

The arithmetic Knowledge of prospective teachers. Strengths and Weaknesses¹

Conocimiento de aritmética de futuros maestros. Debilidades y fortalezas

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Miguel Ángel Montes
Luis Carlos Contreras

University of Huelva. Faculty of Education Sciences. Department of Didactics and Philosophy. Huelva. España.

María del Mar Liñán

University of Seville. Faculty of Education Sciences. Department of Didactics of Mathematics. Seville. España.

María Cinta Muñoz-Catalán

Nuria Climent

José Carrillo

University of Huelva. Faculty of Education Sciences. Department of Didactics of Sciences and Philosophy. Huelva. España.

Abstract

The media has recently alerted public opinion to a situation which research into mathematical education has been highlighting for the last two decades: the poor quality of primary teachers' mathematical training. This failing of prospective primary teachers (PPTS), which has been noted at an informal level in our universities for some time, is the focus of this study. Against this background, the paper describes an exploratory study using a survey about the mathematical knowledge required for teaching with 737 trainee primary teachers at three

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Andalusian university training centres, carried out under the auspices of a Teaching Innovation Project at one of them. Using the framework of the Mathematics Teachers' Specialised Knowledge (MTSK), and specifically, the sub-domain concerning knowledge of mathematics topics, a questionnaire was developed which contained items relating to fractions, decimals and percentages, chosen as much for their inherent importance as for their application to other mathematical contents and other disciplines within the scope of primary education. By this means we were able to explore the prospective teachers' knowledge of these contents. The results highlighted a significant number of weaknesses, some already described in the literature, and some strengths. In both cases, the findings represent baseline data which can be compared with the situation in other primary training centres. We hope they also provide food for thought for the educational authorities in regard to university entrance selection procedures. More specifically, the study should be a starting point for training centres to redesign their programmes.

Keywords: MTSK, Prospective Primary Teachers, Initial Training, Professional Knowledge, Arithmetic

Resumen

Los medios de comunicación han alertado recientemente a la opinión pública acerca de una realidad que la investigación en educación matemática llevaba evidenciando las dos últimas décadas: la deficiente formación matemática de los maestros. Estas deficiencias, que habíamos venido constatando de manera informal en nuestras universidades con los estudiantes para Maestro (EPM), son el objeto de estudio de nuestra investigación. En ese contexto, este artículo describe un estudio exploratorio tipo *survey* sobre el conocimiento matemático necesario para la enseñanza que tienen 737 estudiantes para Maestro de tres centros de formación de maestros de universidades andaluzas, realizado en el contexto de un Proyecto de Innovación Docente de una de ellas. Usando el marco de conocimiento especializado del profesor de Matemáticas (MTSK) y, dentro de él, el subdominio relativo al conocimiento de los temas matemáticos, se elaboró un cuestionario que contenía ítems relativos a fracciones, decimales y porcentajes, contenidos que fueron elegidos tanto por su trascendencia intrínseca como por su aplicación a otros contenidos matemáticos y a otras disciplinas en el ámbito de la Educación Primaria. Esto nos ha permitido explorar el conocimiento que estos futuros profesores poseen sobre esos contenidos. Los resultados han mostrado un número importante de debilidades, algunas de las cuales ya habían sido descritas por la literatura de investigación, y también algunas fortalezas. En ambos casos, la información obtenida supone un referente para explorar la realidad de otros centros de formación de maestros; un elemento de reflexión para las autoridades académicas acerca de los procesos de selección para el acceso a la universidad y,

más concretamente, a estos centros de formación; y un punto de partida para el rediseño de sus programas de formación.

Palabras clave: MTSK, estudiantes para Maestro, formación inicial, conocimiento profesional, aritmética.

Introduction

Out of all the primary level topics relating to arithmetic, the most important, in terms of both the actual content and its application to other mathematical content and other disciplines, is that coming under the umbrella of fractions, decimals and percentages. In consequence, primary teachers' mathematical knowledge requires special attention regarding the structure, contents and associations relating to this area. It is the area in which algorithms and procedures are frequently covered, and where pupils often display significant gaps in their knowledge at the end of their period of compulsory education (PISA; TIMSS).

There are no specific entry requirements for Primary Education courses in Spain. Nor is the area amongst the most popular, which means that for many students, the degree was not their first choice. Added to this, it must be recognised that a significant number of students have had little contact with mathematics for many years, which may be the reason why their grasp of elementary mathematics has proved to be lacking (Contreras, Carrillo, Zakaryan, Muñoz-Catalán and Climent, 2012).

Evidence of this can be found in both national and international studies. In the area of fractions, decimals and percentages, specifically, we can cite the work of Putt (1995) with regard to the representation and ordering of numbers as decimals; Post, Harel, Behr and Lesh (1991) with respect to decimal and fractional representations; Castro, Castro and Segovia (2004), and Zazkis and Campbell (1994) regarding decimals less than one, calculation rules for decimal numbers, and adjusting positional value in estimates; and Muñoz-Catalán and Carrillo (2007) in terms of the use of formal methods of simplifying fractions, and the concept of fraction itself.

It should be added that, in our model of teachers' knowledge, this area of mathematical knowledge plays a very significant role, for which reason

it is important to have a very clear idea of each student's starting point before undertaking any training.

