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Analysis of dynamical systems via normal forms

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Antonio Algaba Durán

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UNIVERSITY OF HUELVA
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Memoria presentada por Natalia Fuentes Díaz para optar al grado de Doctor en Ciencias Matemáticas por la Universidad de Huelva.

Fdo. Natalia Fuentes Díaz.

Huelva, Septiembre de 2015.

ANALYSIS OF DYNAMICAL SYSTEMS VIA NORMAL FORMS

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CERTIFICAN: que la presente memoria ha sido realizada por Natalia Fuentes Díaz bajo nuestra supervisión, y constituye su Tesis Doctoral para aspirar al grado de Doctor en Matemáticas por la Universidad de Huelva.

V^{o} B^{o} de los Directores

Antonio Algaba Durán   Cristóbal García García.

Huelva, Septiembre de 2015.

A mis padres y hermana, Vicente, Marisa y Raquel.
A mis hijos, Lucas y Valeria.
A Manolo, mi compañero.

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Introducción

Desde hace muchos años los problemas de la dinámica han sido objeto de estudio por científicos de distintas épocas. Podemos decir que los sistemas dinámicos se ocupan del estudio de los modelos de evolución de los sistemas en un cierto espacio (espacio de fases). Los más conocidos, y estudiados por Newton, son los problemas de la mecánica celeste, es decir, el estudio de movimientos de cuerpos dentro del sistema solar. Fue en este momento cuando surgió el planteamiento de estudiar estos modelos, que describían problemas dinámicos, por medio de ecuaciones diferenciales ordinarias. Estas ecuaciones, a pesar de tener en algunos casos aspecto simple, resultaban notablemente difíciles de solucionar al aplicarlas a problemas específicos. Esto ocupó las mentes de los más grandes matemáticos de los siglos XVIII y XIX. Mientras que para la teoría de ecuaciones diferenciales ordinarias lineales fue desarrollada una teoría relativamente completa, fue la teoría de sistemas no lineales la que permaneció especialmente inaccesible. Los orígenes del desarrollo de la teoría de sistemas dinámicos, tal y como la conocemos actualmente fue iniciada por el matemático francés Henri J. Poincaré y se remonta a los años 1892-1899, en sus trabajos sobre el problema de los tres cuerpos de la mecánica celeste. Poincaré desarrolla una serie de nuevas técnicas de donde se originan lo que son la geometría y la topología modernas. Estas técnicas se basaban, principalmente, en la descripción global de todas las soluciones (o *retrato de fases*), particularmente describe la utilidad del llamado *mapa de Poincaré* en el estudio de soluciones periódicas y en el efecto de pequeñas perturbaciones de las condiciones iniciales (o *estabilidad*). En este sentido fue él quién desarrolla los conceptos de *variedad estable e inestable*. Después de Poincaré, fueron matemáticos como Birkhoff (1927), ver [24], y, un poco más tarde, Andronov y Pontryagin (1937), quiénes impulsan el trabajo iniciado por éste. En el caso de Andronov y Pontryagin, introducen el concepto de *estabilidad estructural* (ver [18]). Finalmente fueron matemáticos como Kolmogorov, Arnold y Moser quiénes probaron la estabilidad de ciertas soluciones periódicas en el problema de los tres cuerpos, dando origen a la teoría conocida con sus tres nombres (teoria de Kolmogorov-Arnold-Moser) y al famoso *Teorema KAM* , denominado así por las iniciales de sus nombres. Más tarde serían Lorenz, meteórologo norteamericano y alumno de Birkhoff, y Smale quiénes enriquecieron la teoría de sistemas dinámicos, mediante sus estudios en los ya conocidos *atractor de Lorenz y herradura de Smale*, dando el punto de partida a la *teoría del caos*.

Una vez hecho este breve recorrido por los inicios de la teoría moderna de sistemas dinámicos, nos centraremos más en describir qué problemas abordaremos en esta memoria. Hay una gran diversidad de disciplinas entre las que podemos citar la electronica, mecánica, química, etc, donde existen magnitudes que sirven para describir determinados fenómenos. Estos fenómenos son descritos mediante el uso de sistemas dinámicos, es decir, modelos matemáticos que describen los fenómenos antes mencionados, mediantes sus derivadas con respecto al tiempo, que son expresadas en función de unas variables de estado y, en ocasiones, del tiempo. En un sentido amplio, el objetivo de la teoría de sistemas dinámicos es determinar la estructura del conjunto de soluciones de estos modelos.

En concreto, si denotamos el correspondiente estado del sistema en un cierto instante de tiempo t mediante una variable $\mathbf{x}(t)$ (que podrá tomar valores vectoriales), la evolución con el tiempo de $\mathbf{x}(t)$ viene descrita por una ecuación diferencial ordinaria de la forma

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t, \mathbf{x}(t)).$$

Nos centraremos en esta memoria en el caso de sistemas autónomos, que son aquellos de la forma

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t)), \tag{0.0.1}$$

es decir, en los que el segundo miembro no depende explícitamente de t.

La variable $\mathbf{x}(t)$ se mueve en un cierto espacio, denominado *espacio de fases o espacio de estados*. Las gráficas de las soluciones de este tipo de sistemas, pueden interpretarse como curvas en el espacio de fases. Nuestro objetivo será determinar cómo queda estructurado el espacio de fases por las curvas solución.

Las soluciones más simples son las constantes (puntos fijos, o estacionarios, o de equilibrio), dados por $\frac{d\mathbf{x}(t)}{dt} = 0$. Con frecuencia, éstas son las únicas soluciones que pueden ser conocidas con precisión en las aplicaciones. Otro tipo de soluciones son las periódicas, que vuelven al punto inicial después de un cierto tiempo T, denominado periodo. Las gráficas de las soluciones periódicas (en el espacio de fases) describen trayectorias que son curvas cerradas.

La determinación de la estructura topológica global de las curvas solución en el espacio de estados puede resultar inabordable en la práctica. Un planteamiento menos ambicioso es limitar el estudio de la dinámica de un sistema a un entorno de un cierto elemento crítico, generalmente un punto fijo.

En este sentido existen herramientas para clasificar todas las trayectorias de un sistema dinámico y con ello obtener una representación del espacio de fases. Entre estas herramientas estan la búsqueda de *integrales primeras o de factores integrantes inversos*, problemas que abordaremos en distintos capítulos de esta memoria.

Una complicación añadida al estudio de los sistemas dinámicos es cuando dicho sistema tiene dependencia de ciertos parámetros y pretendemos analizar el cambio en la estructura topológica de las soluciones al variar los parámetros, es decir, caracterizar los *fenómenos de bifurcación.* Este problema también puede ser abordado desde el punto de vista *local*, es decir, analizar las bifurcaciones locales de la familia de sistemas, esto es, los posibles cambios que se produzcan en la estructura del espacio de fases, en un entorno de un cierto elemento crítico, considerando además que los parámetros varían localmente en un entorno de un cierto valor.

Un paso previo al estudio de estos fenómenos de bifurcación, o a cualquier propiedad dinámica que queramos estudiar, es intentar buscar una simplificación de la expresión analítica del campo vectorial $\mathbf{F}(\mathbf{x})$ del sistema. La más importante de ellas es la reducción a forma normal.

La teoría de formas normales (también llamada *forma normal clásica*) fue introducida por Poincaré, utilizando cambios de variables de la forma identidad más términos no lineales, utilizados más tarde por Dulac y Liapunov, y desarrollados posteriormente por Birkhoff.

La idea básica de la teoría de formas normales es usar cambios de variables para simplificar tanto como sea posible las expresiones analíticas de los campos vectoriales bajo estudio. Más concretamente, pretendemos eliminar aquellos términos no lineales, que no son esenciales para la determinación del comportamiento dinámico o la conducta de bifurcación local. Tradicionalmente, las mencionadas simplificaciones se obtienen a través de transformaciones en las variables de estados, es decir, mediante el uso de $\mathcal{C}^\infty$-conjugación. Para ciertos tipos de problemas, es posible considerar las mejoras proporcionadas por el uso de transformaciones no sólo en las variables de estados sino también en el tiempo, a través del proceso denominado $\mathcal{C}^\infty$-equivalencia.

Nos centraremos, en esta memoria, en el estudio de formas normales para equilibrios en sistemas autónomos. La simplificación en los términos no lineales se alcanza grado a grado: si hacemos un cambio no lineal de coordenadas de la forma identidad mas términos de grado k, es fácil comprobar que los términos hasta grado $k-1$ de $\mathbf{F}(\mathbf{x})$ no se ven alterados, mientras que los de grado k cambian de forma lineal a través del denominado operador homológico de grado k, que se define mediante el producto de Lie en el que interviene la parte lineal del sistema.

El primer problema que surge aquí es estudiar si es posible transformar el sistema en uno lineal mediante cambios del tipo mencionado. Ello es posible si las correspondientes ecuaciones homológicas resultan ser compatibles. Si éste no es el caso, existirán ciertos términos no lineales que no podremos eliminar. No obstante, mediante una elección adecuada del cambio siempre podremos eliminar en los términos de orden k de $\mathbf{F}(\mathbf{x})$ la parte que está en la imagen del operador homológico. En este sentido, la simplificación en los términos no lineales consiste en reducir los términos de orden k de nuestro sistema a un subespacio complementario a la imagen (co-rango) del operador homológico.

Es importante señalar que las simplificaciones anteriormente indicadas dependen de la parte lineal del sistema, la cual determina el operador homológico. Por otra parte, la consideración de la solución general de la ecuación homológica permite introducir ciertos grados de libertad en el procedimiento de reducción a forma normal. Así, en la forma normal a orden superior aparecerán ciertas constantes arbitrarias que, seleccionadas adecuadamente, pueden proporcionar simplificaciones adicionales.

A la hora de estudiar dichas simplificaciones adicionales, es preciso tener en cuenta la información que proporcionan los términos no lineales. Se llega así al concepto de forma hipernormal (también denominada forma normal única).

En esta memoria analizamos y estudiamos distintos problemas dinámicos a través de la formas normales quasi-homogéneas, las cuales desarrollamos a partir de la teoría de forma normal clásica. Para ello utilizamos, en lugar de desarrollos en series de Taylor en un entorno de un punto singular, desarrollos quasi-homogéneos del campo vectorial que describe el sistema ($\mathbf{F}(\mathbf{x})$).

A continuación hacemos un breve resumen de la estructura de esta tesis.

En el capítulo primero estudiamos los centros de campos vectoriales quasi-homogéneos planos de grado 0, 1, 2, 3 y 4, realizando una clasificación de los mismos. Dicho estudio nos ha permitido extender el trabajo realizado por Libre & Pessoa [66]. Además, en cada caso, realizamos un estudio de la reversibilidad y la integrabilidad analítica de cada uno de los centros. Debemos destacar que hemos encontrado centros que no son reversibles ni analíticamente integrables, lo cual representa un nuevo escenario respecto del estudio de los centros no degenerado y los centros nilpotentes. En la parte final del Capítulo 1, hacemos una introducción a la estabilidad estructural, definiendo este concepto para campos quasi-homogéneos. En el teorema principal de esta parte, *Theorem 1.5.43*, caracterizamos los campos quasi-homogéneos planos que son estructuralemte estables. Dicho teorema generaliza los resutados obtenidos por Llibre *et al.* [65] (Theorem A) y por Oliveira & Zhao [71] (Theorem 2). Algunos resultados obtenidos en este capítulo han sido publicados en:

A. ALGABA, N. FUENTES, C. GARCÍA. *Centers of quasi-homogeneous polynomial planar systems.* Nonlinear Analysis: Real World Applications. Volumen 13, Edición 1, 2012, Páginas 419–431.

En el capítulo segundo hacemos una revisión de la teoría de formas normales. En primer lugar, realizamos un breve resumen de la formas normales clásicas y describimos las formas normales quasi-homogéneas bajo $\mathcal{C}^\infty$-conjugación y $\mathcal{C}^\infty$-equivalencia. Hacemos uso del *Poliedro de Newton* para describir la elección de un tipo adecuado, mostrando diferentes ejemplos para la elección de dicho tipo. Describimos la forma normal de paso cero (concepto análogo al de forma canónica de Jordan en el caso lineal). Describimos el uso del triángulo de Lie en el caso quasi-homogéneo y, por último, aplicamos estas técnicas al cálculo de la forma normal de un caso particular de la singularidad Takens-Bogdanov.

En el capítulo tercero, dedicado a las formas quasi-homogéneas planas, introducimos dos descomposiciones de campos vectoriales planos. La primera, la ya conocida, *descomposición conservativa-disipativa*, descompone cualquier campo vectorial quasi-homogéneo en dos componentes: una conservativa y otra disipativa. La segunda, que deriva de la anteriormente citada, descompone cuaquier campo vectorial quasi-homogéneo en tres componentes. Dicha descomposición nos ha permitido mostrar en este capítulo, de una forma relativamente simple, la expresión de la forma normal a orden infinito, de familias de campos vectoriales cuyas componentes quasi-homogénea principales poseen una singularidad nilpotente o son degeneradas.

En el capítulo cuarto, hacemos uso de estas formas normales para estudiar la integrabilidad analítica de campos vectoriales planos haciendo uso de un resultado conocido de Algaba *et al.* [9]. Además, caracterizamos teóricamente la existencia de factor integrante inverso, tanto algebraico como formal, (esto es, damos condiciones necesarias y suficientes de existencia) para campos vectoriales planos que son perturbaciones de sistemas hamiltonianos degenerados. En la última sección de este capítulo, aplicamos dichos resultados al estudio de la existencia de factores integrantes inversos en distintas familias del tipo anteriormente citado.

Algunos resultados obtenidos en los Capítulos 3 y 4 han dado lugar a los siguientes artículos:

A. ALGABA, N. FUENTES, C. GARCÍA, M. REYES. *A class of non-integrable systems admitting an inverse integrating factor.* Journal of Mathematical Analysis and Applications. Volumen 420, Edición 2, 2014, Páginas 1439–1454.

A. ALGABA, N. FUENTES, C. GARCÍA, M. REYES. *Non-formally integrable centers admitting an algebraic inverse integrating factor.* Journal of Dynamics and Differential Equations, en segunda fase de revisión.

Finalmente, en el capítulo cinco, generalizamos el cálculo de las formas normales planas a campos vectoriales en $\mathbb{R}^3$. Esto incluye una generalización, para campos vectoriales en $\mathbb{R}^3$, de la nueva descomposición para campos quasi-homogéneos introducida en el capítulo tercero. Como aplicación realizamos el cálculo, usando nuestras técnicas, de la forma normal de un caso de la singularidad *Hopf-zero*, comparándola con los resultados obtenidos en los trabajos de Chen *et al.* [33,34] y Gazor & Mokthari [52] entre otros. También desarrollamos la forma normal de un caso particular de la singularidad *Triple-zero*, a orden infinito. Finalmente, en la última parte del capítulo hacemos una discusión del papel que juegan los parámetros en el cálculo de las formas normales.

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CHAPTER 1

Some results about quasi-homogeneous planar systems.

1.1 Introduction

In the study of the planar systems, one of the classic problems in the qualitative theory of the analytical systems, is the study of the phase portrait in a neighborhood of a singular point and, in particular, to characterize when the singular point is a center or a focus. A singular point is called center-focus type, also called monodromic, when the orbits rotate around the singular point, i.e. it is a center or a focus in the analytic case. Once established the monodromy, the center problem determines, by studying the Poníncaré map, when, all the orbits in a neighborhood of the singular point are closed. This chapter deals with the classification of the centers for a class of polynomial differential system with null linear part (degenerate centers). The classification of the centers of polynomial differential system with linear part $(-y, x)^T$ started with the works of Dulac [42], Bautin [22], Kapteyn [62, 63] and Zoladek [87] for quadratic system, and continued with the works of Sibirskii [77] and Zoladek [86] for symmetric cubic system. A lot of work has been performed about non-degenerate centers (see Giné [58]), but only partial results have been reached and we are very far to obtain a complete classification of all non-degenerate centers for the polynomial differential systems of degree greater than or equal to three. The nilpotent centers (i.e.,with linear part $(y,0)^T$) are characterized theoretically (see Berthier & Moussu [23], Moussu [70], Giacomini et al. [55]) but only a few families of nilpotents centers are known (see Gassul & Torregrosa [51], Sadovskii [75], Algaba et al. [12]). However, the case of degenerate centers (i.e., with null linear part) is not characterized theoretically (see Gassull et al. [50]).

Another problem related to the previous one is the problem of the reversibility, i.e., when a vector field is invariant to an involution in the state variables and the change of sign in the time variable. (see Berthier and Moussu [23], Moussu [70], Algaba at al. [17], Teixeira and Yang [81])

Next, the third problem we study in this chapter is the analytical integrability of a quasi-homogeneous vector field, more specifically, to determine when a planar vector field has an analytical first integral, i.e., a function that remains constant along the trajectories of the system. (see Chavarriga et al. [30, 32], Algaba et al. [1, 9], Cairó and Llibre [28], Llibre and Pessoa [66], Llibre and Zhang [67])

We consider system

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}). \tag{1.1.1}$$

with $\mathbf{F}_r = (P, Q)$, with P and Q coprime and $\mathbf{F}_r$ a quasi-homogeneous vector field. In this case, it is known that the origin is the unique (real and finite) singularity of $\mathbf{F}_r$. Therefore, if $\mathbf{F}_r$ has a center at the origin its period annulus is $\mathbb{R}^2 \setminus \{0\}$, i.e., the center is global.

Our two motivations for the study of the quasi-homogeneous centers are by one hand, Theorem 1.3.32 which assures us that: *a necessary condition so that the origin of a perturbation of a quasi-homogeneous vector field be a center is that the origin of the quasi-homogeneous vector field be a center.* By other hand, a recent work Llibre and Pessoa [66], which shows a classification of these centers up to fourth degree (third degree for us). Notice that our definition of degree of a quasi-homogeneous vector field disagrees with the one given in Llibre and Pessoa [66] in one unit.

In this chapter, by using others techniques, we extend this study up to fifth degree (fourth degree for us), and besides, we characterize the integrability and the reversibility of each one of the centers found. It is know that all non-degenerate centers are reversible and analytically integrable (see Poincaré [72, 73]). Nilpotent centers are orbitally reversible (see Berthier and Moussu [23]) but there exist nilpotent centers that are not analytically integrable (see Moussu [70]). In this chapter is shown a new situation, we find polynomial centers which are neither orbitally reversible nor analytically integrable (see Theorems 1.6.54, 1.6.57).

Finally, the last problem discussed in this chapter is the *structural stability*. The beginnings in the study of the structural stability of homogeneous polynomial planar vector fields are dated 1960. In this year Markus [68] presents his work which classifies the quadratic planar homogeneous polynomial vector fields such that their components have no common factors. Later, in 1968 Argemí [19] completed the cassification of Markus. Moreover he furnished the classification of the cubic vector field that have no common factors. At the same time, he obtained upper and lower bounds for the number of phase portraits of the planar homogeneous polynomial vector fields of degree m which have no common factors. Subsequent results, relative to an algebraic classification of the quadratic planar homogeneous vector fields, can be found in the work of Date [41] and Sibirsky [78], where the two authors show, using different techniques, the classification of quadratic vector fields with common factors. In 1990, Cima & Llibre [38] obtain a topological classification of the cubic homogeneous polynomial vector fields with or without common factors and they present an algorithm for studying the phase portraits of homogeneous polynomial vector fields of degree $m \leq 3$ and find an algebraic classification for the planar homogeneous polynomial of degree $m = 3$. This classification was extended later by Collins [40] for the planar homogeneous polynomial of degree $m \leq 1$. In this chapter we study the structural stability of planar quasi-homogeneous vector fields with respect to perturbations in the space of the planar quasi-homogeneous vector fields. We apply it in several examples.

This chapter is structured as follows. In the next section we give some definitions and previous concepts. In section 3, we characterize the monodromy of a quasi-homogeneous vector field. In sections 4 and 5 we characterize the reversibility, integrability and center problem and describe the classes of topological equivalence in $\varepsilon_r^{\mathbf{t}}$, (being $\varepsilon_r^{\mathbf{t}}$ the vectorial space of the quasi-homogeneous vector field of type $\mathbf{t}$ and degree k, that are structurally stables). In section 6 we describe, as application, the topological equivalence class $\varepsilon_1^{(1,2)}$ and $\varepsilon_2^{(1,3)}$ and describes the centers for degenerated quasi-homogeneous vector fields up to four degree, extending the results reached in Llibre & Pesoa [66]. We also study the centers that are reversible and analytically integrable detecting cases of centers which are neither reversible nor integrable.

1.2 Quasi-homogeneous vector field: definitions and properties.

Through this memory, we will consider a fixed type $\mathbf{t} = (t_1, t_2, \cdots, t_n)$ with $t_i \in \mathbb{N}$ (here, $\mathbb{N}$ is the set of natural numbers not including zero, whereas $\mathbb{N}_0$ will denote the set of natural numbers including zero). We will use standard multi-index notations: a *multi-index* is an element $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{N}_0^n$. Moreover, we will write the monomials as $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Finally, the canonical basis will be denoted as $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

We will deal with smooth vector fields, which we assume that can be formally expanded in terms of the canonical basis as follows,

$$\mathcal{B} = \{\mathbf{x}^{\mathbf{a}}\mathbf{e}_j : \mathbf{a} \in \mathbb{N}_0^n, 1 \leq j \leq n\}.$$

(The qualification of *formal* indicates that we will not address any question about the convergence of the expansions).

Definition 1.2.1. *Let* $\mathbf{t} = (t_1, t_2, \ldots, t_n)$*, we define, module of* $\mathbf{t}$ *as,* $|\mathbf{t}| = t_1 + t_2 + \cdots + t_n$

Definition 1.2.2. *An scalar function f is said quasi-homogeneous of type* $\mathbf{t}$ *and degree k if its monomials satisfy*

$$a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = k. \tag{1.2.2}$$

The vector space of quasi-homogeneous polynomials in n variables of type $\mathbf{t}$ *and degree k will be denoted by* $\mathcal{P}_k^{\mathbf{t}}$.

Remark 1. *Observe that a quasi-homogeneous function* f*, of type* $\mathbf{t}$ *and degree* k*, can be expressed as follows,*

$$f(\mathbf{x}) = \sum_{i=1}^{s} \alpha_i \mathbf{x}^{a_i}$$

being $\mathbf{x}^{a_i} = x_1^{a_i^{(1)}} x_2^{a_i^{(2)}} \cdots x_n^{a_i^{(n)}}$ *verifing that* $a_i^{(1)} t_1 + a_i^{(2)} t_2 + \cdots + a_i^{(n)} t_n = k$*, for all* $i = 1 \ldots s$

Definition 1.2.3. *A vector field* $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ *is said quasi-homogeneous of type* $\mathbf{t}$ *and degree* k *if its components* $F_j \in \mathcal{P}^{\mathbf{t}}_{k+t_j}$ *for all* $j = 1, 2, \ldots, n$*. We will denote* $\mathcal{Q}^{\mathbf{t}}_k$ *the vector space of quasi-homogeneous vector fields of type* $\mathbf{t}$ *and degree* k*. (Notice that deg*$(\mathbf{F}) = r - 1$ *respect to the usual homogenous degree.)*

Definition 1.2.4. *Given a vector field* $\mathbf{F} = (P_1, P_2, \ldots, P_n)^T$*, we define, the divergence of* $\mathbf{F}$*, div*$(\mathbf{F}) := \frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \cdots + \frac{\partial P_n}{\partial x_n}$.

The following lemmas show some properties about quasi-homogeneity. Previously to its proofs we show an alternative characterization for quasi-homogeneous functions and vector fields.

