Excited-State Phase Transition and Onset of Chaos in Quantum Optical Models

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We study critical behavior of excited states and its relation to order and chaos in the Jaynes-Cummings and Dicke models of quantum optics. We show that both models exhibit a chain of excited-state quantum phase transitions demarcating the upper edge of the superradiant phase. For the Dicke model, the signatures of criticality in excited states are blurred by the onset of chaos which happens in the same energy region.

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One of the goals of many-body physics is the understanding of quantum critical phenomena [1]. A quantum phase transition (QPT) appears in systems with Hamiltonian $H(\lambda)$ which show a sudden change of the ground-state form with a control parameter λ varying across a critical value λ_c. In the last few years the connection of a QPT with the entanglement [2–4] and with the emergence of chaotic behavior [2, 4, 5] has been investigated. This is a hot topic also because all the mentioned areas play important roles in the current research related to quantum information technologies [6]. Very recently, a quantum critical phenomenon of a new type—the one related to excited states rather than to the ground state—has been discussed for several model systems [7–9]. An excited-state quantum phase transition (ESQPT) represents a non-analytic evolution of individual excited states with the control parameter. It can also be observed as a singular variation of the state density with energy and entails dramatic dynamical consequences [10, 11].

As ESQPTs have so far been identified in simple, mostly integrable systems, two natural questions arise: First, are the ESQPTs also generic in more complex, prevailingly chaotic systems? Second, how the collapse of quantum energy levels, a typical ESQPT signature, affects the level repulsion, which is inherent in quantum chaos? The aim of this Letter is to address these questions by analyzing a model of collective interactions of matter and light. The model is formulated in a simplified, integrable version, the so-called Jaynes-Cummings model [12], and in a more complex, non-integrable version, the Dicke model [13]. Both versions have recently stirred up great interest since the implementation of a tunable matter-light coupling represents a route to study quantum critical effects [14–16]. The QPT to a superradiant phase within the Dicke model was studied theoretically [2, 17–19] and recently realized experimentally using a superfluid gas in an optical cavity [16].

Both the Dicke and Jaynes-Cummings models assume a set of two-level atoms interacting by a dipole coupling of strength λ with a single-mode bosonic field (cavity photons). The models are expressed via the creation and annihilation operators $b^\dagger$ and b, describing a bosonic mode with frequency ω (with $N_b = b^\dagger b$ the number of photons), and the SU(2) generators $\{J_\pm, J_z\}$, describing the ensemble of N_a two-level atoms with the level splitting ω_0 in terms of a pseudospin of length $J = N_a/2$. A useful realization of the SU(2) algebra can be built through an array of spin-$\frac{1}{2}$ particles located on $2J$ sites,

$$J_+ = \sum_{i=1}^{2J} a^\dagger_{\uparrow i} a_{\downarrow i} = J_-^\dagger \ , \ J_z = \tfrac{1}{2} \sum_{i=1}^{2J} \left(a^\dagger_{\uparrow i} a_{\uparrow i} - a^\dagger_{\downarrow i} a_{\downarrow i} \right) , \tag{1}$$

where $a^\dagger_{\uparrow i}$ or $a_{\uparrow i}$ and $a^\dagger_{\downarrow i}$ or $a_{\downarrow i}$ create or annihilate spin-up and spin-down states of the fermion on site i (the upper and lower state of the ith atom) and the ladder operators $J_\pm$ describe collective spin flips along the array. $N_\uparrow \equiv J_z + J$ counts the number of atoms excited to the upper level. The system has two degrees of freedom associated, e.g., with N_b and J_z.

