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Research Article

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Local analytic integrability for a class of degenerate planar vector fields

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Abstract: Using the normal form theory and the existence of an algebraic inverse integrating factor we characterize the local analytic integrability of the systems whose quasi-homogeneous leading term is $(a_1y^3 + a_2x^3y, b_1x^5 + b_2x^2y^2)$. More specifically we prove that the analytic integrable vector fields inside such family are orbitally equivalent to a semi-quasi-homogeneus system, that is, are not orbitally equivalent to its lowest-degree quasi-homogeneous term.

Keywords: local analytic integrability; normal form; algebraic inverse integrating; analytic vector fields

AMS Mathematics Subject Classification: 34C20; 34C14

1 Introduction

A differential system is analytically integrable at the origin when it has an analytic first integral, that is, a non-constant analytic function that is constant on the solution curves in a neighborhood of the origin. It is not always possible to explicitly write the solutions and the analytic first integrals in terms of explicit functions. Here, our objective is to determine when a differential system has an analytic first integral. In [1] it was proved the existence of a non-invertible map at the origin that transforms any integrable system into a linear one. The extension of this result to n-dimensional systems was made in [2]. Therefore any integrable differential system is equivalent to a linear differential system in a full Lebesgue measure subset of the domain of definition of the differential system. In particular, this equivalence is not defined at the singular points of the differential system.

Normal form theory provides an approach for determining the integrability of a vector field. We recall that two systems (or vector fields $\mathbf{F}$ and $\mathbf{G}$) are *orbitally equivalent* if by means of a near-identity change of variable $\mathbf{x} = \phi(\mathbf{y})$ and a formal rescaling of the time $\mathrm{d}t/\mathrm{d}\tau = \eta(\mathbf{x})$, with $\eta(\mathbf{0}) = 1$, system $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{F}(\mathbf{x})$ is transformed into $\mathbf{y}' = \mathrm{d}\mathbf{y}/\mathrm{d}\tau = \mathbf{G}(\mathbf{y})$. In this sense, there are some relevant results. The analytic vector field whose origin is an isolated singular point with non-zero linear part (saddles and linear centers) is analytically integrable if, and only if, it is orbitally equivalent to its linear part, [3], [4]. We say that it is *orbitally linearizable*. In [5], [6] was given a method to compute necessary and sufficient conditions of analytic integrability for such singular points with non-zero linear part. The nondegenerate centers [7]–[11] and resonant saddles are the most studied systems and especially the Lotka-Volterra systems, see [12]–[20] and references therein. It is worth

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pointing out that for analytic vector fields, by ([21], Theorem A), the existence of a formal first integral is equivalent to the existence of an analytic first integral. For this reason, when we use Taylor expansions of functions and vector fields, we do not consider questions of convergence.

In [22] was solved the analytic integrability problem around a nilpotent singularity of a planar vector field under generic conditions. We recall that in a nilpotent singularity its linear part is nonzero but whose eigenvalues are zero. Later, in [23] was solved the remaining case completing the analytic integrability problem for such singularity. In both cases, the vector field has an analytic local first integral if, and only if, it is *orbitally quasi-homogenerizable* (orbitally equivalent to its leading quasi-homogeneous term).

For analytic perturbations of quadratic or cubic homogeneous vector field, there exists a similar result. More specifically, the analytic vector field whose origin is an isolated singular point whose leading homogeneous term is quadratic or cubic, is analytically integrable if, and only if, it is formally orbitally equivalent to its leading homogeneous term, see [24]–[26].

However, not all the analytically integrable vector field with a degenerate singularity (zero linear part) are orbitally equivalent to its first quasi-homogeneous term. For instance, in [27], [28] are given vector fields whose leading term is Hamiltonian and the analytic integrable perturbations are not orbitally quasi-homogenerizable.

Here, we study a generic family of analytic vector fields whose origin is degenerate. We prove that whether Hamiltonian or not its leading quasi-homogeneous term the vector field is analytically integrable if, and only if, it is orbitally equivalent to a semi-quasi-homogeneous vector field (sum of two quasi-homogeneous vector fields). We note that the origin of some of these integrable systems are centers.

From all the results obtained up to now the analytic integrability problem of any vector field is a very difficult problem that depends on choosing the good normal form that can solve the problem. In general this problem remains open as well as the integrable center problem associated.

Consider the polynomial differential system

$$\dot{x} = a_1y^3 + a_2x^3y, \qquad \dot{y} = b_1x^5 + b_2x^2y^2, \tag{1.1}$$

and assume that the origin is an isolated singular point, *i.e.* a_1, a_2, b_1, b_2 real numbers and $a_1b_1 \neq 0$. The associated vector field of system (1.1) is a quasi-homogeneous vector field of weight (2, 3). By a linear scaling, the system can be expressed as

$$\dot{x} = -4y^3 + 2(d-a)x^3y, \qquad \dot{y} = 6\sigma x^5 + 3(d+a)x^2y^2, \tag{1.2}$$

with a, d real numbers and $\sigma = \pm 1$.

In this paper, we solve the analytic integrability problem of the analytic systems whose leading quasi-homogeneous term is the quasi-homogeneous vector field (1.2), that is

$$\dot{x} = -4y^3 + 2(d-a)x^3y + \cdots, \qquad \dot{y} = 6\sigma x^5 + 3(d+a)x^2y^2 + \cdots, \tag{1.3}$$

where the dots mean terms of higher quasi-homogeneous order. Concretely, we characterize the systems (1.3) with an analytic first integral at origin through the existence of a normal form orbitally equivalent and through the existence of an algebraic inverse integrating factor.

The analytic integrability problem of system (1.3) has been solved in [28] for the case $d = 0$ and $\sigma = 1$ (Hamiltonian system whose origin is a center-focus) and in [29] for the case $a = 0$ and $\sigma = -1$. Here, we solve the analytic integrability problem of the analytic system (1.3) for all the cases, see Theorems 2.2 and 2.4 below. Moreover, we give the expression of its primitive first integral.

We also give a characterization of the analytical integrability of the system (1.3) through the existence of an algebraic inverse integrating factor, see Theorems 2.3 and 2.6 in the Section 2.4 and Section 2.5.

2 Main results

We introduce some notation and concepts. Given $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 natural numbers without common factors, a scalar function f of two variables is *a quasi-homogeneous function* of type or weight exponent $\mathbf{t}$ and degree j if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^j f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree j is denoted by $\mathcal{P}_j^{\mathbf{t}}$. A vector field $\mathbf{F} = (P, Q)^T$ is *a quasi-homogeneous vector field* of type $\mathbf{t}$ and degree j if $P \in \mathcal{P}_{j+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{j+t_2}^{\mathbf{t}}$. We denote the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree j by $\mathcal{Q}_j^{\mathbf{t}}$.

From ([30], Prop. 2.7), every $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ can be uniquely written as $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0^{\mathbf{t}}$ with $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ and $\mu \in \mathcal{P}_r^{\mathbf{t}}$ where $\mathbf{D}_0^{\mathbf{t}} = (t_1x, t_2y)^T \in \mathcal{Q}_0^{\mathbf{t}}$ (dissipative quasi-homogeneous vector field) and $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to the polynomial h).

Following this notation, the polynomial system (1.2) is $\dot{\mathbf{x}} = \mathbf{F}_7 = \mathbf{X}_h + \mu\mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}_7^{\mathbf{t}}$, with $\mathbf{t} = (2, 3)$, $h(x, y) = y^4 + \sigma x^6 + ax^3y^2$, $\mu(x, y) = \mathrm{d}x^2y$ and $\mathbf{D}_0^{\mathbf{t}} = (2x,\ 3y)^T$, that is,

$$\mathbf{F}_7 = \begin{pmatrix} -4y^3 - 2ax^3y \\ 6\sigma x^5 + 3ax^2y^2 \end{pmatrix} + dx^2y\begin{pmatrix} 2x \\ 3y \end{pmatrix}.$$

We recall the concepts of invariant curve and integral first of a differential system (or a vector field). A function $C \in \mathbf{C}[[x, y]]$ (algebra of formal power series in x, y over $\mathbf{C}$), with $C(\mathbf{0}) = 0$, is an invariant curve at the origin of the vector field $\mathbf{F} = (P, Q)^T$, if there exists $K \in \mathbf{C}[[x, y]]$, named *cofactor* of C, such that $F(C) := P\partial C/\partial x + Q\partial C/\partial y = KC$. Moreover, if $K \equiv 0$, the vector field $\mathbf{F}$ is *formally integrable* and C is a *formal first integral* of $\mathbf{F}$.