In this respect, we have been working on a Project for Teaching Innovation at the University of Huelva entitled "Prospective Primary Teachers' Knowledge for Teaching Mathematics: Analysis of Difficulties" (PIE 1101), with the aim of identifying the strengths and weaknesses of prospective primary teachers (PPTS) at the start of their studies, and of gaining insights into the possible causes, such that the training they are offered can fill the gaps at the same time as developing the specialised mathematical knowledge they are to need.

In the first stage of the project, we developed a process for the identification and analysis of the students' incomplete and erroneous primary level mathematical knowledge. That is to say, we started from the mathematical knowledge held in common by the PPTS. However, the subsequent analysis of this knowledge brings with it two further considerations: awareness of the points of difficulty and frequently occurring errors associated with specific topics, and familiarity with learning strategies that can be applied to these, both of interest to primary teacher training.

In this paper, we focus on the task of identifying the PPTS' strengths and weaknesses with respect to fractions, decimals and percentages.

Theoretical Background

Teachers' professional knowledge has been the object of much debate and theorisation, and has given rise to multiple programmes aimed at its improvement. In Shulman's (1986) seminal work, this knowledge is divided into two large macro-components: knowledge about the subject to be taught and pedagogical knowledge associated with this content. These macro-components are then further sub-divided into seven sub-domains, among which the teacher's knowledge can be classified. This initial distinction is of special interest when it comes to developing initial and in-service training programmes in that, whilst recognising the integrated nature of teachers' knowledge, it suggests that it is possible to achieve a detailed characterisation of each of the two components. Building

on Shulman's work, various such characterisations of teachers' knowledge have been developed (Fennema and Franke, 1992; Davis and Simmt, 2006; Rowland, Turner, Thwaites and Huckstep, 2009), and in keeping with Shulman's decision to separate out the two macro-components above, the research group at the University of Michigan developed the model, "Mathematical Knowledge for Teaching" (MKT: Ball, Thames and Phelps, 2008), not only as an instrument of analysis, but also as the foundation on which to build teacher-training courses. The model defines three sub-domains for each macro-component defined by Shulman. Of particular interest for this paper are those associated with knowledge of the subject, that is, Common Content Knowledge (CCK), Specialised Content Knowledge (SCK), and Horizon Content Knowledge (HCK). CCK is defined as the kind of knowledge of the subject that anyone using mathematics might require in their profession, including those working outside the field of teaching. It is thus the knowledge associated with the theory and practice of mathematics, and its potential application to other fields; in short, the kind of mathematics to be found in textbooks. SCK is one of the chief contributions of the Michigan group, as it recognises the difference in the mathematical knowledge required by a mathematics teacher, as opposed to, say, a physicist or mathematics researcher. In the paper in which the model was presented, the specialist nature of this knowledge is given shape in terms of the actions it enables the teacher to perform: "*responding to students why questions*", [...] *and giving or evaluating mathematical explanations*" (Ball et al., 2008, p. 400), amongst others. Finally, HCK, defined as "an orientation to, and a familiarity with, mathematical knowledge" (Jakobsen, Thames and Ribeiro, 2013), is the knowledge enabling the teacher to understand how mathematics works (from which the 'familiarity with'), and to take a prospective view of the mathematical topics to be dealt with in class (from which the 'orientation to'). Nevertheless, this model has encountered problems in the definition between the different sub-domains (Silverman and Thompson, 2008), which led the research group at the University of Huelva to develop a refinement. This new model, denominated "Mathematics Teachers' Specialised Knowledge" (Carrillo, Climent, Contreras, Muñoz-Catalán, 2013), starts from the premise that the mathematics teacher's knowledge is specialised by virtue of being part of the knowledge required to give lessons, irrespective of whether other professions might draw on it. Nevertheless, in the design of the model, only the knowledge specific to

mathematics teachers is considered, all other general knowledge of use to teaching, but alien to mathematics, being excluded. To this extent, despite its synthetic nature, the mathematics teacher's knowledge is considered amenable to an atomistic analysis. In the course of conducting their teaching, the teacher understands, puts into action and reflects on different elements. These elements comprise, in terms of mathematical knowledge, the concepts, procedures and structures inherent in mathematical thinking, and in terms of pedagogical content knowledge, systems for organising content for teaching, the pupils' ways of approaching content, and curricular constraints. These six elements of teachers' knowledge and reflection give rise to six different sub-domains.

We will first describe those fitting the characterisation of the macro-component described by Shulman relating to subject knowledge. The sub-domain *Knowledge of Topics* (KOT) takes mathematical content as its object, and contains the knowledge of different dimensions (which we organise into categories) associated with each topic, including properties and their theoretical underpinnings, the procedures which can be undertaken with the topic, the phenomenology (Freudenthal, 1983) and applications of the content to real or mathematical situations (such as different examples in which the topic becomes evident), the different meanings and definitions of the concept, and the representations of the content. These categories will enable us to design the structure of the data-collection instrument and subsequent analysis.