The Jaynes-Cummings Hamiltonian [12]

$$H_1(\lambda) = \omega_0 J_z + \omega b^\dagger b + \frac{\lambda}{\sqrt{4J}} \left[b J_+ + b^\dagger J_- \right] \tag{2}$$

conserves the quantity $M/2 = N_b + N_\uparrow$, and is therefore integrable. We work with a fixed value $M/2 = N_a$, which implies a finite dimension of the Hilbert space, and set $\omega > \omega_0$. The Dicke Hamiltonian [13]

$$H_2(\lambda) = \omega_0 J_z + \omega b^\dagger b + \frac{\lambda}{\sqrt{4J}} \left[\left(b + b^\dagger \right) \left(J_+ + J_- \right) \right] \tag{3}$$

conserves the parity $\Pi = (-1)^{M/2}$, but not the number M itself. Therefore, it is not integrable. The Hilbert-space dimension for any N_a is infinite since states with unlimited photon numbers N_b are coupled by the interaction term. From a practical point of view this means

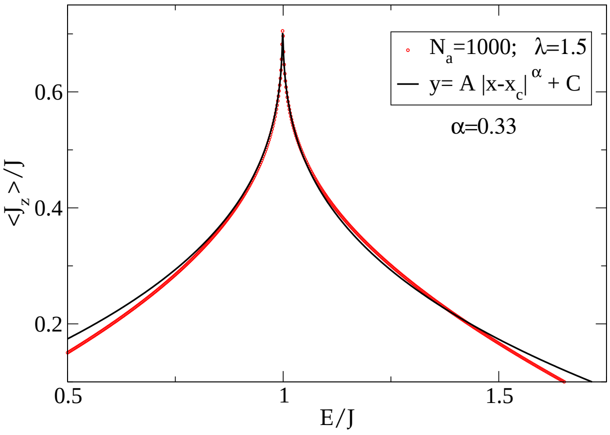

FIG. 1: (color online) Scaled atomic inversion as a function of energy for the Jaynes-Cummings model. Calculations are done for $\omega_0 = 1$, $\omega = 2$, $\lambda = 1.5$, and $J = 500$.

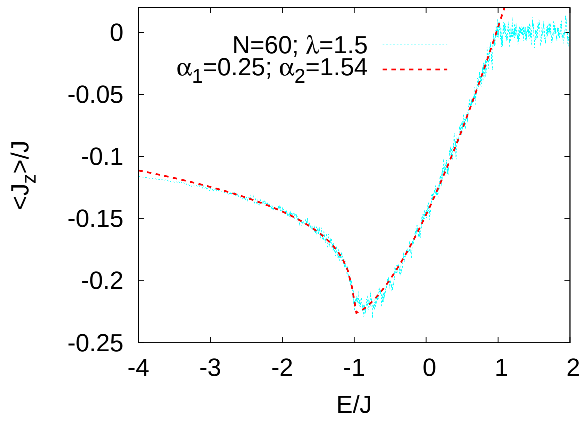

FIG. 2: (color online) Scaled atomic inversion as a function of energy for the Dicke model. Calculations done for $\omega_0 = \omega = 1$, $\lambda = 1.5$, $J = 30$ and results smoothed over 20 points.

that the photon space needs to be truncated by a certain maximal value $N_b^{\max}$. In the numerical calculations presented below we checked that an increase of $N_b^{\max}$ did not cause noticeable changes of results. In these calculations we set $\omega = \omega_0$, which corresponds to the resonance absorption and emission of photons by the atoms.

The models serve as toy examples of the maser phase transition. Indeed, in the thermodynamic limit, $J \to \infty$, both Hamiltonians yield a QPT of the second order [2] whose properties are obtained from a relevant semiclassical analysis [11]. The critical values of the coupling strength are $\lambda_c = \sqrt{(\omega_0 - \omega)^2/2}$ for the Jaynes-Cummings model and $\lambda_c = \sqrt{\omega\omega_0/2}$ for the the Dicke model. Below the critical point a *normal phase* exists, in which the ground state is similar to that at $\lambda = 0$, given by the photon vacuum ($N_b = 0$) combined with a maximally excited ($J_z = +J$) or a totally unexcited ($J_z = -J$) state of the atom array. The first case is valid for the Jaynes-Cummings model with $\omega > \omega_0$ and $M = 4J$, the second one for the Dicke model. For the coupling strength below λ_c, the $\lambda = 0$ ground-state form is preserved in the sense of expectation values: we have $\langle N_b \rangle = 0$, $\langle J_z \rangle = \pm J$, and the ground-state energy $E_0^{\rm norm} = \pm J\omega_0$ in the thermodynamic limit. When crossing the critical value λ_c, the ground state eventually flips to a form with varying expectation values $\langle N_b \rangle > 0$ and $-J < \langle J_z \rangle < +J$ and decreasing energy $E_0(\lambda) < E_0^{\rm norm}$, in which both the photon field and the atomic array acquire partial, macroscopic excitations. This regime can be interpreted as a *superradiant phase* [2, 17]. The quantities $\langle J_z \rangle$ (or $\langle N_\uparrow \rangle$) and $\langle N_b \rangle$ represent suitable order parameters of the superradiant phase transition.