If the cofactor of C is the divergence of the system, we say that C is an inverse integrating factor. This name for C comes from the fact that C^{-1} defines an integrating factor of $\mathbf{F}$, *i.e.* $\mathbf{F}/C$ is divergence-free on $\{V \neq 0\}$.

According the number of invariant curves of the system (1.2), we distinguish two cases:

For $\sigma = 1$, it has the following situations: if $a \in (-2, 2)$, system (1.2) has one irreducible invariant curve $(y^2 + \frac{a}{2}x^3)^2 + (1 - \frac{a^2}{4})x^6$. In this case, the origin is a focus or a center. If $a = 2$ (or $a = -2$), system (1.2) has one irreducible invariant curve $y^2 + x^3$ (or $y^2 - x^3$).

For ($\sigma = -1$) or ($\sigma = 1$ and $a \notin [-2, 2]$), system (1.2) has two irreducible invariant curves $y^2 + \frac{1}{2}(a - b)x^3$ and $y^2 + \frac{1}{2}(a + b)x^3$, with $b = +\sqrt{a^2 - 4\sigma}$.

The following result provides the system (1.2) having a polynomial first integral.

Proposition 2.1 *The following statements hold:*

(i) *Assume that $\sigma = 1$ and $a \in [-2, 2]$, then system* (1.2) *is polynomially integrable if, and only if, $d = 0$. In such a case, system* (1.2) *is Hamiltonian system whose Hamiltonian function is $(y^2 + \frac{a}{2}x^3)^2 + (1 - \frac{a^2}{4})x^6$.*

(ii) *Assume that $\sigma = -1$ with $a \in \mathbb{R}$ or $\sigma = 1$ with $a \notin [-2, 2]$, then system* (1.2) *is polynomially integrable if, and only if, $d/b \in \mathbb{Q} \cap (-1, 1)$, with $b = +\sqrt{a^2 - 4\sigma}$. Moreover, in such a case, if we write $d/b = (m_2 - m_1)/(m_1 + m_2)$ with m_1, m_2 coprime natural numbers, then $I_M = (y^2 + \frac{1}{2}(a - b)x^3)^{m_1}(y^2 + \frac{1}{2}(a + b)x^3)^{m_2}$ is the polynomial primitive first integral of system* (1.2).

Proof. From ([31], Theorem 16) if the origin of system (1.2) is an isolated singular point of $\mathbf{F}_7$ then the irreducible factors of $h = y^4 + \sigma x^6 + ax^3y^2$ are irreducible invariant curves at the origin. We study the factors of h:

For $\sigma = 1$ and $a \in [-2, 2]$ h has an irreducible polynomial only. From ([32], Theorem 3.1), $\mathbf{F}_7$ is polynomially integrable if, and only if, it is a Hamiltonian vector field, i.e., $d = 0$.

For either $\sigma = -1$ and $a \in \mathbb{R}$, or $\sigma = 1$ and $a \notin [-2, 2]$, the irreducible factors of h are $y^2 + \frac{1}{2}(a - b)x^3$ and $y^2 + \frac{1}{2}(a + b)x^3$. So, a polynomial primitive first integral of $\mathbf{F}_7$ is, if it exists, $I_M = (y^2 + \frac{1}{2}(a - b)x^3)^{m_1}(y^2 + \frac{1}{2}(a + b)x^3)^{m_2}$ with $M = 6(m_1 + m_2)$. The equation $F_7(I_M) = 0$ becomes

$$0 = 6\left[(m_1 - m_2)b + d(m_1 + m_2)\right]I_Mx^2y.$$

Therefore, system (1.2) has a polynomial first integral if $d = \frac{(m_2 - m_1)}{m_1 + m_2}b$. Therefore we have that $d/b \in \mathbb{Q} \cap (-1, 1)$. ■

Any analytic vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t} = (2, 3)$ of successive degrees. Thus, the analytic system (1.3) can be written in the form $\dot{\mathbf{x}} = \mathbf{F}_7 + \mathbf{F}_8 + \cdots$, where $\mathbf{F}_j = (P_{j+2}, Q_{j+3})^T \in \mathcal{Q}_j^{\mathbf{t}}$. The polynomial integrability of the lowest-degree quasi-homogeneous term of an analytic vector field is a necessary condition of formal integrability of the analytic vector field. Moreover by ([21], Theorem A), the existence of a formal first integral assures the existence of an analytic first integral, Therefore, from view of Proposition 2.1, to solve the analytic integrability of system (1.3) it is enough to study the following two cases:

Case A. The system $\dot{\mathbf{x}} = \mathbf{F}_7 + \mathbf{F}_8 + \cdots$, with

$$\mathbf{F}_7 = \begin{pmatrix} -4y^3 - 2ax^3y \\ 6x^5 + 3ax^2y^2 \end{pmatrix} \in \mathcal{Q}_7^{(2,3)} \tag{2.4}$$

and $a \in [-2, 2]$.

Case B. The system $\dot{\mathbf{x}} = \mathbf{F}_7 + \mathbf{F}_8 + \cdots$, with

$$\mathbf{F}_7 = \begin{pmatrix} -4y^3 + 2(d-a)x^3y \\ 6\sigma x^5 + 3(a+d)x^2y^2 \end{pmatrix} \in \mathcal{Q}_7^{(2,3)} \tag{2.5}$$

and either $\sigma = -1$ and $a \in \mathbb{R}$ or $\sigma = 1$ and $a \notin [-2, 2]$ with $d = \frac{m_2 - m_1}{m_1 + m_2} b$, and $b = +\sqrt{a^2 - 4\sigma}$ where m_1, m_2 are coprime natural numbers.

2.1 Analytic integrability of system (2.4)

The following result was given in [28]. In such work the authors characterize when system (2.4) is analytically integrable providing an orbitally equivalent normal form.

Theorem 2.2 ([28], Theorem 5.25) *The system* (2.4) *is analytically integrable if, and only if, it is orbitally equivalent to a Hamiltonian system*

$$\begin{aligned} \dot{x} &= -4y^3 - 2ax^3y - 2\beta_9 x^4 y, \\ \dot{y} &= 6x^5 + 3ax^2y^2 + 4\beta_9 x^3 y^2, \end{aligned} \tag{2.6}$$

being β_9 a real number introduced by Proposition 4.12*. Moreover, the primitive first integral is of the form* $(y^2 + \frac{a}{2}x^3)^2 + (1 - \frac{a^2}{4})x^6 + \cdots$.

Here we prove that the existence of a formal inverse integrating factor for system (2.4) is also a necessary and sufficient condition for the analytic integrability of system (2.4).

Theorem 2.3 *The system* (2.4) *is analytically integrable if, and only if, it has a formal inverse integrating factor* $h + \cdots \in \mathbf{C}(x, y)$ *with* $h = (y^2 + \frac{a}{2}x^3)^2 + (1 - \frac{a^2}{4})x^6$.

Proof. First we see the necessary condition. We assume that system (2.4) is analytically integrable. From Theorem 2.2, it is orbitally equivalent to the system (2.6) with $h^* = (y^2 + \frac{a}{2}x^3)^2 + (1 - \frac{a^2}{4})x^6 + \beta_9 x^4 y^2$ a polynomial inverse integrating factor of system (2.6). So, performing the near-identity change which transforms system (2.6) into system (2.4), it transforms h^* into an inverse integrating factor $h + \cdots$ of system (2.4).