We recognise that this knowledge can be shared with other professions, but that certain dimensions, such as the meanings of a topic, are particularly relevant to the work of the mathematics teacher, making it specialised, to the extent that they represent a tool for exercising the profession of teacher. *Knowledge of the Structure of Mathematics* concerns understanding the mathematical context of an item to be taught. It is the kind of structural knowledge which enables the teacher to consider content prospectively, in the sense that he or she approaches "basic mathematics from an advanced perspective," or conversely, approaches "advanced mathematics from a basic perspective". It is these considerations, together with the idea of Felix Klein of understanding mathematics from a higher perspective, providing the teacher with a synthesised view of the structure of mathematics, which constitute this sub-domain. Finally, *Knowledge of the Practice of Mathematics* (KPM) consists of knowledge about how mathematics is constructed. It involves elements of a general nature, such

as knowing different kinds of demonstrations or forms of reasoning, like syntax, and notions of classification and generalisation, and elements associated with a specific topic, such as the logic underpinning the idea of equivalence class, fundamental to considering equivalent fractions. These three sub-domains together comprise, in this new perspective, the content which in Shulman's terms would be subject matter knowledge, and which in this case, given the mathematical contextualisation of the teacher to be considered, we denominate mathematical knowledge.

In the domain of pedagogical content knowledge (Shulman's second macro-component), if we focus attention on the means of organising mathematical content for teaching, the sub-domain *Knowledge of Mathematics Teaching* (KMT) emerges. This consists of different strategies, materials, resources and aids that enable mathematical content to be organised in the way the teacher would like. Included within this sub-domain are theories of teaching, such as the features of problem-solving as a methodological gambit. In the same way, if we focus on the teacher's knowledge of the pupil as learner, we arrive at the sub-domain *Knowledge of the Features of Learning* (KFLM), which includes the teacher's knowledge of the pupils' difficulties, errors and obstacles, conceptions and initial ideas about concepts, the language and vocabulary typically employed by pupils to talk about certain concepts, and knowledge of the steps along the way to learning a particular item. Also included in this sub-domain are theories of learning, such as APOS. Thirdly, based on curricular questions, there is the sub-domain *Knowledge of Mathematics Learning Standards* (KMLS), which contains guidelines that the teacher might take into account when sequencing content, such as research literature, documents supplied by professional associations (e.g. NCTM), and of course the national curriculum of the country in question.

Method

Given our interest in identifying the strengths and weaknesses of the PPTS' knowledge of fractions, decimals and percentages, we settled on using a *Survey* methodology (Colás and Buendía, 1998), designing a multiple-choice questionnaire. The analysis of the data thus obtained first involved

a frequency count of five categories of knowledge about fractions, decimals and percentages²: phenomena and applications; meanings and definitions (including images of the concepts); properties and their fundamental principles; representations; procedures.

As primary teacher trainers in three university centres –the University of Huelva, the Cardenal Spínola CES Seville CEU (University Centre for Teacher Training), and the University of Seville– gave us access to 737 primary trainees (or around 1,500 in total). None had received tuition in Mathematics Education relating to arithmetic in the course of their degree, such that their previous contact with the topics took place during the compulsory education phase, or in some cases from 16 to 18 years of age.

The questionnaire as data collection instrument

A pilot questionnaire was put to the test with a group of 52 students in their second year of a degree in Primary Education at the Cardenal Spínola CES Seville CEU. This gave us the opportunity to refine the data collection instrument. The questions selected for inclusion were drawn chiefly from the work of Dickson, Brown and Gibson (1991), Hill, Schilling and Ball (2004), Ball (1990 a, 1990 b), Hernández, Noda, Palarea and Socas, (2003), Contreras et al. (2012), but also included items written by the participating teachers themselves, based on frequently occurring errors they had noted in previous years.

Bearing in mind the results of the pilot, especially in terms of estimated and real turn-around times, the definitive questionnaire was given the form of 17 multiple-choice questions of four options each and only one correct answer. In addition, amongst the three incorrect responses, there was an *expected answer* drawn from the above literature on the topic.

Questions 4 and 8 dealt with discounts as applications of percentages. Questions 1 and 13 considered the meanings of fractions (as part of a whole), understood as phenomenology. Likewise, questions 1, 11, 13, 15 and 16 demanded knowledge of the definitions of fraction (and the role of the unit), improper fractions, rational numbers, decimals, and meaning of positional value. In terms of properties and their fundamental features, questions 3, 9, 10, 16 and 17 focused on the density of rational numbers, the hierarchy of operations, and the nature of ordering in rational numbers (with an emphasis on negative numbers). Regarding the

⁽²⁾ We assume that different organisational criteria are possible. We apply these due to their adjustment to our theoretical framework.

category of representations and their interpretation, all the questions required an understanding of the expression of different registers for rational numbers and irrational decimals (including percentages), with questions 1, 12, 13, 15 and 16 particularly focussed on this aspect. Finally, with respect to the category of procedures, questions 4, 5, 6, 9, 10, 11, 12, 14 and 17 involved the ordering of decimals and fractions, and operations on fractions and decimals (including basic operations and changes of register).

Key amongst the instructions that the students were given for completing the questionnaire was the injunction against using a calculator.