Consider the following matrix,

$$E = \begin{pmatrix} \varepsilon^{t_1} & 0 & \cdots & 0 \\ 0 & \varepsilon^{t_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon^{t_n} \end{pmatrix}.$$

Proposition 1.2.5. *It holds the following items,*

(a) *The function* f *is quasi-homogeneous of type* $\mathbf{t}$ *and degree* k *if and only if*

$$f(E\mathbf{x}) = \varepsilon^k f(\mathbf{x}). \tag{1.2.3}$$

(b) *The vector field* $\mathbf{F}$ *is quasi-homogeneous of type* $\mathbf{t}$ *and degree* k *if and only if*

$$\mathbf{F}(E\mathbf{x}) = \varepsilon^k E\mathbf{F}(\mathbf{x}). \tag{1.2.4}$$

Proof.

a) Let consider $f \in \mathcal{P}_k^{\mathbf{t}}$, then $E\mathbf{x} = \begin{pmatrix} \varepsilon^{t_1}x_1 \\ \varepsilon^{t_2}x_2 \\ \vdots \\ \varepsilon^{t_n}x_n \end{pmatrix}$. In consequence $f(E\mathbf{x}) =$

$= \sum_{i=1}^{s} \alpha_i(\varepsilon^{t_1}x_1)^{a_i^{(1)}}(\varepsilon^{t_2}x_2)^{a_i^{(2)}} \cdots (\varepsilon^{t_n}x_n)^{a_i^{(1)}} = \sum_{i=1}^{s} \alpha_i \varepsilon^{t_1 a_i^{(1)}} \varepsilon^{t_2 a_i^{(2)}} \cdots \varepsilon^{t_n a_i^{(n)}} x_1^{a_i^{(1)}} x_2^{a_i^{(2)}} \cdots x_n^{a_i^{(n)}} = \varepsilon^k \sum_{i=1}^{s} \alpha_i \mathbf{x}^{a_i} = \varepsilon^k f(\mathbf{x})$.

b) Let consider $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$, then $\mathbf{F}(E\mathbf{x}) = (F_1, F_2, \ldots F_n)^T(E\mathbf{x}) = (F_1(E\mathbf{x}), F_2(E\mathbf{x}), \ldots, F_n(E\mathbf{x}))$ with $F_j \in \mathcal{P}_{k+t_j}$. Using item **a)**, $\mathbf{F}(E\mathbf{x}) = (\varepsilon^{k+t_1}F_1, \varepsilon^{k+t_2}F_2, \ldots, \varepsilon^{k+t_n}F_n) = \varepsilon^k \cdot (\varepsilon^{t_1}\varepsilon^{t_2} \ldots \varepsilon^{t_n})(F_1, F_2, \ldots, F_n)^T = \varepsilon^k \cdot E\mathbf{F}(\mathbf{x})$.

■

Lemma 1.2.6. *Consider $f \in \mathcal{P}_k^{\mathbf{t}}$. Then, its gradient ∇f verifies,*

$$\nabla f(E\mathbf{x}) = \varepsilon^k E^{-1} \nabla f(\mathbf{x}).$$

Proof. It is sufficient to differentiate (1.2.3) respect to x. ■

Lemma 1.2.7. *Consider $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$. Then, $D\mathbf{F}(E\mathbf{x}) = \varepsilon^k ED\mathbf{F}(\mathbf{x})E^{-1}$.*

Moreover, the column j of the matrix $D\mathbf{F}$ is a quasi-homogeneous vectorial space of degree $k - t_j$ respect the type $\mathbf{t}$.

Proof. To get the first equality, it is sufficient to differentiate (1.2.4). In addition, the column j of $D\mathbf{F}(\mathbf{x})$ is

$$D\mathbf{F}(E\mathbf{x})\mathbf{e}_j = \varepsilon^k ED\mathbf{F}(\mathbf{x})E^{-1}\mathbf{e}_j = \varepsilon^k ED\mathbf{F}(\mathbf{x})\varepsilon^{-t_j}\mathbf{e}_j = \varepsilon^{k-t_j} ED\mathbf{F}(\mathbf{x})\mathbf{e}_j.$$

The result follows from (b) of Proposition 1.2.5. ■

Lemma 1.2.8. *Let be $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$, $\mathbf{G} \in \mathcal{Q}_l^{\mathbf{t}}$. Then, $[\mathbf{F}, \mathbf{G}] \in \mathcal{Q}_{k+l}^{\mathbf{t}}$.*

Proof.

$$\begin{aligned}
[\mathbf{F},\mathbf{G}](E\mathbf{x}) &= D\mathbf{F}(E\mathbf{x})\mathbf{G}(E\mathbf{x}) - D\mathbf{G}(E\mathbf{x})\mathbf{F}(E\mathbf{x}) \\
&= \varepsilon^l ED\mathbf{F}(\mathbf{x})E^{-1}\varepsilon^k E\mathbf{G}(\mathbf{x}) - \varepsilon^k ED\mathbf{G}(\mathbf{x})E^{-1}\varepsilon^l E\mathbf{F}(\mathbf{x}) \\
&= \varepsilon^{k+l}E\left(D\mathbf{F}(\mathbf{x})\mathbf{G}(\mathbf{x}) - D\mathbf{G}(\mathbf{x})\mathbf{F}(\mathbf{x})\right) = \varepsilon^{k+l}E[\mathbf{F},\mathbf{G}](\mathbf{x}).
\end{aligned}$$

Just use paragraph (b) of Proposition 1.2.5 to complete the proof. ■

Lemma 1.2.9. *Consider $\mu \in \mathcal{P}_k^{\mathbf{t}}$ and $\mathbf{F} \in \mathcal{Q}_l^{\mathbf{t}}$. Then, $\mu\mathbf{F} \in \mathcal{Q}_{k+l}^{\mathbf{t}}$.*

Proof. We have

$$\mu(E\mathbf{x})\mathbf{F}(E\mathbf{x}) = \varepsilon^k\mu(\mathbf{x})\varepsilon^l E\mathbf{F}(\mathbf{x}) = \varepsilon^{k+l}E\mu(\mathbf{x})\mathbf{F}(\mathbf{x}).$$

Again, from (b) of Proposition 1.2.5, we get the result. ■

Lemma 1.2.10. *Consider $f \in \mathcal{P}_k^{\mathbf{t}}$ and $\mathbf{F} \in \mathcal{Q}_l^{\mathbf{t}}$. Then, $\nabla f \cdot \mathbf{F} \in \mathcal{P}_{k+l}^{\mathbf{p}}$.*

Proof. From lemma 1.2.6, we obtain $\nabla f(E\mathbf{x}) = \varepsilon^k E^{-1}\nabla f(\mathbf{x})$. Therefore,

$$\nabla f(E\mathbf{x}) \cdot \mathbf{F}(E\mathbf{x}) = \varepsilon^k(\nabla f(\mathbf{x}))^T E^{-1}\varepsilon^r E\mathbf{F}(\mathbf{x}) = \varepsilon^{r+k}\nabla f(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}).$$

From (a) of proposition 1.2.5, we obtain the result. ■

The following lemma is a version of *Euler's theorem* for quasi-homogeneous case.

Lemma 1.2.11. *Let $f \in \mathcal{P}_k^{\mathbf{t}}$ and $\mathbf{D}_0 = (t_1x_1, t_2x_2, \cdots, t_nx_n)^T \in \mathcal{Q}_0^{\mathbf{t}}$ be. Then*

$$\nabla f \cdot \mathbf{D}_0 = kf.$$

Proof. For being $f \in \mathcal{P}_k^{\mathbf{t}}$, $f(E\mathbf{x}) = \varepsilon^k f(\mathbf{x})$, deriving respect to ε we obtain that

$$\nabla f(E\mathbf{x}) \cdot (D_\varepsilon E)\mathbf{x} = k\varepsilon^{k-1}f(\mathbf{x}).$$

For $\varepsilon = 1$ we obtain the result. ■

Lemma 1.2.12. *Consider* $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$. *Then* $\mathrm{div}\,(\mathbf{F}_k) \in \mathcal{P}_k^{\mathbf{t}}$.

Proof. Using el Lemma 1.2.7 we have:

$$\begin{aligned} \mathrm{div}\,(\mathbf{F}_k)\,(E\mathbf{x}) &\overset{\mathrm{def}}{=} \mathrm{tr}\,(D\mathbf{F}_k\,(E\mathbf{x})) = \mathrm{tr}\left(\varepsilon^k ED\mathbf{F}_k(\mathbf{x})E^{-1}\right) \\ &= \varepsilon^k \mathrm{tr}\,(D\mathbf{F}_k(\mathbf{x})) = \varepsilon^k \mathrm{div}\,(\mathbf{F}_k)\,(\mathbf{x}) \end{aligned}$$

■

Lemma 1.2.13. *Given two vectorial spaces* $\mathbf{F}$, $\mathbf{G}$, *and a smooth scalar function* μ, *then*

$$[\mu\mathbf{F}, \mathbf{G}] = (\nabla\mu \cdot \mathbf{G})\mathbf{F} + \mu[\mathbf{F}, \mathbf{G}]$$

Proof. It is easily shown using the definition of the Lie bracket. ■

Lemma 1.2.14. *Let* $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ *and* $\mathbf{D}_0 = (t_1x_1, t_2x_2, \cdots, t_nx_n)^T \in \mathcal{Q}_0^{\mathbf{t}}$. *Then,* $[\mathbf{F}_k, \mathbf{D}_0] = k\mathbf{F}_k$. *particularly, if* $\mathbf{F}_0 \in \mathcal{Q}_0^{\mathbf{t}}$ *then* $[\mathbf{F}_0, \mathbf{D}_0] = 0$.

Proof. Consider $[\mathbf{F}_k, \mathbf{D}_0]\,\mathbf{e}_j$, the j-th component of the Lie bracket, $j = 1, \cdots, n$. Using the Euler lemma 1.2.11 we obtain,

$$[\mathbf{F}_k, \mathbf{D}_0]\,\mathbf{e}_j = \nabla\mathbf{F}_k\mathbf{e}_j \cdot \mathbf{D}_0 - \nabla\mathbf{D}_0\mathbf{e}_j \cdot \mathbf{F}_k = (k + t_j)\mathbf{F}_k\mathbf{e}_j - t_j\mathbf{F}_k\mathbf{e}_j = k\mathbf{F}_k\mathbf{e}_j$$

■

An important property of the quasi-homogeneous vector fields is its invariance respect to certain changes of scale in the state variables and time. This property determines the dynamics of the quasi-homogeneous systems as shown below.

Lemma 1.2.15. *If* $\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ *is a solution of the system* $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$, $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ *then* $\delta(t) = (u^{t_1}x_1(t), u^{t_2}x_2(t), \ldots, u^{t_n}x_n(t))^T$, *with* $u \in \mathbb{R}\setminus\{0\}$, *is solution of* $\mathbf{x}' = \mathbf{F}_r(\mathbf{x})$ *where* $\mathbf{x}' = \frac{d\mathbf{x}}{d\tau}$, *being* $\tau = \frac{t}{u^r}$.

Proof. If $\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is a solution of the system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ it is verified that $\dot{\gamma}(t) = \mathbf{F}_r(\gamma(t))$.

$$\begin{aligned}
\delta'(t) &= \dot{\delta}(t)u^r = u^r(u^{t_1}\dot{x}_1(t), u^{t_2}\dot{x}_2(t), \ldots, u^{t_n}\dot{x}_n(t))^T \\
&= u^r \begin{pmatrix} u^{t_1} & 0 & 0 & \ldots & 0 \\ 0 & u^{t_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & u^{t_n} \end{pmatrix} \dot{\gamma}(t) = u^r \begin{pmatrix} u^{t_1} & 0 & 0 & \ldots & 0 \\ 0 & u^{t_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & u^{t_n} \end{pmatrix} \mathbf{F}_r(\gamma(t)) \\
&= \mathbf{F}_r(u^{t_1}x_1(t), u^{t_2}x_2(t), \ldots, u^{t_n}x_n(t)) = \mathbf{F}_r(\delta(t))
\end{aligned}$$

■

From this result, the following dynamical consequence is derived.

Proposition 1.2.16. *Let $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$ be and consider that the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is isolated equilibrium. Then the unique equilibrium of the system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is the origin. In addition, this quasi-homogeneous system has no limit cycles and homoclinic orbits.*

Proof. The proof is a consequence of Lemma 1.2.15 because, if $\gamma(t) = (x_1, x_2, \ldots, x_n)$ is an equilibrium of the system, different from the origin, then $\delta(t) = (u^{t_1}x_1, u^{t_2}x_2, \ldots, u^{t_n}x_n)$ is other equilibrium for all value of $u \in \mathbb{R} \setminus \{0\}$. This defines a curve of singular points of the system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ which approaches to the origin when u tends to zero, and this is inconsistent with the fact that the origin is an isolated equilibrium.

Moreover, the system can not have limits cycles or homoclinic, since it, if $\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is one of these orbits then, applying Lemma 1.2.15, there would be a continuum of periodic orbits or homoclinic cycles which is contradictory. ■

1.2.1 Planar quasi-homogeneous vector fields.

In this subsection, we present a decomposition of any quasi-homogeneous vector field. This decomposition is a generalization of that given, for the homogeneous case, by Baider [21] and Collins [39]. It was introduced by Algaba *et* *al.* [9]. Given a type $\mathbf{t}$, any quasi-homogenous vector field can be decomposed uniquely as the sum of two quasi-homogeneous vector field: one of them having zero divergence (*conservative part*) and the other one with divergence equal to the original vector field (*dissipative part*). The proof of this result can be seen in [14].

Before continuing we introduce some definitions.

Definition 1.2.17.

1. *We denote* $\mathbf{X}_h := (-\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x})^T$ *to the Hamiltonian vector field with Hamilton function* h.
2. *We define, the wedge product of two vector fields,* $\mathbf{F} \wedge \mathbf{G} := P\tilde{Q} - Q\tilde{P}$, *where* $\mathbf{F} = (P, Q)^T$ *and* $\mathbf{G} = (\tilde{P}, \tilde{Q})^T$.

Proposition 1.2.18. *Assume that* $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$, *then there exist unique polynomials* $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$ *and* $h_{k+|\mathbf{t}|} \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$ *such that:*

$$\mathbf{P}_k = \mathbf{X}_{h_{k+|\mathbf{t}|}} + \mu_k \mathbf{D}_0, \tag{1.2.5}$$

where $h_{k+|\mathbf{t}|} = \frac{1}{k+|\mathbf{t}|}(\mathbf{D}_0 \wedge \mathbf{P}_k)$ *and* $\mu_k = \frac{1}{k+|\mathbf{t}|} div(\mathbf{P}_k)$.

This decomposition is called, conservative-dissipative splitting.

Proof.

In first time we prove the unicity. Suppose there exists $\mu \in \mathcal{P}_k^t$ and $h \in \mathcal{P}_{k+|t|}^t$ verifying the relationship (1.2.5) then, using the Euler Theorem for quasi-homogeneous vector fields, is obtained,

▶ $div(\mathbf{P}_k) = div(\mathbf{X}_h) + div(\mu \mathbf{D}_0) = \frac{\partial}{\partial x}(\mu t_1 x) + \frac{\partial}{\partial y}(\mu t_2 y) = \frac{\partial \mu}{\partial x} t_1 x + + \frac{\partial \mu}{\partial y} t_2 y + \mu|t| = \nabla\mu \cdot \mathbf{D}_0 + \mu|t| = \mu \cdot k + \mu|t| = (k + |\mathbf{t}|)\mu$.

▶ $\mathbf{D}_0 \wedge \mathbf{P}_k = \mathbf{D}_0 \wedge \mathbf{X}_h = \nabla h \cdot \mathbf{D}_0 = (k + |t|)h$.

Now we prove the existence. Using again the Euler Theorem

$(\mathbf{X}_h + \mu\mathbf{D}_0) \cdot \mathbf{e_1} = -\frac{\partial h}{\partial y} + \mu t_1 x = t_2 \cdot \mathbf{P}_k \cdot \mathbf{e_1} + (k+t_1)\mathbf{P}_k \cdot \mathbf{e_1}] = (k+t_1+t_2)\mathbf{P}_k \cdot \mathbf{e_1}$ $= \mathbf{P}_k \cdot \mathbf{e_1}$.

The result for the second component of the vector field follows analogously.

■

Remark 2. *We want to clarify that, throughout this chapter and in the subsequent, we will refer to* $h_{k+|\mathbf{t}|}$ *as the conservative part of* $\mathbf{P}_k$ *and* μ_k *as the dissipative part of it.*

Alternative proof of Proposition (1.2.16):

Proof. If $\gamma(t) = (x(t), y(t))$ is a limit cycle, taking into acount that h is an inverse integrating factor of (1.1.1), by applying [56, Theorem 9] it is deduced that $h(\gamma(t)) = 0$, this implies that $\gamma(t)$ is solution of a real factor of h, that is, it is solution of $f(x, y) = 0$ where $f = x$, $f = y$ or $f = y^{t_1} - ax^{t_2}$ with $a \in \mathbb{R} \setminus \{0\}$ and this is inconsistent with the fact that $\gamma(t)$ is a limit cycle because $f(x, y) = 0$ are not ovals. ■

The following result is deduced from Proposition (1.2.16)

Corolario 1.2.19. *If* $\mathbf{F}_r$ *is a planar quasi-homogeneous vector field and the origin of* $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ *is a center, then it is a global center.*

Proof. Let consider $\gamma(t) = (x(t), y(t))$ a center of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$, using Lemma (1.2.15) then $\gamma(t) = (u^{t_1}x(t), u^{t_2}y(t))$ with $u \in \mathbb{R}\setminus\{0\}$ is a center. For differents values of u, we obtain that, the center, is in all real plane. ■

1.3 Monodromy for planar quasi-homogeneous vector fields.

As it has been noted in the introduction, a singular point is called monodromic when the orbits rotate around itself, that is, in the case of analytical systems a monodromic singular point is either a center or a focus. As we will see later, the factors of h, the function of the conservative part of $\mathbf{F}_r$, characterize the monodromy of the singular point. With the end of show a factorization of h we will introduce, in the next subsection, some concepts about the vectorial space of the quasi-homogeneous polinomial $\mathcal{P}_k^t$.

1.3.1 The vectorial space $\mathcal{P}_k^{\mathbf{t}}$.

Next, we describe some aspects of the vector space of the quasi-homogenous polynomials. Our purpose is to provide a basis for the vector space $\mathcal{P}_k^{\mathbf{t}}$.

Lemma 1.3.20. *Fixed $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 prime numbers with each other, and $k \in \mathbf{N}_0$ such that $\mathcal{P}_k^{\mathbf{t}} \neq 0$. Then it is possible to determine k_1, k_2 and k_3 so that $k = k_1 t_1 + k_2 t_2 + k_3 t_1 t_2$, being k_1, k_2, $k_3 \in \mathbf{N}_0$ with $k_1 < t_2$, $k_2 < t_1$. Moreover, k_1, k_2 and k_3 are uniqueness.*

Proof. If $\mathcal{P}_k^t \neq \{0\}$ then, for all monomial $x^m y^n \in \mathcal{P}_k^{\mathbf{t}}$ we can write $mt_1 + nt_2 = k$. Considering $m = \tilde{m} t_2 + k_1$ with $0 \leq k_1 < t_1$ and $n = \tilde{n} t_1 + k_2$ with $0 \leq k_2 < t_2$, we obtain $k = (\tilde{m} + \tilde{n}) t_1 t_2 + k_2 t_2 + k_1 t_1$. Moreover, if we denote $k_3 = \tilde{m} + \tilde{n}$, k_1, k_2 and k_3 are uniqueness because if there exist others $k_3^{(1)}, k_2^{(1)}, k_1^{(1)}$ verifying the conditions, we obtain that,

$$(k_1 - k_1^{(1)}) t_1 = [(k_3^{(1)} - k_3) t_1 + (k_2^{(1)} - k_2)] t_2$$

and, as t_1, t_2 are relatively prime, where $0 \leq k_1 < t_2$ and $0 \leq k_1^{(1)} < t_2$ then $k_1 = k_1^{(1)}$ and, therefore, $|k_3^{(1)} - k_3| t_1 = |k_2 - k_2^{(1)}|$. Since $0 \leq k_2 < t_1$ and $0 \leq k_2^{(1)} < t_1$ we obtain that $0 \leq |k_2 - k_2^{(1)}| < t_1$, consequently $k_3^{(1)} = k_3$ and $k_2^{(1)} = k_2$. ∎

The following lemma summarizes these results, providing a basis of the vectorial space $\mathcal{P}_k^{\mathbf{t}}$.

Proposition 1.3.21. *Let be $k \in \mathbb{N}_0$ such that $\mathcal{P}_k^{\mathbf{t}} \neq 0$ and consider $\mathbf{t} = (t_1, t_2)$. Then,*

(a) $\mathcal{P}_0^{\mathbf{t}} = \text{Span}\,\{1\}$.

(b) $\mathcal{P}_k^{\mathbf{t}} = \mathit{span}\left\{ x^{k_1 + t_2(k_3 - j)} y^{k_2 + t_1 j} : j = 0, \ldots, k_3 \right\}$, *if* $k \in \mathbf{N}$.

1.3.2 Monodromy of the Hamiltonian function h.

Below we show a factorization on $\mathbb{C}[x, y]$ for quasi-homogeneous polynomial functions, (where $\mathbb{C}[x, y]$ is the ring of the polynomials with coefficients in $\mathbb{C}$).

Taking into account Proposition 1.3.21, we can write any quasi-homogeneous polynomial of type $\mathbf{t}$ and degree $k \in \mathbb{N}$, i.e., $p_k \in \mathcal{P}_k^{\mathbf{t}}$, as follows,

$$p_k(x,y) = x^{k_1} y^{k_2} \sum_{j=0}^{k_3} \alpha_j x^{t_2(k_3-j)} y^{jt_1}.$$

where $k_1, k_2, k_3 \in \mathbb{N} \setminus \{0\}$ and $k_1 < t_2,\ k_2 < t_1,\ k = k_1 t_1 + k_2 t_2 + k_3 t_1 t_2$.

On the other hand, if we consider the polinomial $\sum_{j=0}^{k_3} \alpha_j x^{t_2(k_3-j)} y^{jt_1}$ in the variables $X = x^{t_2}$, $Y = y^{t_1}$, we obtain a new homogeneous polinomial with the form, $\sum_{j=0}^{k_3} \alpha_j X^{(k_3-j)} Y^j$ and we can write,

$$p_k(x,y) = x^{k_1} y^{k_2} \sum_{j=0}^{k_3} \alpha_j X^{(k_3-j)} Y^j.$$

and this allows us to afirm that $p_k(x,y)$ is associated with a homogenous polynomial $p_k^{\text{hom}}(X,Y)$ of degree k_3 in the variables $X = x^{t_2}$ and $Y = y^{t_1}$, so that,

$$p_k(x,y) = x^{k_1} y^{k_2} p_k^{\text{hom}}(X,Y) \tag{1.3.6}$$

being $p_k^{\text{hom}}(X,Y) = \sum_{j=0}^{k_3} \alpha_j X^{(k_3-j)} Y^j$.

Therefore, using the above expression, we can factorizate $p_k(x,y)$ as follows,

1. Let consider $\alpha_{k_3} = \cdots = \alpha_{k_3-l+1} = 0$ and $\alpha_{k_3-l} \neq 0$. Then, we can write $p_k(x,y) = x^{k_1} y^{k_2} X^l \sum_{j=0}^{k_3-l} \alpha_j (Y/X)^j$.
2. Let be $\lambda_j \in \mathbb{C}$ the roots of the polinomial $\sum_{j=0}^{k_3-l} \alpha_j (Y/X)^j$

Therefore, the factorization mentioned is of the form,

$$p_k(x,y) = \alpha_{k_3-l} x^{k_1+lt_2} y^{k_2} \prod_{j=1}^{k_3-l} (y^{t_1} - \lambda_j x^{t_2}),\ \lambda_j \in \mathbb{C} \tag{1.3.7}$$

And using (1.3.7) hlwe can get an expression of h as product of irreducible factors in $\mathbb{C}[x,y]$. Also, by scaling in the time, we can assume that the leader coefficient of h in the variable y is one, that is

$$h(x,y) = x^{m_x} y^{m_y} \prod_{j=1}^{m} (y^{t_1} - \lambda_j x^{t_2})^{m_j}, \tag{1.3.8}$$

with $m, m_x, m_y \in \mathbb{N}\cup\{0\}$; $m_j \in \mathbb{N}$ for $j = 1, \cdots, m$; $\lambda_j \in \mathbb{C}\backslash\{0\}$ for $1 \leq j \leq m$, and $\lambda_i \neq \lambda_j$ $i \neq j$. (for more details see Algaba et al. [14]).

The next proposition provides a condition of non-monodromy for quasi-homogeneous vector fields with the form $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$.