Now we prove the sufficiency. From Proposition 4.12, system (2.4) is orbitally equivalent to system (4.17),

$$\dot{\mathbf{x}} = \mathbf{G} = \mathbf{F}_7 + \mu_8 \mathbf{D}_0^{\mathbf{t}} + \mathbf{G}_9 + \sum_{j \geq 10} \mu_j \mathbf{D}_0^{\mathbf{t}}$$

with $\mathbf{F}_7 = \mathbf{X}_{(y^2 + \frac{a}{2}x^3)^2 + (1 - \frac{a^2}{4})x^6}$, $\mathbf{G}_9 = \mathbf{X}_{\beta_9 x^4 y^2} + \alpha_9 x^3 y \mathbf{D}_0^{\mathbf{t}}$ and $\mu_j \in \mathrm{Cor}(\ell_j)$ where ℓ_j is the linear operator defined in (4.16). The quasi-homogeneous polynomial h is a polynomial primitive first integral of system $\dot{\mathbf{x}} = \mathbf{F}_7$ and h is also an inverse integrating factor of system since it is a Hamiltonian system. So, if H^* is a formal inverse integrating factor of $\mathbf{G}$, then $H^* = H_{12}^* + H_{13}^* + \cdots$ with $H_{12}^* = h$. It holds that H^* satisfies the equation $G(H^*) - \operatorname{div}(\mathbf{G})H^* = 0$. We impose the above equation degree to degree. The equation to degree 20 is

$$0 = F_7\left(H_{13}^*\right) + \mu_8 D_0^{\mathbf{t}}\left(H_{12}^*\right) - 13H_{12}^*\mu_8 = F_7\left(H_{13}^*\right) - \mu_8 h,$$

that is $\ell_{20}\left(H_{13}^*\right) = \mu_8 h$. From Lemma 4.11, we can choose $\operatorname{Cor}\left(\ell_{20}\right)$ such that $\mu_8 h \in \operatorname{Cor}\left(\ell_{20}\right)$. Moreover, the above equation holds if $\mu_8 h \in \operatorname{Range}\left(\ell_{20}\right) \cap \operatorname{Cor}\left(\ell_{20}\right)$ which implies $\mu_8 = 0$. On the other hand, $H_{13}^* \in \operatorname{Ker}\left(\ell_{20}\right)$. Let note that $\operatorname{Ker}(\ell_{12j+k}) = \operatorname{span}\left\{h^j\right\}$; otherwise, $\operatorname{Ker}(\ell_{j+7}) = \{0\}$. Therefore, $\operatorname{Ker}\left(\ell_{13}\right) = \{0\}$ i.e., $H_{13}^* = 0$.

If we write $H_{14}^* = c_{70}x^7 + c_{42}x^4y^2 + c_{14}xy^4$, the condition for degree 21 it arrives to $c_{70} = 0, c_{42} = \beta_9, c_{14} = 0$ and $\alpha_9 = 0$. So, the polynomial $h^* = h + \beta_9 x^4y^2$ is an inverse integrating factor and a primitive first integral of $\mathbf{F}_7 + \mathbf{G}_9$. Therefore, the inverse integrating factors of $\mathbf{F}_7 + \mathbf{G}_9$ starting by h are of the form $\hat{H} = h^* f(h^*)$ with $f(\mathbf{0}) = 1$. From the equation $(F_7 + \mathbf{G}_9)(\hat{H}) = \operatorname{div}(\mathbf{G}_9)\hat{H}$ with $\hat{H} = H_{12} + \cdots$, we obtain the relation

$$F_7(H_{N+2}) + \mathbf{G}_9(H_N) = 0, \quad \forall N \geq 12. \tag{2.7}$$

Now we prove that $\mu_i = 0$ for all $i > 9$. Otherwise, let $j_0 = \min\{i > 9\colon \mu_i \neq 0\}$. Denoting by $\mathcal{J}^k H$ and $\mathcal{J}^k H^*$ the k-jet quasi-homogeneous of H and H^* and taking into account that H^* is an inverse integrating factor of $\mathbf{G} = \mathbf{F}_7 + \mathbf{G}_9 + \mu_{j_0}\mathbf{D}_0^{\mathbf{t}} + \cdots$, we obtain that

$$\mathcal{J}^{j_0+4}H^* = \mathcal{J}^{j_0+4}(H), \quad \mathcal{J}^{j_0+5}H^* = \mathcal{J}^{j_0+5}(H) + H_{j_0+5}^*,$$

and $\mathcal{J}^{j_0+5}H^*$ satisfies $(F_7 + G_9)(H^*) = 0$ up to order $j_0 + 11$. Moreover

$$F_7\left(H_{j_0+5}^*\right) + F_7\left(H_{j_0+5}\right) + G_9(H_{j_0+3}) + \mu_{j_0}D_0(h) = \operatorname{div}(\mu_{j_0}D_0)h.$$

By (2.7), we obtain that $F_7\left(H_{j_0+5}^*\right) = (j_0 - 7)\mu_{j_0}h$, that is, $\ell_{j_0+12}\left(H_{j_0+5}^*\right) = (j_0 - 7)\mu_{j_0}h$. Therefore, $\mu_{j_0} = 0$ and we arrive to contradiction. Thus, we have that system (2.4) is orbitally equivalent to $\dot{\mathbf{x}} = \mathbf{F}_7 + \mathbf{G}_9$. Applying Theorem 2.2, the result follows. ■

2.2 Analytic integrability of system (2.5)

Here we characterize the analytically integrable system (2.5). Note that the leading quasi-homogeneous term is a non-Hamiltonian vector field. In what follows, we denote $f_6 = y^2 + \frac{1}{2}(a-b)x^3 \in \mathcal{P}_6^{\mathbf{t}}$ and $g_6 = y^2 + \frac{1}{2}(a+b)x^3 \in \mathcal{P}_6^{\mathbf{t}}$.

Theorem 2.4 *The system* (2.5) *is analytically integrable if, and only if, it is orbitally equivalent to*

$$\begin{aligned} \dot{x} &= -4y^3 + 2(d-a)x^3y - \frac{14m_1\beta_9}{3m_1 + 4m_2}x^4y, \\ \dot{y} &= 6\sigma x^5 + 3(d+a)y^2x^2 + \frac{7\beta_9(a-b)}{2}x^6 + \frac{28m_2\beta_9}{3m_1 + 4m_2}y^2x^3, \end{aligned} \tag{2.8}$$

with β_9 a real number introduced by Proposition 4.13*. Moreover, the primitive first integral is of the form $(f_6 + \cdots)^{m_1}(g_6 + \cdots)^{m_2}$.*

Proof. First we see the sufficient condition. We assume that system (2.5) is orbitally equivalent to the system (2.8). It is easy to check that

$$J = f_6^{m_1}\left(g_6 + c_{40}x^4\right)^{m_2} \tag{2.9}$$

with $c_{4,0} = \frac{7(m_1+m_2)}{2(3m_1+4m_2)}\beta_9$, is a polynomial first integral of system (2.8). Hence, performing the near-identity change that transforms system (2.8) into system (2.5), we have that $I = (f_6 + \cdots)^{m_1}(g_6 + \cdots)^{m_2}$ is a formal first integral of system (2.5). By ([21], Theorem A), system (2.5) has also an analytic first integral, see also [11].

Now we prove the necessity. From Proposition 4.13, system (2.5) is orbitally equivalent to system (4.18), $\dot{\mathbf{x}} = \mathbf{G} = \mathbf{F}_7 + \mu_8\mathbf{D}_0^{\mathbf{t}} + \mathbf{G}_9 + \sum_{j\geq 10}\mu_j\mathbf{D}_0^{\mathbf{t}}$ with $\mathbf{F}_7 = \mathbf{X}_{f_6 g_6} + \mathrm{d}x^2 y\mathbf{D}_0^{\mathbf{t}}$, $\mathbf{G}_9 = \mathbf{X}_{\beta_9 x^4 f_6} + \alpha_9 x^3 y\mathbf{D}_0^{\mathbf{t}}$ and $\mu_j \in \mathrm{Cor}(\ell_j)$ where ℓ_j is the linear operator defined by (4.16). By Proposition 2.1, we have that $I_M = f_6^{m_1} g_6^{m_2}$ is a polynomial primitive first integral of system $\dot{\mathbf{x}} = \mathbf{F}_7$. Moreover, if system (2.5) is analytically integrable, then system (4.18) is formally integrable and from Lemma 4.11, we also can assume that $\mathrm{Cor}(\ell_{j+M}) = I_M\mathrm{Cor}(\ell_j)$.

From ([31], Theorem 17), the primitive first integral of $\mathbf{G}$ is of the form $I = (f_6 + \cdots)^{m_1}(g_6 + \cdots)^{m_2}$, *i.e.* $I = I_M + \sum_{i>M} I_i$ with $I_M = f_6^{m_1} g_6^{m_2}$, $M = 6m_1 + 6m_2$ and $I_i \in \mathcal{P}_i^{\mathbf{t}}$.

