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# Weighing hierarchical power and active contribution in cooperative games with authorization structure

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## Abstract
Cooperative games model situations in which a group of players work together to make a profit. Frequently, in cooperative situations there are dependency or hierarchical relationships between the players, which must be taken into account when allocating the common profit obtained by the grand coalition. Multiple structures have been used in the literature to model those relationships, and several values have been proposed, but there is something in common in all of them: if a player can veto the participation of another in any coalition, then both players will receive the same share of the profit derived from the active cooperation of the vetoed player. In other words, actively cooperating and giving permission to cooperate are equally valued. In many situations this is neither fair nor realistic. In this paper we introduce a family of allocation rules for cooperative games with authorization structure, which reward positional power less than active cooperation.

**Keywords** Cooperative games · Shapley value · Allocation rules · Permission structures · Hierarchical structures · Authorization structures

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## 1 Introduction

Cooperative game theory studies situations in which a group of players collaborate to obtain a common profit. In this paper we will deal with cooperative games in characteristic function form, simply referring to them as cooperative games. A cooperative game assigns to each subset of players (coalition) the profit that these players can achieve by themselves. The main problem that arises in this context is how to allocate the profit obtained by the grand coalition. Different solutions to this problem have been proposed in the literature, the Shapley value (Shapley 1953) being the best known of them. In most of these solutions it is assumed that the players are socially symmetric, and, therefore, the only fact to be taken into account when allocating the profit is the capacity of each player to modify the gain achievable by each coalition. Nevertheless, in practice it is not uncommon for there to be asymmetries between the players. These asymmetries usually entail restrictions in cooperation. Especially interesting cases are those in which hierarchical relationships between the players are considered. In such cases, it seems reasonable that those relationships must be taken into account when allocating the total profit. In this sense, Bessey (2023) investigated the effects of hierarchical structures on outcomes in economic experiments. Given a cooperative situation in a hierarchical structure, if a coalition is formed it is possible that not all players within the coalition can cooperate, since some of them may need the permission from players that are not in the coalition. In this line, Gilles et al. (1992) studied cooperative situations in which a hierarchical structure imposes veto relations between the players. They introduced the concept of game with permission structure, which consists of a set of players, a cooperative game and a mapping that assigns to each player a set of subordinates. They also defined and characterized a solution, the permission value, for these games. The payoff that this solution assigns to each player depends not only on their ability to actively generate more profit when joining a coalition, but also on their position in the hierarchy, that is, on their power to veto the cooperation of other players. Derks and Peters (1993) generalized the approach considered by Gilles et al. by introducing the so-called restrictions. A restriction assigns to each coalition the subset of players that can cooperate when this coalition is formed. Using this concept, they modeled dependency relationships which are not necessarily hierarchical. They also defined and characterized a solution for games with restrictions. Since then, cooperative situations with hierarchical relationships have been modeled through different mathematical structures: antimatroids (Algaba et al. 2004), matroids (Bilbao et al. 2001), augmenting systems (Bilbao and Ordóñez 2009), levels structures (Winter 1989), authorization structures (Gallardo et al. 2018), etc. A compilation of these models is given in van den Brink (2017). There is something in common in all of them: if a player $i$ has veto power over a player $j$ (that is, $j$ cannot cooperate in any coalition that does not contain $i$) then it is considered that $i$ and $j$ deserve the same share of the profit derived from the active cooperation of $j$. This fact, which does not seem fair or realistic, is the motivation of the present paper. Our goal is to obtain allocation rules that reward hierarchical power with a reasonable proportion of the profits generated by cooperation. We will use multichoice games (Hsiao and Raghavan 1993; Ayoshin and Tanaka 2000) to separately compute the gains derived from active cooperation and those that come from hierarchical power.

The paper is organized as follows. In Sect. 2 we recall some preliminaries concerning cooperative games and authorization operators. In Sect. 3 we propose a multichoice approach to obtain a new family of values for games with authorization structure and in Sect. 4 we provide them with an axiomatization. Section 5 presents conclusions of the results.

## 2 Preliminaries

Throughout this paper, $N$ denotes a fixed finite set. The subsets of $N$ will be called coalitions. The family of all coalitions will be denoted by $2^N$.

### 2.1 Cooperative games

A *cooperative game (with transferable utility)* on $N$ is a function (called characteristic function) $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Given a coalition $E$, $v(E)$ is the worth of $E$, and it is interpreted as the maximal gain that the players in $E$ can achieve when they cooperate. The family of cooperative games on $N$ is denoted by $\mathcal{G}$. This set is a $(2^{|N|} - 1)$-dimensional real vector space. One basis of this space is the collection $\{u_F : F \subseteq N, F \neq \emptyset\}$ where for a coalition $F \in 2^N \setminus \{\emptyset\}$ the *unanimity game* $u_F$ is defined as $u_F(E) = 1$ if $F \subseteq E$ and $u_F(E) = 0$ otherwise. Every game $v \in \mathcal{G}$ can be written as a linear combination of unanimity games:

$$v = \sum_{\{E \in 2^N : E \neq \emptyset\}} \triangle_v(E) u_E, \tag{1}$$

where $\triangle_v(E)$ is the dividend of the coalition $E$ in the game $v$. A game $v \in \mathcal{G}$ is *monotone* if $v(F) \leqslant v(E)$ for every $F \subseteq E \subseteq N$. A player $i \in N$ is a *null player* in $v \in \mathcal{G}$ if $v(E) = v(E \backslash \{i\})$ for all $E \subseteq N$. A player $i \in N$ is a *necessary player* in $v \in \mathcal{G}$ if $v(E) = 0$ for every $E \subseteq N \backslash \{i\}$. Observe that if there exists a null and necessary player then $v = 0$ and all the players are null and necessary.

A value on $\mathcal{G}$ is a function $\psi : \mathcal{G} \to \mathbb{R}^N$ that assigns to each game $v \in \mathcal{G}$ a vector $(\psi_i(v))_{i \in N}$ where the real number $\psi_i(v)$ is the payoff to player $i$ in the game $v$. Multiple values have been defined in the literature. The best-known of them is the *Shapley value* (Shapley 1953), which is defined, for every $v \in \mathcal{G}$ and every $i \in N$, as

$$Sh_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left[ v(E) - v(E \setminus \{i\}) \right], \tag{2}$$

where

$$p_E = \frac{(|N| - |E|)!(|E| - 1)!}{|N|!}. \tag{3}$$

van den Brink (1994) characterized the Shapley value as the unique value $\psi$ that satisfies: (S1) *Efficiency*, $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \mathcal{G}$; (S2) *Additivity*, $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)$ for all $v_1, v_2 \in \mathcal{G}$; (S3) *Null player property*, if $i \in N$ is a null player in $v \in \mathcal{G}^N$ then $\psi_i(v) = 0$; and (S4) *Necessary player property*, if $i$ is a necessary player in a monotone game $v \in \mathcal{G}$, then $\psi_i(v) \geq \psi_j(v)$ for all $j \in N$.

### 2.2 Authorization operators

A permission structure on $N$ is given by a mapping $S : N \to 2^N$ where the players in $S(i)$ are the successors of player $i \in N$, that is, $S(i)$ contains all agents that are dominated directly by agent $i$. Let $\hat{S}$ denote the transitive closure of $S$, i.e. $j \in \hat{S}(i)$ if and only if there exists a sequence $\{i_p\}_{p=0}^{q}$ such that $i_0 = i, i_q = j$ and $i_p \in S(i_{p-1})$ for $1 \leq p \leq q$. The players in $\hat{S}(i) \backslash S(i)$ are those dominated indirectly by $i$. The set of predecessors of player $i$ is $P_S(i) = \{j \in N : i \in S(j)\}$ and the set of superiors of $i$ in $S$ is denoted by $\hat{P}_S(i) = \{j \in N : i \in \hat{S}(j)\}$. Several assumptions can be made about how a permission structure restricts the formation of coalitions in a game. In the conjunctive approach (Gilles et al. 1992) it is assumed that every player needs permission from all their superiors in order to be allowed to cooperate. The conjunctive sovereign part of a coalition $E$ contains those players in $E$ whose superiors are in $E$, that is $A_c^S(E) = \{i \in E : \hat{P}_S(i) \subseteq E\}$. In the disjunctive approach (van den Brink 1997) it is assumed that each player only needs permission from one of their predecessors. According to this approach, a coalition is autonomous if every player in the coalition either has no predecessors or at least one of their predecessors is in the coalition. The disjunctive sovereign part of a coalition $E$, denoted by $A_d^S(E)$, is the largest autonomous subset of $E$.