Proposition 1.3.22. *Consider $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$. If in the decomposition of h given in (1.3.8) there exists any real factor, then the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is non-monodromic.*

Proof. If $f := y^{t_1} - a x^{t_2} \in \mathcal{P}^{\mathbf{t}}_{t_1 t_2}$, $a \in \mathbb{R}$ is a real factor of h then there exists g such that $h = fg$ and therefore $\mathbf{X}_h = f\mathbf{X}_g + g\mathbf{X}_f$, in this way, we obtain:

$$\begin{aligned} \nabla f \cdot \mathbf{F}_r &= \nabla f \cdot \mathbf{X}_h + \mu \nabla f \cdot \mathbf{D}_0 = f \nabla f \cdot \mathbf{X}_g + g \nabla f \cdot \mathbf{X}_f + t_1 t_2 \mu f \\ &= f\left(\nabla f \cdot \mathbf{X}_g + t_1 t_2 \mu\right) \end{aligned}$$

i.e., $f = 0$ is an invariant curve, and therefore, the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is non-monodromic. The same happens if $f = x$ or $f = y$. ∎

This property suggests to state the following definition.

Definition 1.3.23. *The polynomial $h \in \mathcal{P}_k^t$ is monodromic, if the decomposition (1.3.8) of h on $\mathbb{C}[x,y]$ verifies that m_x, m_y are nulls, $\lambda_j \in \mathbb{C}\setminus\mathbb{R}$ and h is non-constant, that is, h is non-constant and only has complex (no real) factors in its decomposition on $\mathbb{C}[x,y]$. Observe that, a consequence of this definition, is that $h > 0$).*

Remark 3. *Observe that, a consequence of the above definition, is that h is defined positive or defined negative.*

1.3.3 The trigonometric blow-up.

Next we show a change of coordinates called *blow-up*. In this case we describe a blow-up trigonometric which is a generalization of the changes in polar coordinates. These changes expand the non-hyperbolic equilibrium in a curve on which it has a finite number of singularities and whose topological type is determined through the Hartam-Grobman Theorem. For this task we define the trigonometric generalized function $Cs(t)$, $Sn(t)$ which are solutions of the following initial value problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2t_1 y^{2t_1-1} \\ 2t_2 x^{2t_2-1} \end{pmatrix}, \text{ with } x(0)=1,\ y(0)=0. \tag{1.3.9}$$

Denoting $H(x,y) = x^{2t_2} + y^{2t_1}$, it has to $(Cs(\theta), Sn(\theta))$ are solutions of the hamiltonian system $(\dot{x},\dot{y})^T = \mathbf{X}_H$ whose origin is a center, therefore these functions are periodic, with minimal period T and also verifies that $Cs^{2t_2}(\theta) + Sn^{2t_1}(\theta) = 1$, $\forall \theta \in [0,T]$.

Proposition 1.3.24. *Let $Cs(\theta)$, $Sn(\theta)$ be the previously defined functions. The change of variables,*

$$\begin{aligned} x &= u^{t_1} Cs(\theta), \\ y &= u^{t_2} Sn(\theta), \end{aligned} \tag{1.3.10}$$

the reparametrization in the time $t = \frac{2t_1t_2}{u^r}\tau$, and applying the change $u = \frac{\rho}{1-\rho}$, system (1.1.1) is transformed into

$$\begin{aligned} \rho' &= \rho(1-\rho)(2t_1t_2\mu(\theta) - h'(\theta)), \\ \theta' &= (r + |\mathbf{t}|)h(\theta). \end{aligned} \tag{1.3.11}$$

In addition, this change becomes the straights $\rho = 0$ in the origin, the straights $\rho = 1$ in the infinite of the plane (x,y) and the region $D = \{0 \le \rho < 1, 0 \le \theta < T\}$ is transformed in the open disk $\{(x,y) \in \mathbb{R}^2\ |0 \le x^{2t_2} + y^{2t_1} < +\infty\}$

Proof.

Differentiating (1.3.10) respect the time, we obtain,

$$\begin{aligned} \dot{x} &= t_1 u^{t_1-1}\mathrm{Cs}(\theta)\dot{u} + u^{t_1}\tfrac{d\mathrm{Cs}(\theta)}{d\theta}\dot{\theta}, \\ \dot{y} &= t_2 u^{t_2-1}\mathrm{Sn}(\theta)\dot{u} + u^{t_2}\tfrac{d\mathrm{Sn}(\theta)}{d\theta}\dot{\theta}. \end{aligned}$$

And this is equivalent to the following vectorial equation,

$$\dot{\mathbf{x}} = \frac{1}{u}\mathbf{D}_0\dot{u} + \frac{1}{u^{2t_1t_2-|\mathbf{t}|}}\mathbf{X}_H\dot{\theta}.$$

Therefore, we obtain that,

1. $\dot{\mathbf{x}} \wedge \mathbf{X}_H = \frac{1}{u}(\mathbf{D}_0 \wedge \mathbf{X}_H)\dot{u}$.

2. $\mathbf{D}_0 \wedge \dot{\mathbf{x}} = \frac{1}{u^{2t_1t_2-|\mathbf{t}|}}(\mathbf{D}_0 \wedge \mathbf{X}_H)\dot{\theta}$,

   where $\mathbf{D}_0 \wedge \mathbf{X}_H = t_1 x\left(2t_2 x^{2t_2-1}\right) - t_2 y\left(-2t_1 y^{2t_1-1}\right) = 2t_1t_2u^{2t_1t_2} \neq 0$.

In addition, using conservative-dissipative decomposition we get

$$\begin{aligned}
\dot{\mathbf{x}} \wedge \mathbf{X}_H &= \mathbf{F}_r \wedge \mathbf{X}_H = [\mathbf{X}_h + \mu\mathbf{D}_0] \wedge \mathbf{X}_H \\
&= \mathbf{X}_h \wedge \mathbf{X}_H + 2t_1t_2H(x,y)\mu(x,y), \\
\mathbf{D}_0 \wedge \dot{\mathbf{x}} &= \mathbf{D}_0 \wedge \mathbf{X}_h(x,y) = (r+|\mathbf{t}|)h(x,y).
\end{aligned}$$

Moreover

$$\begin{aligned}
h_{r+|\mathbf{t}|}(x,y) &= \frac{1}{r+|\mathbf{t}|}(\mathbf{D}_0 \wedge \mathbf{X}_h) = \frac{1}{r+|\mathbf{t}|}\nabla h_{r+|\mathbf{t}|} \cdot \mathbf{D}_0 \\
&= u^{r+|\mathbf{t}|}h(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta)) \stackrel{\mathrm{def}}{=} u^{r+|\mathbf{t}|}h(\theta), \\
\mu(x,y) &= u^r\mu(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta)) \stackrel{\mathrm{def}}{=} u^r\mu(\theta), \\
\mathbf{X}_h \wedge \mathbf{X}_H &= u^{r+2t_1t_2-|\mathbf{t}|}\left[-\frac{\partial h(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta))}{\partial \mathrm{Sn}(\theta)}\frac{d\mathrm{Sn}(\theta)}{d\theta} - \frac{\partial h(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta))}{\partial \mathrm{Cs}(\theta)}\frac{d\mathrm{Cs}(\theta)}{d\theta}\right] \\
&= -u^{r+2t_1t_2-|\mathbf{t}|}h'(\theta).
\end{aligned}$$

After applying the reparametrization in the time $dt = \frac{2t_1t_2}{u^r}d\tau$, and the new change of variables $u = \frac{\rho}{1-\rho}$, we can conclude that system (1.1.1) is transformed into (1.3.11). ■

Remark 4. *Since $\rho' = 0$ in (1.3.11) when $\rho = 1$, then the boundary of D, $\partial D = \{(\rho,\theta) : \rho = 1\}$ is an invariant circle of the flow of (1.3.11). This circle corresponds to the infinite of the system (1.1.1), therefore the vector field $E(\mathbf{F}_r)$, asociated to the system (1.3.11), and defined in an open neighborhood U of $\bar{D}$, is an analytical extension of the vector field $\mathbf{F}_r$ to the infinity. $\bar{D}$ denotes the closure of D in $\mathbb{R}^2$. Although we only study the phase portrait of $E(\mathbf{F}_r)$ on the closed disk $\bar{D}$, we can also consider $E(\mathbf{F}_r)$ defined in the open neighborhood U. In this way, be applied the local study of the equilibrium of $E(\mathbf{F}_r)$ on ∂D.*

Proposition 1.3.25. *Let $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ be, with h non-monodromic and verifying all real factors are simple, then the equilibria of $E(\mathbf{F}_r)$ are $(\rho,\theta) =$*

$(0, \theta_i)$ *y* $(\rho, \theta) = (1, \theta_i)$, $i = 1, \cdots, 2s$ *where* s *is the number of real factors,* θ_i *is associated with each real factor of* h*, i.e.,* $h(\theta_i) = 0$ *and each real factor is associated with two values* θ_i*. Moreover all equilibria are hyperbolic saddles or nodes, verifing the following propierty, if* $(\rho, \theta) = (1, \theta_i)$ *is saddle (node) then* $(\rho, \theta) = (0, \theta_i)$ *is a node (saddle).*

Proof. It follows from (1.3.11) that the equilibria are the value θ such that $h(\theta) = 0$. In this case, as h is non-monodromic, the equilibria are the value $\theta = \theta_i$ associated with the real factors of h. Using the Proposition 1.2.16 there are no equilibria, in the finite plane, different from the origin. Therefore, the equilibria of (1.3.11) are $(\rho, \theta) = (0, \theta_i)$ and $(\rho, \theta) = (1, \theta_i)$, $i = 1, \cdots, N$ where θ_i is associated with each real factor of h, i.e., $h(\theta_i) = 0$. Moreover, each irreducible factor of h, either x, y or $y^{t_1} - ax^{t_2}$, has two associated values θ_i. Therefore there are an even number of values θ_i.

The matrix of the linear part of the system (1.3.11) in the equilibrium $(\rho, \theta) = (1, \theta_i)$ is:

$$\begin{pmatrix} -(2t_1t_2\mu(\theta_i) - h'(\theta_i)) & 0 \\ 0 & (r + |\mathbf{t}|)h'(\theta_i) \end{pmatrix}$$

where $2t_1t_2\mu(\theta_i) - h'(\theta_i) \neq 0$ since, otherwise, the curve $\theta = \theta_i$ is a curve of singular point of the system (1.3.11) which means, that the origin of the system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is not an isolated equilibrium. Neither can it be $h'(\theta_i) = 0$ since, otherwise, the factor h associaded with θ_i be a multiple factor, which is contradictory. Therefore, the equilibrium $(\rho, \theta) = (1, \theta_i)$ is a hyperbolic saddle or a node.

Respect to the equilibrium $(\rho, \theta) = (0, \theta_i)$, the matrix of the linear part of the system (1.3.11) is:

$$\begin{pmatrix} (2t_1t_2\mu(\theta_i) - h'(\theta_i)) & 0 \\ 0 & (r + |\mathbf{t}|)h'(\theta_i) \end{pmatrix}$$

By the same above reasoning, the equilibrium is a hyperbolic saddle or a node.

Furthermore, from the form of the linearization matrix in the equilibria $(\rho, \theta) = (0, \theta_i)$ and $(\rho, \theta) = (1, \theta_i)$ for all θ_i it follows that, if the equilibrium $(\rho, \theta) = (1, \theta_i)$ is a saddle (node), then the equilibrium $(\rho, \theta) = (0, \theta_i)$ is a node (saddle). ∎

Proposition 1.3.26. *If* $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{P}_r^{\mathbf{t}}$ *and* h *is monodromic then there exists* $d \in \mathbb{N}$ *such that* $r = t_1(t_2 - 1) + t_2(t_1 - 1) + 2(d-1)t_1 t_2$

Proof. If h is monodromic, by (1.3.8) is verified that, $m_x = m_y = 0$, $m_j = 0$ for all $1 \le j \le m$. Therefore, denoting $d = \sum_{j=1}^{m} m_j$, it is obtained that $r + |\mathbf{t}| = 2d\, t_1 t_2$ and, consequently: $r = t_1(t_2 - 1) + t_2(t_1 - 1) + 2(d-1)t_1 t_2$. ∎

Proposition 1.3.27. *Let* $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$, *with* h *monodromic and* $\mathbf{G}_r := \mathbf{X}_g + \lambda \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ *then, the origin of system* $\dot{\mathbf{x}} = \mathbf{F}_r + \varepsilon \mathbf{G}_r$ *is monodromic, for all sufficiently small* ε.

Proof. The system (1.3.11) asociated with the vector field $\mathbf{F}_r + \varepsilon \mathbf{G}_r = \mathbf{X}_{h+\varepsilon g} + (\mu + \varepsilon\lambda)\, \mathbf{D}_0$ is:

$$\begin{aligned} \rho' &= \rho(1-\rho)(2t_1t_2(\mu(\theta) + \varepsilon\lambda(\theta) - h'(\theta) - \varepsilon g'(\theta)), \\ \theta' &= (r + |\mathbf{t}|)\,(h(\theta) + \varepsilon g(\theta)) \end{aligned}$$

As, the origin of $\dot{\mathbf{x}} = \mathbf{F}_r$ is monodromic, then $h(\theta) > 0$ for all $\theta \in [0, T]$. Therefore, for all small enough ε, it is verified that $h(\theta) + \varepsilon g(\theta) > 0$. In consequence, the origin of $\dot{\mathbf{x}} = \mathbf{F}_r + \varepsilon \mathbf{G}_r$ is monodromic. ∎

1.3.4 Monodromy and center condition.

Next result characterizes the monodromy of a system by means of the monodromy of the conservative part of its first quasi-homogeneous component. We denote, in the following theorem and in the rest of this memory, for the first quasi-homogeneous component $\mathbf{F}_r$ of a system with the form $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, the conservative part $h_{r+|\mathbf{t}|} := h$ and the dissipative part $\mu_r := \mu$.

Theorem 1.3.28. *Let* $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ *and* $\mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \sum_{j>0} \mathbf{F}_{r+j}(\mathbf{x})$, $\mathbf{F}_{r+j} \in \mathcal{Q}_{r+j}^{\mathbf{t}}$. *Then, it is verified:*

(a) *If* h *is monodromic then the origin of the system* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ *is monodromic.*

(b) *If h is not monodromic and all its real factors are simple then the origin of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is non-monodromic.*

Proof.

a) Using conservative-dissipative decomposition of each one of the quasi-homogeneous components of $\mathbf{F}$, this field can be written as

$$\mathbf{F} = \sum_{j=0}^{\infty} \left[\mathbf{X}_{h_{r+j+|\mathbf{t}|}} + \mu_{r+j}\mathbf{D}_0 \right]. \tag{1.3.12}$$

with $h_{r+j+|t|} \in \mathcal{P}^t_{r+j+|t|}$ and $\mu_{r+j} \in \mathcal{P}_{r+j}$.

We consider the change of variables

$$\begin{aligned} x &= u^{t_1} Cs(\theta), \\ y &= u^{t_2} Sn(\theta), \end{aligned} \tag{1.3.13}$$

where $(Cs(\theta), Sn(\theta))$ are the solutions of initial values problem:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{X}_H(\mathbf{x}), \\ \mathbf{x}(0) = (1,0)^T, \end{cases}$$

with $\mathbf{x} = (x, y)$ and $H(x,y) = y^{2t_1} + x^{2t_2}$.

We should note that the point $(1,0)^T$ belongs to the periodic ring, since $\mathbf{X}_H$ is a quasi-homogeneous global center. Therefore $Cs(\theta)$ and $Sn(\theta)$ are periodic functions of period T.

Differentiating with respect to the time we obtain $\dot{\mathbf{x}} = \frac{1}{u}\mathbf{D}_0\dot{u} + \frac{1}{u^{2t_1t_2-|\mathbf{t}|}}\mathbf{X}_H\dot{\theta}$.

Therefore, $\dot{\mathbf{x}} \wedge \mathbf{X}_H = \frac{1}{u}\mathbf{D}_0 \wedge \mathbf{X}_H \dot{u}$ and $\mathbf{D}_0 \wedge \dot{\mathbf{x}} = \frac{1}{u^{2t_1t_2-|\mathbf{t}|}}\mathbf{D}_0 \wedge \mathbf{X}_H\dot{\theta}$, where $\mathbf{D}_0 \wedge \mathbf{X}_H = 2t_1t_2u^{2t_1t_2} \neq 0$.

In addition, using conservative-dissipative decomposition of each quasi-homogeneous terms, we get

$$\begin{aligned} \dot{\mathbf{x}} \wedge \mathbf{X}_H &= \sum_{j\geq 0} \mathbf{F}_{r+j} \wedge \mathbf{X}_H = \sum_{j\geq 0} [\mathbf{X}_{h_{r+|\mathbf{t}|+j}} + \mu_{r+j}\mathbf{D}_0] \wedge \mathbf{X}_H \\ &= \sum_{j\geq 0} \mathbf{X}_{h_{r+j+|\mathbf{t}|}} \wedge \mathbf{X}_H + 2t_1t_2 H(x,y) \sum_{j\geq 0} \mu_{r+j}(x,y), \\ \mathbf{D}_0 \wedge \dot{\mathbf{x}} &= \sum_{j\geq 0} \mathbf{D}_0 \wedge \mathbf{X}_{h_{r+j+|\mathbf{t}|}}(x,y) = \sum_{j\geq 0} (r+j+|\mathbf{t}|) h_{r+j+|\mathbf{t}|}(x,y). \end{aligned}$$

Moreover, for each $j \geq 0$, we have

$$
\begin{array}{rcl}
h_{r+j+|\mathbf{t}|}(x,y) & = & u^{r+j+|\mathbf{t}|}h_{r+j+|\mathbf{t}|}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)) \stackrel{\text{def}}{=} u^{r+j+|\mathbf{t}|}h_{r+j+|\mathbf{t}|}(\theta), \\
\mu_{r+j}(x,y) & = & u^{r+j}\mu_{r+j}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta)) \stackrel{\text{def}}{=} u^{r+j}\mu_{r+j}(\theta), \\
\mathbf{X}_{h_{r+j+|\mathbf{t}|}} \wedge \mathbf{X}_H & = & u^{r+j+2t_1t_2-|\mathbf{t}|}\Big[-\frac{\partial h_{r+j+|\mathbf{t}|}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta))}{\partial \mathrm{Sn}(\theta)}\frac{d\mathrm{Sn}(\theta)}{d\theta} \\
& - & \frac{\partial h_{r+j+|\mathbf{t}|}(\mathrm{Cs}(\theta),\mathrm{Sn}(\theta))}{\partial \mathrm{Cs}(\theta)}\frac{d\mathrm{Cs}(\theta)}{d\theta}\Big] \\
& = & -u^{r+j+2t_1t_2-|\mathbf{t}|}h'_{r+j+|\mathbf{t}|}(\theta).
\end{array}
$$

Taking into acount that $\mathrm{Cons}(\mathbf{F}_r) = h$ and $\mathrm{Diss}(\mathbf{F}_r) = \mu$ we obtain, after applying the reparametrization time $dt = \frac{2t_1t_2}{u^r}d\tau$, that system (1.3.12) is transformed into:

$$
\begin{array}{rcl}
u' & = & [2t_1t_2\mu(\theta) - h'(\theta)]u + \mathcal{O}(u^2), \\
\theta' & = & (r+|\mathbf{t}|)h(\theta) + \mathcal{O}(u),
\end{array} \tag{1.3.14}
$$

with $u > 0$ and $' = \frac{d}{d\tau}$.

Since h is monodromic then $h(\theta) \neq 0$ for all $\theta \in [0,T]$. Therefore, $\theta' \neq 0$ for all $|u| << 1$, consequently the origin is a monodromic singular point of system (1.3.12).

b) Considering the transformed system 1.3.14, if h is non-monodromic, there exists θ_0 simple root of h, i.e. $h(\theta_0) = 0$. In this case, the matrix of the linear part of the system 1.3.14 in the equilibrium $(u,\theta) = (0,\theta_0)$ is of the form,

$$
\begin{pmatrix} 2\mathbf{t}_1t_2\,\mu(\theta_0) - h'(\theta_0) & 0 \\ 0 & h'(\theta_0) \end{pmatrix}
$$

From the above matrix is possible to deduce that, the equilibrium $(0,\theta_0)$ is always an hyperbollic equilibrium or semi-hyperbolic which implies that there are always invariant curves that come into the equilibrium.

■

Remark 5. *Considering* $\mathbf{F}(\mathbf{x}) = \sum_{j\geq 0}\mathbf{F}_{r+j}(\mathbf{x})$, $\mathbf{F}_{r+j} := \mathbf{X}_{h_{r+|\mathbf{t}|+j}} + \mu_{r+j}\mathbf{D}_0 \in \mathcal{Q}^{\mathbf{t}}_{r+j}$*,is proposed the following problem:*

There exists any condition on the vector field $\mathbf{F}^{(N)} := \sum_{j=0}^{N} \mathbf{F}_{r+j}$ *such that, the monodromic condition of* $H = \sum_{j=0}^{N} h_{r+|\mathbf{t}|+j}$*, implies the monodromy of the field* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$*?*

For $N = 0$*, Theorem 1.3.28 prove that is enough that* h *be monodromic.*

Corolario 1.3.29. *h is monodromic if and only if $\mathbf{X}_h$ has a center at the origin.*

Proof. Let consider $\mathbf{F}_r = \mathbf{X}_h$ with h monodromic, using the Theorem 1.3.28, then the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is monodromic. Therefore, since $\mathbf{F}_r$ is a hamiltonian, is possible to conclude that the origin is a center.

To prove the sufficient condition, we take h non-monodromic, then h has factors of the form $f = x$, $f = y$ or $f = y^{t_1} - ax^{t_2}$, with $a \in \mathbb{R}$ therefore $f = 0$ is an invariant curve of $\mathbf{X}_h$, and consequently, the origin is not a center.

■

Next statement stablishes a sufficient and necessary condition for monodromy of a quasi-homogeneous vector field.

Theorem 1.3.30. *Let* $\mathbf{F}_r \in \mathcal{Q}_r^t$.

The origin of $\mathbf{F}_r$ is monodromic if and only if $\mathbf{X}_h$ has a center at the origin.

Proof. From Lemma 1.2.18, $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$ with $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ and $\mu \in \mathcal{P}^{\mathbf{t}}_r$. The change of variables (1.3.13) and the reparametrization in the time $dt = \frac{2t_1t_2}{u^r} d\tau$ transform system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ into

$$\begin{aligned} u' &= [2t_1t_2\mu(\theta) - h'(\theta)]u, \\ \theta' &= (r + |\mathbf{t}|)h(\theta), \end{aligned} \tag{1.3.15}$$

with $u > 0$ and $' = \frac{d}{d\tau}$.

To prove the necessary condition we assume that $\mathbf{X}_h$ has not a center at the origin. From Corollary 1.3.29, h is not monodromic then h has a real factor in its decomposition on $\mathbb{C}[x,y]$, therefore there exists a $\theta_0 \in [0,T)$ such that $h(\theta_0)=0$ that is, $\theta=\theta_0$ is invariant for (1.3.15), therefore, the real factor is a solution of $\dot{\mathbf{x}}=\mathbf{F}_r(\mathbf{x})$, in consequence, the origin of $\mathbf{F}_r$ is non-monodromic and this is a contradiction. The sufficient condition is followed from Theorem 1.3.28 and Corollary 1.3.29.

■

From Theorem 3.7 of [8] and Theorem 3.3 of [14] can be obtained the following result that we will use in next section.

Theorem 1.3.31. *Assume that* $\mathbf{F}_r=\mathbf{X}_h+\mu\mathbf{D}_0$*, with* $h\in\mathcal{P}_r^{\mathbf{t}}$ *monodromic. Then:*

a) the origin of (1.1.1) is a center if and only if $|\mathbf{t}|$ *is odd or* $|\mathbf{t}|$ *is even and* $I_{\mathbf{F}_r}=0$.

b) the origin of (1.1.1) is an unstable focus if and only if $|\mathbf{t}|$ *is even and* $\operatorname{sign}(h)I_{\mathbf{F}_r}>0$.

c) the origin of (1.1.1) is an stable focus if and only if $|\mathbf{t}|$ *is even and* $\operatorname{sign}(h)I_{\mathbf{F}_r}<0$.

with

$$I_{\mathbf{F}_r} \;=\; -2\pi \sum_{\substack{Im(\lambda_j)>0\\ h^{hom}(1,\lambda_j)=0}} Im(Res[\frac{\mu^{hom}(1,y)}{h^{hom}(1,y)},\lambda_j]), \qquad (1.3.16)$$

where $h^{hom}(x,y)$ *and* $\mu^{hom}(x,y)$ *are defined in (1.3.6).*

The following theorem give us a necessary condition so that a perturbated vector field of $\mathbf{F}_r$ have a center at the origin.