We claim that the ground-state QPT in both models is followed for $\lambda > \lambda_c$ by a chain of *excited-state* phase transitions located at the critical energy coinciding with the ground-state energy of the normal phase, $E_c = E_0^{\rm norm}$. A detailed discussion of the ESQPT effects and their semiclassical roots is given elsewhere [11]. Here we only demonstrate a singular dependence of the order parameter on energy around the critical value E_c. The dependence has the form

$$\langle J_z \rangle = \langle J_z \rangle_c + A\,|E - E_c|^{\alpha} \ , \qquad (4)$$

which is characterized, apart from the $E = E_c$ value $\langle J_z \rangle_c$ and a coefficient A, by a critical exponent α. Note that we analyze now only the behavior of the expectation value $\langle J_z \rangle$, but related dependences are obtained also for $\langle N_b \rangle$. Results for the Jaynes-Cummings model with 1000 atoms are plotted in Fig. 1 for $\lambda = 1.5$. The points (red online) correspond to numerical data, while the solid black curve shows a fit using Eq. (4) with the critical exponent $\alpha = 0.33$. The cusp singularity at $E_c/J = 1$ is characterized by the value $\langle J_z \rangle_c = J$, hence $\langle N_\uparrow \rangle = N_a$ (not seen in the figure). This can be described as the $\lambda < \lambda_c$ ground-state structure propagating through the spectrum along the line $E = E_c$, where we indeed observe multiple avoided crossings of individual levels [11].

Similar results for the Dicke model are plotted in Fig. 2. They are obtained with 60 atoms for $\lambda = 1.5$. Again, the points represent numerical data and the dashed curve a fit by Eq. (4). The number of atoms is much smaller than the one used in the previous calculation because of a rather large value of $N_b^{\max}$ needed to get convergence for the levels above the critical energy. In particular, results for $E/J > 1$ (the flat part of the numerical dependence in Fig. 2) are *not* converged.

A comparison with Fig. 1 shows that the results for the Dicke model are fuzzier than those for the Jaynes-Cummings model. They exhibit sizable fluctuations around a smooth dependence. This is so although the raw data were smoothed by the method of a moving average: each value $\langle J_z \rangle/J$ shown in the figure is the mean value for 20 consecutive energy levels centered at the given energy. Another peculiarity of the Dicke model is that the order parameter is described by *two* critical exponents: $\alpha_1 = 0.25$ for $E < E_c$, and $\alpha_2 = 1.54$ for

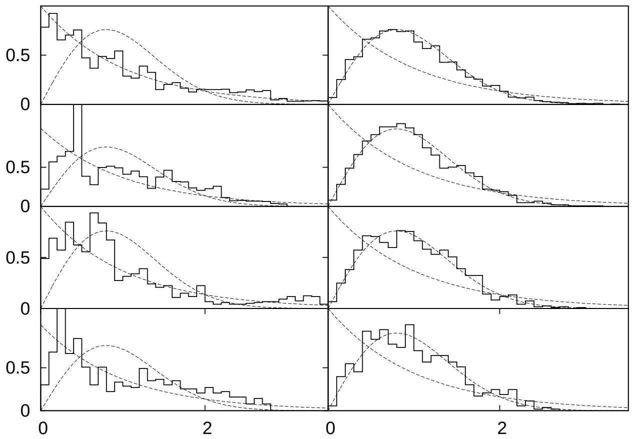

FIG. 3: Nearest-neighbor spacing distribution $P(s)$ for the Dicke model with $\lambda = 3$. The left column shows the histograms for $E < E_c$, the right column for $E > E_c$. The rows from bottom to top correspond to $N_a = 22$, 30, 38, and 46.