Mathematics teachers' specialised knowledge of fractions, decimal numbers and percentages as support to the questionnaire

A mathematics teacher's specialised knowledge, especially of the content in hand, matters not only in terms of its mathematical value, but also in that it provides instruction with tools for organising, making sense of, and communicating this content. Ma (1999) claims that lack of content knowledge (in this case, the division of fractions) causes problems when it comes to creating useful representations in teaching, and that "*even pedagogical knowledge may not compensate for their ignorance of the concept*" (p. 89). We will thus make a connection between the required knowledge and the corresponding sub-domains within the MTSK model.

With respect to Knowledge of Topics (KOT), we were interested to find out the meaning which the PPTS gave to fractions (for example, part-whole), particularly to improper fractions, as part of their phenomenology. Equally, we were interested in their knowledge about definitions, representations and the ordering of, and operations using, fractions, decimals and percentages (associated with the category "procedures"). Taking these three latter items as alternative ways of representing rational numbers, we were keen to discover the extent to which the students were able to move from one to another while retaining the numerical value, and to learn the meaning they attached to this value (Llinares and Sánchez, 1988), ordering rational numbers, and taking into account throughout the property of density of rational numbers. Connected to the foregoing, is the concept

of rational number, which we associate with knowledge of its definition. In the same way, the location of rational numbers along the real number line is also relevant, with the additional difficulty of identifying and locating the decimal numbers, and the problem of the negative decimal numbers. Another point of interest was that of percentages. In phenomenological applications (such as calculating discounts), we were interested in students ability not only to calculate the direct percentage of a number, but also to reverse the procedure of “making a percentage discount”, and the means of chaining together two percentages as just one, a procedure very closely related to the meaning of the product of fractions, and to which we would add other procedures such as the addition of fractions, the obtaining of equivalent fractions, and the hierarchy of operations. One element which we believe should form a fundamental part of primary education students’ KOT is the Decimal Number System (DNS), which can be explored through questions about its properties and fundamental features, tackling the issue of place value and the meaning of each one of the elements.

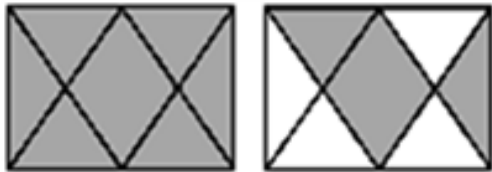
Additionally, we were interested in the knowledge relating to the relation between fractions and the area of a shape through the identification of fractions in models representing area as part of the whole, as well as Knowledge of the Structure of Mathematics (KSM), in working with generic quantities, expressed in the form of an unknown or variable item. We were also interested in the students’ Knowledge of the Practice of Mathematics (KPM) linked to inducing the need to generalise a procedure. Nevertheless, although we believe that these aspects of a PPT’s potential knowledge are fundamental to the putting into effect of their work, they did not originally receive special attention during the development of the questionnaire. It was during the *post hoc* analysis that we realised that this knowledge might be brought into play in answering the questions, given the way they were formulated. For this reason, the questionnaire which we explain in detail below is chiefly focussed on exploring the KOT associated with fractions, decimals and percentages (relative to the elements highlighted in the foregoing paragraphs), aiming to identify both the strengths and weaknesses of the prospective teachers’ knowledge.

Below, we present the questions forming the questionnaire, along with the data we hoped it would provide about the KOT of the PPTs.

FIGURE I. Question I in the questionnaire

I. If the shaded parts of the drawing below represent the part of two chocolate bars that we have eaten, and the unshaded what we have not eaten, indicate which of the following options represents what we have eaten:

a) $10/14$ of a bar
b) $12/16$ of a bar
c) $6/8$ of a bar
d) $6/4$ of a bar

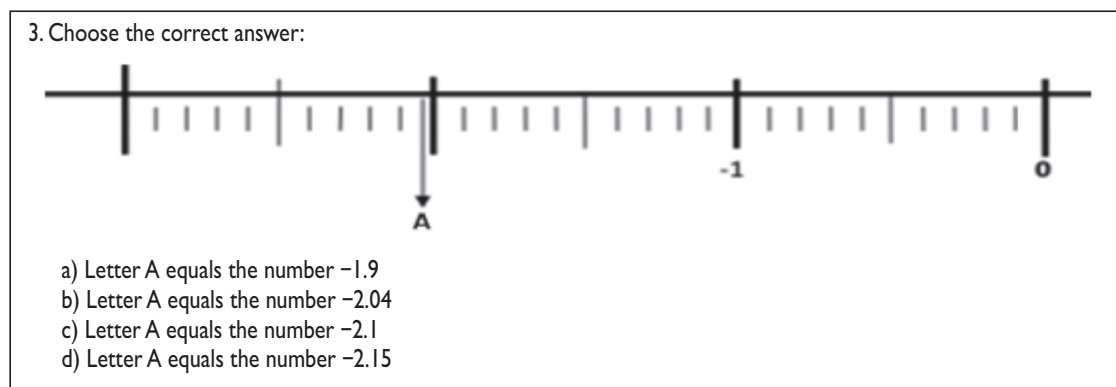


This question is based on problem number 5 posed by Hill et al. (2004), with the intention of calibrating responders' conception of unity and focussing attention on the improper fraction. Llinares (2003) notes that the idea of unity appears when we need to reconstruct it given the representation of the part.