Next, we impose that the equation $G(I) = 0$ is satisfied degree by degree. Integrability condition to degree $M + 8$ is

$$0 = \left(G(I)\right)_{M+8} = F_7(I_{M+1}) + \mu_8 D_0^{\mathbf{t}}(I_M) = F_7(I_{M+1}) + M\mu_8 I_M.$$

On one side, the above equation holds, if $\mu_8 I_M \in \mathrm{Range}(\ell_{M+8}) \cap \mathrm{Cor}(\ell_{M+8}) = \{0\}$ which implies $\mu_8 = 0$. On the other hand, $I_{M+1} \in \mathrm{Ker}(\ell_{M+8})$. Let note that $\mathrm{Ker}(\ell_{jM+7}) = \mathrm{span}\left\{I_M^j\right\}$; otherwise, $\mathrm{Ker}(\ell_{j+7}) = \{0\}$. Therefore, $\mathrm{Ker}(\ell_{M+8}) = \{0\}$ i.e., $I_{M+1} = 0$.

Taking into account that f_6 is an invariant curve of system (4.18), we obtain that the primitive first integral is $I = f_6^{m_1}(g_6 + c_{21}x^2 y + (c_{40}x^4 + c_{12}xy^2) + \cdots)^{m_2}$. Thus, $I_M = f_6^{m_1} g_6^{m_2}$, $I_{M+1} = m_2 f_6^{m_1} g_6^{m_2-1} c_{21}x^2 y$. As $I_{M+1} = 0$ we obtain that $c_{21} = 0$. Moreover, $I_{M+2} = m_2 f_6^{m_1} g_6^{m_2-1}(c_{40}x^4 + c_{12}xy^2)$.

The integrability condition for degree $M + 9$, $F_7(I_{M+2}) + G_9(I_M) = 0$ give us

$$c_{12} = 0,\ c_{40} = \frac{7(m_1 + m_2)}{2(3m_1 + 4m_2)}\beta_9,\ \alpha_9 = \frac{4(m_2 - m_1)}{3m_1 + 4m_2}\beta_9.$$

We denote $\hat{\mathbf{G}}_9 = \mathbf{X}_{\beta_9(y^2 + \frac{1}{2}(a-b)x^3)x^4} + \frac{4(m_2-m_1)}{3m_1+4m_2}\beta_9 x^3 y\mathbf{D}_0^{\mathbf{t}}$. The analytic first integrals of $\mathbf{F}_7 + \hat{\mathbf{G}}_9$ are $I = \Psi(J) = J + a_2 J^2 + a_3 J^3 + \cdots$ where J is defined in (2.9) with Ψ any analytic function with $\Psi(\mathbf{0}) = 1$. Thus

$$F_7\left(\Psi(J)_{N+2}\right) + \hat{G}_9\left(\Psi(J)_N\right) = 0,\quad \forall N \geq M. \tag{2.10}$$

We now prove that $\mu_i = 0$ for all $i > 9$. Otherwise, let $j_0 = \min\{i > 9 \colon \mu_i \neq 0\}$.Now we suppose that $\mathbf{G} = \mathbf{F}_7 + \hat{\mathbf{G}}_9 + \mu_{j_0}\mathbf{D}_0^{\mathbf{t}} + \cdots$ is analytically integrable, satisfying $\mathcal{J}^{j_0-1}\mathbf{G} = \mathcal{J}^{j_0-1}(\mathbf{F}_7 + \hat{\mathbf{G}}_9)$, then there exists a first integral I^* of $\mathbf{G}$, such that

$$\mathcal{J}^{j_0+M-8}I^* = \mathcal{J}^{j_0+M-8}(\Psi(J)),$$
$$\mathcal{J}^{j_0+M-7}I^* = \mathcal{J}^{j_0+M-7}(\Psi(J)) + I^*_{j_0+M-7},$$

and $\mathcal{J}^{j_0+M-7}I^*$ is a first integral of $\mathbf{F}_7 + \hat{\mathbf{G}}_9$ up to order $j_0 + M - 1$. Moreover

$$F_7\left(I^*_{j_0+M-7}\right) + F_7\left(\Psi(J)_{j_0+M-7}\right) + \hat{G}_9\left(\Psi(J)_{j_0+M-9}\right) + \mu_{j_0}D_0(I_M) = 0.$$

From (2.10), $\hat{G}_9\left(\Psi(J)_{j_0+M-9}\right) = -F_7\left(\Psi(J)_{j_0+M-7}\right)$. Therefore, $F_7\left(I^*_{j_0+M-7}\right) = -MI_M\mu_{j_0}$, that is, $\ell_{j_0+M}\left(I^*_{j_0+M-7}\right) = -MI_M\mu_{j_0}$. We arrive to a contradiction since the term on the left side of the above equation belongs to $\mathrm{Range}(\ell_{j_0+M})$ and the term on the right side belongs to $\mathrm{Cor}(\ell_{j_0+M})$. Therefore, $\mu_{j_0} = 0$ and system (2.5) is orbitally equivalent to system (2.8). ■

Remark. The existence of polynomial orbitally equivalent normal forms for the analytically integrable system (2.4) and (2.5) allows us to solve the analytical integrability problem, see Theorems 2.2 and 2.4. These normal forms have been essential to the success of our research. In view of these results, we propose the following problem as the objective of our research: Is it true that any analytically integrable vector field is orbitally equivalent to a polynomial normal form?

The quasi-homogeneous polynomials $h = f_6 g_6 \in \mathcal{P}^{\mathbf{t}}_{12}$ and $I_M = f_6^{m_1} g_6^{m_2} \in \mathcal{P}^{\mathbf{t}}_M$, with $M = 6(m_1 + m_2)$, are an inverse integrating factor and a first integral of $\mathbf{F}_7$. Therefore, the inverse integrating factors of $\mathbf{F}_7$ are hI_M^q for any q. In particular, if q satisfies that $(1 + m_1 q)(1 + m_2 q) \geq 0$ and Mq is not natural, the function

$$V = f_6^{(1+m_1 q)} g_6^{(1+m_2 q)}$$

is an *irrational* inverse integrating factor of $\mathbf{F}_7$, which is a quasi-homogeneous function of rational (non-natural) degree $12 + Mq$. So, if m is an integer such that Mqm is a natural number, then $W_s := (V)^m$ is a quasi-homogeneous polynomial of degree $s = m(12 + Mq)$.

Fixing a integer m, we define the following linear operator which we will use in the proof of Theorem 2.6,

$$\tilde{\ell}_j : \mathcal{P}^{\mathbf{t}}_{j-7} \to \mathcal{P}^{\mathbf{t}}_j \tag{2.11}$$
$$p_{j-7} \to F_7(p_{j-7}) - m\mathrm{div}(\mathbf{F}_7) \cdot p_{j-7}, \ \ j > 7.$$

Now we present the following technical result.

Lemma 2.5 *Consider system* (2.5)*. We can choose the complementary subspaces to* $\mathrm{Range}(\tilde{\ell}_{j+s})$ *and* $\mathrm{Range}(\ell_j)$ *such that* $\mathrm{Cor}\left(\tilde{\ell}_{j+s}\right) = W_s \mathrm{Cor}\left(\ell_j\right)$*, for* $j \geq 10$.

Proof. We remark that $\tilde{\ell}_j(p_{j-7}) = \left(F_7 - \frac{12md}{j-7} x^2 y D_0\right)(p_{j-7})$, i.e., the operator $\tilde{\ell}_{j+s}$ is the operator ℓ_{j+s} changing d by $d - \frac{12md}{j+s-7}$. We distinguish two cases: For $j \neq 6l + 7$, by Proposition 4.14, $\mathrm{Cor}\left(\ell_{j+s}\right) = \mathrm{Cor}\left(\tilde{\ell}_{j+s}\right)$. Moreover, if we take the same basis as in the proof of Proposition 4.14, the matrices associated with the operator $\tilde{\ell}_j$ are upper triangular matrices and all their elements are distinct. Consequently, all diagonal elements are zero, except for at most one. Therefore, it always is possible to choose a basis of $\mathrm{Cor}\left(\tilde{\ell}_{j+s}\right)$ such that its elements are a multiple of W_s.

Moreover, by Proposition 4.14, $\mathrm{Cor}\left(\tilde{\ell}_{j+s}\right)$ and $W_s \mathrm{Cor}\left(\ell_j\right)$ have the same dimension since s is a multiple of 6.