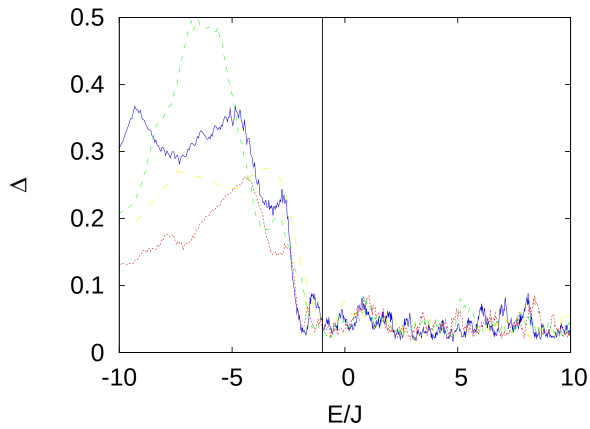

FIG. 4: (color online) Onset of chaos in the Dicke model with $\lambda = 3$ measured by the distance (5) of actual level statistics from the Wigner distribution. The distance is plotted as a function of the mean scaled energy of the relevant fraction of the spectrum, the vertical line indicating the critical energy. Calculation done for various atom numbers: $N_a = 22$ (very light long-dashed line, yellow online), 30 (light medium-dashed line, green online), 38 (dark short-dashed line, red online), and 46 (very dark solid line, blue online).

$E > E_c$. Despite these differences, Fig. 2 shows a qualitatively similar dependence as Fig. 1, demonstrating a related type of singularity in the Dicke model at the critical energy $E_c/J = -1$. Moreover, we checked that the raw data (not shown in the figure) yield $\langle J_z \rangle_c = -J$, hence $N_\uparrow = 0$ for $E = E_c$, as expected from the semiclassical analysis. We therefore conclude that for $\lambda > \lambda_c$ both models exhibit an ESQPT at $E_c = E_0^{\rm norm}$. This critical energy terminates the domain of the superradiant phase present at low temperatures [17].

We know that the Dicke model is nonintegrable and partly chaotic. The superradiant transition at zero temperature was shown [2] to be correlated with a crossover from ordered to chaotic behavior. Hence one may ask whether quantum chaos is also somehow related to the critical behavior of excited states. It can be anticipated that chaotic properties of the spectrum introduce large fluctuations that partly hide the singular dependence at the critical point in Fig. 2. However, chaos and ESQPTs have some properties which are difficult to conciliate. On one hand, the most significant feature of quantum chaos is the level repulsion, which entails a null probability of finding two levels at the same energy [20]. On the other hand, an ESQPT as a rapid restructuring of excited states is typically connected with a rather close approach of levels (numerous sharp avoided crossings), often with a singular accumulation of levels at $E = E_c$ [7–10]. Therefore, the relation between the level repulsion and spectral signatures of an ESQPT constitutes an interesting theoretical challenge.

Let us consider the Dicke Hamiltonian with a value of λ above λ_c, where the system is partly chaotic [2]. To analyze how chaos and the ESQPT can dwell together, we calculate the spacing distribution $P(s)$, where s is a normalized ($\overline{s} = 1$) distance between two neighboring levels, on both sides of the critical energy E_c. This distribution is known to interpolate between the Poissonian and Wigner forms, $P^{\rm P} = e^{-s}$ and $P^{\rm W} = \frac{\pi}{2} s e^{-\pi s^2/4}$, respectively, as the system transforms from regular to a chaotic regime [20]. In order to have a large number of levels below the critical energy, we present results for $\lambda = 3$; different choices lead to similar pictures.