The solution requires the responder to be aware that there are two units and that the result is larger than a unit: the fact that each of the options includes the phrase "of a bar" should make it clear that this is to be taken as the unit. Once this premise is assimilated, options *a*, *b* and *c* can be discarded, and *d* checked. To do this it is sufficient to observe that each bar is composed of 8 triangles of equal size, and that half of the second bar is shaded. It is also necessary to know the equivalence of fractions and to know how to decompose $6/4$ as $1 (4/4)$ plus $4/8$, and even the mixed number, $1 \frac{4}{8}$.

From amongst the wrong answers, the more expected are $12/16$ and $6/8$, as in each the idea of improper fraction is not present. The option $10/14$ suggests the responder has difficulties in identifying an appropriate unit of measurement to calculate the shaded area.

FIGURE II. Question 3 in the questionnaire



This question is posed in Castro (2001), in which it is noted that the comparison of decimal numbers can be problematic because there is a tendency to consider the decimal part as a natural number. What is more, the difficulty is augmented by being located on the negative axis of the number line; as González Marí (2001) points out about whole numbers, notions associated with negative numbers are not easy to assimilate, even though they can be met with in everyday situations. This question demands the responder to identify A as a negative number smaller than -2 , to recognise that each line between the units represents a tenth, and to know how to put both decimals and negative numbers in order so as to locate A between -2 and -2.1 . For their part, Castro and Torralbo (2001) note that this kind of representation of rational numbers along the number line cause doubts to emerge when several numbers are involved.

Option a suggests the responder is unaware that A is less than -2 (or, which is the same, believe that $-1.9 < -2$). Options c and d might suggest that the responder is not aware that the evenly spaced vertical bars represent tenths of a unit. All three wrong answers imply difficulties in putting decimals and negative numbers in order.

FIGURE III. Question 4 in the questionnaire

4. If I paid 12.710€ for a second-hand car, what was its factory price if the previous owner sold it to me with an 18% discount?
- a) The original price was 15.600€
 - b) The original price was 15.500€
 - c) The original price was 14.997.80€
 - d) None of the above

This question draws on Dickson et al. (1991) observation underlining the “clear importance that an understanding of percentages exerts in everyday life and in commercial activities” (p. 323).

In this question it is essential to recognise the fact that we do not have the starting price, but that if we have had an 18% discount, the price that we have actually paid is 82% of the original price. In a recent study, Contreras et al. (2012) demonstrated that the most frequently occurring error consists in applying the percentage to the known price, that is the figure which has already been discounted, and adding the stipulated percentage to this amount, that is, assuming that the final price determines the percentage discount of the original price.

Solving the proportionality (which is, in one way, the inverse of applying a discount) requires an ability to handle basic operations with rational numbers. Option c is the most expected of the wrong answers as it stems from the idea that “undoing” the discount means applying the percentage to the final (i.e., the discounted) price and adding the amount thus obtained to the same, a frequently occurring error (Ariza, Sánchez, Trigueros, 2011).

FIGURE IV. Question 5 in the questionnaire

5. Choose the correct answer corresponding to the final result of the following series of operations:

200 $\xrightarrow{75\%}$ $\xrightarrow{6/10}$ $\xrightarrow{90\%}$ $\xrightarrow{:0.3}$

a) 2,700
b) 2.70
c) 27
d) 270

This question supposes being able to handle the different expressions of a rational number, its equivalents and the operations on them (Llinares and Sánchez, 1988), and probably estimating, given that the calculations are to be done without a calculator. It is foreseeable that the greatest difficulty lies with the final division, as difficulties often accompany understanding that a higher number than the dividend can be obtained after division. Likewise, if the division is achieved by transforming the decimal number into a fraction, difficulties again arise, since, as Ball (1990b) and Ma (1999) assert, the algorithm for dividing by a fraction is known to most people, but not the mathematical foundations underpinning it.

FIGURE V. Question 8 in the questionnaire

8. If some goods go on sale with a discount on the manufacturing price X, and the retailer applies another discount to this price, the purchasing price is obtained:

a) By applying the sum of both discounts to the original value X
b) By applying the product of the two discounts to the original value X
c) By applying only the second discount to the original value X
d) None of the above is correct

As in question 4, we are again faced with a non-direct application of percentages, the essence of which is the correct identification of the values to which these should be applied in each case (Contreras et al., 2012). The question seeks to identify a widespread error in applying consecutive percentages: the natural tendency to add both percentages together to arrive at the final figure. We also sought to check whether the students

were able to interpret the percentages as fractions, in which case identifying the solution as the product of both should be immediate.

FIGUREVI. Question 9 in the questionnaire

9. Indicate the correct answer:

- a) Between 0.21 and 0.22 there is no decimal number
- b) Between 0.21 and 0.22 there are more than 10 decimal numbers
- c) Between 0.21 and 0.22 there are exactly 10 decimal numbers
- d) Between 0.21 and 0.22 there are exactly 9 decimal numbers

FIGUREVII. Question 10 in the questionnaire

10. Between $\frac{3}{7}$ and $\frac{4}{7}$:

- a) There is no fraction
- b) It is not possible to know whether or not there are fractions
- c) There is an infinite number of fractions
- d) There is a finite number of fractions

Drawing on Ruiz and Castro (2011), in these two questions (9 and 10) we aimed to identify knowledge of the process which allows us to find a rational number between two given numbers, determining as the chief error the notion of a following number in $\mathbb{Z}$ (answer *a*), question 9 being in the decimal register.