Theorem 1.3.32. *If the origin of* $\dot{\mathbf{x}}=\mathbf{F}_r(\mathbf{x})$ *is a focus, then the origin of system*

$$\dot{\mathbf{x}}=\mathbf{F}_r(\mathbf{x})+\mathbf{F}_{r+1}(\mathbf{x})+\dots \quad \textit{with } \mathbf{F}_{r+j}(\mathbf{x})\in\mathcal{Q}_{r+j}^{\mathbf{t}}, \qquad (1.3.17)$$

is also a focus with the same stability.

Proof.

Orbits of system (1.3.14) in generalized polar coordinates are defined by generalized Abel equation

$$\frac{du}{d\theta} = \left[\frac{2t_1t_2\mu(\theta)}{(r+|\mathbf{t}|)h(\theta)} - \frac{h'(\theta)}{(r+|\mathbf{t}|)h(\theta)}\right] u + \mathcal{O}(u^2).\,, \qquad (1.3.18)$$

with $h(\theta) \neq 0$ for all $\theta \in [0,T]$ because it is monodromic.

If we denote $u(\theta, u_0) = \sum_{n\geq 1} a_n(\theta)u_0^n$, the solution of the equation (1.3.18) satisfying $u(0,u_0) = u_0$ and by replacing $u(\theta, u_0)$ in the equation (1.3.18) we have

$$a_1(\theta) = \exp\left(\int_0^{\theta}\left[\frac{2t_1t_2\mu(\alpha)}{(r+|\mathbf{t}|)h(\alpha)} - \frac{h'(\alpha)}{(r+|\mathbf{t}|)h(\alpha)}\right] d\alpha\right).$$

The Poincare map of system (1.3.17) is given by

$$P(u_0) = u(T,u_0) = \sum_{n\geq 1} a_n(T)u_0^n, \text{ defined for } u_0 > 0,$$

and, for system $\dot{\mathbf{x}} = \mathbf{F}_r$, is given by $P(u_0) = a_1(T)u_0$, with

$$a_1(T) = e^{-\frac{1}{r+|\mathbf{t}|}\int_0^T \frac{h'(\theta)}{h(\theta)}d\theta + \frac{2t_1t_2}{r+|\mathbf{t}|}\int_0^T \frac{\mu(\theta)}{h(\theta)}d\theta} = e^{\frac{2t_1t_2}{r+|\mathbf{t}|}I_{\mathbf{F}_r}}. \qquad (1.3.19)$$

where $I_{\mathbf{F}_r} = \int_0^T \frac{\mu(\theta)}{h(\theta)}d\theta$.
Therefore, the first Lyapunov constant is the same for both systems (1.1.1) and (1.3.17). In the case that the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is a focus we have $a_1(T) \neq 1$, otherwise the origin is a center. Therefore, the origin of the perturbed system is also a focus with the same stability as the focus of the non-perturbed system.

■

Remark 6. *This result shows that if the origin of system (1.3.17) is a center, then the origin of (1.1.1) must be a center. This has been a motivation for studying the centers of vector fields of the form (1.1.1).*

In [13] can be seen the following example. In this article is proved the monodromy of the vector field described bellow. This example, shows that, there exists monodromic vector fields with the form, $\dot{\mathbf{x}} = \mathbf{F}_r + \cdots$, being $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ and however h is non-monodromic.

Example: Consider $\mathbf{F} = (y^3 + x^2y, xy^2 - x^5)^T$. Respect to the type, $\mathbf{t} = (t_1, t_2) = (1, 1)$, $\mathbf{F}$ can be written as $\mathbf{F} = \mathbf{F}_2 + \mathbf{F}_4$, where $\mathbf{F}_2 = (y^3 + x^2y, xy^2)^T$ and $\mathbf{F}_4 = (0, -x^5)^T$. In page 5408, example 3 of [13], can be seen that $\mathbf{F}$ is monodromic. However, h is non-monodromic.

1.4 Integrability, reversibility and center problem for planar quasi-homogeneous vector fields.

At first we give some necessary definitions.

- An involution is a function $\sigma \in \mathcal{C}^\omega(U_0 \subset \mathbb{R}^2, \mathbb{R}^2)$, such that $\sigma \circ \sigma = Id$, where U_0 is a neighborhood of the origin.

- A system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ (or a vector field $\mathbf{F}$) is reversible, if there exists an involution σ ($\sigma \neq \pm Id$), $\sigma(\mathbf{0}) = \mathbf{0}$, such that $\sigma_*\mathbf{F} = -\mathbf{F}$, where $\sigma_*\mathbf{F}$ denotes the pull-back of $\mathbf{F}$ by the transformation σ.

  Obviously, if $\mathbf{F}$ is reversible with respect to the involution σ and Φ is a diffeomorfism, such that, $\Phi(0) = 0$, then $\Phi_*\mathbf{F}$ is reversible with respect to the involution $\Phi \circ \sigma \circ \Phi^{-1}$.

- A system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ (or a vector field $\mathbf{F}$) is orbitally reversible if there exists an analytical scalar function f, $f(0) = 1$, such that $f\mathbf{F}$ is a reversible vector field.

  Remark: For quasi-homogeneous vector fields, the concepts of reversibility and orbital reversibility agree.

- A system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ (or a vector field $\mathbf{F}$) is axis-reversible if either it is reversible to the involution $\sigma(x, y) = (-x, y)$ ($\mathbb{R}_x - reversible$) or to the involution $\sigma(x, y) = (x, -y)$ ($\mathbb{R}_y - reversible$).

The following result, about integrability, can be seen in [14].

Proposition 1.4.33. *If $\mathbf{F}_r$ is irreducible and h has multiple factors in decomposition (1.3.8), $h \neq$cte. and $\mu \not\equiv 0$, then $\mathbf{F}_r$ is non-analytically integrable.*

Theorem 1.4.34. *Assume that $\mathbf{F}_r$ is irreducible, h is monodromic and h has more than two factors, all of then simples, in its decomposition (1.3.8). System $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ has a first integral if and only if either $\mu \equiv 0$ or there exists n_x, n_y, n_i, $i = 1, \cdots, m$ non-negative numbers, not all zeros, such that*

$$\begin{cases} Res[\eta^{\text{hom}}(X,1),0] = \frac{(n_x+1)(r+|\mathbf{t}|)-M_0}{t_2 M_0}, & \text{if } m_x = 1, \\ Res[\eta^{\text{hom}}(1,Y),0] = -\frac{(n_y+1)(r+|\mathbf{t}|)-M_0}{t_1 M_0}, & \text{if } m_y = 1, \\ Res[\eta^{\text{hom}}(1,Y),\lambda_i] = -\frac{(n_i+1)(r+|\mathbf{t}|)-M_0}{M_0}, & i = 1, \cdots, m, \end{cases} \tag{1.4.20}$$

with $\eta^{\text{hom}}(X,Y) = \frac{\mu^{\text{hom}}(X,Y)}{X^{\delta_x} Y^{\delta_y} h^{\text{hom}}(X,Y)}$ and $M_0 = t_1(n_x+1)\delta_x + t_2(n_y+1)\delta_y + t_1 t_2 \sum_{j=1}^{m}(n_j+1)$.

Moreover, in this case, a first integral of degree M_0 is

$$U(x,y) = x^{(n_x+1)\delta_x} y^{(n_y+1)\delta_y} \prod_{i=1}^{m} (y^{t_1} - \lambda_i x^{t_2})^{n_i+1}.$$

The following proposition and corollary can be deduced from the works developed by Montgomery & Zippin (see [69]). For sake completeness we give the proofs.

Next result characterizes the canonical involutions in $\mathbb{R}^2$.

Proposition 1.4.35. *Let σ be an involution, $\sigma \in \mathcal{C}^\omega$ and $\sigma \neq \pm Id$. Then there exists $\Psi(x,y) = (x+by+\cdots, ax+y+\cdots)$ such that $\Psi \circ \sigma \circ \Psi^{-1}(x,y)$ is $(-x,y)$ or $(x,-y)$*

Proof. We consider $\sigma(\mathbf{x}) = A\mathbf{x} + \cdots$ with $\mathbf{x} = (x,y)$. Since σ is an involution $A^2 = I_2$. Using linear transformation $\phi(x,y) = (x+by, ax+y)$ this involutions can be expressed in the new variables as:

1. $\sigma(x,y) = \pm(x + f(x,y), y + g(x,y))$,
2. $\sigma(x,y) = (x + f(x,y), -y + g(x,y))$,
3. $\sigma(x,y) = (-x + f(x,y), y + g(x,y))$,

The only involution of type 1 are $\pm Id$. Just consider $f(x,y) = A_{20}x^2 + A_{11}xy + A_{02}y^2 + \cdots$ and $g(x,y) = B_{20}x^2 + B_{11}xy + B_{02}y^2 + \cdots$ Therefore $\sigma(x,y) = (x + A_{20}x^2 + A_{11}xy + A_{02}y^2 + \cdots, y + B_{20}x^2 + B_{11}xy + B_{02}y^2 + \cdots)$, imposing $\sigma^2(x,y)=(x,y)$ it is easy to deduce that $f\equiv 0\equiv g$, in consequence the involutions that correspond to type 1 are $\pm Id$.

For the involutions of type 2, we consider the change of variables $\Psi(x,y)=\left(\frac{x+\sigma_1(x,y)}{2},\frac{y-\sigma_2(x,y)}{2}\right)=(u,v)$ with $\sigma=(\sigma_1,\sigma_2)$. Then

$$\begin{aligned}(\Psi\circ\sigma\circ\Psi^{-1})(u,v) &= \Psi\circ\sigma(x,y)=\Psi(\sigma_1(x,y),\sigma_2(x,y))\\ &= \left(\frac{x+\sigma_1(x,y)}{2},\frac{-y+\sigma_2(x,y)}{2}\right)=(u,-v).\end{aligned}$$

The process is similar for the involutions of type 3. ■

From Proposition 1.4.35 it is easy to prove that

Corolario 1.4.36. *Let* $\mathbf{F}$ *reversible be. Then there exists* $\Psi(x,y)=(x+by+\cdots,ax+y+\cdots)\in\mathcal{C}^\omega$ *such that* $\Psi_*\mathbf{F}$ *is axis-reversible.*

Next statement provides a sufficient and necessary condition about the reversibility of a quasi-homogeneous vector field.

Theorem 1.4.37. *Let* $\mathbf{F}_r\in\mathcal{Q}_r^{\mathbf{t}}$.

$\mathbf{F}_r$ *is reversible if and only if there exists* $\Psi=Id+\Psi_0$, $\Psi_0\in\mathcal{Q}_0^{\mathbf{t}}$, *where* $diag(D\Psi_0(0))=0$ *such that* $\Psi_*\mathbf{F}_r$ *is axis-reversible.*

Proof. The sufficient condition is clear. We prove the necessary condition depending on types $\mathbf{t}$:

1. If $\mathbf{t}=(1,1)$ we are in the case described in Corollary 1.4.36.

2. If $\mathbf{t}=(1,n)$ with $n>1$, from Corollary 1.4.36, the change of variables Ψ, that transforms the vector field into an axis-reversible system, can be decomposed in the form $\Psi=(id+\Psi_{\le 0})\circ(id+\Psi_{>0})$, where

   $$\Psi_{\le 0}=\textstyle\sum_{i=-M}^{0}\Psi_i,\ \ \Psi_i\in\mathcal{Q}_i^{\mathbf{t}}$$
   $$\Psi_{>0}=\textstyle\sum_{i=1}^{\infty}\Psi_i,\ \ \Psi_i\in\mathcal{Q}_i^{\mathbf{t}}$$

   with $diag(D\Psi_0(0))=0$.

   In this case, $m=n-1$ and the change of variables with negative degree and its inverse is

$$id + \Psi_{\leq 0} = \begin{cases} u = x, \\ v = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n + y, \end{cases}$$

$$(id + \Psi_{\leq 0})^{-1} = id - \Psi_{\leq 0} = \begin{cases} u = x, \\ v = -\alpha_1 x - \alpha_2 x^2 - \cdots - \alpha_n x^n + y. \end{cases}$$

From Corollary 1.4.36, $\Psi_* \mathbf{F}_r$ is axis-reversible, therefore $(id + \Psi_{\leq 0}) \circ (id + \Psi_{>0})_* \mathbf{F}_r = \mathbf{G}$ with $\mathbf{G}$ axis-reversible. Considering the degrees of both vector fields, $\mathbf{G} = \mathbf{G}_s + \cdots + \mathbf{G}_r + \cdots$ with $s < r$, $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$. Therefore

$$(id + \Psi_{>0})_* \mathbf{F}_r = (id - \Psi_{\leq 0})_* \mathbf{G},$$

and decomposing it into quasi-homogeneous components.

$$\mathbf{F}_r + \widehat{\mathbf{F}}_{r+1} + \cdots = \widehat{\mathbf{G}}_l + \cdots + \widehat{\mathbf{G}}_s + \cdots + \widehat{\mathbf{G}}_r + \cdots ,$$

we conclude $\widehat{\mathbf{G}}_l = \cdots = \widehat{\mathbf{G}}_s = \cdots = \widehat{\mathbf{G}}_{r-1} \equiv 0$. Moreover

$$(id - \Psi_{\leq 0})_* \mathbf{G} = \widehat{\mathbf{G}}_r + \cdots ,$$

then $\Psi_{-1} = \cdots = \Psi_{1-n} \equiv 0$, consequently $id + \Psi_{\leq 0} = id + \Psi_0$ and $(id + \Psi_0) \circ (id + \Psi_{>0})_* \mathbf{F}_r = \mathbf{G_r} + \ldots$ is axis-reversible. Therefore, in particular, $(id + \Psi_0)_* \mathbf{F}_r = \mathbf{G}_r$ is axis-reversible.

3. If $\mathbf{t} = (m, n)$, $1 < m < n$ proceed in the same way. It is sufficient to consider that

$$id + \Psi_{\leq 0} = \begin{cases} u = x, \\ v = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_M x^M + y, \end{cases}$$

with $M = \lfloor \frac{n}{m} \rfloor$ where $\lfloor x \rfloor$ denotes integer part of x.

■

The following proposition establishes the axis-reversibility of a quasi-homogeneous vector field in function of its conservative and dissipative part.

Proposition 1.4.38. *Assume that* $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{Q}_r^t$

a) $\mathbf{F}_r$ *is* $\mathbb{R}_x$*-reversible if and only if* $h(-x, y) = h(x, y)$ *and* $\mu(-x, y) = -\mu(x, y)$.

b) $\mathbf{F}_r$ *is* $\mathbb{R}_y$*-reversible if and only if* $h(x,-y) = h(x,y)$ *y* $\mu(x,-y) = -\mu(x,y)$.

Proof. We prove item **a)**, the other case is similar. $\mathbf{F}_r = (P,Q)^T$ is $\mathbb{R}_x$-reversible if and only if $P(-x,y) = P(x,y)$ and $Q(-x,y) = -Q(x,y)$, then

► $h(-x,y) = \frac{1}{r+|t|}\mathbf{D}_0(-x,y) \wedge \mathbf{F}_r(-x,y) = \frac{1}{r+|t|}(-t_1 x Q(-x,y) - t_2 y P(-x,y)) = h(x,y)$.

► $\mu(-x,y) = \frac{1}{r+|t|}(\frac{\partial P(-x,y)}{\partial(-x)} + \frac{\partial Q(-x,y)}{\partial y}) = \frac{1}{r+|t|}(-\frac{\partial P(x,y)}{\partial x} - \frac{\partial Q(x,y)}{\partial y}) = -\mu(x,y)$.
This proves the necessary condition.

If $h(-x,y) = h(x,y)$ and $\mu(-x,y) = -\mu(x,y)$, then

► $P(-x,y) = -\frac{\partial h(-x,y)}{\partial y} + t_1(-x)\mu(-x,y) = \frac{\partial h(x,y)}{\partial y} + t_1 x \mu(x,y) = P(x,y)$.

► $Q(-x,y) = \frac{\partial h(-x,y)}{\partial(-x)} + t_2 y \mu(-x,y) = -\frac{\partial h(x,y)}{\partial x} - t_2 y \mu(x,y) = -Q(x,y)$.

This finishes the proof. ■

From the above proposition is possible to establish the following property. This gives us a condition of reversibility according to the type $\mathbf{t} = (t_1, t_2)$.

Proposition 1.4.39. *Let* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ *and* h *monodromic. It holds the following propierties:*

a) *If* t_1 *is odd and* t_2 *is even, then* $\mathbf{F}_r$ *is* R_x*-reversible.*

b) *If* t_1 *is even and* t_2 *is odd, then* $\mathbf{F}_r$ *is* R_y*-reversible.*

Proof.

a) We prove paragraph **(a)**. Since h is monodromic, using (1.3.8) is obtained that $h(x,y) = \prod_{j=1}^{N}\left((y^{t_1} - a_j x^{t_2})^2 + b_j^2 x^{2t_2}\right)^{n_j}$, therefore $h(-x,y) = h(x,y)$. From Proposition 1.3.26, there exists $d \in \mathbb{N}$ such that $r = t_1(t_2-1) + t_2(t_1-1) + 2(d-1)t_1 t_2$ with $d = \sum_{j=1}^{N} n_j$. Hence, the dissipative part of $\mathbf{F}_r$ can be written in the form $\mu(x,y) = x^{t_2-1}y^{t_1-1}\sum_{j=0}^{2(d-1)} \alpha_j x^{t_2(2(d-1)-j)} y^{j t_1}$. Taking into accout that t_2 is even, $\mu(-x,y) = -\mu(x,y)$, can be obtained. Using Proposition 1.4.38, item **(a)** is achieved that $\mathbf{F}_r$ is R_x-reversible.

b) The paragraph **(b)** is proved analogously using now Proposition 1.4.38 item **(b)**.

■

Remark 7. *The hypothesis h monodromic, is essential. Considering the field $\mathbf{F}_r = \mathbf{X}_{yx^3} \in \mathcal{Q}_2^{(1,2)}$, in this case $h = yx^3$ is non-monodromic, $t_1 = 1$ is odd and $t_2 = 2$ is even. However $\mathbf{F}_r = (-x^3, 3x^2y)^T$ is not R_x-reversible.*

Corolario 1.4.40. *Let $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ be, h monodromic and $|\mathbf{t}|$ odd. Then, the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is a center.*

Proof. Since h is monodromic and $|\mathbf{t}|$ is odd, using Proposition 1.4.39 it is obtained that $\mathbf{F}_r$ is R_x-reversible or R_y-reversible. Considering that the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is monodromic, is obtained that also is a center. ■

1.5 Structural stability of quasi-homogeneous planar vector fields.

The first definition of *structural stability* for planar vector fields was shown by Andronov & Pontriaguin [18] who, in 1937, studied the structural stability for analytic vector fields on the closed 2-dimensional disc. Roughly speaking, we say that a vector field $\mathbf{F}$ is structurally stable if there are no substantial changes in the dynamics of the field and anyone of its neighbors. In other words, a vector field $\mathbf{F}$ is structurally stable if its phase portrait is topologically equivalent (via homeomorfhism) to the phase portrait of all of its neighbors in a suitable topology. Next, we adapt these ideas to quasi-homogeneous vector fields.

Definition 1.5.41. *We say that two vector fields $\mathbf{F}$, $\mathbf{G}$ are topologically equivalent if there are a homeomorphism Φ and a change of scale in time, that transforms orbits of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ into orbits of $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ without the need to keep the sense or parameterization.*

Definition 1.5.42. *Let consider a quasi-homogeneous vector field $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. We say that $\mathbf{F}_r$ is structurally stable, with respect to perturbation in $\mathcal{Q}_r^{\mathbf{t}} \neq \{0\}$,* *if $\mathbf{F}_r$ is topologically equivalent to any quasi-homogeneous vector field with the form $\mathbf{F}_r + \varepsilon \mathbf{G}_r$, where ε is a sufficient small parameter and $\mathbf{G}_r \in \mathcal{Q}_r^{\mathbf{t}}$.*

We denoted by $\mathcal{E}_r^{\mathbf{t}}$ the set of all vector fields in $\mathcal{Q}_r^{\mathbf{t}}$ which are structurally stable respect to perturbations in $\mathcal{Q}_r^{\mathbf{t}} \neq \{0\}$.

The following theorem characterizes the planar quasi-homogeneous vector fields that are structurally stable.

Theorem 1.5.43. *It holds that, $\mathbf{F}_r := \mathbf{X}_h + \mu \mathbf{D}_0 \in \mathcal{E}_r^{\mathbf{t}}$ if and only if one of the following conditions is satisfied:*

(a) *h is monodromic, $|\mathbf{t}|$ is even and $I_{\mathbf{F}_r} \neq 0$.*

(b) *h is monodromic and $|\mathbf{t}|$ is odd.*

(c) *h is not monodromic and all real factor of $h(x, y)$ are simple, i.e. the integer numbers given by (1.3.8), $m_x, m_y \in \{0, 1\}$ and $m_j = 1$ for $1 \leq j \leq M$.*

Proof. Firstly we provide the sufficient condition.

(a) If h is monodromic, $|\mathbf{t}|$ is even and $I_{\mathbf{F}_r} \neq 0$ then, from Theorem 1.3.31, the origin of $\mathbf{F}_r$ is a focus. Using the Corolary 1.2.19, it can be prove that this focus is global. Next we will prove that the origin of $\mathbf{F}_r + \varepsilon \mathbf{G}_r$ being $\mathbf{G}_r$ any small deformation of $\mathbf{F}_r$ with the form $\mathbf{G}_r = \mathbf{X}_g + \eta \mathbf{D}_0$, remains a global focus with the same stability. Considering $I_{\mathbf{F}_r + \varepsilon \mathbf{G}_r}$, this is a continuous function of ε such that for $\varepsilon = 0$, it is verified that $I_{\mathbf{F}_r} \neq 0$. Therefore, for ε sufficiently small is obtained $I_{\mathbf{F}_r + \varepsilon \mathbf{G}_r} \neq 0$. Also, using the same ideas is possible to obtain $sign\left((h + \varepsilon g) I_{\mathbf{F}_r + \varepsilon \mathbf{G}_r}\right) = sign\left(h I_{\mathbf{F}_r}\right)$. Hence, it is a focus with the same stability, i.e., $\mathbf{F}_r \in \mathcal{E}_r^{\mathbf{t}}$.

(b) Consider $\mathbf{F}_r + \varepsilon \mathbf{G}_r$ with $\mathbf{G}_r \in \mathcal{Q}_r^{\mathbf{t}}$ a small perturbation of $\mathbf{F}_r$. Since h is monodromic, the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is monodromic and using Proposition 1.3.27 is obtained that, also, the origin of $\dot{\mathbf{x}} = \mathbf{F}_r + \varepsilon \mathbf{G}_r$ is monodromic. Taking into account that $|\mathbf{t}|$ is odd, from Corolary 1.4.40 is deduced that the origin of both systems are centers, therefore $\mathbf{F}_r \in \mathcal{E}_r^{\mathbf{t}}$.

(c) If h is non-monodromic and all real factors of $h(x,y)$ are simple factors, using Proposition 1.3.25 is obtained that all equilibria of (1.3.11) are hyperbolics.

From Proposition 1.2.16 is obtained that, the system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ has no limit cycles nor other equilibria outside of the origin, hence, the dynamic of the system (1.3.11) is determined by the topological type of equilibria in $\rho = 0$ and $\rho = 1$ which are hyperbolic. Any perturbations sufficiently small of $\mathbf{F}_r$ preserves the same equilibriums and with the same topological type, therefore, $\mathbf{F}_r \in \mathcal{E}_r^{\mathbf{t}}$.

Now we provide the necessary condition.

- If h is monodromic, $|\mathbf{t}|$ is even and $I_{\mathbf{F}_r} = 0$, from Theorem 1.3.31 the origin of $\mathbf{F}_r$ is a center. Next we will prove that, there exists a small deformation of $\mathbf{F}_r$ which transform the origin in a focus, i.e., $\mathbf{F}_r \notin \mathcal{E}_r^{\mathbf{t}}$. It is enough to consider a small perturbation of $\mathbf{G}_r := \mathbf{F}_r + \varepsilon\lambda_r\mathbf{D}_0$ where $\lambda_r \in \mathcal{P}_r^{\mathbf{t}}$. Applying the Proposition 1.3.26 is obtained that, there exists $d \in \mathbb{N}$ such that $r = t_1(t_2-1) + t_2(t_1-1) + 2(d-1)t_1t_2$, therefore, it is possible consider $\lambda_r = x^{t_2-1+2(d-1)t_2}y^{t_1-1}$.