In Fig. 3 we show the $P(s)$ distributions calculated for both subcritical ($E < E_c$) and supercritical ($E > E_c$) parts of the spectrum and for various atom numbers (see caption for details). In all cases we used the same number of levels for building the histograms on both sides of the critical energy. For $E > E_c$, the spectral statistics closely follows the Wigner surmise. In contrast, for $E < E_c$ the shape of histograms is not clear: while the $N_a = 46$ case is rather close to the Poissonian distribution, lower atom numbers yield neither Wigner, nor Poissonian form. The most relevant fact is that below E_c we observe $P(s = 0) > 0$ while above E_c we get $P(s = 0) \approx 0$. In other words, the levels start to repel each other when crossing the ESQPT critical energy.

To obtain a more quantitative description, we analyze how the spectral statistics changes as one moves in the spectrum. To increase the accuracy of the procedure, we rely on the accumulated spacing distribution $F(s) = \int_0^s dx\, P(x)$. Then, given a numerical sequence of spacings $\{s_i\}$, we measure its "distance" from the Wigner surmise by means of the following quantity

$$\Delta = \frac{\sum_i \left[F^{\rm W}(s_i) - F(s_i)\right]^2}{\sum_i \left[F^{\rm W}(s_i) - F^{\rm P}(s_i)\right]^2}, \qquad (5)$$

where $F^{\rm W}$ and $F^{\rm P}$ are accumulated distributions derived from $P^{\rm W}$ and $P^{\rm P}$, respectively. Therefore, $\Delta = 0$ if the numerical sequence follows the Wigner distribution, while $\Delta = 1$ for a Poissonian sequence (note however that $\Delta = 1$ does not imply the Poisson statistics).

Results are shown in Fig. 4. We have constructed sequences of 200 consecutive spacings and, using Eq. (5), calculated the distance Δ of each sequence from the Wigner statistics. These distances are plotted as a function of the mean energy of the respective sequence. We can see that a quite abrupt transition from $\Delta > 0$ to $\Delta \sim 0$ takes place just below the critical energy E_c. The range of the covered values of N_a is not large enough to infer what happens as one approaches the thermodynamic limit, but it suffices to conjecture that the critical behavior in the order parameter is accompanied by a change in the spectral statistics.

This result refines on the hypothesis of Ref. [2] stating that the spectrum of the Dicke model with $\lambda > \lambda_c$ is regular at low energies. Our numerical results suggest that a transition to chaos takes place around the critical energy E_c. A compact picture of various dynamical regimes implicit in the Dicke model can therefore be put as follows: We start at $\lambda = 0$ with the ground state having $N_b = N_\uparrow = 0$. Increasing λ, the ground-state keeps fixed averages $\langle N_b \rangle = \langle N_\uparrow \rangle = 0$ until we cross the critical point λ_c for the superradiant transition, where the averages start increasing. For any value of $\lambda > \lambda_c$, there exists a region above the ground state in which we find no level repulsion, $P(s = 0) > 0$. This seems to be a common feature of almost the whole superradiant domain in the $\lambda \times E$ phase diagram. If the energy is increased closely below the critical value $E_c = -J\omega_0$, where $\langle N_\uparrow \rangle$ drops sharply, the level repulsion sets in, leading the spectrum to the Wigner type of statistics. Above E_c, the system becomes fully chaotic.

In this Letter, the existence of an excited-state quantum phase transition is demonstrated in two models describing the collective matter-light interaction. In the integrable Jaynes-Cummings model, the ESQPT leads to a neat nonanalyticity of the order parameter $\langle J_z \rangle$ at the critical energy E_c. The nonintegrable Dicke models exhibits a similar type of ESQPT, but with signatures blurred by the onset of chaotic behavior in the spectrum. From our numerical calculations we conclude that a crossover from the regime with no level repulsion to the one with the Wigner level statistics takes place just below the critical energy. Level repulsion, a fundamental feature of quantum chaotic systems, and a cumulation of sharp avoided crossings, a typical signature of an ESQPT, are difficult to conciliate in general. We anticipate to observe a similar qualitative behavior in other nonintegrable systems with ESQPTs. Moreover, also the other dynamical effects of ESQPTs, as for instance anomalous decoherence factors obtained in the Lipkin model [10], are expected to be fuzzier in quantum chaotic systems.

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