Notice that the hierarchy given by a permission structure is actually determined by the mappings $A_c^S$ (in the conjunctive approach) or $A_d^S$ (in the disjunctive approach). That is, it is enough to give a mapping $A : 2^N \to 2^N$ that assigns to each coalition $E \in 2^N$ the set $A(E)$ of players who are allowed to cooperate when coalition $E$ is formed. Gallardo et al. (2018) defined the mapping $A$ as an authorization operator. A function $A : 2^N \to 2^N$ is an *authorization operator* on $N$ if it satisfies the following requirements: 1) $A(E) \subseteq E$ for any $E \subseteq N$, and 2) if $E \subseteq F$ then $A(E) \subseteq A(F)$. The set of all authorization operators on $N$ will be denoted by $\mathcal{A}$. Notice that the conditions that define authorization operators are reasonable. The first condition says that if a coalition $E$ is formed, then the players who can actively cooperate within the coalition belong to $E$. The second condition says that if a player is authorized to cooperate within a coalition $E$, then they are also authorized to cooperate in any coalition that contains $E$. If $A \in \mathcal{A}$, we say that a player $j \in N$ *depends partially on* $i \in N$ (according to $A$) if there exists $E \subseteq N$ with $j \in A(E) \backslash A(E \backslash \{i\})$. And we say that player $i$ has *veto power* over $j$ (according to $A$) if $j \notin A(N \setminus \{i\})$.

### 2.3 Games with authorization structure

A *game with authorization structure* on $N$ is a pair $(v, A) \in \mathcal{G} \times \mathcal{A}$. A *value for games with authorization structure* on $N$ is a mapping $\Psi : \mathcal{G} \times \mathcal{A} \to \mathbb{R}^N$ that assigns to every game with authorization structure a payoff vector. The classic Shapley value is also a value for games with authorization structure, which establishes an allocation of the profit regardless of the dependency relationships. But it is clear that in order to obtain fair payoff vectors it is necessary to take into account the structure. In order to obtain solutions for games with permission structure, Gilles et al. (1992) and van den Brink (1997) proposed to modify the characteristic function taking into account the restrictions in cooperation determined by the permission structure. In this sense, they define the conjunctive and disjunctive restricted game, respectively, as $v_c^S(E) = v\left(A_c^S(E)\right)$ and $v_d^S(E) = v\left(A_d^S(E)\right)$ for every permission structure $S : N \to 2^N$ and every $E \subseteq N$. The conjunctive permission value and the disjunctive permission value are defined as $\phi^c(v, S) = Sh\left(v_c^S\right)$ and $\phi^d(v, S) = Sh\left(v_d^S\right)$ respectively. Following this idea, Gallardo et al. (2018) introduced the *authorization value* $\phi : \mathcal{G} \times \mathcal{A} \to \mathbb{R}^N$ defined as

$$\phi(v, A) = Sh(v^A) \tag{4}$$

where $v^A(E) = v(A(E))$ for every $E \subseteq N$. Notice that if we consider a permission structure $S : N \to 2^N$, then $\phi^c(v, S) = \phi(v, A_c^S)$ and $\phi^d(v, S) = \phi(v, A_d^S)$. Therefore, any cooperative situation that can be modeled by a game with permission structure can also be modeled by a game with authorization structure. Let us see an example. Let $N = \{1, 2, 3, 4\}$ and let $S$ be the permission structure on $N$ given by the following digraph in Fig. 1.

We will consider the disjunctive approach for this permission structure. This means that player 3 needs permission from either player 1 or player 2, and player 4 needs permission from player 3 and from either player 1 or player 2. Notice that this cooperative situation can be described by the authorization operator $A \in \mathcal{A}$ defined in Table 1 (in which we have omitted the trivial equality $A(\emptyset) = \emptyset$).

Let us consider the game $u_{\{4\}}$. The disjunctive permission value of $(u_{\{4\}}, S)$ is equal to the authorization value of $(u_{\{4\}}, A)$. Whereas $Sh(u_{\{4\}}) = (0, 0, 0, 1)$, it is easy to check that

$$\phi(u_{\{4\}}, A) = \frac{1}{12}(1, 1, 5, 5).$$

## 3 The proportional authorization values

### 3.1 Motivating example

The following example shows that the authorization value does not satisfactorily resolve all cooperative situations with dependency relationships between the players.

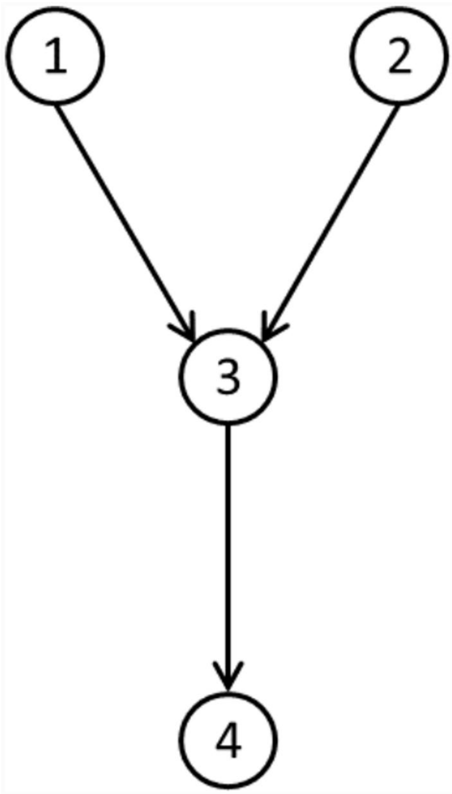


**Fig. 1** Example of authorization structure

**Table 1** Example of authorization structure

| E | A(E) | E | A(E) | E | A(E) |
|---|---|---|---|---|---|
| $\{1\}$ | $\{1\}$ | $\{2\}$ | $\{2\}$ | $\{3\}$ | $\emptyset$ |
| $\{4\}$ | $\emptyset$ | $\{1,2\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,3\}$ |
| $\{1,4\}$ | $\{1\}$ | $\{2,3\}$ | $\{2,3\}$ | $\{2,4\}$ | $\{2\}$ |
| $\{3,4\}$ | $\emptyset$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,4\}$ | $\{1,2\}$ |
| $\{1,3,4\}$ | $\{1,3,4\}$ | $\{2,3,4\}$ | $\{2,3,4\}$ | $\{1,2,3,4\}$ | $\{1,2,3,4\}$ |

**Table 2** Cooperative game of the motivating example

| E | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|---|---|---|---|---|---|---|---|
| $v(E)$ | 1 | 1 | 0 | 3 | 1 | 1 | 3 |

Consider two companies 1 and 2. Each one of them can produce, let us say, one million units of a certain electronic component, that can be sold at a profit of one monetary unit each. Both companies can decide to assemble their components (one unit for one unit) and produce a new component that can be sold at a profit of three units

**Table 3** Authorization structure of the motivating example

| E | {1} | {2} | {3} | {1, 2} | {1, 3} | {2, 3} | {1, 2, 3} |
|---|---|---|---|---|---|---|---|
| $A(E)$ | {1} | ∅ | {3} | {1} | {1, 3} | {2, 3} | {1, 2, 3} |

each. Besides, there is a third company which does not produce any component. This situation can be modeled by a classical cooperative game $v$, where $v(E)$ is the profit (in millions of monetary units) that the companies in $E$ can generate when they cooperate (Table 2).

Now consider the following scenario. Company 3, as a patent holder, sues company 2 for patent infringement. Suppose that this infringement is evident and, therefore, it is certain that company 3 will win the lawsuit. Notice that company 3 has veto power over company 2. This situation can be modeled by the authorization structure $A$ defined in Table 3.

If we calculate the authorization value of $(v, A)$ we obtain

$$\phi(v, A) = \frac{1}{6}(8, 5, 5).$$

Observe that, although company 3 is a null player in $v$, the vector $\phi(v, A)$ assigns company 3 the same payoff as company 2. But this is not realistic. In practice, company 2 would end up giving company 3 a percentage of the profits, but this percentage would almost certainly not exceed 25 (Goldscheider 2018). If we propose exactly 25 as said percentage, the proportion between their payoffs should be 1/3, that is, if $\psi(v, A)$ is the payoff vector then

$$\psi_3(v, A) = \frac{1}{3}\psi_2(v, A).$$

Our goal in this paper is to introduce a solution for cooperative games with authorization structure that allows to weigh the value of positional power. To this end, we will use multichoice games to define the proportional authorization values.