Question 10 aims to identify the knowledge required to locate a fraction between two given fractions, beyond merely comparing the numerators (answer *a*), by means of equivalent fractions, or by first converting the given fractions into decimal expressions. As in the previous question, we also wanted to explore the erroneous idea that there is a finite number of fractions between two given fractions, indicating a lack of awareness of the density of rational numbers (Pehkonen, Hannula, Maijala and Soro, 2006). In this instance, we expected a higher percentage of errors than for question 9, as the representation of rational numbers through coefficients between whole numbers tends to accentuate the misconception.

Answer *a* draws on the intuitive idea that between two fractions consisting of the same denominator and consecutive numerators (en $\mathbb{Z}$), there cannot be any further fractions. Answer *b* represents a distracter,

playing on the doubt arising from the conflict between the intuitive idea expressed in *a* and an intuition of the property of density of rational numbers. Answers *c* and *d* concern the ability of the PPTS to generalise the process of making fractions which lie between those given, opting either for a limited number, answer *d*, or an infinite number, answer *c*, the correct one.

FIGURE VIII. Question 11 in the questionnaire

11. The number which is three thousandths from 2.347 is:

- a) 2.344
- b) 2.647
- c) 2.317
- d) None is correct

This question deals with understanding the DNS (basic ideas, Ma, 1999), but this time through the concept of distance. The question was designed with a subtraction (answer *a*), as opposed to an addition, in order to make the association with the concept of distance less evident. The wrong answers were designed to highlight a lack of awareness of the positional value of the thousandths, with the value being mistakenly interpreted as a tenth or hundredth. Konic, Godino and Rivas (2010) note that, as an essential item on the primary curriculum, difficulties in grasping the concept of positional value are frequently detected in the course of learning decimal numbers.

FIGURE IX. Question 12 in the questionnaire

12. Indicate which one, or ones, of the representations below can be written as 1.75:

- a) $15/10 \times 5/10$
- b) 100×0.75
- c) $3/4 + 100/25$
- d) $12/8 + 50/200$

As in Muñoz-Catalán and Carrillo (2007), this question aims to analyse –although from a different perspective– the ability to estimate and operate, using fractions and decimals in combination, without the use of a

calculator. It requires a certain flexibility of thought in recognising the underlying structures, and demands a sound understanding of the equivalent forms of expressing a rational number. Some of the answers can be discarded through the process of estimation, such as *c*, in which a fraction equivalent to $\frac{1}{4}$ is added to a positive number.

FIGURE X. Question 13 in the questionnaire

13. Indicate the diagram equivalent to the fraction $\frac{7}{8}$

a) 

b) 

c) Both a and b are correct

d) None of the above is correct

In the same way that the first question analysed improper fractions, this question considers the concept of proper fraction. In this case, however, the numerical expression is provided and it is the corresponding diagrammatic equivalent that must be identified. The expected error is answer *c*, which also gives further information on the conceptualisation of improper fractions.

FIGURE XI. Question 14 in the questionnaire

14. Indicate which percentage is equivalent to $\frac{2}{10}$:

a) 2%

b) 20%

c) 0.2%

d) 0.02%

This question requires a simple identification of the equivalence between fractional and percentage registers for a number. The expected most likely error in this case is offered in option *c*, which is decimal equivalent of $\frac{2}{10}$.

In this question, we aim to explore the common difficulty pupils have in associating a fraction with a percentage, something which Dickson et al.

(1991) mention as one of the weaknesses detected in studies into the area. We included the possibility of difficulties with DNS arising from the position of the decimal point after dividing by 10 and then converting it to a percentage.

FIGURE XII. Question 15 in the questionnaire

15. In 1.237 there are

- a) 23 tenths
- b) 12 tenths
- c) 237 tenths
- d) There are only 2 tenths

As in question 11, the focus is on understanding the DNS, in this case in respect to decimal expressions and the meaning of each of the terms, in particular the positional value of the elements within the non-whole part.

Konic et al. (2010) note that the way certain problems are expressed in primary textbooks can lead children not to consider the relation between the position a number occupies and the value that is assigned to it; in our case, the expected answer is *d* 'There are only two tenths', echoing this situation of frequently badly constructed knowledge.

FIGURE XIII. Question 16 in the questionnaire

16. Choose the correct answer:

- a) $1,23 < 1,23\hat{2} < 1,2\hat{3} < 1,2\hat{3}$
- b) $1,2\hat{3} < 1,23\hat{2} < 1,2\hat{3} < 1,23$
- c) $1,23 < 1,2\hat{3}\hat{2} < 1,23\hat{2} < 1,2\hat{3}$
- d) None of the answers is correct

In this question it is necessary not only to know how to put decimal numbers in order, but also to understand the system of recurring numbers. In effect, knowing how to sequence the numbers by comparing their positional value here demands full understanding of how this positionality operates.

