is nonzero and the second one is zero. Hence the condition (B) is satisfied and the center condition is $b = 0$. Finally if $b = 0$ the origin of system (18) is a degenerate center because the system is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$, that is, the system is time-reversible, see [21].

In this last example we see that V_1 cannot be computed by (7) and the heuristic method proposed here gives some center conditions but not all the necessary center conditions. Therefore we find a subset of the center variety of the corresponding differential system studied.

Example 4.

Finally we study the differential system

$$\dot{x} = Pxy^2 - y^3 + Qx^5, \qquad \dot{y} = xy^2 + Py^3 + (1 + \frac{9}{4}Q^2)x^7, \tag{21}$$

where P and $Q \in \mathbb{R}$. System (21) is a particular case of system (13) in [24, page 697]. The monodromy of the singular point at the origin this system was studied using the results established in [15]. System (21) has two edges in the Newton polygon with $(p, q) = (1, 1)$ and $(p, q) = (1, 3)$. In the edge of type $(1, 1)$ the system is written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} Pxy^2 - y^3 \\ xy^2 + +Py^3 \end{pmatrix} + \begin{pmatrix} Qx^5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (1 + \frac{9}{4}Q^2)x^7 \end{pmatrix}. \tag{22}$$

Using the type $(1, 1)$, we take polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Next we substitute the symbolic expansion of the form (5) into the expansion of form (4) and we get an equation for $u_1(\theta)$ given by $-Pu_1(\theta) + u_1'(\theta) = 0$. We recall that this differential equation does not define an initial value problem. First we assume that we can compute $u_1(2\pi)$ through the expression

$$u_1(2\pi) = \exp\left\{GPV \int_0^{2\pi} P\, d\theta\right\} = e^{2\pi P}. \tag{23}$$

We continue the computations, firstly calculating

$$u_1(\theta) = \exp\left\{\int_0^{\theta} P d\theta\right\} = e^{P\theta}.$$

The next equation for $u_2(\theta)$ is $-Pu_2(\theta) + u_2'(\theta) = 0$. Hence we take $u_2(\theta) = c_2 e^{P\theta}$. For the next equation we made the change $u_3(\theta) = u_1(\theta)v_3(\theta)$ and the equation for $v_3(\theta)$ is

$$v_3'(\theta) = e^{P\theta} Q \cos^4\theta \cot\theta (P + \cot\theta)$$

Consequently the expression for $u_3(2\pi)$ can be written as

$$\begin{aligned} u_3(2\pi) &= QP\left(GPV \int_0^{2\pi} e^{P\theta} \cos^4\theta \cot\theta d\theta\right) \\ &+ Q\left(GPV \int_0^{2\pi} e^{P\theta} \cos^4\theta \cot^2\theta d\theta\right). \end{aligned}$$

From the expression of $u_1(2\pi)$ we get $P = 0$ and the second Cauchy global principal value is nonzero because we integrate a positive function. Hence the condition (A) is satisfied and the found center conditions are $P = Q = 0$. Moreover in the such case the origin of system (21) is a degenerate center because the system is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$, that is, the system is time-reversible system, see [21]. The computations can be also made using the other type $(1, 3)$ but are more involved. Indeed taking weight polar coordinates $x = r\cos\theta$, $y = r^3 \sin\theta$ we have

$$u_1(2\pi) = \exp\left\{GPV \int_0^{2\pi} \frac{4Q\cos^6\theta + (4 + 9Q^2)\cos^7\theta \sin\theta + 4\cos\theta \sin^3\theta}{(4 + 9Q^2)\cos^8\theta - 12Q\cos^5\theta \sin\theta + 4\cos^2\theta \sin^2\theta} d\theta\right\}.$$

We cannot compute this Cauchy principal value directly but we can made numerical integration of the Cauchy principal value for different values of Q and we obtain the same numerical value given by $e^{2\pi Q}$. Therefore, as in Example 3, we arrive to different results depending of the type used. However it seems that if we could continue with the computations using the type $(1, 3)$ we would arrive at the previous condition $P = 0$.

System (21) was studied in [24, page 697]. There was found that the Poincaré return map is expressed as $\Pi(x) = e^{2\pi(P+Q)}x + o(x)$. This implies that for $P + Q > 0$ the origin is an unstable focus and for $P + Q < 0$ the origin is an stable focus. However the center problem is not completely solved in [24]. The value of the first generalized Liapunov constant is not in contradiction with the conjecture because the conditions defined by $P = Q = 0$ is a subset of the center variety. We clarify that the system has only the origin as a finite singular point. Moreover for the values $P = -0.625$ and $Q = 2$, we have detected numerically that the system has a limit cycle.

4. Conclusions

For monodromic degenerate singular points with characteristic directions Eq. (5) is not true. In the classical works of Medvedeva [25,26] is proved that the return map has the form $\Pi(x) = V_1 x + o(x)$. But this does not means that the flow of the system is $r(\theta, r_0) = a_1(\theta)r_0 + o(r_0)$ for all θ, and when we evaluate $r(\theta, r_0)$ in 2π more linear terms from the queue can perturb the first linear term in r_0. Since $r(\theta, \rho_0)$ is a Dulac function, and it could happen that in a neighborhood of the characteristic directions, the non-regular terms of the Dulac's expansion (perhaps expressible in terms of so-called Roussarie compensators could dominate, see [29, page 141]). For that reason, in some cases, the return map is computed by decomposing it into different transitions maps in local charts associated to the desingularization processes, [14,15,24].

In [14,24] was proved that the first Poincaré–Liapunov constant may not be always computable using Eq. (7) and it is shown that it can also depend on the higher order terms of Eq. (4). These terms appear by integrating the variational equations associated with the regular maps that appear in some local charts of the different blow-ups implemented.

The methodology presented in this work can be applied to regular and non-regular return maps and if the hypothesis (A) or (B) are satisfied no obstructions appear to find certain conditions (perhaps not all the necessary center conditions). In all the examples presented the vanishing these conditions give a subset of the center variety. Therefore this is established as Conjecture 1.

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