### 3.2 Proposed methodology

Multichoice games were introduced by Hsiao and Raghavan (1993) to model cooperative situations in which each player, when cooperating within a coalition, can choose between different levels of cooperation. Each player is allowed to take $(m + 1)$ actions $\sigma_0, \sigma_1, \ldots, \sigma_m$, where $\sigma_0$ is the action of doing nothing and $\sigma_k$ is the action of cooperating at level $k$ for each $k = 1, \ldots, m$. If $\beta = \{0, 1, \ldots, m\}$ then the action space is given by $\beta^N$. Each element $\mathbf{x} = (x_i)_{i \in N} \in \beta^N$ is called an action of $N$, and $x_i = k$ if and only if player $i$ takes action $\sigma_k$. For every $\mathbf{x} \in \beta^N$, $l \in \beta$ and $H \subseteq N$ the *l-level set* in $H$ of $\mathbf{x}$ is

$$H_l(\mathbf{x}) = \{i \in H : x_i = l\} \tag{5}$$

A *multichoice cooperative game* over $N$ with level set $\beta$ is a function $V : \beta^N \to \mathbb{R}$ with $V(\mathbf{0}) = 0$. The set of multichoice games over $N$ with $m$ levels is denoted by

$\mathcal{MG}^{\beta^N}$. A value on $\mathcal{MG}^{\beta^N}$ is a mapping $\Psi : \mathcal{MG}^{\beta^N} \to \mathbb{R}^{\beta\times N}$. If $V \in \mathcal{MG}^{\beta^N}$, $k \in \beta$ and $i \in N$, the number $\Psi_{k,i}(V)$ is interpreted as the payoff that is allocated to player $i$ for taking action $\sigma_k$. It is considered that $\psi_{0,i}(V) = 0$. In order to obtain a value for multichoice games, Hsiao and Raghavan consider weights $0 = w(0) \leqslant w(1) \leqslant \ldots \leqslant w(m)$ for the different actions that the players can take. These weights determine the ratios that are considered to be fair when allocating the payment obtained by a group of players who take different actions within a coalition. The weight of any $\mathbf{x} \in \beta^N$ is $\|\mathbf{x}\|_w = \sum_{j\in N} w(x_j)$. Once these weights are fixed, Hsiao and Raghavan introduce a value $\Phi^w$ on $\mathcal{MG}^{\beta^N}$, defined as

$$\Phi^w_{k,i}(V) = \sum_{\substack{l=1 \\ w(l)>0}}^{k} \sum_{\substack{\mathbf{x}\in\beta^N \\ x_i = l}} w(l) P^w_{\mathbf{x}}(i)(V(\mathbf{x}) - V(\mathbf{x}-\mathbf{e}^i)) \tag{6}$$

where $\mathbf{e}^i \in \{0,1\}^N$ is given by $e^i_i = 1$ and $e^i_j = 0$ for every $j \in N\backslash\{i\}$, and

$$P^w_{\mathbf{x}}(i) = \sum_{F\subseteq N\backslash(N_m(\mathbf{x})\cup\{i\})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + \sum_{j\in F}(w(x_j+1) - w(x_j))}. \tag{7}$$

We aim to define a new value for games with authorization structure. To this end, we will follow a procedure widely used in the literature to obtain values for cooperative games with combinatorial structure. Firstly, from the characteristic function and the combinatorial structure, a new characteristic function (usually called the restricted game) is obtained. Then, a solution concept is applied to the restricted game. In our case, the key point is that the restricted game will not be a cooperative game in coalitional form, but a multichoice game. We detail the procedure below.

Firstly, given a game with authorization structure $(v, A)$, we will define a multichoice game $M^A_v$. In $M^A_v$, when a coalition is formed, players in $N$ will be able to take tree different actions. The action $\sigma_0$ consists of doing nothing, that is, neither actively cooperating nor giving permission to other players to actively cooperate within the coalition. The action $\sigma_1$ consists of not actively cooperating but giving permission to any player to cooperate within the coalition. The action $\sigma_2$ consists of both actively cooperating (as long as the player is authorized to do so) and giving permission to any other players to cooperate within the coalition. Therefore, the action space is $\{0,1,2\}^N$, and it will be denoted by $3^N$. The payment that will be obtained by the multichoice coalition $\mathbf{x} \in 3^N$ will be equal to the payment attainable in the original game $v$ by the players who are willing to actively cooperate (i.e., those in $N_2(\mathbf{x})$) and, at the same time, are allowed to cooperate (i.e., those in $A(N_1(\mathbf{x}) \cup N_2(\mathbf{x}))$). This leads to the following definition of the multichoice game $M^A_v$:

$$M^A_v(\mathbf{x}) = v(N_2(\mathbf{x}) \cap A(N_1(\mathbf{x}) \cup N_2(\mathbf{x}))) \tag{8}$$

for every $\mathbf{x} \in 3^N$.

Now, we will apply the multichoice value $\Phi^w$ to the game $M^A_v$. To this end, we will use the weights $w(0) = 0, w(1) = r \in [0,1]$ and $w(2) = 1$. Later on we will show that $r$ can be interpreted as the ratio between the payoff derived from veto power and that derived from active cooperation. Since it is assumed that the grand coalition will eventually form, we are interested in the payoffs that the players will receive when they decide to actively cooperate. In this way we obtain the payoff vector $\Phi^w_{2,i}(M^A_v)$ given by

$$\Phi^w_{2,i}(M^A_v) = \sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_i = 1}} rP^r_{\mathbf{x}}(i)(M^A_v(\mathbf{x}) - M^A_v(\mathbf{x} - \mathbf{e}^i)) + \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 2}} P^r_{\mathbf{x}}(i)(M^A_v(\mathbf{x}) - M^A_v(\mathbf{x} - \mathbf{e}^i). \tag{9}$$

with

$$P^r_{\mathbf{x}}(i) = \sum_{F \subseteq (N_0(\mathbf{x}) \cup N_1(\mathbf{x})) \setminus \{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|}. \tag{10}$$

### 3.3 A family of values for games with authorization structure

**Definition 1** Let $r \in [0,1]$. The $r$-authorization value is the value $\xi^r$ for games with authorization structure defined as

$$\xi^r_i(v, A) = \Phi^w_{2,i}\left(M^A_v\right)$$

for every $(v, A) \in \mathcal{G} \times \mathcal{A}$ and every $i \in N$. The family of proportional authorization values is the set of the $r$-authorization values with $r \in [0,1]$.

Our first goal will be to determine which values are obtained for $r = 0$ and $r = 1$. The following two propositions answer this.

**Proposition 1** *The* 1*-authorization value is equal to the authorization value*, *that is*, $\xi^1 = \phi$.

***Proof*** Suppose $r = 1$. Let $v \in \mathcal{G}$, $A \in \mathcal{A}$ and $i \in N$. Each vector $\mathbf{y} \in 3^N$ with $y_i = 2$ is identified to another one $\mathbf{x}$ with $x_i = 1$ by $\mathbf{y} = \mathbf{x} + \mathbf{e}^i$, moreover $\|y\|_w = \|x\|_w$. It is easy to check that

$$P^1_{\mathbf{y}}(i) = P^1_{\mathbf{x}}(i) = \sum_{F \subseteq (N_0(\mathbf{x}) \cup N_1(\mathbf{x})) \setminus \{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_0(\mathbf{x})|}$$

and then

$$\xi_i^1(v,A) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 1}} \sum_{F \subseteq (N_0(\mathbf{x}) \cup N_1(\mathbf{x})) \setminus \{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_0(\mathbf{x})|} (M_v^A(\mathbf{x}+\mathbf{e}^i) - M_v^A(\mathbf{x}-\mathbf{e}^i)).$$