For $j = 6l + 7$, if $l \neq Mj_0$, then $\mathrm{Ker}\left(\ell_{j+7}\right) = \{0\}$ and $\mathrm{Ker}\left(\tilde{\ell}_{j+s+7}\right) = \{0\}$. Otherwise, if $l = Mj_0$, then $\mathrm{Ker}\left(\ell_{6Mj_0+7}\right) = \left\{I_M^{j_0}\right\}$ and $\mathrm{Ker}\left(\tilde{\ell}_{6Mj_0+s+7}\right) = \left\{W_s I_M^{j_0}\right\}$. Therefore, $\mathrm{Cor}\left(\tilde{\ell}_{j+s}\right) = W_s \mathrm{Cor}\left(\ell_j\right)$. ■

In the next result we characterize when system (2.5) is analytically integrable through the existence of an algebraic inverse integrating factor.

Theorem 2.6 *The system* (2.5) *is analytically integrable if, and only if, there exists an inverse integrating factor* $V^* = (f_6 + \cdots)^{1+m_1 q}(g_6 + \cdots)^{1+m_2 q}$ *with* $6(m_1 + m_2)q \notin \mathbb{N}$, $(1 + m_1 q)(1 + m_2 q) \geq 0$ *and* $q \in \mathbb{Q}$.

Proof. First we see the necessary condition. We assume that system (2.5) is analytically integrable. From Theorem 2.4, it is orbitally equivalent to the system (2.8) and $J = f_6^{m_1}\left(g_6 + c_{4,0}x^4\right)^{m_2}$ is a first integral of system (2.8) with $c_{4,0} = \frac{7(m_1+m_2)}{2(3m_1+4m_2)}\beta_9$. It is easy to check that hJ^q with $h = f_6 g_6$ is an inverse integrating factor of system (2.8), for any q. We can choose q such that hJ^q be irrational function and $Mq \notin \mathbb{N}$, $(1 + m_1 q)(1 + m_2 q) \geq 0$. So, performing the near-identity change of variables and the scaling of time which transform system (2.8) into system (2.5), also transform hJ^q into $V^* = (f_6 + \cdots)^{1+m_1 q}(g_6 + \cdots)^{1+m_2 q}$ where V^* is a non-formal inverse integrating factor of system (2.5).

Now we prove the sufficiency. From Proposition 4.13, system (2.5) is orbitally equivalent to system (4.18), $\dot{\mathbf{x}} = \mathbf{G} = \mathbf{F}_7 + \mu_8 \mathbf{D}_0^{\mathbf{t}} + \mathbf{G}_9 + \sum_{j\geq 10} \mu_j \mathbf{D}_0^{\mathbf{t}}$ with $\mathbf{F}_7 = \mathbf{X}_h + \mathrm{d}x^2 y \mathbf{D}_0^{\mathbf{t}}$, $\mathbf{G}_9 = \mathbf{X}_{\beta_9 x^4 f_6} + \alpha_9 x^3 y \mathbf{D}_0^{\mathbf{t}}$ and $\mu_j \in \mathrm{Cor}\left(\ell_j\right)$ where ℓ_j is the linear operator defined by (4.16). If system (2.5) has a non-formal inverse integrating factor then system (4.18) has a non-formal inverse integrating factor V^*.

The function $W^* := (V^*)^m = W_s + \cdots$ is a formal function and the leading quasi-homogeneous term of W^* is $W_s = h^m (I_M)^{mq}$ with $s = m(12 + Mq) \in \mathbb{N}$.

V^* is an inverse integrating factor of $\mathbf{G}$ if, and only if, W^* satisfies the equation $G(W^*) - m\mathrm{div}(G)W^* = 0$. We impose the above equation degree by degree. The equation of degree $s+8$ is

$$\begin{aligned} 0 &= F_7\left(W^*_{s+1}\right) + \mu_8 D_0^{\mathbf{t}}(W_s) - m\mathrm{div}(\mathbf{F}_7)W^*_{s+1} - 13mW_s\mu_8 \\ &= F_7\left(W^*_{s+1}\right) - m\mathrm{div}(\mathbf{F}_7)W^*_{s+1} + (s-13m)\mu_8 W_s = \tilde{\ell}_{s+8}\left(W^*_{s+1}\right) + (s-13m)\mu_8 W_s, \end{aligned}$$

with $s - 13m = m(1-Mq) \neq 0$. From Lemma 2.5, $\mu_8 W_s \in \mathrm{Cor}\left(\tilde{\ell}_{s+8}\right)$. Hence, on one side, the above equation holds, if $\mu_8 W_s \in \mathrm{Range}\left(\tilde{\ell}_{s+8}\right) \cap \mathrm{Cor}\left(\tilde{\ell}_{s+8}\right) = \{0\}$ which implies $\mu_8 = 0$. On the other hand, $W^*_{s+1} \in \mathrm{Ker}\left(\ell_{s+8}\right)$. Notice that $\mathrm{Ker}(\tilde{\ell}_{jM+s+7}) = W_s\mathrm{span}\left\{I^j_M\right\}$; otherwise, $\mathrm{Ker}(\tilde{\ell}_{j+7}) = \{0\}$. Therefore, $\mathrm{Ker}\left(\tilde{\ell}_{s+8}\right) = \{0\}$ i.e., $W^*_{s+1} = 0$.

Taking into account that f_6 is an invariant curve of system (4.18), we obtain that $W^* = f_6^{m(1+m_1q)}(g_6 + c_{21}x^2y + \left(c_{40}x^4 + c_{12}xy^2\right) + \cdots)^{m(1+m_2q)}$. Moreover $c_{21} = 0$ since $W^*_{s+1} = 0$. The condition for degree $s+9$ give us

$$c_{12} = 0,\ c_{4,0} = \frac{7(m_1+m_2)}{2(3m_1+4m_2)}\beta_9,\ \alpha_9 = \frac{4(m_2-m_1)}{3m_1+4m_2}\beta_9.$$

We denote by $\hat{\mathbf{G}}_9 = \mathbf{X}_{\beta_9 x^4 f_6} + \frac{4(m_2-m_1)}{3m_1+4m_2}\beta_9 x^3 y \mathbf{D}_0^{\mathbf{t}}$. The polynomial inverse integrating factor of $\mathbf{F}_7 + \hat{\mathbf{G}}_9$ is $V = hJ^q$. Thus $W = h^m J^{mq}$ satisfies

$$F_7(W_{N+2}) + \hat{G}_9(W_N) = m\mathrm{div}(\mathbf{F}_7)W_{N+2} + m\mathrm{div}(\mathbf{G}_9)W_N,\quad \forall N \geq s. \tag{2.12}$$

We now prove that $\mu_i = 0$ for all $i > 9$. Otherwise, let $j_0 = \min\{i > 9\colon \mu_i \neq 0\}$.

We suppose that V^* is an inverse integrating factor of $\mathbf{G} = \mathbf{F}_7 + \hat{\mathbf{G}}_9 + \mu_{j_0}\mathbf{D}_0^{\mathbf{t}} + \cdots$, then

$$\begin{aligned} \mathcal{J}^{j_0+s-8}W^* &= \mathcal{J}^{j_0+s-8}(W), \\ \mathcal{J}^{j_0+s-7}W^* &= \mathcal{J}^{j_0+s-7}(W) + W^*_{j_0+s-7}, \end{aligned}$$

and $\mathcal{J}^{j_0+s-7}W^*$ satisfies $(F_7 + \hat{G}_9)(W^*) - m\mathrm{div}(\mathbf{F}_7 + \hat{\mathbf{G}}_9)W^* = 0$ up to order $j_0 + s - 1$. Moreover,

$$\begin{aligned} &F_7\left(W^*_{j_0+s-7}\right) + F_7\left(W_{j_0+s-7}\right) + \hat{G}_9(W_{j_0+s-9}) + \mu_{j_0}D_0(W_s) \\ &\quad = m\mathrm{div}(\mathbf{F}_7)W^*_{j_0+s-7} + m\mathrm{div}(\mathbf{F}_7)W_{j_0+s-7} + m\mathrm{div}(\mathbf{G}_9)W_{j_0+s-9} + m\mathrm{div}(\mu_{j_0}D_0)W_s. \end{aligned}$$

From (2.12), we have that

$$F_7\left(W^*_{j_0+s-7}\right) + \mu_{j_0}D_0(W_s) = m\mathrm{div}(\mathbf{F}_7)W^*_{j_0+s-7} + m\mathrm{div}(\mu_{j_0}D_0)W_s.$$

So, $F_7\left(W^*_{j_0+s-7}\right) - m\mathrm{div}(\mathbf{F}_7)W^*_{j_0+s-7} = (m(j_0+5)-s)\mu_{j_0}W_s$, *i.e.* $\tilde{\ell}_{j_0+s}\left(W^*_{j_0+s-7}\right) = (m(j_0+5)-s)\mu_{j_0}W_s$. As $m(j_0+5) - s = m(j_0 - 7 - Mq) \neq 0$, we obtain that $\mu_{j_0} = 0$ and we arrive to contradiction.