  Applying to the system $\dot{\mathbf{x}} = \mathbf{G}_r(\mathbf{x}) = \mathbf{F}_r + \varepsilon\lambda_r\mathbf{D}_0$ the change of variables $x = u^{t_1}\mathrm{Cs}(\theta)$, $y = u^{t_2}\mathrm{Sn}(\theta)$ where $\mathrm{Cs}(t)$, $\mathrm{Sn}(t)$ are the solutions of the initial value problem $(\dot{x},\dot{y})^T = \mathbf{F}_r$, $(x(0),y(0))^T = (1,0)^T$, taking into accout that $\mathbf{F}_r$ is a global center, it is obtained that $(\mathrm{Cs}(t),\mathrm{Sn}(t))$ is a periodic solution, of minimun period T. Moreover,

$$\mathbf{G}_r \quad = \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \tfrac{1}{u}\mathbf{D}_0(x,y)\dot{u} + \tfrac{1}{u^r}\mathbf{F}_r(x,y)\dot{\theta}$$

  Hence,

$$\begin{aligned} (r+|\mathbf{t}|)\varepsilon\lambda_r h(x,y) &= \mathbf{G}_r \wedge \mathbf{F}_r = \tfrac{1}{u}\left(\mathbf{D}_0 \wedge \mathbf{F}_r\right)\dot{u} = \tfrac{r+|\mathbf{t}|}{u}h(x,y)\dot{u} \\ (r+|\mathbf{t}|)h(x,y) &= \mathbf{D}_0 \wedge \mathbf{G}_r = \tfrac{1}{u^r}\mathbf{D}_0 \wedge \mathbf{X}_h(x,y)\dot{\theta} = \tfrac{r+|\mathbf{t}|}{u^r}h(x,y)\dot{\theta} \end{aligned}$$

  Therefore, as $h(x,y) \neq 0$:

$$\begin{aligned} \dot{u} &= u\varepsilon\lambda_r(x,y) = u^{r+1}\varepsilon\lambda_r(\theta), \\ \dot{\theta} &= u^r \end{aligned}$$

That is, $\frac{du}{d\theta} = u\varepsilon\lambda_r(\theta)$ and consequently

$$u(\theta) \;\; = \;\; u(0)\exp\left(\varepsilon\int_0^\alpha \lambda_r(\alpha)d\alpha\right)$$

The origin will be an attractive focus (repulsive) if $u(T) < u(0)$ ($u(T) > u(0)$) and a center if $u(T) = u(0)$. Since t_1 y t_2 are odds $\lambda_r(\theta) = \mathrm{Cs}^{t_2-1+2(d-1)t_2}(\theta)\mathrm{Sn}^{t_1-1}(\theta) > 0$ for all $\theta \in [0,T]$, hence $\int_0^T \lambda_r(\alpha)d\alpha > 0$ therefore, if $\varepsilon > 0$ ($\varepsilon < 0$), the origin of $\dot{\mathbf{x}} = \mathbf{G}_r(\mathbf{x})$ is a repulsive (attractive) focus, consequently $\mathbf{F}_r \notin \mathcal{E}_r^{\mathbf{t}}$.

- If h is non-monodromic and $y^{t_1} - ax^{t_2}$ with $a \in \mathbb{R}\setminus\{0\}$ is a multiple factor of h with multiplicity order $n > 1$ then, there exists f such that $h = (y^{t_1} - ax^{t_2})^n f$. Considering the vector field $\mathbf{G}_r := \mathbf{F}_r + \varepsilon\mathbf{X}_g$ which is a perturbation of $\mathbf{F}_r$ where $g = x^{t_2}(y^{t_1} - ax^{t_2})^{n-1}f$, i.e. $\mathbf{G}_r = \mathbf{X}_{(y^{t_1}-(a-\varepsilon)x^{t_2})(y^{t_1}-ax^{t_2})^{n-1}f} + \mu\mathbf{D}_0$. The vector field $\mathbf{G}_r$ is not topologically equivalent to $\mathbf{F}_r$ since $y^{t_1} - (a-\varepsilon)x^{t_2}$ is a new invariant curve for $\varepsilon \neq 0$ and sufficiently small, in consequence $\mathbf{F}_r \notin \mathcal{E}_r^{\mathbf{t}}$.

∎

Remark 8. *This result generalizes the homogeneous case studied in [65, Theorem A]. In [71, Theorem 2] is studied the quasi-homogeneos structural stability of vector fields, but do not consider the case* **b)** *of the above theorem.*

1.5.1 Classes of topological equivalence in $\mathcal{E}_r^{\mathbf{t}}$

Proposition 1.5.44. *If $\mathbf{F}_r \in \mathcal{E}_r^{\mathbf{t}}$ then the vector field $E(\mathbf{F}_r)$ has no equilibria in ∂D (case h monodromic) or it has $2s := 2(m_x + m_y + M) > 0$ equilibriums in ∂D, (caso h non-monodromic) where $m_x, m_y \in \{0,1\}$ and $M \in \mathbb{N}\cup\{0\}$.*

Proof. If $\mathbf{F}_r \in \mathcal{E}_r^{\mathbf{t}}$ then using Theorem 1.5.43 it has that h is monodromic and, therefore $E(\mathbf{F}_r)$ does not have equilibria or h is non-monodromic and all real factors of h are simple factors. From Proposition 1.3.25, $s = \delta_x + \delta_y + M$ is the number of simple real factors, where $\delta_x \in \{0,1\}$, $\delta_x = 1$ if x is factor of h, $\delta_y \in \{0,1\}$, $\delta_y = 1$ if y is factor of h and M is the number of real factors of type $y^{t_1} - ax^{t_2}$ with $a \in \mathbb{R} \setminus \{0\}$. Moreover, each of them determine two equilibria of $E(\mathbf{F}_r)$ on ∂D. ■

The following definition is needed to determine the topologically equivalent classes in $\mathcal{E}_r^{\mathbf{t}}$.

Definition 1.5.45. *Let $\mathbf{F}_r \in \mathcal{E}_r^{\mathbf{t}}$ and h non-monodromic. Consider $2s$ the number of equilibria in ∂D of $E(\mathbf{F}_r)$. It is defined*

$$\mathcal{S}(\mathbf{F}_r) = \{\sigma_i \ : \ 1 \leq i \leq 2s \ \ \text{and} \ \ \sigma_i \in \{S, N_r, N_a\}\}$$

the sequence of symbols is defined as

- $\sigma_i = S$*, if the equilibrium is a saddle.*
- $\sigma_i = N_r$ *if the equilibrium is a repulsive node.*
- $\sigma_i = N_a$ *if the equilibrium is an atractive node.*

Obviously, the number of equilibria and their topological types, determine the class of equivalence, so the number of symbols S, N_r y N_a of $\mathcal{S}(\mathbf{F}_r)$ characterizes the class of equivalence.

Moreover, two vector fields $\mathbf{F}_r$ and $\mathbf{G}_r$, having the same number of equilibria, with the same topological types but distributed differently, can belong to the same class, if an equilibrium can be brought to the other, using a translation. This amounts to saying that,

There exists $n \in \mathbb{N}$ an a permutation Ψ such that $\Psi^n(\mathcal{S}(\mathbf{F}_r)) = \mathcal{S}(\mathbf{G}_r)$ where $\Psi(\sigma_i) = \sigma_{i+1}$ for all $i = 1, \cdots, 2s-1$ and $\Psi(\sigma_{2s}) = \sigma_1$.

Proposition 1.5.46. *Consider $\mathbf{F}_r, \mathbf{G}_r \in \mathcal{E}_r^{\mathbf{t}}$. It holds that, $\mathbf{F}_r$ y $\mathbf{G}_r$ are topologically equivalent if and only if they satisfy any of the following conditions:*

a) *$\mathbf{F}_r$ and $\mathbf{G}_r$ are both monodromics.*

b) *$\mathbf{F}_r$ and $\mathbf{G}_r$ are non-monodromics with $E(\mathbf{F}_r)$ y $E(\mathbf{G}_r)$ having the same number of equilibriums in ∂D and there exist $n \in \mathbb{N}$ and a permutation Ψ such that $\Psi^n(\mathcal{S}(\mathbf{F}_r)) = \mathcal{S}(\mathbf{G}_r)$, where $\Psi(\sigma_i) = \sigma_{i+1}$ for $i = 1, \cdots, 2s-1$ and $\Psi(\sigma_{2s}) = \sigma_1$.*

Proof. The statement establishes that to prove that two vector fields $\mathbf{F}_r, \mathbf{G}_r \in \mathcal{E}_r^{\mathbf{t}}$ are topologically equivalent it is enough to prove that the dynamic of $E(\mathbf{F}_r)$ and $E(\mathbf{G}_r)$ on a neighborhood of ∂D are topologically equivalents. Clearly, the necessary condition is trivial. The sufficient condition will be proved.

The topological class of a vector field, is determined by the topological structure of its orbits. Thus, to determine when two quasi-homogeneous vector fields are topologically equivalents, it must be studied if the solutions of the associated systems (1.3.11) are equivalents. The Proposition 1.2.16 ensures that these systems have no limit cycles nor homoclinic orbits and the Proposition 1.3.25 afirms that the only equilibria of this system are in (ρ^*, θ^*) where θ^* is a solution of $h(\theta) = 0$ and ρ^* is 0 or 1. Furthermore, its topological type is a saddle or a node. Being, the topological type of the equilibriums, in $\rho = 0$ converse that $\rho = 1$. Thus, the topology class for a vector field, is determined by the number of equilibria in $\rho = 1$, of their topological types and its disposal within the invariant circle $\rho = 1$

a) $\mathbf{F}_r$ and $\mathbf{G}_r$ are monodromics. In this case $E(\mathbf{F}_r)$ and $E(\mathbf{G}_r)$ do not have critical points in ∂D.

- If $|\mathbf{t}|$ is even, from Theorem 1.5.43, item **(a)** and Theorem 1.3.31, the origin of the two associated systems, is a focus. Therefore they are topologically equivalents.
- If $|\mathbf{t}|$ is odd, using Corolary 1.4.40, it is obtained that the origin of the two associated systems is a center. Consequently they are topologically equivalents.

b) $\mathbf{F}_r$ and $\mathbf{G}_r$ are non-monodromics. Then, the following cases are considered:

- If the number of equilibriums of $E(\mathbf{F}_r)$ and $E(\mathbf{G}_r)$ in ∂D are equal, and $\mathcal{S}(\mathbf{F}_r) = \mathcal{S}(\mathbf{G}_r)$ then, $\mathbf{F}_r$ and $\mathbf{G}_r$ are topologically equivalents since, in this case, the equilibria and the topological types on $\rho = 0$, agree. In the case $\mathcal{S}(\mathbf{F}_r) \neq \mathcal{S}(\mathbf{G}_r)$ but a sequence of symbols can be transformed into another by a translation, it has the same result.

■

1.6 Applications

1.6.1 Study of the topological equivalence classes $\mathcal{E}_1^{(1,2)}$ and $\mathcal{E}_2^{(1,3)}$.

Now, the topological equivalence classes of $\mathcal{E}_1^{(1,2)}$ and $\mathcal{E}_2^{(1,3)}$ are studied. For this, first, a simplified canonical form of the quasi-homogeneous vector fields, belonging to $\mathcal{E}_1^{(1,2)}$ and $\mathcal{E}_1^{(1,3)}$ is determined.

Proposition 1.6.47. *Let* $\tilde{\mathbf{F}}_1 := (a_1x^2 + a_2y, b_1x^3 + b_2xy)^T \in \mathcal{E}_1^{(1,2)}$, $a_2(a_2b_1 - a_1b_2) \neq 0$. *The system* $\dot{\mathbf{x}} = \tilde{\mathbf{F}}_1(\mathbf{x})$ *is conjugated to* $\dot{\mathbf{x}} = \mathbf{F}_1(\mathbf{x})$, *where:*

$$\mathbf{F}_1 \quad := \quad (y + dx^2, \sigma x^3 + 2dxy)^T, \tag{1.6.21}$$

being $\sigma = \operatorname{sign}(\Delta)$, $\Delta = (b_2 - 2a_1)^2 + 8a_2b_1 \neq 0$ *and* $d = \frac{2a_1+b_2}{\sqrt{2|\Delta|}}$, $d \neq \pm\frac{1}{\sqrt{2}}$.

Proof. Note that,

- $a_2 \neq 0$ otherwise, x would be a factor of reducibility of $\tilde{\mathbf{F}}_1$.
- $a_2b_1 - a_1b_2 \neq 0$ otherwise $\tilde{\mathbf{F}}_1$ is reducible and, consequently, the origin of $\dot{\mathbf{x}} = \tilde{\mathbf{F}}_1(\mathbf{x})$ is not an isolated equilibrium.
- The conservative part of $\mathbf{F}_1$, $h = \frac{1}{4}(-2a_2y^2 + (b_2 - 2a_1)xy + b_1x^4)$, must be simple factors, therefore its discriminant $\Delta := (b_2 - 2a_1)^2 + 8a_2b_1 \neq 0$, otherwise h would have double factors.

To prove the statement just apply the change $x = \frac{\sqrt{8}}{\sqrt{|\Delta|}}u$, $y = \frac{\sqrt{8}}{a_2\sqrt{|\Delta|}}v - \frac{2a_1-b_2}{4a_2}u^2$ ∎

Proposition 1.6.48. *System* $\dot{\mathbf{x}} = \mathbf{F}_1(\mathbf{x})$ *where* $\mathbf{F}_1$ *is given in (1.6.21) with* $d \neq 2\sqrt{2}$, *has the following three classes of topological equivalence:*

a) *If* $\sigma = -1$, *the origin is a global center.*

b) *If* $\sigma = 1$, $d < -2\sqrt{2}$ *the origin is non-monodromic and it has four elliptic sectors.*

c) *If $\sigma = 1$, $d > -2\sqrt{2}$ the origin is non-monodromic and it has four hyperbolic sectors.*

Proof.

a) If $\sigma = -1$, then $h = -\frac{1}{4}x^4 - \frac{1}{2}y^2$ is monodromic and $|\mathbf{t}| = 3$ is odd. Using Theorem 1.3.31, the system is a global center. From Proposition 1.5.46, paragraph **b)**, defines a topological equivalence class.

b) If $\sigma = 1$, $h = \frac{1}{4}x^4 - \frac{1}{2}y^2 = -\frac{1}{2}\left(y - \frac{1}{\sqrt{2}}x^2\right)\left(y + \frac{1}{\sqrt{2}}x^2\right)$ is non-monodromic. The equilibriums in ∂D are $(\rho, \theta) = (1, \theta_i)$ where, the values θ_i, $i = 1, 2, 3, 4$ verifies: $\mathrm{Sn}^2(\theta_i) + \mathrm{Cs}^4(\theta_i) = 1$ y $\mathrm{Sn}(\theta_i) = \pm\frac{1}{\sqrt{2}}\mathrm{Cs}^2(\theta_i)$ i.e.: $\mathrm{Sn}^2(\theta_i) = \frac{1}{2}\mathrm{Cs}^4(\theta_i)$ and therefore $\mathrm{Cs}^4(\theta_i) = \frac{2}{3}$, that is, $\mathrm{Cs}(\theta_i) = \pm\frac{\sqrt[4]{2}}{\sqrt[4]{3}}$. The two values of θ associated to the factor $(y - \frac{1}{\sqrt{2}}x^2)$ are: $(\mathrm{Cs}(\theta_i), \mathrm{Sn}(\theta_i)) = \left(\pm\frac{\sqrt[4]{2}}{\sqrt[4]{3}}, \frac{1}{\sqrt{3}}\right)$ and the values associated to $(y + \frac{1}{\sqrt{2}}x^2)$ are: $(\mathrm{Cs}(\theta_i), \mathrm{Sn}(\theta_i)) = \left(\pm\frac{\sqrt[4]{2}}{\sqrt[4]{3}}, -\frac{1}{\sqrt{3}}\right)$. Ordered in the decreasing sense of θ, is obtained:

$$(\mathrm{Cs}(\theta_1), \mathrm{Sn}(\theta_1)) = \left(\frac{\sqrt[4]{2}}{\sqrt[4]{3}}, -\frac{1}{\sqrt{3}}\right) \quad (\mathrm{Cs}(\theta_2), \mathrm{Sn}(\theta_2)) = \left(-\frac{\sqrt[4]{2}}{\sqrt[4]{3}}, -\frac{1}{\sqrt{3}}\right)$$
$$(\mathrm{Cs}(\theta_3), \mathrm{Sn}(\theta_3)) = \left(-\frac{\sqrt[4]{2}}{\sqrt[4]{3}}, \frac{1}{\sqrt{3}}\right) \quad (\mathrm{Cs}(\theta_4), \mathrm{Sn}(\theta_4)) = \left(\frac{\sqrt[4]{2}}{\sqrt[4]{3}}, \frac{1}{\sqrt{3}}\right)$$

That is, $(\mathrm{Cs}(\theta_i), \mathrm{Sn}(\theta_i)) = \left((-1)^{i+1}\frac{\sqrt[4]{2}}{\sqrt[4]{3}}, (-1)^{\lfloor (i+1)/2 \rfloor}\frac{1}{\sqrt{3}}\right)$, $i = 1, 2, 3, 4$ Taking into account that $h'(\theta) = \mathrm{Cs}^3(\theta)(-2\mathrm{Sn}(\theta)) - \mathrm{Sn}(\theta)(4\mathrm{Cs}^3(\theta)) = -6\mathrm{Cs}^3(\theta)\mathrm{Sn}(\theta)$. Then $h'(\theta_i) = 2(-1)^{i+1}\frac{\sqrt[4]{8}}{\sqrt[4]{3}}$ and $\mu(\theta) = \frac{d}{4}\mathrm{Cs}(\theta)$, hence $\mu(\theta_i) = (-1)^{i+1}\frac{d}{4}\frac{\sqrt[4]{2}}{\sqrt[4]{3}}$. The matrix of the linear part of the system (1.3.11) in the equilibrium $(\rho, \theta) = (1, \theta_i)$ is:

$$\begin{pmatrix} (-1)^{i+1}\frac{\sqrt[4]{2}}{\sqrt[4]{3}}(-d + 2\sqrt{2}) & 0 \\ 0 & (-1)^{i+1}8\sqrt{2}\frac{\sqrt[4]{2}}{\sqrt[4]{3}} \end{pmatrix}$$

Therefore,

- For i odd. In this case the above matrix has the form,

$$\begin{pmatrix} \frac{\sqrt[4]{2}}{\sqrt[4]{3}}(-d + 2\sqrt{2}) & 0 \\ 0 & 8\sqrt{2}\frac{\sqrt[4]{2}}{\sqrt[4]{3}} \end{pmatrix}$$

- For i even. In this case the above matrix has the form,

$$\begin{pmatrix} \frac{\sqrt[4]{2}}{\sqrt[4]{3}}(d-2\sqrt{2}) & 0 \\ 0 & -8\sqrt{2}\frac{\sqrt[4]{2}}{\sqrt[4]{3}} \end{pmatrix}$$

where, in both cases, if $d < -2\sqrt{2}$ all equilibria are nodes and, in this case, there exist four parabolic sectors.

c) In the case $d > -2\sqrt{2}$ all equilibriums are saddles and, in this case, there exist four hyperbolic sectors. The case $d = -2\sqrt{2}$ is not considered because, for this value, the quasi-homogeneous vector field (1.6.21) is not structurally stable.

The conclusion is shown in the following table,

	θ_1	θ_2	θ_3	θ_4
$d < -2\sqrt{2}$	unstable node	stable node	unstable node	stable node
$d > -2\sqrt{2}$	saddle	saddle	saddle	saddle

Table 1.1: Summary of the topological equivalence classes of $\varepsilon_1^{(1,2)}$.

Therefore, there exist two classes of equivalence in the coordinates (x, y). These are: four elliptic sector (for $d < -2\sqrt{2}$) and four hyperbolic sector (for $d > -2\sqrt{2}$). ■

Proposition 1.6.49. *Consider* $\tilde{\mathbf{F}}_2 := (a_1x^3 + a_2y, b_1x^5 + b_2x^2y)^T \in \mathcal{E}_2^{(1,3)}$, $a_2(a_2b_1 - a_1b_2) \neq 0$. *The system* $\dot{\mathbf{x}} = \tilde{\mathbf{F}}_2(\mathbf{x})$ *is conjugated to* $\dot{\mathbf{x}} = \mathbf{F}_2(\mathbf{x})$, *where:*

$$\mathbf{F}_2 \ := \ (y + dx^3, \sigma x^5 + 3dx^2y)^T, \tag{1.6.22}$$

being $\sigma = \operatorname{sign}(\Delta)$, $\Delta = (b_2 - 3a_1)^2 + 12a_2b_1 \neq 0$ *and* $d = \frac{2\sqrt{3}(3a_1+b_2)}{6\sqrt{|\Delta|}}$, $d \neq \pm\frac{1}{\sqrt{3}}$.

Proof. Note that

- $a_2 \neq 0$, otherwise x would be a factor of reducibility $\tilde{\mathbf{F}}_2$.

- $a_2b_1 - a_1b_2 \neq 0$, otherwise $\tilde{\mathbf{F}}_2$ is reducible and, consequently, the origin of $\dot{\mathbf{x}} = \tilde{\mathbf{F}}_1(\mathbf{x})$ is not an isolated equilibrium.

- The conservative part of $\tilde{\mathbf{F}}_2$, $h = \frac{1}{6}(-3a_2y^2 + (b_2 - 3a_1)x^3y + b_1x^6)$, must be simple factors therefore its discriminant $\Delta := (b_2-3a_1)^2+12a_2b_1 \neq 0$, otherwise h would have double factors.

To prove the enunciate of this proposition, is sufficient to apply the following change $x = \frac{\sqrt[4]{12}}{\sqrt[4]{|\Delta|}}u$, $y = \frac{\sqrt[4]{12}}{\sqrt[4]{|\Delta|}}v + \frac{b_2-3a_1}{6a_2}\frac{\sqrt[4]{12}}{\sqrt[4]{|\Delta|}}u^3$ ∎

Proposition 1.6.50. *System $\dot{\mathbf{x}} = \mathbf{F}_2(\mathbf{x})$ where $\mathbf{F}_2$ is given in (1.6.22) has the following three classes of topological equivalence:*

a) *If $\sigma = -1$, we have two options:*

1. *$d = 0$, the origin is a global center.*
2. *$d \neq 0$, the origin is a focus.*

b) *If $\sigma = 1$, $d < -\sqrt{2}$ the origin is non-monodromic and it has four elliptic sectors.*

c) *If $\sigma = 1$, $d > -\sqrt{2}$ the origin is non-monodromic and it has four hyperbolic sectors.*

Proof.

a) If $\sigma = -1$, then $h = -\frac{1}{6}x^6 - \frac{1}{2}y^2$ is monodromic and $|\mathbf{t}| = 4$ is even. Moreover, $I_{\mathbf{F}_r} = \frac{\sqrt{3}}{2}d$. Using Theorem 1.3.31, it is possible diferenciate two cases,

1. If $d = 0$, the origin is a global center. In this case $\mathbf{F}_2 \notin \mathcal{E}_2^{(1,3)}$.
2. If $d \neq 0$, the origin is a unstable focus (for $d < 0$) and a stable focus (for $d > 0$).

b) If $\sigma = 1$, $h = \frac{1}{6}x^6 - \frac{1}{2}y^2 = -\frac{1}{2}\left(y - \frac{1}{\sqrt{3}}x^3\right)\left(y + \frac{1}{\sqrt{3}}x^3\right)$ is non-monodromic. In this case the equilibria in ∂D are $(\rho, \theta) = (1, \theta_i)$ where, the values θ_i, $i = 1, 2, 3, 4$ verifies: $\mathrm{Sn}^2(\theta_i) + \mathrm{Cs}^6(\theta_i) = 1$ and $\mathrm{Sn}(\theta_i) = \pm\frac{1}{\sqrt{3}}\mathrm{Cs}^3(\theta_i)$ i.e.: $\mathrm{Sn}^2(\theta_i) = \frac{1}{3}\mathrm{Cs}^6(\theta_i)$ and therefore $\mathrm{Cs}^6(\theta_i) = \frac{3}{4}$, that is, $\mathrm{Cs}(\theta_i) = \pm\frac{\sqrt[6]{3}}{\sqrt[6]{4}}$.