Given $\mathbf{x} \in 3^N$ with $x_i = 1$, if we write each $F \subseteq (N_0(\mathbf{x}) \cup N_1(\mathbf{x}))\setminus\{i\}$ as $F = H \cup R$ with $H \subseteq N_0(\mathbf{x})\setminus\{i\}$ and $R \subseteq N_1(\mathbf{x}) \setminus \{i\}$, we obtain

$$\sum_{F \subseteq (N_0(\mathbf{x}) \cup N_1(\mathbf{x})) \setminus \{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_0(\mathbf{x})|} = \sum_{H \subseteq N_0(\mathbf{x}) \setminus \{i\}} \frac{(-1)^{|H|}}{\|\mathbf{x}\|_w + |H|} \sum_{R \subseteq N_1(\mathbf{x}) \setminus \{i\}} (-1)^{|R|}.$$

Notice that if $N_1(\mathbf{x}) \setminus \{i\} \neq \emptyset$ then $\sum_{R \subseteq N_1(\mathbf{x}) \setminus \{i\}} (-1)^{|R|} = 0$. Therefore,

$$\begin{aligned} \xi_i^1(v,A) &= \sum_{\substack{\mathbf{x} \in 3^N \\ N_1(\mathbf{x}) = \{i\}}} \sum_{F \subseteq N_0(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F|} (M_v^A(\mathbf{x}+\mathbf{e}^i) - M_v^A(\mathbf{x}-\mathbf{e}^i)) \\ &= \sum_{\{E \subseteq N : i \in E\}} \sum_{F \subseteq N \setminus E} \frac{(-1)^{|F|}}{|E| + |F|} (v(A(E)) - v(A(E \setminus \{i\}))) \qquad (11) \\ &= \sum_{\{E \subseteq N : i \in E\}} \sum_{F \subseteq N \setminus E} \frac{(-1)^{|F|}}{|E| + |F|} (v^A(E) - v^A(E \setminus \{i\})). \end{aligned}$$

We calculate[1]

$$\sum_{F \subseteq N \setminus E} \frac{(-1)^{|F|}}{|E| + |F|} = \sum_{r=0}^{n-|E|} \binom{n-|E|}{r} \frac{(-1)^r}{|E| + r} = \frac{(|E|-1)!(n-|E|)!}{n!}. \qquad (12)$$

Substituting in (11) we obtain $\xi_i^1(v,A) = \phi_i(v,A)$. □

**Proposition 2** *If $v \in \mathcal{G}$ and $A \in \mathcal{A}$, then*

$$\xi^0(v,A) = Sh(v|_{A(N)})$$

*where $v|_{A(N)} \in \mathcal{G}$ is defined as $v|_{A(N)}(E) = v(E \cap A(N))$ for every $E \subseteq N$. In particular, if $A(N) = N$, then*

$$\xi^0(v,A) = Sh(v).$$

***Proof*** Suppose $r = 0$. Let $v \in \mathcal{G}$, $A \in \mathcal{A}$ and $i \in N$. We have

[1] The equality follows from the expression of the beta function, for all $a, b \in \mathbb{N}$,

$$\frac{(a-1)!(b-1)!}{(a+b-1)!} = \beta(a,b) = \sum_{r=0}^{b-1} \frac{(-1)^r}{a+r} \binom{b-1}{r}.$$

$$\xi_i^0(v,A) = \sum_{\substack{\mathbf{x}\in 3^N\\ x_i=2}} \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_1(\mathbf{x})|}(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x}-\mathbf{e}^i)),$$

because

$$P_{\mathbf{x}}^0(i) = \sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_1(\mathbf{x})|}.$$

Following a reasoning similar to that used in Proposition 1, we can obtain that if $\mathbf{x}\in 3^N$, $x_i = 2$ and $N_0(\mathbf{x}) \neq \emptyset$, then

$$\sum_{F\subseteq N_0(\mathbf{x})\cup N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F_1(\mathbf{x})|} = 0.$$

Therefore,

$$\begin{aligned}
\xi_i^0(v,A) &= \sum_{\substack{\mathbf{x}\in 3^N\\ N_0(\mathbf{x})=\emptyset\\ x_i=2}} \sum_{F\subseteq N_1(\mathbf{x})} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + |F|}(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x}-\mathbf{e}^i))\\
&= \sum_{\{E\subseteq N: i\in E\}} \sum_{F\subseteq N\setminus E} \frac{(-1)^{|F|}}{|E|+|F|}(v(E\cap A(N)) - v((E\setminus\{i\})\cap A(N)))\\
&= \sum_{\{E\subseteq N: i\in E\}} \sum_{F\subseteq N\setminus E} \frac{(-1)^{|F|}}{|E|+|F|}(v|_{A(N)}(E) - v|_{A(N)}(E\setminus\{i\}))
\end{aligned}$$

and it suffices to apply (12). □

Next, we will give a result that illustrates the meaning of the parameter $r$ in the sense of the motivating example. We need a previous definition.

**Definition 2** Let $A \in \mathcal{A}$ and $i \in N$. We say that $i$ has no positional power (in $A$) if $A(E)\setminus A(E\setminus\{i\}) \subseteq \{i\}$ for every $E \subseteq N$.

**Proposition 3** *Let $v \in \mathcal{G}$, $A \in \mathcal{A}$ and $i, j \in N$ be such that*

1. *$j$ has no positional power,*
2. *$i$ is a null player,*
3. *$i$ has veto power over $j$,*
4. *no player other than $i$ and $j$ depends partially on $i$.*

Then, $\xi_i^r(v,A) = r\xi_j^r(v,A)$ for every $r \in [0,1]$.

***Proof*** Let $v \in \mathcal{G}$, $A \in \mathcal{A}$ and $i, j \in N$ be such that the conditions stated in the theorem are satisfied. Let $r \in [0,1]$.

Since $j$ has no positional power, it is clear that

$$\xi_j^r(v,A) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_j = 2}} P_{\mathbf{x}}^r(j)(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x} - \mathbf{e}^j))$$

which, taking into account that $i$ has veto power over $j$, is equal to

$$\begin{aligned} \xi_j^r(v,A) = & \sum_{\substack{\mathbf{x} \in 3^N \\ x_j = 2, x_i = 1}} P_{\mathbf{x}}^r(j)(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x} - \mathbf{e}^j)) \\ & + \sum_{\substack{\mathbf{y} \in 3^N \\ y_j = y_i = 2}} P_{\mathbf{y}}^r(j)(M_v^A(\mathbf{y}) - M_v^A(\mathbf{y} - \mathbf{e}^j)) \end{aligned}$$

For each $\mathbf{x} \in \beta^N$ with $x_j = 2$ and $x_i = 1$ we get

$$\begin{aligned} P_{\mathbf{x}}^r(j) = & \sum_{\substack{F \subseteq N_0(\mathbf{x}) \cup N_1(\mathbf{x}) \\ i \notin F}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \\ & + \sum_{\substack{H \subseteq N_0(\mathbf{x}) \cup N_1(\mathbf{x}) \\ i \notin H}} \frac{(-1)^{|H \cup \{i\}|}}{\|\mathbf{x}\|_w + r|H_0(\mathbf{x})| + (1-r)|H_1(\mathbf{x}) \cup \{i\}|} \\ & = P_{\mathbf{x}}^r(i) - \sum_{\substack{H \subseteq N_0(\mathbf{x}) \cup N_1(\mathbf{x}) \\ i \notin H}} \frac{(-1)^{|H|}}{\|\mathbf{x}\|_w + 1 - r + r|H_0(\mathbf{x})| + (1-r)|H_1(\mathbf{x})|} \\ & = P_{\mathbf{x}}^r(i) - \sum_{\substack{H \subseteq N_0(\mathbf{x}) \cup N_1(\mathbf{x}) \\ i \notin H}} \frac{(-1)^{|H|}}{\|\mathbf{x} + \mathbf{e}^i\|_w + r|H_0(\mathbf{x} + \mathbf{e}^i)| + (1-r)|H_1(\mathbf{x} + \mathbf{e}^i)|} \end{aligned}$$

Let $\mathbf{y} \in \beta^N$ with $y_j = y_i = 2$,

$$P_{\mathbf{y}}^r(j) = \sum_{F \subseteq N_0(\mathbf{y}) \cup N_1(\mathbf{y})} \frac{(-1)^{|F|}}{\|\mathbf{y}\|_w + r|F_0(\mathbf{y})| + (1-r)|F_1(\mathbf{y})|}.$$