Thus, system (2.5) is orbitally equivalent to system (2.8) and the result follows applying Theorem 2.4. ∎

Remark. The irrational function $V^* = (y^2 - x^3)^{4/3}(y^2 + x^3)^{3/2}$ is an inverse integrating factor of the system $\dot{x} = -4y^3 + \frac{4}{5}x^3y + 2x^7,\ \dot{y} = -6x^5 + \frac{6}{5}x^2y^2 + 3x^6y$. This system can be written as a system (2.5), $\dot{\mathbf{x}} = \mathbf{F}_7 + \mathbf{F}_{12}$ with $\sigma = -1, a = -2, m_1 = 2, m_2 = 3$ and $\mathbf{F}_{12} = x^6\mathbf{D}_0^t$ where $x^6 \in \mathrm{Cor}\left(\ell_{12}\right)$. From Theorem 2.4, the system does not have an analytic first integral. In this case, V^* is an inverse integrating factor provided by Theorem 2.6 with $q = \frac{1}{6}$ and thus $6(m_1 + m_2)q = 5 \in \mathbb{N}$. Therefore, the condition on the degree of the lowest degree quasi-homogeneous term of V^* is a necessary condition.

If we take $q = -\frac{1}{m_2}$ (or $q = -\frac{1}{m_1}$) the inverse integrating factor provided in Theorem 2.6 is a power of one invariant curve.

Corollary 2.7 *The system* (2.5) *is analytically integrable if, and only if, the system has an inverse integrating factor* $(f_6 + \cdots)^{\frac{m_2-m_1}{m_2}}$.

Proof. First we see the necessary condition. We assume that system (2.5) has an analytic first integral. On one hand, by ([31], Theorem 17), the system has only two invariant curves at origin, $f_6^* = f_6 + \cdots$ and $g_6^* = g_6 + \cdots$, and the primitive first integral is of the form $I = \left(f_6^*\right)^{m_1}\left(g_6^*\right)^{m_2}$. On the other hand, from Theorem 2.6, the system has an inverse integrating factor $V^* = \left(f_6^*\right)^{1+m_1q}\left(g_6^*\right)^{1+m_2q}$ with $6(m_1+m_2)q \notin \mathbb{N}$, $(1+m_1q)(1+m_2q) \geq 0$ and $q \in \mathbb{Q}$. Recalling that the product of a first integral and an inverse integrating factor is also an inverse integrating factor, we obtain that V^*I^{-q-1/m_2} is an inverse integrating factor of the form $(f_6+\cdots)^{\frac{m_2-m_1}{m_2}}$.

The sufficient condition follows applying Theorem 2.6 for $q = -\frac{1}{m_2}$. ■

3 Applications

Theorem 3.8 *The polynomial differential system*

$$\begin{aligned}\dot{x} &= -4y^3 - 2x^3y + a_{50}x^5 + a_{22}x^2y^2\\ \dot{y} &= 6x^5 + 3x^2y^2 + b_{41}x^4y + b_{13}xy^3\end{aligned} \tag{3.13}$$

is analytically integrable if, and only if, $b_{41} + 5a_{50} = 3b_{13} + 2a_{22} = 0$.

Proof. The system (3.13) can be written into the form $\mathbf{\dot{x}} = \mathbf{F}_7 + \mathbf{F}_8$ with

$$\mathbf{F}_7 = \mathbf{X}_{y^4+x^3y^2+x^6}, \qquad \mathbf{F}_8 = \begin{pmatrix} a_{50}x^5 + a_{22}x^2y^2\\ b_{41}x^4y + b_{13}xy^3\end{pmatrix}.$$

Indeed, it is a system (2.4) case A with $a = 1 \in [-2,2]$. From Proposition 2.1, $\mathbf{F}_7$ has a polynomial first integral since $d = 0$ and $a \in [-2,2]$.

To prove the sufficiency, it enough to check that when $b_{13} = -\frac{2}{3}a_{22}$ and $b_{41} = -5a_{50}$ the polynomial $I = y^4 + x^3y^2 + x^6 - \frac{1}{3}a_{22}x^2y^3 - a_{50}x^5y$ is a first integral of system (3.13).

Now we prove the necessary condition. A normal form of system (3.13) is

$$\mathbf{\dot{x}} = \mathbf{F}_7 + \left(\alpha_8^{(1)}x^4 + \alpha_8^{(2)}xy^2\right)\mathbf{D}_0^{\mathbf{t}} + \mathbf{X}_{\beta_9x^4y^2} + \alpha_9x^3y\mathbf{D}_0^{\mathbf{t}} + \cdots$$

The first coefficients are $\alpha_8^{(1)} = \frac{1}{13}(5a_{50} + b_{41})$ and $\alpha_8^{(2)} = \frac{1}{13}(2a_{22} + 3b_{13})$. Imposing $\alpha_8^{(1)} = \alpha_8^{(2)} = 0$, we obtain that $b_{13} = -\frac{2}{3}a_{22}$ and $b_{41} = -5a_{50}$. ■

Theorem 3.9 *The polynomial differential system*

$$\begin{aligned}\dot{x} &= -4y^3 - 4x^3y + a_{50}x^5 + a_{22}x^2y^2\\ \dot{y} &= 6x^5 + 9x^2y^2 + b_{41}x^4y + b_{13}xy^3\end{aligned} \tag{3.14}$$

is analytically integrable if, and only if, one of the following conditions is satisfied:

(i) $b_{41} + 8a_{50} = a_{50} + b_{13} = 2a_{22} - 5a_{50} = 0$,

(ii) $2b_{41} + 7a_{50} = b_{13} + 4a_{50} = a_{22} - 4a_{50} = 0$.

Proof. The system (3.14) can be written as $\mathbf{\dot{x}} = \mathbf{F}_7 + \mathbf{F}_8$ with

$$\mathbf{F}_7 = \mathbf{X}_{\left(y^2+\frac{1}{2}x^3\right)(y^2+2x^3)} - \frac{1}{2}x^2y\mathbf{D}_0^{\mathbf{t}}, \qquad \mathbf{F}_8 = \begin{pmatrix} a_{50}x^5 + a_{22}x^2y^2\\ b_{41}x^4y + b_{13}xy^3\end{pmatrix}.$$

In fact, it is a system (2.5) case B with $d = \frac{1}{2}$, $a = \frac{5}{2}$, $\sigma = 1$, $b = \frac{3}{2}$. From Proposition 2.1, $\mathbf{F}_7$ has a polynomial first integral since $\frac{d}{b} = \frac{1}{3} \in (-1,1)$. Moreover, the primitive polynomial first integral of $\mathbf{F}_7$ is $I_{18} = (x^3 + 2y^2)(y^2 + 2x^3)^2$.

First we prove the necessary condition. A normal form of system (3.14) is

$$\dot{\mathbf{x}} = \mathbf{F}_7 + \left(\alpha_8^{(1)}x^4 + \alpha_8^{(2)}xy^2\right)\mathbf{D}_0^{\mathbf{t}} + \mathbf{X}_{\beta_9 x^4(2y^2+x^3)} + \alpha_9 x^3 y \mathbf{D}_0^{\mathbf{t}} + \cdots$$

The first coefficients are

$$\alpha_8^{(1)} = \frac{1}{380}(169a_{50} - 18a_{22} + 12b_{13} + 14b_{41}),$$
$$\alpha_8^{(2)} = \frac{1}{95}(22a_{22} - 6a_{50} + 4b_{41} + 17b_{13}).$$

Imposing $\alpha_8^{(1)} = \alpha_8^{(2)} = 0$, we obtain that $a_{22} = \frac{1}{2}(4a_{50} - b_{13})$ and $b_{41} = -\frac{1}{2}(19a_{50} + 3b_{13})$.
For these values, α_9 and β_9 are given by

$$\alpha_9 = \frac{1}{420}\left(514a_{50}^2 + 445a_{50}b_{13} + 91b_{13}^2\right)$$
$$\beta_9 = \frac{1}{840}\left(3541a_{50}^2 + 3340a_{50}b_{13} + 679b_{13}^2\right).$$

So, by Theorem 2.4, if system (3.14) has an analytic first integral then

$$11\alpha_9 - 4\beta_9 = (a_{50} + b_{13})(4a_{50} + b_{13}) = 0.$$

If $b_{13} = -a_{50}$ we obtain the case **(i)**. If $b_{13} = -4a_{50}$ we have the case **(ii)**.
To prove the sufficiency, we provide an analytic first integral for each family.
For the case **(i)**, an analytic first integral is $I = (2x^3 + y^2)^2\left(2x^3 + 4y^2 - 3a_{50}x^2y\right)$.
For the case **(ii)**, an analytic first integral is $I = (x^3 + 2y^2)(4x^3 + 2y^2 - 3a_{50}x^2y)^2$. ■

4 Appendix: Orbital normal form for systems whose leading term is a quasi-homogeneous vector field

First we present the following orbital normal form provided in [22].