Consequently,

- The equilibriums associated with the factor $\left(y-\frac{1}{\sqrt{3}}x^3\right)$ are $\left(\frac{\sqrt[6]{3}}{\sqrt[6]{4}},\frac{1}{2}\right)$ and $\left(-\frac{\sqrt[6]{3}}{\sqrt[6]{4}},-\frac{1}{2}\right)$
- The equilibriums associated with the factor $\left(y+\frac{1}{\sqrt{3}}x^3\right)$ are $\left(\frac{\sqrt[6]{3}}{\sqrt[6]{4}},-\frac{1}{2}\right)$ and $\left(-\frac{\sqrt[6]{3}}{\sqrt[6]{4}},\frac{1}{2}\right)$

Ordered it, in the decreasing sense of θ, is obtained:

$$(\mathrm{Cs}(\theta_1),\mathrm{Sn}(\theta_1))=\left(\frac{\sqrt[6]{3}}{\sqrt[6]{4}},-\frac{1}{2}\right)\quad(\mathrm{Cs}(\theta_2),\mathrm{Sn}(\theta_2))=\left(-\frac{\sqrt[6]{3}}{\sqrt[6]{4}},-\frac{1}{2}\right)$$
$$(\mathrm{Cs}(\theta_3),\mathrm{Sn}(\theta_3))=\left(-\frac{\sqrt[6]{3}}{\sqrt[6]{4}},\frac{1}{2}\right)\qquad(\mathrm{Cs}(\theta_4),\mathrm{Sn}(\theta_4))=\left(\frac{\sqrt[6]{3}}{\sqrt[6]{4}},\frac{1}{2}\right)$$

Taking into account that $h=\frac{1}{6}\mathrm{Sn}^6(\theta)-\frac{1}{2}\mathrm{Cs}^2(\theta)$, then $h'(\theta)=\mathrm{Cs}^5(\theta)(-2\mathrm{Sn}(\theta))-\mathrm{Sn}(\theta)(6\mathrm{Cs}^5(\theta))=-8\mathrm{Cs}^5(\theta)\mathrm{Sn}(\theta)$ and $\mu(\theta)=\frac{d}{4}\mathrm{Cs}(\theta)$ and consequently $\mu(\theta_i)=(-1)^{i+1}\frac{d}{4}\frac{\sqrt[4]{2}}{\sqrt[4]{3}}$, we can write the matrix of the linear part of the system (1.3.11) in the equilibrium $(\rho,\theta)=(1,\theta_i)$ as follows:

$$\begin{pmatrix} \frac{\sqrt[6]{3^2}}{\sqrt[6]{4^2}}(-3d+(-1)^i\,2\sqrt{3}) & 0 \\ 0 & (-1)^{i+1}\,12\sqrt{3} \end{pmatrix}$$

Therefore,

- For i odd, the matrix of the above matrix has the form,

$$\begin{pmatrix} \frac{\sqrt[6]{3^2}}{\sqrt[6]{4^2}}(-3d-2\sqrt{3}) & 0 \\ 0 & 12\sqrt{3} \end{pmatrix}$$

- For i even, the matrix of the above matrix has the form,

$$\begin{pmatrix} \frac{\sqrt[6]{3^2}}{\sqrt[6]{4^2}}(-3d+2\sqrt{3}) & 0 \\ 0 & -12\sqrt{3} \end{pmatrix}$$

Therefore, if i is odd, for $d<-\frac{2\sqrt{3}}{3}$, the equilibria are unstable nodes and, for $d>-\frac{2\sqrt{3}}{3}$, the equilibria are saddles. In the case that i is even, for $d>\frac{2\sqrt{3}}{3}$, the equilibria are stable nodes and, for $d<-\frac{2\sqrt{3}}{3}$, the equilibria are saddles. The cases $d=-\frac{2\sqrt{3}}{3}$ and $d=\frac{2\sqrt{3}}{3}$ are not considered because, for this value, the quasi-homogeneous vector field (1.6.22) is not structurally stable.

These results are resumed in the following table:

	θ_1	θ_2	θ_3	θ_4
$d < \frac{-2\sqrt{3}}{3}$	unstable node	saddle	unstable node	saddle
$\frac{-2\sqrt{3}}{3} < d < \frac{2\sqrt{3}}{3}$	saddle	saddle	saddle	saddle
$d > \frac{2\sqrt{3}}{3}$	saddle	stable node	saddle	stable node

Table 1.2: Summary of the topological equivalence classes of $\varepsilon_2^{(1,3)}$.

Therefore, there exist two classes of equivalence in the coordinates (x, y). These are: four hyperbolic sectors (for $\frac{-2\sqrt{3}}{3} < d < \frac{2\sqrt{3}}{3}$) and four parabolic sectors if $d < \frac{-2\sqrt{3}}{3}$ or $d > \frac{2\sqrt{3}}{3}$. ∎

1.6.2 Applications to the degenerate center problem.

In this section, we study the center set (center problem) of the vector field $\mathbf{F}_r \in \mathcal{Q}_r^t$ for $0 \leq r \leq 4$ (i.e., to decide when the origin of $\mathbf{F}_r$ is a center). It is known that if $\mathbf{F}_r$ is monodromic (i.e., the origin of $\mathbf{F}_r$ is monodromic) and either it is reversible or is analytically integrable then $\mathbf{F}_r$ is a center (i.e. the origin of $\mathbf{F}_r$ is a center). In this section we also study the subsets of centers which are neither reversible nor analytically integrable. The analysis of these systems will be done classifying them by degrees and for each degree we consider the expression of $\mathbf{F}_r$ depending on the type $\mathbf{t}$:

(**Case** $\mathbf{r} = \mathbf{0}$) The system $\dot{\mathbf{x}} = \mathbf{F}_0(\mathbf{x})$, depending on the type $\mathbf{t}$, can be expressed as

$$\mathbf{t} = (1,1), \qquad \begin{cases} \dot{x} = a_{10}x + a_{01}y, \\ \dot{y} = b_{10}x + b_{01}y, \end{cases} \tag{1.6.23}$$

$$\mathbf{t} = (1,t_2), \qquad \begin{cases} \dot{x} = a_{10}x, \\ \dot{y} = b_{t_2 0}x^{t_2} + b_{01}y, \qquad t_2 > 1, t_2 \in \mathbb{N}. \end{cases} \tag{1.6.24}$$

The following theorem solves the center, reversibility and analytical integrability problem for systems (1.6.23) and (1.6.24).

Theorem 1.6.51.

1. *The origin of system (1.6.24) is non-monodromic.*

2. *The origin of system (1.6.23) is monodromic if and only if* $(b_{01}-a_{10})^2+4b_{10}a_{01}<0$ *and in this case, it is a center if and only if* $a_{10}=-b_{01}$.

3. *If the origin of system (1.6.23) is a center, then system (1.6.23) is reversible and analytically integrable.*

Proof.

The Hamiltonian part of the conservative-dissipative decompositions of systems (1.6.23) and (1.6.24) are:

$$\text{System (1.6.23):}\quad h(x,y)=\tfrac{1}{2}[b_{10}x^2+(b_{01}-a_{10})xy-a_{01}y^2].$$

$$\text{System (1.6.24):}\quad h(x,y)=\tfrac{1}{t_2+1}x[b_{t_20}x^{t_2}+(b_{01}-a_{10}t_2)y].$$

1. From Theorem 1.3.28 the origin of system (1.6.24) is non-monodromic because h has the real factor x.

2. From Theorem (1.3.28), imposing the monodromy condition to system (1.6.23), we obtain the relation $(b_{01}-a_{10})^2+4a_{01}b_{10}<0$. In this case, $h(x,y)=-\frac{a_{01}}{2}\left((y-ax^2)+b^2x^2)\right)$ being $a=-\frac{a_{01}}{2}\frac{b_{01}-a_{10}}{2a_{01}}$ and $b=\frac{\sqrt{-(b_{01}-a_{10})^2+4a_{01}b_{10}}}{4a_{01}^2}\neq 0$

   By other hand, if $h(x,y)$ is monodromic then it has the form $h(x,y)=c[(y-ax)^2+b^2x^2]$, $c\cdot b\neq 0$. From Theorem 1.3.31, we obtain $I_{\mathbf{F}_r}=\frac{\pi(a_{10}+b_{01})}{b}$. Therefore the origin of system (1.6.23) is a center if and only if $a_{10}=-b_{01}$.

3. In this case, system (1.6.23) is a Hamiltonian vector field, consequently it is analytically integrable.

   Finally, by the Theorem 1.4.37, the change of variables $u=x$, $v=-ax+y$ transform the system (1.6.23) into an axis-reversible system.

■

(**Case** $\mathbf{r}=\mathbf{1}$) The system $\dot{\mathbf{x}}=\mathbf{F}_1(\mathbf{x})$, depending on the type $\mathbf{t}$, can be expressed as

$$\mathbf{t} = (1,1), \qquad \begin{cases} \dot{x} = a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} = b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{cases} \tag{1.6.25}$$

$$\mathbf{t} = (2,3), \qquad \begin{cases} \dot{x} = a_{01}y, \\ \dot{y} = b_{20}x^2, \end{cases} \tag{1.6.26}$$

$$\mathbf{t} = (1,2), \qquad \begin{cases} \dot{x} = a_{20}x^2 + a_{01}y, \\ \dot{y} = b_{30}x^3 + b_{11}xy, \end{cases} \tag{1.6.27}$$

The following result deals with the center problem for systems (1.6.25), (1.6.26) and (1.6.27).

Theorem 1.6.52.

1. *The origin of (1.6.25) and (1.6.26) is non-monodromic. The origin of (1.6.27) is monodromic if and only if* $(b_{11} - 2a_{20})^2 + 8b_{30}a_{01} < 0$.
2. *If the origin of system (1.6.27) is monodromic, then it is a center if and only if* $2a_{20} = -b_{11}$.
3. *If the origin of system (1.6.27) is a center, then system (1.6.27) is reversible and analytically integrable.*

Proof. Systems (1.6.25), (1.6.26) and (1.6.27) can be expressed in the form $\mathbf{X}_h + \mu\mathbf{D}_0$ with

$$\text{System (1.6.25)} \begin{cases} h(x,y) = \frac{1}{3}[b_{20}x^3 + (b_{11} - a_{20})x^2y + (b_{02} - a_{11})xy^2 - a_{02}y^3], \\ \mu(x,y) = \frac{1}{3}[x(2a_{20} + b_{11}) + y(a_{11} + 2b_{02})], \end{cases}$$

$$\text{System (1.6.26)} \begin{cases} h(x,y) = \frac{1}{6}(2b_{20}x^3 - 3a_{01}xy) = \frac{1}{6}x(2b_{20}x^2 - 3a_{01}y), \\ \mu(x,y) = 0, \end{cases}$$

$$\text{System (1.6.27)} \begin{cases} h(x,y) = \frac{1}{4}[b_{30}x^4 + (b_{11} - 2a_{20})x^2y - 2a_{01}y^2], \\ \mu(x,y) = \frac{1}{4}[(2a_{20} + b_{11})x], \end{cases}$$

1. The Hamilton function of systems (1.6.25) and (1.6.26) is a three degree polynomial and it has a real factor in its decomposition (1.3.8). Therefore, from Theorem 1.3.28, the origin system (1.6.25) is non-monodromic. By the other hand, the Hamilton function of the decomposition of system (1.6.27) has only complex factors if and only if $(b_{11} - 2a_{20})^2 + 8b_{30}a_{01} < 0$.

2. Assuming that system (1.6.27) is monodromic then, the Hamiltonian part of decomposition of this system can be expressed in the form $h(x,y) = c[(y-ax^2)^2 + b^2x^4]$, $c \cdot b \neq 0$ and using Theorem 1.3.31 we obtain $I_{\mathbf{F}_r} = \frac{-(2a_{20}+b_{11})}{2b}$. Therefore the origin of system (1.6.27) is a center if and only if $2a_{20} + b_{11} = 0$.

3. In the case of system (1.6.27) has a center at the origin, then it is Hamiltonian and consequently analytically integrable. By other hand, the change of variables $u = x$, $v = -ax^2 + y$ transforms the system (1.6.27) into an axis-reversible system, from Theorem 1.4.37 the system is reversible.

■

(**Case r = 2**) Next we show any systems $\dot{\mathbf{x}} = \mathbf{F}_2(\mathbf{x})$, depending on the type $\mathbf{t}$, can be expressed as

$$\mathbf{t} = (1,1), \qquad \begin{cases} \dot{x} = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} = b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{cases} \tag{1.6.28}$$

$$\mathbf{t} = (1,2), \qquad \begin{cases} \dot{x} = a_{30}x^3 + a_{11}xy, \\ \dot{y} = b_{40}x^4 + b_{21}x^2y + b_{02}y^2, \end{cases} \tag{1.6.29}$$

$$\mathbf{t} = (1,3), \qquad \begin{cases} \dot{x} = a_{30}x^3 + a_{01}y, \\ \dot{y} = b_{50}x^5 + b_{21}x^2y, \end{cases} \tag{1.6.30}$$

To study family (1.6.28), for simplicity, we reduce the number of parameters.

Proposition 1.6.53. *The origin of system (1.6.28) is monodromic if and only if there exists a degree zero change of variables and a time reparametrization that transforms it into*

$$\begin{cases} \dot{x} = \mu_0x^3 + (\mu_1 - 2(B^2+1))x^2y + \mu_2xy^2 - 4y^3, \\ \dot{y} = 4B^2x^3 + \mu_0x^2y + (\mu_1 + 2(B^2+1))xy^2 + \mu_2y^3, \end{cases} \tag{1.6.31}$$

Proof.

The functions h and μ of the conservative and dissipative part of the decomposition of system (1.6.28) are

$$\begin{cases} h(x,y) = \frac{1}{4}[b_{30}x^4 + (b_{21} - a_{30})x^3y + (b_{12} - a_{21})x^2y^2 + (b_{03} - a_{12})xy^3 - a_{03}y^4], \\ \mu(x,y) = \frac{1}{4}[(3a_{30} + b_{21})x^2 + (2a_{21} + 2b_{12})xy + (a_{12} + 3b_{03})y^2], \end{cases}$$

From Theorem 1.3.28 the origin of (1.6.28) is monodromic if and only if h is monodromic. In this case, we can assume that $a_{03} \neq 0$, otherwise h is non-monodromic. By scaling in the time we can assume the coefficient of y^4 in the polynomial h is 1.

We will distinguish two different possibilities in the case h monodromic

a) h has simple factors on $\mathbb{C}[x, y]$. Then h is the form $h(x, y) = [(y-a_1x)^2 + b_1^2x^2][(y - a_2x)^2 + b_2^2x^2]$, $b_i \neq 0$, $i = 1, 2$, and $(a_1, b_1) \neq (a_2, b_2)$. The change of variables $u = (|b_1| - a_1A)x + Ay$, $v = y - (a_1 + A|b_1|)x$, where $A = 0$ if $a_1 = a_2$ or A verifies the equation $\frac{a_2-a_1}{b_1}A^2 + (1 - \frac{(a_2-a_1)}{b_1^2} + \frac{b_2^2}{b_1^2})A - \frac{a_2-a_1}{b_1} = 0$ in other case, transforms system (1.6.28) into $\dot{\mathbf{x}} = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0$ where

$$\tilde{h}(x, y) = (y^2 + x^2)(y^2 + B^2x^2),$$
$$\tilde{\mu}(x, y) = \mu_0x^2 + \mu_1xy + \mu_2y^2,$$

and writing the system, in the usual form, we obtain (1.6.31).

b) h has multiple factors on $\mathbb{C}[x, y]$. Then h can be written in the form $h(x, y) = [(y - a_1x)^2 + b_1^2x^2]^2$. Using the change of variables $x_1 = |b_1|x$, $x_2 = y - a_1x$, h and μ can be transformed into:

$$\tilde{h}(x, y) = (y^2 + x^2)^2,$$
$$\tilde{\mu}(x, y) = \mu_0x^2 + \mu_1xy + \mu_2y^2,$$

and writing the system in the usual form, we obtain (1.6.31) in the particular case $B = 1$.

■

Next, to characterize the centers of family (1.6.28) we will work with the equivalent canonical family (1.6.31).

The following result classifies the centers for systems (1.6.28), (1.6.29) and (1.6.30).

Theorem 1.6.54.

1. *The origin of system (1.6.29) is non-monodromic.*

2. *The origin of system (1.6.30) is monodromic if and only if* $(b_{21}-3a_{30})^2+12a_{01}b_{50}<0$. *In this case, the origin of (1.6.30) is a center if and only if* $3a_{30}+b_{21}=0$. *These centers are reversible and analytically integrable.*

3. *The origin of system (1.6.31) is a center if and only if* $\mu_0=-B\mu_2$. *These centers are reversible if and only if either,* $B=1$ *or* $\mu_0=\mu_2=0$ *and are analytically integrable if and only if* $B\neq 1$ *and* $\mu_0=\mu_2=\mu_3=0$.

Remark 9. *Notice that there exit systems in the family (1.6.31) which are centers non-reversible and non-integrable. Moreover, there exist systems which are reversible and non-integrable and vice versa. This is a new situation respect to the one of non-degenerate and nilpotent centers.*

In the proof of this theorem we will use Proposition 1.4.33 and Theorem 1.4.34.

Proof of Theorem The functions h and μ of the conservative and dissipative part of the decomposition of systems (1.6.29), (1.6.30) and (1.6.31) are:

$$\text{System (1.6.29)} \begin{cases} h(x,y)=\frac{1}{5}[b_{40}x^5+(b_{21}-2a_{30})x^3y+(b_{02}-2a_{11})xy^2], \\ \mu(x,y)=\frac{1}{5}[(3a_{30}+b_{21})x^2+(a_{11}+2b_{02})y], \end{cases}$$

$$\text{System (1.6.30)} \begin{cases} h(x,y)=\frac{1}{6}[b_{50}x^6+(b_{21}-3a_{30})x^3y-3a_{01}y^2], \\ \mu(x,y)=\frac{1}{6}(3a_{30}+b_{21})x^2, \end{cases}$$

$$\text{System (1.6.31)} \begin{cases} h(x,y)=(y^2+x^2)(y^2+B^2x^2), \\ \mu(x,y)=\mu_0x^2+\mu_1xy+\mu_2y^2, \end{cases}$$

1. The Hamiltonian function of the decomposition of system (1.6.29) has the real factor x in its factorization (1.3.8), therefore, from Theorem 1.3.28, the origin of (1.6.29) is non-monodromic.

2. From Theorem 1.3.28, imposing the monodromy condition to system (1.6.30), we obtain the relation $(b_{21}-3a_{30})^2+12b_{50}a_{01}<0$. Assuming that system (1.6.30) is monodromic, the Hamiltonian part of decomposition of this system can be expressed in the form

$$h(x,y)=c[(y-ax^3)^2+b^2x^6],$$

where $c = 3a_{01}$, $a = \frac{b_{21}-3a_{30}}{6a_{01}}$ and $b = \frac{\sqrt{|(b_{21}-3a_{30})^2+12b_{50}a_{01}|}}{6a_{01}}$. From Proposition 1.3.31, we obtain $I_{\mathbf{F}_r} = \frac{\pi}{b}(3a_{30}+b_{21})$ and we can conclude the origin of system (1.6.30) is a center if and only if $3a_{30}+b_{21} = 0$.

In the cases of the origin of system (1.6.30) is a center, using Theorem 1.4.37, the change of variables $u = |b|x$, $v = y - ax^3$ transforms the system into an axis-reversible system. Moreover, this centers are integrable because the conditions of center cancels the dissipative part and the system is Hamiltonian.

3. To study system (1.6.31) we distinguish two subcases:

   i) $B \neq 1$ From Theorem 1.3.31, we obtain $I_{\mathbf{F}_r} = \frac{-(\mu_0+B\mu_2)}{2B(B+1)}$. Therefore the origin of system (1.6.31) is a center if and only if $\mu_0 + B\mu_2 = 0$.

   In the cases that the origin of system (1.6.31) is a center, if $\mu_0 = \mu_2 = 0$ then system (1.6.31) is axis-reversible, consequently, it is reversible. In other case there not exists a change of variables with the form $(id+\Psi_0)$, $\Psi_0 \in \mathcal{Q}_0^{\mathbf{t}}$ such that $(id+\Psi_0)_*\mathbf{F}_2$ is axis-reversible, from Theorem 1.4.37 the system is non-reversible.

   From Theorem 1.4.34, system (1.6.31) is integrable if it is verified

   a) $Res[\eta^{hom}(1,Y),i] = \frac{\mu_1-(\mu_0-\mu_2)i}{2(B^2-1)} = -\frac{4(n_1+1)-M_0}{M_0}$.
   b) $Res[\eta^{hom}(1,Y),-i] = \frac{\mu_1+(\mu_0-\mu_2)i}{2(B^2-1)} = -\frac{4(n_2+1)-M_0}{M_0}$.
   c) $Res[\eta^{hom}(1,Y),Bi] = \frac{\mu_1 B-(\mu_0-B^2\mu_2)i}{2B(B^2-1)} = -\frac{4(n_3+1)-M_0}{M_0}$.
   d) $Res[\eta^{hom}(1,Y),-Bi] = \frac{\mu_1 B+(\mu_0-B^2\mu_2)i}{2B(B^2-1)} = -\frac{4(n_4+1)-M_0}{M_0}$.

   where $M_0 = (n_1+1)+(n_2+1)+(n_3+1)+(n_4+1)$. From here we can conclude the unique condition of integrability is $\mu_0 = \mu_1 = \mu_2 = 0$ i.e., when system (1.6.31) is a Hamiltonian system.

   ii) $B = 1$. From Theorem 1.3.31, $I_{\mathbf{F}_r} = \frac{-\pi}{2}(\mu_0+\mu_2)$. Therefore the origin of system (1.6.31) is a center if and only if $\mu_0 + \mu_2 = 0$.

   In this case, to study the reversibility we consider two subcases. If $\mu_0 = 0$ then $\mu_2 = 0$ and, by applying Proposition 1.4.38 the system is axis-reversible. If $\mu_0 \neq 0$, the change of variables $(id+\Psi_0) = (x+\beta y, y-\beta x)$ with $\beta = \frac{\mu_1 \pm \sqrt{\mu_1^2+4\mu_0^2}}{2\mu_0}$ transform the system into an axis-reversible system. From Theorem 1.4.37 it is reversible.

By applying Proposition 1.4.33, we can observe that h has double factors, consequently, the centers are non-integrable.

■

(**Case r = 3**) Some systems $\dot{\mathbf{x}} = \mathbf{F}_3(\mathbf{x})$, depending on the type $\mathbf{t}$, can be expressed as

$$\mathbf{t} = (1,1) \qquad \begin{cases} \dot{x} = a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4, \\ \dot{y} = b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4, \end{cases} \tag{1.6.32}$$

$$\mathbf{t} = (1,2) \qquad \begin{cases} \dot{x} = a_{40}x^4 + a_{21}x^2y + a_{02}y^2, \\ \dot{y} = b_{50}x^5 + b_{31}x^3y + b_{12}xy^2, \end{cases} \tag{1.6.33}$$

$$\mathbf{t} = (1,3) \qquad \begin{cases} \dot{x} = a_{40}x^4 + a_{11}xy, \\ \dot{y} = b_{60}x^6 + b_{31}x^3y + b_{02}y^2, \end{cases} \tag{1.6.34}$$

$$\mathbf{t} = (1,4) \qquad \begin{cases} \dot{x} = a_{40}x^4 + a_{01}y, \\ \dot{y} = b_{70}x^7 + b_{31}x^3y, \end{cases} \tag{1.6.35}$$

$$\mathbf{t} = (2,3) \qquad \begin{cases} \dot{x} = a_{11}xy, \\ \dot{y} = b_{30}x^3 + b_{02}y^2, \end{cases} \tag{1.6.36}$$

$$\mathbf{t} = (2,5) \qquad \begin{cases} \dot{x} = a_{01}y, \\ \dot{y} = b_{40}x^4, \end{cases} \tag{1.6.37}$$

The following result analyzes the center problem for systems (1.6.33) - (1.6.37).

Theorem 1.6.55.

1. *The origin of the families (1.6.32) - (1.6.34), (1.6.36) and (1.6.37) is non-monodromic.*

2. *The origin of (1.6.35) is monodromic if and only if* $(b_{31} - 4a_{40})^2 + 16b_{70}a_{01} < 0$.

3. *In the case of the origin of system (1.6.35) is monodromic, it is a center if and only if* $4a_{40} + b_{31} = 0$. *These centers are reversible and analytically integrable.*

Proof.

1. From Theorem 1.3.28, it is enough to verify that the Hamiltonian functions of the conservative-dissipative decomposition of these systems has a real factor.