If we identify $\mathbf{y}$ with $\mathbf{x} = \mathbf{y} - \mathbf{e}^i$ with $x_i = 1$, then $P_{\mathbf{y}}^r(i)$ coincides with the negative term in $P_{\mathbf{x}}^r(i)$ but positive. As $i$ is a null player, we have

$$M_v^A(\mathbf{x}) - M_v^A(\mathbf{x} - \mathbf{e}^j) = M_v^A(\mathbf{x} + \mathbf{e}^i) - M_v^A(\mathbf{x} + \mathbf{e}^i - \mathbf{e}^j) = M_v^A(\mathbf{y}) - M_v^A(\mathbf{y} - \mathbf{e}^j),$$

so we reduce $\xi_j^r(v,A)$ to

$$\xi_j^r(v,A) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_j = 2, x_i = 1}} P_{\mathbf{x}}^r(i)(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x} - \mathbf{e}^j)). \tag{13}$$

Besides, since $i$ is a null player and no other player than $i$ and $j$ depends partially on $i$, it is clear that

$$\xi_i^r(v,A) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_j = 2, x_i = 1}} rP_{\mathbf{x}}^r(i)(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x} - \mathbf{e}^i)) \tag{14}$$

Let $\mathbf{x} \in 3^N$ be such that $x_j = 2, x_i = 1$. Notice that

$$\begin{aligned} M_v^A(\mathbf{x} - \mathbf{e}^i) &= v(N_2(\mathbf{x} - \mathbf{e}^i) \cap A(N_1(\mathbf{x} - \mathbf{e}^i) \cup N_2(\mathbf{x} - \mathbf{e}^i))) \\ &= v(N_2(\mathbf{x}) \cap A((N_1(\mathbf{x}) \cup N_2(\mathbf{x})) \setminus \{i\})) \end{aligned}$$

which, taking into account that $i$ has veto power over $j$, is equal to

$$v((N_2(\mathbf{x}) \setminus \{j\}) \cap A((N_1(\mathbf{x}) \cup N_2(\mathbf{x})) \setminus \{i\}))$$

which, since no other player than $i$ and $j$ depends partially on $i$, is equal to

$$\begin{aligned} &v((N_2(\mathbf{x}) \setminus \{j\}) \cap A(N_1(\mathbf{x}) \cup N_2(\mathbf{x}))) \\ &\quad = v(N_2(\mathbf{x} - \mathbf{e}^j) \cap A(N_1(\mathbf{x} - \mathbf{e}^j) \cup N_2(\mathbf{x} - \mathbf{e}^j))) \\ &\quad = M_v^A(\mathbf{x} - \mathbf{e}^j). \end{aligned}$$

We have proved that if $\mathbf{x} \in 3^N$, $x_j = 2$ and $x_i = 1$, then

$$M_v^A(\mathbf{x} - \mathbf{e}^i) = M_v^A(\mathbf{x} - \mathbf{e}^j). \tag{15}$$

From (13), (14) and (15), it easily follows that $\xi_i^r(v,A) = r\xi_j^r(v,A)$. □

The previous proposition shows that $r$ can be interpreted as the ratio between the payoff derived from veto power and that derived from active cooperation.

### 3.4 Application to the motivating example

Let us go back to the motivating example at the beginning of this section. Let us calculate the game $M_v^A$. Table 4 indicates the value of $M_v^A(\mathbf{x})$ for the vectors $\mathbf{x} \in 3^N$ such that $M_v^A(\mathbf{x}) \neq 0$.

In order to find a realistic payoff vector, we will follow the 25% rule (Goldscheider 2018), which establishes that 25% of the profit from the sale of an infringing good is a reasonable royalty rate. Consequently, we will take $r = \frac{1}{3}$. We obtain that

Table 4 Sets $M_v^A(\mathbf{x})$ of the motivating example

| x | $M_v^A(x)$ |
|---|---|
| (0, 2, 1) | 1 |
| (0, 2, 2) | 1 |
| (1, 2, 1) | 1 |
| (1, 2, 2) | 1 |
| (2, 0, 0) | 1 |
| (2, 0, 1) | 1 |
| (2, 0, 2) | 1 |
| (2, 1, 0) | 1 |
| (2, 1, 1) | 1 |
| (2, 1, 2) | 1 |
| (2, 2, 0) | 1 |
| (2, 2, 1) | 3 |
| (2, 2, 2) | 3 |

$$\xi^{\frac{1}{3}}(v, A) = \left(\frac{40}{28}, \frac{33}{28}, \frac{11}{28}\right).$$

It is worth noting that our payoff vector is not a convex combination of the vectors $\xi^0(v, A) = Sh(v) = \left(\frac{3}{2}, \frac{3}{2}, 0\right)$ and $\xi^1(v, A) = \phi(v, A) = \left(\frac{8}{6}, \frac{5}{6}, \frac{5}{6}\right)$.

## 4 Characterization of the proportional authorization values

Proposition 1 (Subsection 3.3) implies that the $r$-authorization value with $r = 1$ coincides with the authorization value. Gallardo et al. (2018) introduced the following axioms for a value $\psi$ over games with authorization structure.

(A1) EFFICIENCY. If $(v, A) \in \mathcal{G} \times \mathcal{A}$, then $\sum_{i \in N} \psi_i(v, A) = v(A(N))$.

(A2) ADDITIVITY. If $v_1, v_2 \in \mathcal{G}$ and $A \in \mathcal{A}$ then

$$\psi(v_1 + v_2, A) = \psi(v_1, A) + \psi(v_2, A).$$

A player $i \in N$ is *irrelevant* in $(v, A)$ if $i$ is null in $v$ and all the players that depend partially on $i$ are also null players in $v$, that is if $v(E) = v(E \setminus \{j\})$ for every $E \subseteq N$ and $j \in \{i\} \cup \bigcup_{F \subseteq N}(A(F) \backslash A(F \backslash \{i\}))$. Irrelevant players neither have participation power nor hierarchical power in the game with authorization structure.

(A3) IRRELEVANT PLAYER. If $i \in N$ is irrelevant in $(v, A)$ then $\psi_i(v, A) = 0$.

(A4) NECESSARY PLAYER. If $v \in \mathcal{G}$ is monotonic and $i \in N$ is a necessary player in $v$ then $\psi_i(v, A) \geq \psi_j(v, A)$ for all $j \in N$.

Given $(v, A) \in \mathcal{G} \times \mathcal{A}$ and $j \in T \subseteq N$. We define a new authorization structure $A^{T,j} \in \mathcal{A}$ as

$$A^{T,j}(E) = \begin{cases} A(E) \cup \{j\} & \text{if } T \subseteq E, \\ A(E) & \text{otherwise.} \end{cases} \tag{16}$$

for all $E \subseteq N$.

(A5) FAIRNESS. If $(v, A) \in \mathcal{G} \times \mathcal{A}$ and $j \in T \subseteq N$, then for all $i \in T$

$$\psi_i(v, A^{T,j}) - \psi_i(v, A) = \psi_j(v, A^{T,j}) - \psi_j(v, A).$$

In Gallardo et al. (2018) it was proved that the authorization value $\phi$ is the only value for games with authorization structure satisfying axioms A1–A5. In order to characterize the $r$-authorization values we will consider the following properties instead $A4$ and $A5$.

(A4*) EQUAL TREATMENT FOR NECESSARY PLAYERS. If $i, j \in N$ are necessary players in $v \in \mathcal{G}$, then $\psi_i(v, A) = \psi_j(v, A)$ for every $A \in \mathcal{A}$.

Notice that A4 implies A4*.

(A5*) PROPORTIONAL FAIRNESS. There exists a constant $K \in [0, 1]$ such that if $(v, A) \in \mathcal{G} \times \mathcal{A}$, $T \subseteq N$, $i, j \in T$ and $i$ is a null player in $v$, then

$$\psi_i\left(v, A^{T,j}\right) - \psi_i(v, A) = K\left[\psi_j\left(v, A^{T,j}\right) - \psi_j(v, A)\right].$$

Notice that if $j \in A(T)$ then $A^{T,j} = A$. Therefore, the expression above is non trivial only if $j \notin A(T)$. Let us interpret the proportional fairness property. Suppose that we have $v \in \mathcal{G}$, $A \in \mathcal{A}$, $T \subseteq N$, $i \in T$ a null player in $v$ and $j \in T$ such that in the event that coalition $T$ were formed, $j$ would not be allowed to cooperate, that is, $j \notin A(T)$. Imagine now that somehow coalition $T$ gains the power to authorize $j$ to cooperate. Notice that the extra profit that $i$ would get would come exclusively from giving $j$ permission to cooperate within $T$, whereas the extra profit obtained by $j$ would come exclusively from actively cooperating within $T$. The proportional fairness property establishes a fair fixed ratio between both quantities. This ratio must be determined in advance according to what is deemed fair in the type of cooperative situations considered.