Theorem 4.10 *Let $\boldsymbol{F} = \boldsymbol{F}_r + \sum_{j\geq 1}\boldsymbol{F}_{r+j}$ with $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}_r^{\mathbf{t}}$ and $\mathbf{F}_{r+j} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$. If $\mathrm{Ker}\left(\ell_{r+j+|\mathbf{t}|}^{\mathrm{c}}\right) = \{0\}$ for all $j \in \mathbb{N}$ then $\boldsymbol{F}$ is orbitally equivalent to*

$$\mathbf{G} = \mathbf{F}_r + \sum_{j>0}\mathbf{G}_{r+j}, \ \ with\ \mathbf{G}_{r+j} = \mathbf{X}_{\delta_{r+j+|\mathbf{t}|}} + \eta_{r+j}\mathbf{D}_0^{\mathbf{t}} \in \mathcal{Q}_{r+j}^{\mathbf{t}},$$

with $\delta_{r+j+|\mathbf{t}|} \in \mathrm{Cor}\left(\ell_{r+j+|\mathbf{t}|}^{\mathrm{c}}\right)$ and $\eta_{r+j} \in \mathrm{Cor}\left(\ell_{r+j}\right)$, where $\mathrm{Cor}\left(\ell_{r+j+|\mathbf{t}|}^{\mathrm{c}}\right)$ is a complementary subspace to the range of the linear operator $\ell_{r+j+|\mathbf{t}|}^{\mathrm{c}}\colon \Delta_{j+|\mathbf{t}|} \to \Delta_{r+j+|\mathbf{t}|}$ defined by

$$\ell_{r+j+|\mathbf{t}|}^{\mathrm{c}}(\delta) = \mathrm{Proy}_{\Delta_{r+j+|\mathbf{t}|}}\left(\mathbf{F}_r - \frac{\mathrm{div}(\mathbf{F}_r)}{r+j+|\mathbf{t}|}D_0^{\mathbf{t}}\right)(\delta) \tag{4.15}$$

being $\Delta_{j+|\mathbf{t}|}$ a complementary subspace to $h\mathcal{P}_{j-r}^{\mathbf{t}}$, and $\mathrm{Cor}\left(\ell_{r+j}\right)$ is a complementary subspace to the range of the operator

$$\ell_{r+j}\colon \mathcal{P}_j^{\mathbf{t}} \to \mathcal{P}_{r+j}^{\mathbf{t}} \tag{4.16}$$
$$p_j \to F_r(p_j).$$

The following statement, which corresponds to Proposition 3.16 [33], establishes a cyclicity relation between the co-ranges of the operators ℓ_j when $\mathbf{F}_r$ has a polynomial first integral.

Lemma 4.11 *Assume that* $I_M \in \mathcal{P}_M^{\mathbf{t}}$ *is the primitive first integral of* $\mathbf{F}_r$*. Fixed* $j > r$ *with* $\mathcal{P}_{j-r}^{\mathbf{t}} \neq \{0\}$*, we can choose complementary subspaces to* Range(ℓ_j) *such that*

$$\mathrm{Cor}(\ell_{j+M}) = I_M \mathrm{Cor}(\ell_j).$$

Next, we apply Theorem 4.10 to obtain an orbital normal forms of system (2.4) and system (2.5).

Proposition 4.12 *An orbital normal form of system* (2.4) *is*

$$\dot{\mathbf{x}} = \mathbf{F}_7 + \mu_8 \mathbf{D}_0^{\mathbf{t}} + \mathbf{G}_9 + \sum_{j\geq 10} \mu_j \mathbf{D}_0^{\mathbf{t}} \tag{4.17}$$

with $\mathbf{F}_7 = \mathbf{X}_{(y^2+\frac{a}{2}x^3)^2+(1-\frac{a^2}{4})x^6}$, $\mathbf{G}_9 = \mathbf{X}_{\beta_9 x^4 y^2} + \alpha_9 x^3 y \mathbf{D}_0^{\mathbf{t}}$ *and* $\mu_j \in \mathrm{Cor}(\ell_j)$.

Proof. The linear operator (4.15) for system (2.4) corresponds to the operator ℓ_{j+12} restricted to Δ_{j+12} since the divergence of $\mathbf{F}_7$ is zero. Therefore, $\mathrm{Ker}\left(\ell_{j+12}^c\right) = \{\mathbf{0}\}$ for any j, and thus, the hypotheses of Theorem 4.10 are satisfied. We compute $\mathrm{Cor}(\ell_9)$. A basis of $\mathcal{P}_2^{\mathbf{t}}$ is $\mathcal{B}_2 = \{x\}$ and a basis of $\mathcal{P}_9^{\mathbf{t}}$ is $\mathcal{B}_9 = \{x^3 y,\ y^3\}$. As $\ell_9(x) = -4y^3$, we can choose $\mathrm{Cor}(\ell_9) = \mathrm{span}\{x^3 y\}$. Applying Theorem 4.10, we obtain the orbital normal form (4.17). ■

Proposition 4.13 *An orbital normal form of system* (2.5) *is*

$$\dot{\mathbf{x}} = \mathbf{F}_7 + \mu_8 \mathbf{D}_0^{\mathbf{t}} + \mathbf{G}_9 + \sum_{j\geq 10} \mu_j \mathbf{D}_0^{\mathbf{t}} \tag{4.18}$$

with $\mathbf{F}_7 = \mathbf{X}_{y^4+\sigma x^6+a x^3 y^2} + \mathrm{d}x^2 y \mathbf{D}_0^{\mathbf{t}}$, $\mathbf{G}_9 = \mathbf{X}_{\beta_9 (y^2+\frac{1}{2}(a-b)x^3)x^4} + \alpha_9 x^3 y \mathbf{D}_0^{\mathbf{t}}$ *and* $\mu_j \in \mathrm{Cor}(\ell_j)$.

Proof. The linear operator (4.15) for system (2.5) is

$$\ell_{j+12}^{c}\colon \Delta_{j+5} \to \Delta_{j+12}$$

$$p_{j+5} \to \mathrm{Proy}_{\Delta_{j+12}}\left(X_{y^4+\sigma x^6+a x^3 y^2} + \frac{dj}{j+12} x^2 y D_0^{\mathbf{t}}\right)(p_{j+5}),$$

where Δ_{j+5} is a complementary subspace to $h\mathcal{P}_{j-7}^{\mathbf{t}}$ and $h = y^4 + \sigma x^6 + a x^3 y^2$. We prove that the hypothesis of Theorem 4.10 are satisfied, that is $\mathrm{Ker}\left(\ell_{j+12}^c\right) = \{\mathbf{0}\}$ for any j. Indeed, if $j = 2$, we choose $\Delta_7 = \mathrm{span}\{x^2 y\}$ and $\Delta_{14} = \mathrm{span}\left\{x^7, x^4(y^2 + \frac{1}{2}(a-b)x^3)\right\}$. We have that $\ell_{14}^c(x^2 y) = (d-14)x^7 + \mathrm{d}x^4(y^2 + \frac{1}{2}(a-b)x^3)$. So, $\mathrm{Ker}\left(\ell_{14}^c\right) = \{0\}$ and we choose $\mathrm{Cor}\left(\ell_{14}^c\right) = \mathrm{span}\left\{x^4(y^2 + \frac{1}{2}(a-b)x^3)\right\}$. For $j \neq 2$, both subspaces Δ_{j+5} and Δ_{j+12} have dimension 2 and it is easy to check that the associated matrix to the operator ℓ_{j+12}^c is not singular and therefore $\mathrm{Ker}\left(\ell_{j+12}^c\right) = \{0\}$. Consequently, $\mathrm{Cor}\left(\ell_{j+12}^c\right)$ is also a trivial set.