2. The function h y μ of the conservative-dissipative decomposition of system (1.6.35) are:

$$\left\{\begin{array}{l} h(x,y) = \frac{1}{8}[b_{70}x^8 + (b_{31} - 4a_{40})x^4y - 4a_{01}y^2], \\ \mu(x,y) = \frac{1}{8}(4a_{40} + b_{31})x^3, \end{array}\right.$$

   From Theorem 1.3.28, we obtain that system (1.6.35) is monodromic if and only if $(b_{31} - 4a_{40})^2 + 16b_{70}a_{01} < 0$.

3. Assume that the origin of system (1.6.35) is monodromic, the Hamiltonian part of decomposition of system (1.6.35) can be expressed in the form

$$h(x,y) = c[(y - ax^4)^2 + b^2x^8],$$

   where $c = 4a_{01}$, $a = \frac{(b_{31}-4a_{40})^2}{8a_{01}}$ and $b = \frac{\sqrt{|(b_{31}-4a_{40})^2-16b_{70}a_{01}|}}{4a_{01}}$.

   Using Theorem 1.3.31, we obtain $I_{\mathbf{F}_r} = \frac{\pi}{8}(\frac{4a_{40}+b_{31}}{b})$, therefore the origin of system (1.6.35) is a center if and only if $4a_{40} + b_{31} = 0$.

   In this last case, system (1.6.35) is Hamiltonian and, consequently analytically integrable. By other hand, from Theorem 1.4.37, the change of variables $u = |b|x$, $v = y - ax^4$ transforms system (1.6.35) into an axis-reversible system.

■

(**Case r = 4**) Some systems $\dot{\mathbf{x}} = \mathbf{F}_4(\mathbf{x})$, depending on the type $\mathbf{t}$, can be expressed as

$$\mathbf{t}=(1,1),\quad \left\{\begin{array}{rcl}\dot{x} & = & a_{50}x^5+a_{41}x^4y+a_{32}x^3y^2+a_{23}x^2y^3\\ & & +a_{14}xy^4+a_{05}y^5,\\ \dot{y} & = & b_{50}x^5+b_{41}x^4y+b_{32}x^3y^2+b_{23}x^2y^3\\ & & +b_{14}xy^4+b_{05}y^5,\end{array}\right. \tag{1.6.38}$$

$$\mathbf{t}=(1,t_2),\ t_2>1\quad \left\{\begin{array}{rcl}\dot{x} & = & a_{50}x^5+a_{31}\chi_{\{t_2=2\}}x^3y+a_{12}\chi_{\{t_2=2\}}xy^2+\\ & & +a_{11}\chi_{\{t_2=4\}}xy+a_{01}\chi_{\{t_2=5\}}y,\\ \dot{y} & = & b_{t_2+4,0}x^{t_2+4}+b_{41}x^4y+b_{22}\chi_{\{t_2=2\}}x^2y^2\\ & & +b_{03}\chi_{\{t_2=2\}}y^3++b_{02}\chi_{\{t_2=4\}}y^2,\end{array}\right. \tag{1.6.39}$$

where $\chi_{\{p\}}$ is equal to 1 if the proposition p is true, and is equal to 0 otherwise.

Next result studies the monodromy for systems of the family (1.6.38).

Proposition 1.6.56. *The origin of (1.6.38) is monodromic if and only if there exists a change of variables of degree zero and a scaling in the time which transform the system* $\dot{x}=\mathbf{X}_h+\mu\mathbf{D}_0$ *into a system with the form* $\dot{\mathbf{x}}=\mathbf{X}_{\tilde{h}}+\tilde{\mu}\mathbf{D}_0$*, being* $\tilde{h}$ *and* $\tilde{\mu}$ *one and only one of the following:*

$$\left\{\begin{array}{l}\tilde{h}=(x^2+y^2)^3,\\ \tilde{\mu}=\mu_0x^4+\mu_1x^3y+\mu_2x^2y^2+\mu_3xy^3+\mu_4y^4,\end{array}\right. \tag{1.6.40}$$

$$\left\{\begin{array}{l}\tilde{h}=(x^2+y^2)^2(x^2+B^2y^2),\\ \tilde{\mu}=\mu_0x^4+\mu_1x^3y+\mu_2x^2y^2+\mu_3xy^3+\mu_4y^4,\end{array}\right. \tag{1.6.41}$$

$$\left\{\begin{array}{l}\tilde{h}=(x^2+y^2)(x^2+B^2y^2)[(y-Ax)^2+C^2x^2),\\ \tilde{\mu}=\mu_0x^4+\mu_1x^3y+\mu_2x^2y^2+\mu_3xy^3+\mu_4y^4,\end{array}\right. \tag{1.6.42}$$

Proof.

The function h for system (1.6.38) is a homogeneous polynomial of degree 6, without loss of generality we can assume that the coefficient of y^6 in h is 1, otherwise h is non-monodromic. Possible options are:

- h has a pair of conjugate complex factors of multiplicity 3. In this case h can be expressed in the form $h = [(y - a_1x)^2 + b_1^2x^2]^3$. The change of variables $x = b_1x$, $v = y - a_1x$ transform system (1.6.38) into (1.6.40).

- h has a pair of conjugate complex factors of multiplicity 2 and a simple pair of conjugate complex factor or h has three simple pair of complex conjugate factors. By applying the changes of variables described in the proof of item a) of Proposition 1.6.53, system (1.6.38) is transformed into (1.6.41) and (1.6.42) respectively.

■

The following theorem deals with the monodromy, the center problem, reversibility and analytical integrability for systems (1.6.39)-(1.6.42).

Theorem 1.6.57.

1. *The origin of system (1.6.39) is monodromic if and only if* $t_2{=}5$ *and* $(b_{41} - 5a_{50})^2 + 20b_{90}a_{01} < 0$. *In this case, it is a center if and only if* $b_{41} + 5a_{50} = 0$. *These centers are reversible and analytically integrable.*

2. *The origin of system (1.6.40) is a center if and only if* $\mu_2 = -3(\mu_4 + \mu_0)$. *In this case the system is non-integrable. It is reversible if and only if any of the following situations occurs:*

   - $\mu_0 = \mu_2 = \mu_4 = 0$
   - *There exists* $\beta \in \mathbb{R}$, $\beta \neq 0$ *such that,*
     - $\mu_2 = \frac{-3}{2}\frac{\beta^2-1}{\beta}[\mu_1 + \frac{\beta^2-1}{\beta}\mu_0]$
     - $\mu_3 = \frac{1}{2\beta^2}[\frac{(\beta^2-1)(\beta^4-6\beta^2+1)}{\beta}\mu_0 + (\beta^4 - 4\beta^2 + 1)\mu_1]$
     - $\mu_4 = \frac{1}{2\beta}[\frac{\beta^4-4\beta^2+1}{\beta}\mu_0 + (\beta^2 - 1)\mu_1]$

3. *The origin of system (1.6.41) is a center if and only if* $2\mu_4B^2 + (\mu_2 + \mu_4 + \mu_0)B + 2\mu_0 = 0$ *with* $B \neq 0$. *In this case the system is non-integrable. It is reversible if and only if* $\mu_0 = \mu_2 = \mu_4 = 0$.

4. *The origin of system (1.6.42) is a center if and only if* $(C+B+C^3+B^2+3BC+2BC^2+2C^2+B^2C+A^2C)\mu_0+(2ABC+B^2A+AB)\mu_1+(BC^3+A^2BC+B^2A^2+BA^2+B^2C+B^2C^2+BC^2)\mu_2+(AC^2B^2+A^3B^2+AC^2B+A^3B+2\,AB^2C)\mu_3+(A^4B+B^3C+3B^2A^2C+A^2BC+B^2C^4+2B^3C^2+C^4B+2B^2C^2+BC^3+B^3C^3+B^3CA^2+2B^2A^2C^2+3B^2C^3+B^2A^4+2A^2C^2B)\mu_4=0$. *In this case the system is integrable if and only if* $\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0$, *and is reversible if and only if* $A=0$ *y* $\mu_0=\mu_2=\mu_4=0$.

Proof.

1. The Hamiltonian function of the conservative-dissipative decomposition of system (1.6.39) is

   $(5+t_2)h = b_{t_2+4,0}x^{t_2+5}+(b_{41}-t_2a_{50})x^5y+\chi_{\{t_2=2\}}(b_{22}-2a_{31})x^3y^2+\chi_{\{t_2=4\}}(b_{02}-4a_{11})xy^2-5a_{01}\chi_{\{t_2=5\}}y^2+\chi_{\{t_2=2\}}(b_{02}-2a_{12})xy^3$.
   We distinguish two cases:

   - If $t_2\neq 5$, h has the real factor x, from Theorem 1.3.28 system (1.6.39) is non-monodromic.
   - If t_2=5, $10h(x,y) = b_{90}x^{10}+(b_{41}-5a_{50})yx^5-5a_{01}y^2$. In this case, from Theorem 1.3.28, it is monodromic if and only if $(b_{41}-5a_{50})^2+20b_{90}a_{01} < 0$. In this case the Hamiltonian part of decomposition of system (1.6.39) can be expressed in the form $h=[(y-ax^5)^2+b^2x^{10}]$, $b\neq 0$ then, from Theorem 1.3.31, we obtain $I_{\mathbf{F}_r} = \frac{\pi(b_{41}+5a_{50})}{b}$ and we can conclude that the origin of system (1.6.39) is a center if and only if $b_{41}+5a_{50}=0$. From Theorem 1.4.37, system (1.6.39) is reversible because the change of variables $u=x$, $v=y-ax^5$ transforms the system into an axis-reversible system. In the case that the origin of system (1.6.39) is a center, the system is Hamiltonian and consequently analytically integrable.

2. By applying Theorem 1.3.31, $I_{\mathbf{F}_r}=\pi\frac{\mu_2+3\mu_4+3\mu_0}{8}$. Therefore, the origin of system (1.6.40) is a center if and only if $\mu_2+3\mu_4+3\mu_0=0$.

   From Proposition 1.4.33 this system is non-integrable.

In this case, by applying Theorem 1.4.37, the only change of variables that transforms the Hamilton part h into an other Hamiltonian part $\tilde{h}$ which is axis-reversible has the form $u = x + \beta y$, $v = -\beta x + y$. This change transforms h and μ into

- $\tilde{h}(x, y) = (v^2 + u^2)^3/(1 + \beta^2)^2$
- $\tilde{\mu}(x, y) = \frac{\mu_0\beta^4+\beta^3\mu_1+\mu_2\beta^2+\beta\mu_3+\mu_4}{(1+\beta^2)^4}v^4 + \frac{\mu_4\beta^4-\beta^3\mu_3+\mu_2\beta^2-\beta\mu_1+\mu_0}{(1+\beta^2)^4}u^4$
  $+\frac{\mu_2\beta^4+3\beta^3\mu_3-3\beta^3\mu_1-4\mu_2\beta^2+6\mu_0\beta^2+6\mu_4\beta^2+3\beta\mu_1-3\beta\mu_3+\mu_2}{(1+\beta^2)^4}v^2u^2$
  $+\frac{-\beta^4\mu_3-4\mu_4\beta^3+2\mu_2\beta^3-3\beta^2\mu_1+3\beta^2\mu_3+4\mu_0\beta-2\mu_2\beta+\mu_1}{(1+\beta^2)^4}u^3v$
  $+\frac{-\beta^4\mu_1+4\mu_0\beta^3-2\mu_2\beta^3-3\beta^2\mu_3+3\beta^2\mu_1-4\mu_4\beta+2\mu_2\beta+\mu_3}{(1+\beta^2)^4}uv^3$

and we can observe that $\tilde{h}$ is always even in u and v. Therefore, from Proposition 1.4.38, system (1.6.40) is axis-reversible if there exists β such that $\tilde{\mu}$ is odd in u or v, i.e. if there exist β such that annul coefficients u^4, v^4 and u^2v^2 en $\tilde{\mu}$, this is the condition of the theorem.

3. From Theorem 1.3.31, we obtain $I_{\mathbf{F}_r} = \frac{\pi}{2}\frac{2\mu_4B^2+\mu_2B+\mu_4B+\mu_0B+2\mu_0}{(B+1)^2B}$. Therefore, the origin of system (1.6.41) is a center if and only if $2\mu_4B^2 + (\mu_2 + \mu_4 + \mu_0)B + 2\mu_0 = 0$.

   In this case, system (1.6.41) is axis-reversible if and only if $\mu_2 = \mu_4 = \mu_0 = 0$. because, from Proposition 1.4.38

   - $\tilde{h} = (x^2 + y^2)^2(x^2 + B^2y^2)$ is even in x and y.
   - $\tilde{\mu} = \mu_0x^4 + \mu_1x^3y + \mu_2x^2y^2 + \mu_3xy^3 + \mu_4y^4$ is odd in x or y if and only if $\mu_2 = \mu_4 = \mu_0 = 0$

   In other case, from Theorem 1.4.37, system (1.6.41) is non-reversible because there is no change of variables which transforms this system into an axis-reversible system.

   From Proposition 1.4.33 system (1.6.41) is non-integrable.

4. From Theorem 1.3.31 we calculate the expression of $I_{\mathbf{F}_r}$. Imposing $I_{\mathbf{F}_r} = 0$ we obtain the result.

   To study the integrability of system (1.6.42) we use Theorem 1.4.34. We obtain n_1, n_2 y $n_3 \in \mathbb{N}$, such that

$$\begin{aligned}
\text{Res}[\eta^{hom}(1,Y),\, \pm i] &= -\tfrac{6(n_1+1)-M_0}{M_0},\\
\text{Res}[\eta^{hom}(1,Y),\, \pm Bi] &= -\tfrac{6(n_2+1)-M_0}{M_0},\\
\text{Res}[\eta^{hom}(1,Y),\, A \pm Bi] &= -\tfrac{6(n_3+1)-M_0}{M_0},
\end{aligned}$$

where $M_0 = 2(n_1+1) + 2(n_2+1) + 2(n_3+1)$.

These equations are inconsistent so, system (1.6.42) is integrable only when $\mu(x,y) = 0$ i.e. $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$.

From Proposition 1.4.38, system (1.6.42) is axis-reversible if $A = 0$ and $\mu_0 = \mu_2 = \mu_4 = 0$. Otherwise is non-reversible because there is no change of variables which transforms system (1.6.42) into an axis-reversible system.

■

CHAPTER 2

Quasi-homogeneous Normal Forms.

2.1 Introduction

One of the more important questions in any scientific problem is the choice of the variables that we use to model, mathematically, our problem. A crucial step in the study of these problems is to use, among all equivalent expressions, the simplest possible in some sense. In the framework of dynamical systems, is the normal forms theory which is concerned with determining the simplest expressions, i.e., the method of normal forms defines us a way to find the simplest expression. This method has three significant features, firstly we say it is a local method, i.e., the coordinate transformations are generated in a neighborhood of a known solution, in this memory we assume, the known solution will be a fixed point. Secondly, we will mention that, in general, the coordinate transformations will be nonlinear functions of the dependent variables. However, an important point is that these coordinate transformations are found by solving a sequence of linear problems. And finally, it is noteworthy that the structure of a normal form is determined, entirely, by the nature of the principal part of the vector field. Throughout this chapter we consider an autonomous system of the form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \text{with } \mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \tag{2.1.1}$$

where $\mathbf{F}$ is $\mathcal{C}^r$, with r sufficiently large and $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, i.e. the origin is an equilibrium point.

In the following section of this chapter, section 2.2, we show a brief introduction to the classical theory of normal forms. The classical theory of normal forms is due to Poincaré and it applies to systems with non-zero linear part using near-identity transformations to eliminate nonessential terms in the local dynamics. In section 2.3, this classical theory is extended for vector fields developed in quasi-homogeneous terms. In this case to assume that the linear part of $\mathbf{F}$ is non-zero is not necessary (see Algaba *et al.* [6], Baider & Sanders [21], Kokubu *et al.* [64] and Wang *et al.* [85]). In this chapter, the normal forms are obtained by near-identity transformations in the state variables ($\mathcal{C}^\infty$-conjugation) and considering also quasi-homogeneous reparametrizations of the time ($\mathcal{C}^\infty$-equivalence). For more details see Algaba *et al.* [5]. In sections 2.4 and 2.5, are introduced, respectively, the normal form to 0th-step (which carries implicit the right choice for the type $\mathbf{t}$) and the Lie triangle. As an introduction we say that there is not any criterium for the choice of the type $\mathbf{t}$, but nevertheless, it is noteworthy that this choice is very important, since the type $\mathbf{t}$ determines the first quasi-homogeneous component of the vector field. About the Lie triangle, we want to emphasize the importance of this, because it provides an ordered process to obtain simplifications, degree to degree, in the calculation of the normal form. This order has allowed to create algorithms for obtaining the coefficients of the normal form for each degree k.

2.2 Classical Normal Forms.

Before presenting the classical theory of normal forms, we give some definitions and concepts that will be useful.

2.2.1 Preliminaries

The properties and concepts that are presented below are known and can be seen in Chua & Kokubu [37] and Golubitsky & Shaeffer [59].

Definition 2.2.58. *Let $\mathcal{H}_k^n$ be the vectorial space of vector fields with n ho-*

mogeneous components in n variables of degree k. We define the Lie Bracket *of two differentiable vector fields* $\mathbf{F}$ *and* $\mathbf{G}$ *as follows,*

$$[\mathbf{F}, \mathbf{G}] = D\mathbf{F}(\mathbf{x}) \cdot \mathbf{G}(\mathbf{x}) - D\mathbf{G}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}), \quad \textit{for all } x \in \mathbb{R}^n$$

Moreover, it is hold that, given $\mathbf{F} \in \mathcal{H}_i^n$ *and* $\mathbf{G} \in \mathcal{H}_j^n$ *then* $[\mathbf{F}, \mathbf{G}] \in \mathcal{H}_{i+j}^n$

This internal operation in the space of differentiable vector fields has the following properties,

1. **Bilinearity.**

$$\begin{aligned}[a_1\mathbf{F}_1 + a_2\mathbf{F}_2, b_1\mathbf{G}_1 + b_2\mathbf{G}_2] &= a_1b_1[\mathbf{F}_1, \mathbf{G}_1] + a_1b_2[\mathbf{F}_1, \mathbf{G}_2] + a_2b_1[\mathbf{F}_2, \mathbf{G}_1] \\ &+ a_2b_2[\mathbf{F}_2, \mathbf{G}_2],\end{aligned}$$

   for all $a_1,\ a_2,\ b_1,\ b_2 \in \mathbb{R}$ and $\mathbf{F}_1, \mathbf{F}_2, \mathbf{G}_1, \mathbf{G}_2 \in \mathcal{H}_j^n$.

2. **Antisymmetry.**

$$[\mathbf{F}, \mathbf{G}] = -[\mathbf{G}, \mathbf{F}], \quad \text{with } \mathbf{F},\ \mathbf{G} \in \mathcal{H}_j^n.$$

3. **Jacobi Identity.**

$$[[\mathbf{F}, \mathbf{G}], \mathbf{H}] + [[\mathbf{G}, \mathbf{H}], \mathbf{F}] + [[\mathbf{H}, \mathbf{F}], \mathbf{G}] = 0, \quad \text{with } \mathbf{F},\ \mathbf{G},\ \mathbf{H} \in \mathcal{H}_j^n.$$

2.2.2 Normal Forms of Vector Fields

Usually, the normal forms techniques are used to simplify a vector field degree by degree. This requires that the vector field is written in homogeneous components, by its Taylor expansion. Consider system (2.1.1) described as follow,

$$\dot{\mathbf{x}} = A\mathbf{x} + \tilde{\mathbf{F}}_2(\mathbf{x}) + \tilde{\mathbf{F}}_3(\mathbf{x}) + \cdots \tag{2.2.2}$$

where $A = D\mathbf{F}(0)$ is the jacobian matrix at the origin of $\mathbf{F}(\mathbf{x})$ and $\mathbf{F}_i(\mathbf{x})$ represents the order i terms in the Taylor expansion of $\mathbf{F}(\mathbf{x})$.

As mentioned above, the simplification process is performed degree by degree. First, we simplify the lower degree term, using linear transformations, i.e., let T be the matrix that transforms $A = D\mathbf{F}(\mathbf{0})$ into real Jordan canonical form. Then, under the transformation, $\mathbf{x} = T\mathbf{u}$, system (2.2.2) is transformed into,

$$\begin{aligned}\dot{\mathbf{x}} &= T\dot{\mathbf{u}} \\ \dot{\mathbf{u}} &= T^{-1}AT\mathbf{u} + T^{-1}\tilde{\mathbf{F}}_2(T\mathbf{u}) + T^{-1}\tilde{\mathbf{F}}_3(T\mathbf{u}) + \cdots \end{aligned} \tag{2.2.3}$$

Denoting the real Jordan canonical form of A by J, defining $T^{-1}\tilde{\mathbf{F}}_k(T\mathbf{u}) \equiv \mathbf{F}_k(\mathbf{u})$ and renaming u again by x, we obtain,

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{F}_2(\mathbf{x}) + \mathbf{F}_3(\mathbf{x}) + \cdots \tag{2.2.4}$$

From here, near-identity transformations are used to eliminate the terms of degree $k \geq 2$. Fixed a degree k, we apply the transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$, with $\mathbf{P}_k(\mathbf{y}) \in \mathcal{H}_k^n$. The transformed system is,

$$\begin{aligned}\dot{\mathbf{y}} &= (I + D_{\mathbf{y}}\mathbf{P}_k(\mathbf{y}))^{-1}J(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) + \sum_{k\geq 2}(I + D_{\mathbf{y}}\mathbf{P}_k(\mathbf{y}))^{-1}\mathbf{F}_k(\mathbf{x})(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) \\ &= J\mathbf{y} + \sum_{k\geq 2}\mathbf{G}(\mathbf{y}). \end{aligned} \tag{2.2.5}$$

It can be seen in Guckenheimer & Holmes [60], that the transformed vector field does not change up to order $k-1$, i.e.,

$$\begin{aligned}\mathbf{G}_2(\mathbf{y}) &= \mathbf{F}_2(\mathbf{y}), \\ \mathbf{G}_3(\mathbf{y}) &= \mathbf{F}_3(\mathbf{y}), \\ &\vdots \\ \mathbf{G}_{k-1}(\mathbf{y}) &= \mathbf{F}_{k-1}(\mathbf{y}),\end{aligned}$$

and the transformed vector field for degree k is of the form,

$$\mathbf{G}_k(\mathbf{y}) = \mathbf{F}_k(\mathbf{y}) - (D_{\mathbf{y}}\mathbf{P}_k(\mathbf{y})J\mathbf{y} - J\mathbf{P}_k(\mathbf{y})). \tag{2.2.6}$$

At this point, it should be clarified that this process of normal forms is applicable for any matrix A. It is not necessary the previous simplification of the matrix to obtain J, but we consider it is appropriate.

Returning to equal (2.2.6), this induces us to define the following linear operator, called *homological operator*,

$$\begin{aligned}\mathbf{L}_k^J \ : \ &\mathcal{H}_k^n \longrightarrow \mathcal{H}_k^n \\ &\mathbf{P}_k \longrightarrow \mathbf{L}_k^J(\mathbf{P}_k),\end{aligned} \tag{2.2.7}$$

where $\mathbf{L}_k^J(\mathbf{P}_k(\mathbf{x})) = D_{\mathbf{x}}\mathbf{P}_k(\mathbf{x})J\mathbf{x} - J\mathbf{P}_k(\mathbf{x})$. It is easy to prove that $\mathbf{L}_k^J$ is lineal and its expression, written in terms of the Lie bracket, is the following,

$$\mathbf{L}_k^J(\mathbf{P}_k) = [\mathbf{P}_k,\ J]. \tag{2.2.8}$$

Therefore, we can write, the term of degree k of the transformed vector field given in (2.2.6), of the form,

$$\mathbf{G}_k(\mathbf{y}) = \mathbf{F}_k(\mathbf{y}) - \mathbf{L}_k^J(\mathbf{P}_k).$$

At this point, ideally try selecting $\mathbf{P}_k$ so that $\mathbf{L}_k^J(\mathbf{P}_k) = \mathbf{F}_k$, thus we have that $\mathbf{G}_k = 0$ and would have eliminated all the terms of degree k. This is not always possible, since the above equation can be incompatible. In practice, we proceed as follows,

- We consider a subespace $\mathrm{Cor}(\mathbf{L}_k^J)$. This subespace is a complementary subspace of the range of the *homological operator* defined in (2.2.7), i.e., $\mathcal{H}_k^n = \mathrm{Im}(\mathbf{L}_k^J) \oplus \mathrm{Cor}(\mathbf{L}_k^J)$.
- The following step is to descompose $\mathbf{F}_k = \mathbf{F}_k^r + \mathbf{F}_k^c$, where $\mathbf{F}_k^r \in \mathrm{Im}(\mathbf{L}_k^J)$ and $\mathbf{F}_k^c \in \mathrm{Cor}(\mathbf{L}_k^J)$. Then it is possible find a $\mathbf{P}_k \in \mathcal{H}_k^n$ verifing the homological equation,

$$\mathbf{L}_k^J(\mathbf{P}_k) = \mathbf{F}_k^r. \tag{2.2.9}$$

Finally, we obtain,

$$\mathbf{G}_k = \mathbf{F}_k - \mathbf{L}_k^J(\mathbf{P}_k) = \mathbf{F}_k^c.$$

That is, we have simplified $\mathbf{F}_k$, by eliminating the part of the image of the homological operator. If we repeat this procedure for degree $k = 2, 3, 4, \cdots$ we get $\mathbf{G}_k \in \ Cor(\mathbf{L}_k^J)$ for all $k \geq 2$.