Next we will prove that the only values that satisfy A1–A3, A4* and A5* are the proportional authorization values. We need to recall some properties of the multi-choice value $\Phi^w$ (9). Hsiao and Raghavan (1993) proved that $\Phi^w$ satisfies the following properties: (H1) if $\mathbf{y} \in \beta^N$ and $V \in \mathcal{MG}^{\beta^N}$ satisfies $V(\mathbf{x}) = V(\mathbf{x} \wedge \mathbf{y})$ for every $\mathbf{x} \in \beta^N$, then $\sum_{i \in N} \Phi^w_{y_i,i}(V) = V(m, \ldots, m)$, (H2) $\Phi^w(V_1 + V_2) = \Phi^w(V_1) + \Phi^w(V_2)$ for every $V_1, V_2 \in \mathcal{MG}^{\beta^N}$, (H3) if $\mathbf{y} \in \beta^N$ and $V \in \mathcal{MG}^{\beta^N}$ satisfies $V(\mathbf{x}) = 0$ for every $\mathbf{x} \ngeqslant \mathbf{y}$, then $\Phi^w_{k,i}(V) = 0$ for every $i \in N$ and $k < y_i$, and (H4) if $\mathbf{y} \in \beta^N$, $c > 0$ and $V \in \mathcal{MG}^{\beta^N}$ is defined by $V(\mathbf{x}) = c$ if $\mathbf{x} \geqslant \mathbf{y}$ and $V(\mathbf{x}) = 0$ otherwise, then $(\Phi^w_{y_i,i}(V))_{i \in N}$ is proportional to $(w(y_i))_{i \in N}$.

**Theorem 4** *The proportional authorization values are the only values for games with authorization structure which satisfy the properties of efficiency* (A1), *additivity* (A2), *irrelevant player property* (A3), *equal treatment for necessary players* (A4*) *and proportional fairness* (A5*). *Moreover for each* $r \in [0, 1]$, *the r-authorization* *value is the only proportional authorization value that satisfies A5* with constant* $K = r$.

***Proof*** Firstly we will prove that, for each $r \in [0, 1]$, the proportional authorization value $\xi^r$ satisfies the five properties mentioned in the theorem.

*(A1)* Let $v \in \mathcal{G}$ and $A \in \mathcal{A}$. It is clear that $\mathbf{x} \wedge (2, \dots, 2) = \mathbf{x}$ for every $\mathbf{x} \in 3^N$. By property (H1) of the characterization of $\Phi^w$, and the definition of $M_v^A$, we obtain

$$\sum_{i \in N} \xi_i^r(v, A) = \sum_{i \in N} \Phi_{2,i}^w\left(M_v^A\right) = M_v^A(2, \dots, 2) = v(A(N)).$$

*(A2)* Let $v, w \in \mathcal{G}$, $A \in \mathcal{A}$ and $i \in N$. It is clear that $M_{v+w}^A = M_v^A + M_w^A$. If we also use the additivity (H2) of $\Phi^w$, we obtain

$$\begin{aligned}\xi_i^r(v + w, A) &= \Phi_{2,i}^w\left(M_{v+w}^A\right) = \Phi_{2,i}^w(M_v^A + M_w^A) \\ &= \Phi_{2,i}^w(M_v^A) + \Phi_{2,i}^w(M_w^A) = \xi_i^r(v, A) + \xi_i^r(w, A).\end{aligned}$$

*(A3)* Let $v \in \mathcal{G}$, $A \in \mathcal{A}$ and $i \in N$ be such that $i$ is an irrelevant player in $(v, A)$. We must prove that $\xi_i^r(v, A) = 0$. Taking into account the definition of $\xi_i^r(v, A)$, it suffices to prove that for if $\mathbf{x} \in 3^N$ and $x_i \in \{1, 2\}$, then

$$M_v^A(\mathbf{x}) = M_v^A(\mathbf{x} - \mathbf{e}^i). \tag{17}$$

Let $\mathbf{x} \in 3^N$. If $x_i = 2$, equality (17) follows from the facts that $i$ is a null player in $v$ and

$$N_2(\mathbf{x} - \mathbf{e}^i) \cap A(N_2(\mathbf{x} - \mathbf{e}^i) \cup N_1(\mathbf{x} - \mathbf{e}^i)) = \left[N_2(\mathbf{x}) \cap A(N_2(\mathbf{x}) \cup N_2(\mathbf{x}))\right] \setminus \{i\}.$$

Suppose now that $x_i = 1$. Observe that

$$M_v^A(\mathbf{x} - \mathbf{e}^i) = v(N_2(\mathbf{x}) \cap A((N_1(\mathbf{x}) \cup N_2(\mathbf{x})) \setminus \{i\})). \tag{18}$$

Notice that the players in $A(N_1(\mathbf{x}) \cup N_2(\mathbf{x})) \setminus A((N_1(\mathbf{x}) \cup N_2(\mathbf{x})) \setminus \{i\})$ are null players in $v$, since they depend partially on $i$ and $i$ is irrelevant in $(v, A)$. Therefore,

$$v(N_2(\mathbf{x}) \cap A(N_1(\mathbf{x}) \cup N_2(\mathbf{x}))) = v(N_2(\mathbf{x}) \cap A((N_1(\mathbf{x}) \cup N_2(\mathbf{x})) \setminus \{i\})),$$

which, along with (18), leads to (17).

*(A4*)* Let $v \in \mathcal{G}$, $A \in \mathcal{A}$ and $i, j \in N$ be such that $i, j$ are necessary players in $v$. Observe that

$$\xi_i^r(v, A) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 2}} P_{\mathbf{x}}^r(i) M_v^A(\mathbf{x}) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 2 \\ x_j = 2}} P_{\mathbf{x}}^r(i) M_v^A(\mathbf{x}).$$

It is clear that we can obtain the same expression for $\xi_j^r(v,A)$.

*(A5*)* Suppose that $v \in \mathcal{G}$, $A \in \mathcal{A}$, $T \subseteq N$, $i,j \in T$ and $i$ is a null player in $v$. We will prove that

$$\xi_i^r\left(v,A^{T,j}\right) - \xi_i^r(v,A) = r(\xi_j^r\left(v,A^{T,j}\right) - \xi_j^r(v,A)). \tag{19}$$

Namely we will see that $\xi^r$ satisfies A5* with constant $K = r$. On the one hand, we have that

$$\begin{aligned}\xi_i^r\left(v,A^{T,j}\right) - \xi_i^r(v,A) = &\sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_i = 1}} rP_{\mathbf{x}}^r(i)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^{A^{T,j}}(\mathbf{x}-\mathbf{e}^i)\right)\\ &+ \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 2}} P_{\mathbf{x}}^r(i)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^{A^{T,j}}(\mathbf{x}-\mathbf{e}^i)\right)\\ &- \sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_i = 1}} rP_{\mathbf{x}}^r(i)\left(M_v^{A}(\mathbf{x}) - M_v^{A}(\mathbf{x}-\mathbf{e}^i)\right)\\ &- \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 2}} P_{\mathbf{x}}^r(i)\left(M_v^{A}(\mathbf{x}) - M_v^{A}(\mathbf{x}-\mathbf{e}^i)\right)\end{aligned}$$

which, since $i$ is a null player in $v$, is equal to

$$\begin{aligned}\xi_i^r\left(v,A^{T,j}\right) - \xi_i^r(v,A) = &\sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_i = 1}} rP_{\mathbf{x}}^r(i)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^{A^{T,j}}(\mathbf{x}-\mathbf{e}^i)\right)\\ &- \sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_i = 1}} rP_{\mathbf{x}}^r(i)\left(M_v^{A}(\mathbf{x}) - M_v^{A}(\mathbf{x}-\mathbf{e}^i)\right)\end{aligned}$$

which, taking into account that $A^{T,j}(E) = A(E)$ for every $E \subseteq N \backslash \{i\}$, is equal to

$$\xi_i^r\left(v,A^{T,j}\right) - \xi_i^r(v,A) = \sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_i = 1}} rP_{\mathbf{x}}^r(i)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^{A}(\mathbf{x})\right)$$

which, keeping in mind that $A^{T,j}(E) \backslash A(E) \subseteq \{j\}$ for every $E \subseteq N$, is equal to

$$\xi_i^r\left(v, A^{T,j}\right) - \xi_i^r(v, A) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 1 \\ x_j = 2}} rP_{\mathbf{x}}^r(i)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^A(\mathbf{x})\right). \tag{20}$$