FIGURE XIV. Question 17 in the questionnaire

17. Indicate the result of the following operation: $2/10 - [1/4 + 3/2 + (-8 - 1/5)] =$ a) 133/20 b) 1197/180 c) 117/20 d) a and b are both correct

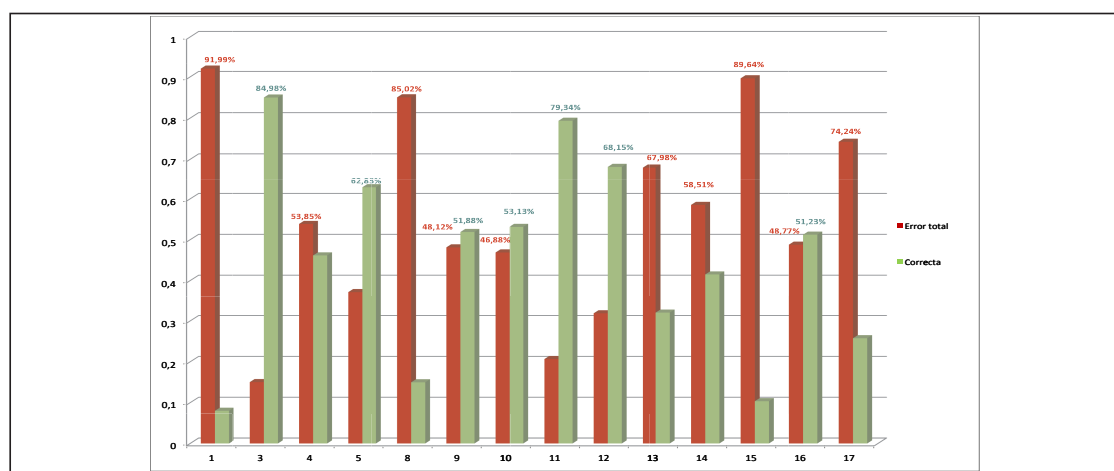
Finally, this last question aims to evaluate the capacity to operate with fractions, focussing in this instance on the addition and subtraction of fractions, the hierarchy of operations and equivalent forms of expressing fractions, as the correct answer requires recognising equivalent solutions.

Results and conclusions

Graph I shows the percentage of correct answers as against incorrect answers for each of the questions in the questionnaire.

Questions 1 and 13 (the meaning and representation of fractions, improper fractions), 8 and 14 (percentages as an application and operator), 15 (meanings of the DNS) and 17 (procedural aspects of fractions) returned the lowest percentage of correct answers; in question 4 (application of percentages) the percentage of wrong answers is also greater than the correct answers. By contrast, in questions 3 (the properties and meanings of negative decimals), 5 (combined operations), 11 (the meaning of distance as subtraction in the DNS) and 12 (the representation of, and operations with, fractions) the percentage of correct answers is significantly high, while in questions 9 and 10 (finding a rational number between two given numbers in decimal and fractional expressions) and 16 (the representations of decimals) the figures are much closer, although the correct answers outnumber the incorrect.

GRAPH I. Distribution by percentage of correct answers against total number of errors



During the obligatory period of schooling, improper fractions are far less frequently taught than proper fractions, and hence the low number of correct answers in questions 1 and 13 was to be expected. Further, given that the interpretation of fraction as parts of whole is most easily understood (Castro and Torralbo, 2001), there is a tendency to present it this way at school; unless the concept of unit is dealt with adequately, a false construction of the concept of fraction can result when the parts are greater than the unit.

Dickson et al. (1991) refer to other studies tackling this question, which highlight the failure to understand improper fractions due to their representation as sub-areas of an area unit. The authors give the example of the addition of two proper fractions which results in an improper fraction, noting the confusion created by a poor acquisition of the concept of unit, which is reinterpreted when they see the improper result.

The meaning of fraction, which can be associated with parts of a whole (continuous or discrete), operators, ratios, and quotient (of two integers), is often approached at primary level from the first of these perspectives, using examples based on parts of a single unit. The fourth approach –quotients– naturally leads to the idea of improper fractions, and is associated with a meaning of fraction which the primary pupils will have seen previously (partitive division), and is not usually associated with parts of a whole. What is more, it is this approach which comes closest to the

formal definition of fraction. Being unaware of this definition typically leads some PPTs to identify $3.5/7$ as the answer to question 10.

In the same way, using percentages for more than the mere application to a specific number tends to challenge the PPTs, as in the case of questions 8 and 14. The result in the former came as no surprise, as this tends to feature prominently amongst the errors we usually find on the teacher training courses. The consecutive nature of the discounts leads trainees to identify addition as the appropriate operation, in the same way that in order to calculate the original price from the discounted price, they tend to apply the percentage discount to this latter and then add to it the resultant amount (option *c* in question 4). In both questions 4 and 8, the key issue (in different ways) is which figure the percentage should be applied to. This is a more cognitively demanding operation than that of simply applying a percentage to a particular figure (which could be identified with the fraction as operator). In question 14, it is precisely the absence of the number on which to operate which usually leads to error (Contreras et al., 2012). If the question had been couched as “What percentage of A (given) corresponds to $2/10$ of this number?” the results would have been the same as those for question 5.