We obtain an expression of $\mathrm{Cor}(\ell_9)$. A basis of $\mathcal{P}_2^{\mathbf{t}}$ is $\mathcal{B}_2 = \{x\}$ and a basis of $\mathcal{P}_9^{\mathbf{t}}$ is $\mathcal{B}_9 = \{x^3 y,\ y^3\}$. The transformed of the basis $\mathcal{B}_2$ is $\ell_9(x) = 2\mathrm{d}x^3 y - 4y^3$. So, we can choose $\mathrm{Cor}(\ell_9) = \mathrm{span}\{x^3 y\}$. Applying Theorem 4.10, we obtain the orbital normal form (4.18). ■

Note that the expression of the normal form depends on the choosing of the expression of the complementary subspaces to range of the operator ℓ_j. Next, we give a expression of them that we use for proving Theorem 2.6.

Proposition 4.14 *Consider system* (2.5) *and denote* $f_6 = y^2 + \frac{1}{2}(a-b)x^3$ *and* $g_6 = y^2 + \frac{1}{2}(a+b)x^3$*. A complementary subspace to the* $\mathrm{Range}(\ell_j)$, $j \geq 10$, *is*

$$\operatorname{Cor}(\ell_j) = \begin{cases} \langle f_6^l x, f_6^{l-1} g_6 x\rangle & si \quad j = 6l+2, \\ \langle f_6^l y\rangle & si \quad j = 6l+3, \\ \langle f_6^l x^2\rangle & si \quad j = 6l+4, \\ \langle f_6^l xy\rangle & si \quad j = 6l+5, \\ \langle f_6^{l+1}, f_6^l g_6\rangle & si \quad j = 6l+6, \\ \{0\} & si \quad j = 6l+7 \text{ and } l \neq (m_1+m_2)k, \text{for all } k \\ \langle f_6^l x^2 y\rangle & si \quad j = 6l+7 \text{ and } l = (m_1+m_2)k_0. \end{cases}$$

Proof. We study the following cases separately:

For $j = 6l+2$. A basis of $\mathcal{P}^{\mathbf{t}}_{j-7}$ is $\mathcal{B}_{6l-5} = \mathcal{B}_{6(l-2)+7} = \{f_6^{l-2-i} g_6^i x^2 y\}_{i=0}^{l-2}$ and a basis of $\mathcal{P}^{\mathbf{t}}_j$ is $\mathcal{B}_{6l+2} = \{f_6^{l-i} g_6^i x\}_{i=0}^{l}$. For $0 \le i \le l-2$, we obtain that

$$F_7(f_6^{l-2-i} g_6^i x^2 y) = A_i f_6^{l-i} g_6^i x + B_i f_6^{l-i-1} g_6^{i+1} x + C_i f_6^{l-i-2} g_6^{i+2} x$$

with

$$A_i = -\frac{(6l-5)(a+b)d}{b^2} + \frac{(12i-6l+5)(a+b)}{b}, \ (a+b)C_i - (a-b)A_i = \frac{14(a^2-b^2)}{b}, \ A_i + B_i + C_i = -16.$$

Thus, we can choose $\operatorname{Cor}(\ell_{6l+2}) = \langle f_6^l x, f_6^{l-1} g_6 x\rangle$.

For $j = 6l+3$. A basis of $\mathcal{P}^{\mathbf{t}}_{j-7}$ is $\mathcal{B}_{6(l-1)+2} = \{f_6^{l-1-i} g_6^i x\}_{i=0}^{l-1}$ and a basis of $\mathcal{P}^{\mathbf{t}}_j$ is $\mathcal{B}_{6l+3} = \{f_6^{l-i} g_6^i y\}_{i=0}^{l}$. In this case, for $0 \le i \le l-1$, we have that

$$F_7(f_6^{l-1-i} g_6^i x) = A_i f_6^{l-i} g_6^i y + (2-A_i) f_6^{l-i-1} g_6^{i+1} y$$

with $A_i = \frac{(3l-2)d}{b} - 6i + 3l - 2$. We can choose $\operatorname{Cor}(\ell_{6l+3}) = \langle f_6^l y\rangle$.

For $j = 6l+4$ with $l-1 < m_1+m_2$. A basis of $\mathcal{P}^{\mathbf{t}}_{j-7}$ is $\mathcal{B}_{6(l-1)+3} = \{f_6^{l-1-i} g_6^i y\}_{i=0}^{l-1}$ and a basis of $\mathcal{P}^{\mathbf{t}}_j$ is $\mathcal{B}_{6l+4} = \{f_6^{l-i} g_6^i x^2\}_{i=0}^{l}$. For $0 \le i \le l-1$, we obtain that

$$F_7(f_6^{l-1-i} g_6^i y) = A_i f_6^{l-i+1} g_6^i x^2 + \left(\frac{b-a}{b+a} A_i - 2(b-a)\right) f_6^{l-i} g_6^{i+1} x^2$$

with $A_i = \frac{(2l-1)(a+b)d}{b} - (4i-2l+1)(a+b)$. So, $\operatorname{Cor}(\ell_{6l+4}) = \langle f_6^l x^2\rangle$.

For $j = 6l+5$ with $l-1 < m_1+m_2$. A basis of $\mathcal{P}^{\mathbf{t}}_{j-7}$ is $\mathcal{B}_{6(l-1)+4} = \{f_6^{l-1-i} g_6^i x^2\}_{i=0}^{l-1}$ and a basis of $\mathcal{P}^{\mathbf{t}}_j$ is $\mathcal{B}_{6l+5} = \{f_6^{l-i} g_6^i xy\}_{i=0}^{l}$. For $0 \le i \le l-1$, we have that

$$F_7(f_6^{l-1-i} g_6^i x^2) = A_i f_6^{l-i+1} g_6^i xy + (4+A_i) f_6^{l-i} g_6^{i+1} xy$$

with $A_i = -\frac{(3l-1)d}{b} - 6i + 3l - 1$ and therefore $\operatorname{Cor}(\ell_{6l+5}) =< f_6^l xy >$.

For $j = 6l+6$ with $l-1 < m_1+m_2$. A basis of $\mathcal{P}^{\mathbf{t}}_{j-7}$ is $\mathcal{B}_{6(l-1)+5} = \{f_6^{l-1-i} g_6^i xy\}_{i=0}^{l-1}$ and a basis of $\mathcal{P}^{\mathbf{t}}_j$ is $\mathcal{B}_{6l+6} = \{f_6^{l+1-i} g_6^i\}_{i=0}^{l+1}$. In this case, for $0 \le i \le l-1$, we obtain that

$$F_7(f_6^{l-1-i} g_6^i xy) = A_i f_6^{l+1-i} g_6^i + B_i f_6^{l-i} g_6^{i+1} + C_i f_6^{l-1-i} g_6^{i+2}$$

with

$$A_i = -\frac{(6l-1)(a+b)d}{b^2} + \frac{(12i-6l+1)(a+b)}{b}, \ A_i + B_i + C_i = -8, \ (a+b)C_i - (a-b)A_i = \frac{10(a^2-b^2)}{b}.$$

We can choose $\mathrm{Cor}(\ell_{6l+6}) = \langle f_6^{l+1}, f_6^l g_6 \rangle$.

For $j = 6l+7$ with $l < m_1 + m_2$. A basis of $\mathcal{P}_{j-7}^{\mathbf{t}}$ is $\mathcal{B}_{6l} = \{f_6^{l-i} g_6^i\}_{i=0}^l$ and a basis of $\mathcal{P}_j^{\mathbf{t}}$ es $\mathcal{B}_{6l+7} = \{f_6^{l-i} g_6^i x^2 y\}_{i=0}^l$. for $0 \le i \le l$, we obtain that

$$F_7(f_6^{l-i} g_6^i) = A_i f_6^{l-i} g_6^i x^2 y$$

with $A_i = -2bi + (b+d)l$. Replacing $d = \frac{m_2 - m_1}{m_1 + m_2} b$, we obtain that $A_i = 0$ if there exists k_0 such that $l = (m_1 + m_2)k_0$ and $i = m_2 k_0$. So, $\mathrm{Cor}(\ell_{6l+7}) = \langle f_6^l x^2 y \rangle$ if $l = (m_1 + m_2)k_0$. Otherwise, $\mathrm{Cor}(\ell_{6l+7}) = \{0\}$. ■

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