And, on the other hand,

$$\begin{aligned}\xi_j^r\left(v, A^{T,j}\right) - \xi_j^r(v, A) = & \sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_j = 1}} rP_{\mathbf{x}}^r(j)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^{A^{T,j}}(\mathbf{x} - \mathbf{e}^j)\right) \\ & + \sum_{\substack{\mathbf{x} \in 3^N \\ x_j = 2}} P_{\mathbf{x}}^r(j)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^{A^{T,j}}(\mathbf{x} - \mathbf{e}^j)\right) \\ & - \sum_{\substack{\mathbf{x} \in 3^N \setminus \{0,1\}^N \\ x_j = 1}} rP_{\mathbf{x}}^r(j)\left(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x} - \mathbf{e}^j)\right) \\ & - \sum_{\substack{\mathbf{x} \in 3^N \\ x_j = 2}} P_{\mathbf{x}}^r(j)\left(M_v^A(\mathbf{x}) - M_v^A(\mathbf{x} - \mathbf{e}^j)\right)\end{aligned}$$

which, taking into consideration that $A^{T,j}(E) \setminus A(E) \subseteq \{j\}$ for every $E \subseteq N$, is equal to

$$\xi_j^r\left(v, A^{T,j}\right) - \xi_j^r(v, A) = \sum_{\substack{\mathbf{x} \in 3^N \\ x_j = 2}} P_{\mathbf{x}}^r(j)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^A(\mathbf{x})\right)$$

which, since $A^{T,j}(E) = A(E)$ for every $E \subseteq N \setminus \{i\}$, is equal to

$$\begin{aligned}\xi_j^r\left(v, A^{T,j}\right) - \xi_j^r(v, A) = & \sum_{\substack{\mathbf{x} \in 3^N \\ x_i = 1 \\ x_j = 2}} P_{\mathbf{x}}^r(j)\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^A(\mathbf{x})\right) \\ & + \sum_{\substack{\mathbf{y} \in 3^N \\ y_i = 2 \\ y_j = 2}} P_{\mathbf{y}}^r(j)\left(M_v^{A^{T,j}}(\mathbf{y}) - M_v^A(\mathbf{y})\right).\end{aligned}$$

Now, if $\mathbf{x}$ satisfies $x_i = 1$ and $x_j = 2$ we can separate in two terms the coefficient $P_{\mathbf{x}}^r(i)$,

$$\begin{aligned} P_{\mathbf{x}}^r(j) &= \sum_{F\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|F|}}{\|\mathbf{x}\|_w + r|F_0(\mathbf{x})| + (1-r)|F_1(\mathbf{x})|} \\ &\quad + \sum_{H\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|H\cup\{i\}|}}{\|\mathbf{x}\|_w + r|H_0(\mathbf{x})| + (1-r)|H_1(\mathbf{x})\cup\{i\}|} \\ &= P_{\mathbf{x}}^r(i) - \sum_{H\subseteq(N_0(\mathbf{x})\cup N_1(\mathbf{x}))\setminus\{i\}} \frac{(-1)^{|H|}}{\|\mathbf{x}\|_w + 1 - r + r|H_0(\mathbf{x})| + (1-r)|H_1(\mathbf{x})|} \\ &= P_{\mathbf{x}}^r(i) - \sum_{H\subseteq N_0(\mathbf{x}+\mathbf{e}^i)\cup N_1(\mathbf{x}+\mathbf{e}^i)} \frac{(-1)^{|H|}}{\|\mathbf{x}+\mathbf{e}^i\|_w + r|H_0(\mathbf{x}+\mathbf{e}^i)| + (1-r)|H_1(\mathbf{x}+\mathbf{e}^i)|} \end{aligned}$$

We identify again each $\mathbf{y} \in \beta^N$ verifying $y_i = y_j = 2$ with $\mathbf{x} = \mathbf{y} - \mathbf{e}^i$ verifying $x_i = 1$. So,

$$P_{\mathbf{y}}^r(j) = \sum_{H\subseteq N_0(\mathbf{x}+\mathbf{e}^i)\cup N_1(\mathbf{x}+\mathbf{e}^i)} \frac{(-1)^{|H|}}{\|\mathbf{x}+\mathbf{e}^i\|_w + r|H_0(\mathbf{x}+\mathbf{e}^i)| + (1-r)|H_1(\mathbf{x}+\mathbf{e}^i)|}$$

which coincides with the above negative term of $P_{\mathbf{x}}^r(i)$ but positive. Moreover, since $i$ is a null player in $v$ we have

$$M_v^{A^{T,j}}(\mathbf{x}) - M_v^A(\mathbf{x}) = M_v^{A^{T,j}}(\mathbf{y}) - M_v^A(\mathbf{y}).$$

So, we get that

$$\xi_j^r\left(v, A^{T,j}\right) - \xi_j^r(v, A) = \sum_{\substack{\mathbf{x}\in 3^N \\ x_i = 1 \\ x_j = 2}} P_{\mathbf{x}}^r\left(M_v^{A^{T,j}}(\mathbf{x}) - M_v^A(\mathbf{x})\right)$$

which, multiplied by $r$, equals (20). This concludes the proof of (19).

We have already seen that all the proportional authorization values satisfy the five properties mentioned in the theorem. Now we will show that such properties uniquely determine these values. Let $\psi$ be a value for games with authorization structure that satisfies the five axioms. Suppose that $\psi$ satisfies A5* for a constant $K \in [0, 1]$. We will prove that $\psi = \xi^K$.

Our first goal will be to show that $\psi(cu_E, A) = \xi^K(cu_E, A)$ for all $c \in \mathbb{R}$, $E \in 2^N\setminus\{\emptyset\}$ and $A \in \mathcal{A}$. To this end, for every $A \in \mathcal{A}$ we denote

$$m(A) = \sum_{F\subseteq N} |A(F)|$$

and we will prove $\psi(cu_E, A) = \xi^K(cu_E, A)$ by induction on $m(A)$. Let $E$ be a non-empty coalition and $c \in \mathbb{R}$.

BASE CASE. Let $A \in \mathcal{A}$ be such that $m(A) = 0$. It follows that $A(F) = \emptyset$ for all $F \subseteq N$. It is clear that all the players in $N \setminus E$ are irrelevant in $(v, A)$. By the irrelevant player property (A3),

$$\xi_i^K(cu_E, A) = 0 \quad \text{for every } i \in N \setminus E. \tag{21}$$

By the property of equal treatment for necessary players (A4*), there exists $b \in \mathbb{R}$ such that

$$\xi_i^K(cu_E, A) = b \quad \text{for every } i \in E. \tag{22}$$

Moreover, by efficiency,

$$\sum_{i \in N} \xi_i^K(cu_E, A) = v(A(N)) = v(\emptyset) = 0. \tag{23}$$

From (21), (22) and (23) it easily follows that $\xi^K(v, A) = 0$. By the same reasoning, $\psi(v, A) = 0$.