Indeed, naming or recognising decimal expressions in non-conventional expressions is a difficulty highlighted in other studies (Ball, 1990 a). There tends to be little importance given to the Decimal Number System, familiarity with which has significant implications at all levels of primary education, such that it is considered of cross-curricular significance. Furthermore, Kamii (1994) notes that learning arithmetic algorithms without a sound understanding of number generates a vicious circle, as it “‘misteaches’ positional value and prevents the correct development of the concept of number” (p. 49), as the positional value is fundamental for an in-depth understanding of the number system which will enable pupils to manipulate it correctly further down the line. Understanding the underlying structure means more than identifying the role of the units in each order. Question 15 focuses on a particular aspect of this understanding, that relating to the reading and writing of numbers, emphasising the order of any unit. Here, the difficulty does not lie in recognising that the number is made up of one unit, two tenths, three hundredths and seven thousandths, but rather reading the amount in different ways, mixing tenths and hundredths, understanding the definition of a decimal number and the sense of units. Another aspect which includes

this sense of understanding of the DNS is tackled in question 11 (the meaning of distance between two rational numbers). However, in this instance the solution involves adding or subtracting 3 thousandths from the given figure, which accounts for the difference in results for both.

It is also interesting to note the closeness in frequencies of the correct and incorrect answers to questions 9 and 10, both designed to study the understanding of the property of density in rational numbers. This finding, irrespective of whether the rational numbers are expressed as decimals or fractions, runs counter to reports of a predominance of incorrect conceptual constructs on the part of the PPTs (Ruiz and Castro, 2011).

Question 17, as mentioned above, requires the responders not only to operate with fractions, but also to recognise hierarchy, to be able to use equivalent fractions and to know how to apply the calculations involved in the lowest common denominator. The confluence of these various aspects determines the difficulty facing the PPTs and reflected in the outcome.

If we now turn to the elements of KOT mentioned in section 4, it seems to us that both the definition and meaning which the PPTs attach to improper and proper fractions (questions 1 and 13), in addition to their understanding of the representations and procedures that can be carried out using fractions, decimals and percentages, in all the aspects dealt with (questions 14, 15, 4 and 8), constitute part of the weaknesses of the PPTs participating in the study. Likewise, both the formation of equivalent fractions, and the hierarchy of operations (question 17) represent significant weaknesses. Knowledge of the Decimal Number System (DNS) would also seem to belong to this group, as, despite the relatively high percentage of correct answers to question 11 about the meaning of the distance between two decimal numbers, questions 14 and 15 –dealing with the structure of the DNS in more depth– returned a significant number of incorrect answers.

With respect to the density of rational numbers and their order, although the number of correct answers to the corresponding questions (9, 10 and 16) exceeds the number of incorrect answers in each case, the difference is so slight that we are inclined to think that they, too, show weaknesses, specifically in regard to knowledge of the properties and foundations of the density of rational numbers, which are perhaps compensated in this instance by the associated procedural knowledge.

On the positive side, in terms of strengths, it is only fair to acknowledge the capacity to place a decimal number on the number line (question 3), even when this is a negative number and as such increases the degree of difficulty and makes the achievement more noteworthy. Also included is the use of the algorithms for adding and multiplying fractions (in the full sense), including the interpretation of distance between decimal expressions as a subtraction, and the processes of estimation and operation using two fractional and/or decimal expressions (questions 11 and 12), this being an area which is amply covered throughout the primary years as well as over the course of the first cycle of secondary.

A final reflection

To finish, we would be gratified if this study was to contribute to the reflection of those with decision-making powers in the educational system. First, the work should help improve the understanding of which aspects relating to fractions, decimals and percentages require attention in Primary Education, and what form that attention should take. Secondly, the educational authorities should define more precisely the mathematical knowledge required of trainee Primary Teachers, as the university is hardly the most appropriate place to go back over knowledge which should have already been assimilated. It is also apposite here to return our attention to the theoretical framework underpinning this study. Although our focus has been KOT, knowledge involving the Structure of Mathematics (KSM) and the Practice of Mathematics (KPM) can also be detected. Together, these three components make up the teacher's Mathematical Knowledge (MK), and are closely related to the four properties proposed by Ma (1999) as the defining features of what she terms as the teacher's *profound understanding of fundamental mathematics*: basic ideas (simple but powerful), related to KOT and KSM, which will enable them to guide their future pupils towards "doing true mathematical activity" (Ma, 1999 p. 148); connectivity, regarding the connections between concepts and procedures (KOT and KSM); multiple perspectives of the same situation or approach to solving a problem (KOT); and longitudinal coherence, which concerns knowledge of the curriculum. The sub-domains comprising our theoretical

framework, and in particular the categories of KOT analysed in this study, together represent a grounded structure of what initial teacher training should concern itself with. For this to happen, it is essential to start with well-founded mathematical knowledge which offers certain guarantees when it comes to tackling the components of Pedagogical Content Knowledge, and developing the specialised mathematical knowledge required by primary teachers if they are to do a good job. One possibility for achieving a better level of mathematical knowledge at the start of initial training, which is being debated at the moment (Castro, Mengual, Prat, Albarracín, Gorogorió, 2014), is the use of specific entry tests for the training courses.

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Contact Address: Miguel Ángel Montes. Facultad de Ciencias de la Educación. Departamento de Didáctica de las Ciencias y Filosofía. Universidad de Huelva. Campus El Carmen. Avda. del 3 de marzo, s/n; Huelva (21071). E-mail: miguel.montes@ddcc.uhu.es