INDUCTIVE STEP. Let $A \in \mathcal{A}$. Let

$$H = \left\{ i \in N : i \text{ is an irrelevant player in } (cu_E, A) \right\}.$$

Since $\xi^K$ and $\psi$ satisfy the irrelevant player property,

$$\xi_i^K(cu_E, A) = \psi_i(cu_E, A) = 0 \quad \text{for all } i \in H. \tag{24}$$

From the necessary player property, there exist $b, b' \in \mathbb{R}$ such that

$$\xi_i^K(cu_E, A) = b \quad \text{and} \quad \psi_i(cu_E, A) = b' \quad \text{for all } i \in E. \tag{25}$$

Now suppose that $i \in N \setminus (H \cup E)$. It is clear that there exists $j \in E$ such that $j$ depends partially on $i$ according to $A$. This means that there exists $F \subseteq N$ such that $j \in A(F) \setminus A(F \setminus \{i\})$. Take $T$ minimal such that $T \subseteq F$ and $j \in A(T)$. It is clear that $i \in T$. Consider $\tilde{A} : 2^N \to 2^N$ defined as

$$\tilde{A}(S) = \begin{cases} A(S) & \text{if } S \neq T, \\ A(T) \setminus \{j\} & \text{if } S = T. \end{cases}$$

It is straightforward to check that $\tilde{A} \in \mathcal{A}$ and $\tilde{A}^{T,j} = A$. By the proportional fairness property,

$$\xi_i^K\left(cu_E, A\right) - \xi_i^K(cu_E, \tilde{A}) = K(\xi_j^K\left(cu_E, A\right) - \xi_j^K(cu_E, \tilde{A})),$$
$$\psi_i\left(cu_E, A\right) - \psi_i(cu_E, \tilde{A}) = K(\psi_j\left(cu_E, A\right) - \psi_j(cu_E, \tilde{A})).$$

Since $j \in E$ we know that $\xi_j^K(cu_E, A) = b$ and $\psi_j(cu_E, A) = b'$. Therefore, we have

$$\xi_i^K\left(cu_E, A\right) = Kb - K\xi_j^K(cu_E, \tilde{A}) + \xi_i^K(cu_E, \tilde{A}),$$
$$\psi_i\left(cu_E, A\right) = Kb' - K\psi_j(cu_E, \tilde{A}) + \psi_i(cu_E, \tilde{A}).$$

Since $m(\tilde{A}) = m(A) - 1$ it follows by induction hypothesis that $\psi(cu_E, \tilde{A}) = \xi^K(cu_E, \tilde{A})$. From this equality and the two equalities above we obtain

$$\xi_i^K\left(cu_E, A\right) - \psi_i\left(cu_E, A\right) = K(b - b').$$

We have proved that

$$\xi_i^K\left(cu_E, A\right) - \psi_i\left(cu_E, A\right) = K(b - b') \quad \text{for all } i \in N \setminus (E \cup H). \tag{26}$$

Now, on the one hand, from (24), (25) and (26), we can obtain

$$\sum_{i \in N} \xi_i^K(cu_E, A) - \sum_{i \in N} \psi_i(cu_E, A) = (b - b')(|E| + K|N \setminus (E \cup H)|), \tag{27}$$

and, on the other hand, as $\xi^K$ and $\psi$ are efficient, we know that

$$\sum_{i \in N} \xi_i^K(cu_E, A) = \sum_{i \in N} \psi_i(cu_E, A). \tag{28}$$

From (27) and (28) it follows that $b = b'$, which leads to $\psi(cu_E, A) = \xi^K(cu_E, A)$.

So we already know that $\psi(cu_E, A) = \xi^K(cu_E, A)$ for all $c \in \mathbb{R}$, $E \in 2^N \setminus \{\emptyset\}$ and $A \in \mathcal{A}$.

Finally, take $v \in \mathcal{G}$ and $A \in \mathcal{A}$. Then,

$$\begin{aligned} \psi(v, A) &= \psi\left(\sum_{\{E \subseteq N : E \neq \emptyset\}} \Delta_v(E) u_E, A\right) = \sum_{\{E \subseteq N : E \neq \emptyset\}} \psi\left(\Delta_v(E) u_E, A\right) \\ &= \sum_{\{E \subseteq N : E \neq \emptyset\}} \xi^K\left(\Delta_v(E) u_E, A\right) = \xi^K\left(\sum_{\{E \subseteq N : E \neq \emptyset\}} \Delta_v(E) u_E, A\right) \\ &= \xi^K(v, A). \end{aligned}$$

□

***Remark 1*** Let us show that the axioms mentioned in the previous theorem are logically independent.

(1) In order to obtain a value that satisfies all the axioms except A1, it suffices to consider $\psi_i^1(v, A) = 0$ for every $(v, A) \in \mathcal{G} \times \mathcal{A}$ and every $i \in N$.
(2) If $v \in \mathcal{G}$ we denote $Y(v) = \{j \in N : j \text{ is not null in } v\}$. Consider

$$\psi_i^2(v, A) = \begin{cases} \dfrac{v(A(N))}{|Y(v)|} & \text{if } i \in Y(v), \\ 0 & \text{if } i \in N \setminus Y(v). \end{cases}$$

It is easy to check that $\psi^2$ satisfies A1, A3, A4* and A5* (for $K=0$). Evidently, $\psi^2$ does not satisfy A2.

(3) We define $\psi_i^3(v,A) = \dfrac{v(A(N))}{|N|}$ for every $(v,A) \in \mathcal{G} \times \mathcal{A}$ and every $i \in N$. It is obvious that $\psi^3$ satisfies A1, A2, A4* and A5* (for $K=1$). Clearly, $\psi^3$ does not satisfy A3.

(4) Kalai and Samet (1987) introduced the weighted Shapley values. A (simple) weighted system on $N$ is a vector $\lambda \in \mathbb{R}^N$ with $\lambda_i > 0$ for all $i \in N$. The weighted Shapley value $Sh^\lambda$ with weighted system $\lambda$ is defined for each $v \in \mathcal{G}$ and $i \in N$ as

$$Sh_i^\lambda(v) = \sum_{\{F \in 2^N : i \in F\}} \frac{\lambda_i \Delta_v(F)}{\sum_{j \in F} \lambda_j}.$$

If $\lambda_i = \lambda_j$ for all $i, j$ then $Sh^\lambda = Sh$. All the weighted Shapley values satisfy S1, S2 and S3 Kalai and Samet (1987). Fix a coalition $F$ with $\emptyset \neq F \subsetneq N$. Consider the value $\psi^4(v,A) = Sh^\lambda(v_{|A(N)})$ where $\lambda_i = 2$ if $i \in F$ and $\lambda_i = 1$ if $i \in N \setminus F$. It is straightforward to prove that $\psi^4$ satisfies A1, A2, A3 and A5* (for $K=0$). Obviously, $\psi^4$ does not satisfy A4*.

(5) We define the value for games with authorization structure $\psi^5 = \frac{1}{2}\xi^0 + \frac{1}{2}\xi^1$. Since $\xi^0$ and $\xi^1$ satisfy A1, A2, A3 and A4*, it is clear that $\psi^5$ also satisfies them. By reduction to absurdity, suppose that $\psi^5$ satisfies A5* for some $K \in [0,1]$. Let $v \in \mathcal{G}$, $A \in \mathcal{A}$, $T \subseteq N$ and $i, j \in T$ be such that $A(N) = N$ and $i$ is a null player in $v$. Notice that $\xi^0(v, A^{T,j}) = \xi^0(v,A) = Sh(v)$. On the one hand, we have that

$$\begin{aligned} \psi_i^5(v,A^{T,j}) - \psi_i^5(v,A) &= \frac{1}{2}\left[\xi_i^0(v,A^{T,j}) + \xi_i^1(v,A^{T,j})\right] \\ &\quad - \frac{1}{2}\left[\xi_i^0(v,A) + \xi_i^1(v,A)\right] \\ &= \frac{1}{2}\left[\xi_i^1(v,A^{T,j}) - \xi_i^1(v,A)\right] \\ &= \frac{1}{2}\left[\xi_j^1(v,A^{T,j}) - \xi_j^1(v,A)\right]. \end{aligned} \tag{29}$$

On the other hand,

$$\begin{aligned} K\left[\psi_j^5(v,A^{T,j}) - \psi_j^5(v,A)\right] &= \frac{K}{2}\left[\xi_j^0(v,A^{T,j}) + \xi_j^1(v,A^{T,j})\right] \\ &\quad - \frac{K}{2}\left[\xi_j^0(v,A) + \xi_j^1(v,A)\right] \\ &= \frac{K}{2}\left[\xi_j^1(v,A^{T,j}) - \xi_j^1(v,A)\right]. \end{aligned} \tag{30}$$

Since (29) and (30) are equal, we can easily conclude that $K=1$, but this implies $\psi^5 = \xi^1$, which leads to $\xi^0 = \xi^1$, which is false. We conclude that $\psi^5$ does not satisfy A5*.

## 5 Conclusions

We have introduced and characterized a family of values $\xi^r$ for games with authorization structure, where the parameter $r \in [0, 1]$ can be interpreted as the ratio between the payoff derived from veto power and that solely derived from active cooperation. To achieve this goal, the use of multichoice games has been crucial, since they have allowed us to weigh the payouts assigned to each potential action by players (doing nothing, granting permission without actively collaborating, and cooperating in all aspects). As future research, we propose applying this methodology to other models of game theory with restricted cooperation in which it may not be feasible to adequately gather all the information of the cooperative situation in a classical TU game.

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