Using a version of *Borel's Teorem*, we obtain the normal form theorem, (see Vanderbauwhede [83]).

Theorem 2.2.59. *There exists a $\mathcal{C}^\infty$-diffeomorphism Φ verifying $\Phi(\mathbf{0}) = \mathbf{0}$ and $D\Phi(\mathbf{0}) = I$ such that, the change of variables $\mathbf{x} = \Phi(\mathbf{y})$ transforms system (2.2.4) into (2.2.5) where $\mathbf{G}_k \in \mathrm{Cor}(\mathbf{L}_k^J)$, for all $k \geq 2$.*

In this case, we say that (2.2.5) is a normal form, under $\mathcal{C}^\infty$-conjugation, for the system (2.2.2).

The homological equation (2.2.9), and consequently, the corresponding normal form, are based on the linear part of the vector field. (see Takens [80], Chow & Hale [35], Guckenheimer & Holmes [60], Elphick *et al.* [45], Iooss & Adelmeyer [61] and Chow *et al.* [36]). This equation does not have, in general, unique solution because it could depend on arbitrary terms that belong to the kernel of the homological operator. These terms can be used later to make simplifications to order higher than k. For more details see Ushiki [82], Chua & Kokubu [37], Baider [20], Algaba *et al.* [3]. For other works dedicated to special cases see Algaba *et al.* [4].

In summary, we have described this procedure for simplifications in the vector field in two stages, the first stage in which we use the linear part of $\mathbf{F}$, to determine the simplifications that are achieved through of the normal form theorem, and a second, in which we will consider the nonlinear terms for further simplifications in the classical normal form. With the perspective that we will adopt in the next section, the above two-step process, may become in only one, because the use of quasi-homogeneous developments allow us to consider linear and nonlinear terms at once (because monomials with different degrees may have the same quasi-homogeneous degree-homogeneous).

2.3 Quasi-homogeneous Normal Forms.

Our first goal is to extend the ideas of the theory of normal forms, for vector fields developed in terms quasi-homogeneous. There are some works that have used similar ideas in the case of the Bogdanov-Takens singularity under $\mathcal{C}^\infty$-conjugation (see Baider & Sanders [21], Kokubu *et al.* [64] and Wang *et al.* [85]). In this section we will discuss the normal forms for vector fields developed in quasi-homogeneous terms under $\mathcal{C}^\infty$-conjugation and $\mathcal{C}^\infty$-equivalence. For developing of this section we will use the lemmas about quasi-homogeneity described in Section 1.2 of Chapter 1. For more details see Algaba *et al.* [5].

2.3.1 The Newton Polyhedron

The Newton polyhedron provides a geometrical representation of the vector space $\mathcal{Q}_k^{\mathbf{t}}$. Let us denote the unit vector as $\mathbf{u} = (1,\ 1, \cdots, 1) \in \mathbb{N}_0^n$. If $x^a \cdot e_j$ is a monomial of F_j, where $\mathbf{F} = (F_1, F_2, \ldots, F_n)^T$, we know that $F_j \in \mathcal{P}_{j+t_j}^{\mathbf{t}}$ and using the relation (1.2.2) can be shortly written as

$$(\mathbf{a} + \mathbf{u} - \mathbf{e}_j) \cdot \mathbf{t} = k + |\mathbf{t}|,$$

where $\cdot$ denotes the usual inner product in $\mathbb{N}_0^n$. Then:

$$\mathcal{Q}_k^{\mathbf{t}} = \operatorname{Span}\left\{\mathbf{x}^{\mathbf{a}}\mathbf{e}_j \in \mathcal{B} : (\mathbf{a} + \mathbf{u} - \mathbf{e}_j) \cdot \mathbf{t} = k + |\mathbf{t}|, j = 1 \cdots, n\right\}.$$

To represent each monomial of $\mathcal{B}$ on $\mathbb{N}_0^n$, we consider the application

$$\begin{aligned} R \ : \ & \mathcal{B} \longrightarrow \mathbb{N}_0^n \\ & \mathbf{x}^{\mathbf{a}}\mathbf{e}_j \to \mathbf{a} + \mathbf{u} - \mathbf{e}_j. \end{aligned}$$

The point of $\mathbb{N}_0^n$ that is image, by R of $\mathbf{x}^{\mathbf{a}}\mathbf{e}_j$ is called *support point* of the monomial. The *Newton polyhedron* of a vector space $\mathbf{F}$ is the subset of $\mathbb{N}_0^n$ consisting of all support points of all monomials of $\mathbf{F}$.

The following result gives a geometrical representation of the vectorial space $\mathcal{Q}_k^{\mathbf{t}}$ on its support set. (see Bruno [27], Dumortier [25]).

Proposition 2.3.60. *It holds the following propierties,*

1. *R is not inyective. In fact, given $\mathbf{a} \in \mathbb{N}_0^n$, all the monomials $\mathbf{x}^{\mathbf{a}-\mathbf{u}+\mathbf{e}_j}\mathbf{e}_j$ ($j = 1, \ldots, n$) are applied in the same point $\mathbf{a}$. Moreover:*
   - (a) *If $\mathbf{a} \in \mathbb{N}_0^n$ has exactly one coordinate equal to zero $a_i = 0$ (i.e., if it belongs to a coordinate plane), then $R^{-1}(\mathbf{a}) = \{\mathbf{x}^{\mathbf{a}-\mathbf{u}+\mathbf{e}_i}\mathbf{e}_i\}$.*
   - (b) *If $\mathbf{a} \in \mathbb{N}_0^n$ has at least two coordinates equal to zero, then $R^{-1}(\mathbf{a}) = \emptyset$.*
   - (c) *If all coordinates $\mathbf{a} \in \mathbb{N}_0^n$ are no-null, then*

   $$R^{-1}(\mathbf{a}) = \left\{\mathbf{x}^{\mathbf{a}-\mathbf{u}+\mathbf{e}_1}\mathbf{e}_1, \mathbf{x}^{\mathbf{a}-\mathbf{u}+\mathbf{e}_2}\mathbf{e}_2, \ldots, \mathbf{x}^{\mathbf{a}-\mathbf{u}+\mathbf{e}_n}\mathbf{e}_n\right\}.$$

2. *R applies the space $\mathcal{Q}_k^{\mathbf{t}}$ on the points of $\mathbb{N}_0^n$ belonging to the hyperplane $\tilde{\mathbf{a}} \cdot \mathbf{t} = k + |\mathbf{t}|$.*

The figure below shows the terms of the development of a planar vector field respect to the type $\mathbf{t} = (1, 2)$. The supports of each one of the terms are in parallel lines, whose normal vector is (1, 2).

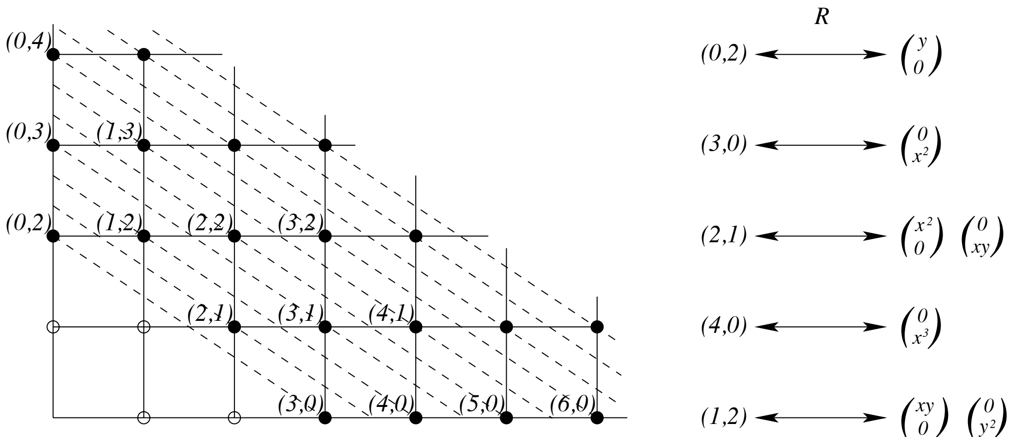

Figure 2.1: Representation of the quasi-homogeneous vector fields of type $(1, 2)$ and degrees $-2, -1, 0, 1, \ldots$

Perhaps the best way to define the *Newton polyhedron* is the given by Dumortier in [44], working with the dual 1-form of a vector field $\mathbf{F}$ given.

Given a vectorial space $\mathbf{F} = \sum_{j=1}^{n} F_j \dfrac{\partial}{\partial x_j}$, its dual 1-form is the 1-form $\omega = i_{\mathbf{F}}\Omega$, with $\Omega = dx_1 \wedge \cdots \wedge dx_n$; i.e.,

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} F_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

where $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ denotes that the differential dx_i doesn't appear. Consider now

$$\mathcal{J}^{\infty}\omega = \sum_{i=1}^{n} \sum_{\alpha_1+\cdots+\alpha_n \geq 1} (-1)^{i+1} a^i_{\alpha_1,\cdots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

The support of ω (or $\mathbf{F}$) is defined as follows,

$$S : \left\{ \bigcup_{i=1}^{n} \bigcup_{\alpha_1+\cdots+\alpha_n \geq 1} (\alpha_1 + 1, \cdots, \alpha_{i-1} + 1, \alpha_i, \alpha_{i+1} + 1, \cdots, \alpha_n + 1) \left| a^i_{\alpha_1,\cdots,\alpha_n} \neq 0 \right. \right\}.$$

On the basis of the set of points S, we can define the *Newton polyhedron* and the *Newton diagram*.

Definition 2.3.61.

1. *The Newton polyhedron of* ω *(or* $\mathbf{F}$*) is the convex hull (that we denoted by* Γ*) of the set,*

$$\mathcal{P} = \bigcup_{(q_1,\cdots,q_n)\in S} \left\{(q_1,\cdots,q_n) + \mathbb{R}^n_+\right\}.$$

2. *The Newton diagram of* ω *(or* $\mathbf{F}$*) is the union* γ *of the compact faces* γ_k *of* Γ*.*

Moreover, we will denominate principal part of ω *(or* $\mathbf{F}$*) respect to the compact face* γ *to:*

$$\omega_\Delta = \sum_{i=1}^{n} \sum_{(\alpha_1,\cdots,\alpha_n)\in\gamma} (-1)^{i+1} a^i_{\alpha_1,\cdots,\alpha_n} x_1^{\alpha_1}\cdots x_n^{\alpha_n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

For example, to fix ideas, if we consider the vector field $\mathbf{F} = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y}$, then its dual 1-form is $\omega = i_{\mathbf{F}}\Omega$, with $\Omega = dx \wedge dy$, i.e.,

$$\omega = F_1\, dy - F_2\, dx$$

For the particular case $\mathbf{F} = (x^3 + 2xy)\frac{\partial}{\partial x} + (y^2 - 4x^2y)\frac{\partial}{\partial y}$, vector field of degree $k = 2$ respect the

$$\omega = (x^3 + 2xy)\, dy - (y^2 - 4x^2y)\, dx$$

The support S of ω (or $\mathbf{F}$) is the set,

$$S = \{(3,1),(1,2)\}$$

In the following figure we show the Newton diagram for the above example.

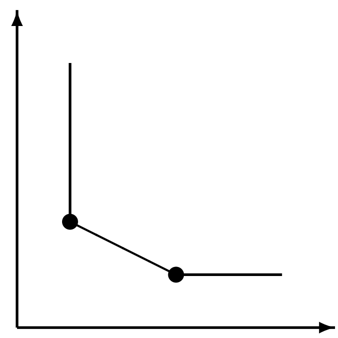

Figure 2.2: Newton diagram of the vector field $\mathbf{F} = (x^3 + 2xy)\frac{\partial}{\partial x} + (y^2 - 4x^2y)\frac{\partial}{\partial y}$

2.3.2 Quasi-homogeneous normal form under $\mathcal{C}^\infty$-conjugation

The theory of normal forms can be applied to quasi-homogeneous vector fields developed in successive degrees respect to a type $\mathbf{t}$. The key idea is summarized in the following commutative diagram,

$$\begin{tikzcd}
\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \arrow[r, "\mathbf{x} = \mathbf{y} + \sum_{k \geq r} \mathbf{P}_k(\mathbf{y})"'] \arrow[d, "\mathbf{x} = E\mathbf{X}"] & \dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}) \\
\dot{\mathbf{X}} = E^{-1}\mathbf{F}(E\mathbf{X}) = \sum_{j \geq r} \mathbf{F}_j(\mathbf{X})\varepsilon^j \arrow[r, "\mathbf{X} = \mathbf{Y} + \sum_{k \geq r} \mathbf{P}_k(\mathbf{Y})\varepsilon^k"'] & \dot{\mathbf{Y}} = \sum_{j \geq r} \mathbf{G}_j(\mathbf{Y})\varepsilon^j = E^{-1}\mathbf{G}(E\mathbf{Y}) \arrow[u, "\mathbf{y} = E\mathbf{Y}, (\varepsilon = 1)"]
\end{tikzcd}$$

Figure 2.3: Commutative diagram for $\mathcal{C}^\infty$-conjugation.

To adapt the procedure for determining a normal form using the quasi-homogeneous terms of successive degrees, first we include a parameter ε using the scaling $\mathbf{x} = E\mathbf{X}$, with $\mathbf{X} \in \mathbb{R}^n$. Thus, we get the system $\dot{\mathbf{X}} = E^{-1}\mathbf{F}(E\mathbf{X})$. Developing in powers of ε, we can write this system in the form

$$\dot{\mathbf{X}} = \mathbf{F}_r(\mathbf{X})\varepsilon^r + \mathbf{F}_{r+1}(\mathbf{X})\varepsilon^{r+1} + \cdots , \tag{2.3.10}$$

where it is easy to prove that $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$. Note that taking $\varepsilon = 1$ in (2.3.10), We were able to develop the system (2.1.1)) in terms respect the type $\mathbf{t}$,

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots . \tag{2.3.11}$$

Remark 10. *The degree r of the first term respect to a type $\mathbf{t}$ is a integer number. It is not necessarily a natural number. For example, if in a two-dimensional vector field with a nilpotent singularity*

$$\begin{aligned} \dot{x} &= y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + \cdots , \\ \dot{y} &= b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2 + \cdots \end{aligned}$$

we consider the type $\mathbf{t} = (1, 50)$, then the vector field $\mathbf{F}$ of the system, developed in quasi-homogeneous terms, is written, about this type, as follows,

$$\mathbf{F} = \mathbf{F}_{-48} + \mathbf{F}_{-47} + \cdots$$

where

$$\begin{aligned} \mathbf{F}_{-48}(x, y) &= (0, b_{20}x^2)^T, \\ \mathbf{F}_{-47}(x, y) &= (0, b_{30}x^3)^T, \\ &\vdots \\ \mathbf{F}_0(x, y) &= (0, b_{50,0}x^{50})^T, \\ \mathbf{F}_1(x, y) &= (a_{2,0}x^2, b_{51,0}x^{51} + b_{1,1}xy)^T, \\ &\vdots \end{aligned}$$

Once the system (2.3.10) has been reduced to normal form, it is enough to undo the scaling with $\mathbf{y} = E\mathbf{Y}$, and taking $\varepsilon = 1$ for obtaining the normal form.

Next, we describe the process to obtain a normal form for system (2.3.11).

In first time, we apply the transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$, where $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ being $k \geq 1$ then, the transformed system is,

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}) = (I + D\mathbf{P}_k(\mathbf{y}))^{-1} \sum_{i \geq r} \mathbf{F}_i (\mathbf{y} + \mathbf{P}_k(\mathbf{y})),$$

and can be expressed in their quasi-homogeneous terms as follows,

$$\dot{\mathbf{y}} = \mathbf{G}_r(\mathbf{y}) + \mathbf{G}_{r+1}(\mathbf{y}) + \cdots, \tag{2.3.12}$$

being $\mathbf{G}_j(\mathbf{y}) \in \mathcal{Q}_j^{\mathbf{t}}$, for all $j \geq r$.

Then, the following proposition is satisfied,

Proposition 2.3.62. *With the above notation,*

- $\mathbf{G}_j(\mathbf{y}) = \mathbf{F}_j(\mathbf{y})$, *for* $j = r, r+1, \ldots, r+k-1$,
- $\mathbf{G}_{r+k}(\mathbf{y}) = \mathbf{F}_{r+k}(\mathbf{y}) - (D\mathbf{P}_k(\mathbf{y})\mathbf{F}_r(\mathbf{y}) - D\mathbf{F}_r(\mathbf{y})\mathbf{P}_k(\mathbf{y}))$.

Proof.

$$\begin{aligned}
\mathbf{G}(E\mathbf{y}) &= (I + D\mathbf{P}_k(E\mathbf{y}))^{-1} \sum_{j \geq r} \mathbf{F}_j (E\mathbf{y} + \mathbf{P}_k(E\mathbf{y})) \\
&= \left(E\left(I + \varepsilon^k D\mathbf{P}_k(\mathbf{y})\right) E^{-1}\right)^{-1} \sum_{j \geq r} E\varepsilon^j \mathbf{F}_j \left(\mathbf{y} + \varepsilon^k \mathbf{P}_k(\mathbf{y})\right) \\
&= E\left(I - \varepsilon^k D\mathbf{P}_k(\mathbf{y}) + \varepsilon^{2k} (D\mathbf{P}_k(\mathbf{y}))^2 - \cdots\right) \sum_{j \geq r} \varepsilon^j \mathbf{F}_j \left(\mathbf{y} + \varepsilon^k \mathbf{P}_k(\mathbf{y})\right).
\end{aligned}$$

Moreover,

$$\mathbf{F}_j\left(\mathbf{y} + \varepsilon^k \mathbf{P}_k(\mathbf{y})\right) = \mathbf{F}_j(\mathbf{y}) + D\mathbf{F}_j(\mathbf{y})\mathbf{P}_k(\mathbf{y})\varepsilon^k + \mathcal{O}\left(\varepsilon^{k+1}\right).$$

Therefore,

$$\begin{aligned}
E^{-1}\mathbf{G}(E\mathbf{y}) &= \left(I - \varepsilon^k D\mathbf{P}_k(\mathbf{y}) + \varepsilon^{2k}\left(D\mathbf{P}_k(\mathbf{y})\right)^2 - \cdots\right)\sum_{j\geq r}\varepsilon^j \mathbf{F}_j\left(\mathbf{y} + \varepsilon^k \mathbf{P}_k(\mathbf{y})\right) \\
&= \left(I - \varepsilon^k D\mathbf{P}_k(\mathbf{y})\right)\sum_{j\geq r}\varepsilon^j\left(\mathbf{F}_j(\mathbf{y}) + D\mathbf{F}_j(\mathbf{y})\mathbf{P}_k(\mathbf{y})\varepsilon^k\right) \\
&\quad +\mathcal{O}\left(\varepsilon^{r+k+1}\right) \\
&= \sum_{j\geq r}\varepsilon^j\left(\mathbf{F}_j(\mathbf{y}) + D\mathbf{F}_j(\mathbf{y})\mathbf{P}_k(\mathbf{y})\varepsilon^k\right) - \varepsilon^{r+k} D\mathbf{P}_k(\mathbf{y})\mathbf{F}_r(\mathbf{y}) \\
&\quad +\mathcal{O}\left(\varepsilon^{r+k+1}\right) \\
&= \mathbf{F}_r(\mathbf{y})\varepsilon^r + \mathbf{F}_{r+1}(\mathbf{y})\varepsilon^{r+1} + \cdots + \mathbf{F}_{r+k-1}(\mathbf{y})\varepsilon^{r+k-1} \\
&\quad + \left(\mathbf{F}_{r+k}(\mathbf{y}) + D\mathbf{F}_r(\mathbf{y})\mathbf{P}_k(\mathbf{y}) - D\mathbf{P}_k(\mathbf{y})\mathbf{F}_r(\mathbf{y})\right)\varepsilon^{r+k} \\
&\quad +\mathcal{O}\left(\varepsilon^{r+k+1}\right).
\end{aligned}$$

■

This result suggests defining the following homological operator,

$$\begin{aligned}
\mathbf{L}_{r+k} \ : \ & \mathcal{Q}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}} \\
& \mathbf{P}_k \to \mathbf{L}_{r+k}(\mathbf{P}_k),
\end{aligned} \qquad (2.3.13)$$

where $\quad \mathbf{L}_{r+k}(\mathbf{P}_k)(\mathbf{y})) = D\mathbf{P}_k(\mathbf{y})\mathbf{F}_r(\mathbf{y}) - D\mathbf{F}_r(\mathbf{y})\mathbf{P}_k(\mathbf{y})$.

Note that the previous operator is linear and only depends on the first term $\mathbf{F}_r$, and can be expressed in terms of the Lie bracket as follows, $\mathbf{L}_{r+k}(\mathbf{P}_k) = [\mathbf{P}_k, \mathbf{F}_r]$.

The Proposition 2.3.62 states that the terms quasi-homogeneous to order $r+k-1$ do not change, and the term of order $r+k$ of the transformed vector field is,

$$\mathbf{G}_{r+k} = \mathbf{F}_{r+k} - [\mathbf{P}_k, \mathbf{F}_r] = \mathbf{F}_{r+k} - \mathbf{L}_{r+k}(\mathbf{P}_k).$$

Following the same ideas of the classical theory of normal forms, we can cancel the part that belongs to the image the linear operator $\mathbf{L}_{r+k}$, selecting suitably $\mathbf{P}_k$.

Performing the changes described above for $k = 1, 2, \cdots$, we obtain the following result.

Theorem 2.3.63. *System (2.3.11) is formally conjugated to (2.3.12), where* $\mathbf{G}_{r+k}$ *belongs to a complementary subspace to the image space of the linear operator* $\mathbf{L}_{r+k}$, $k \geq 1$.

2.3.3 Quasi-homogeneous Normal Form under $\mathcal{C}^\infty$-equivalence

In the above subsection, we have used transformations in the state variables to obtain a normal form ($\mathcal{C}^\infty$-conjugation). In this subsection, we show the advantages that are encountered when we use also changes in the time. ($\mathcal{C}^\infty$-equivalence). As in the case of conjugation, the normal form is based on the information contained in the first quasi-homogeneous term $\mathbf{F}_r$. The key idea is summarized in following commutative diagram:

$$\begin{tikzcd}
\dot{\mathbf{x}} = \sum_{j\geq r} \mathbf{F}_j(\mathbf{x}) \arrow[d, "\frac{dt}{dT} = 1 - \sum_{i\geq 1} \mu_i(\mathbf{x})"] \arrow[dr] & \\
\mathbf{x}' = \sum_{j\geq r} \tilde{\mathbf{F}}_j(\mathbf{x}) \arrow[r, "\mathbf{x} = \mathbf{y} + \sum_{i\geq 1} \mathbf{P}_i(\mathbf{y})"'] \arrow[d, "\mathbf{x} = E\mathbf{X}"] & \mathbf{y}' = \sum_{j\geq r} \mathbf{G}_j(\mathbf{y}) \\
\mathbf{X}' = \sum_{j\geq r} \tilde{\mathbf{F}}_j(\mathbf{X})\varepsilon^j \arrow[r, "\mathbf{X} = \mathbf{Y} + \sum_{i\geq 1} \mathbf{P}_i(\mathbf{Y})\varepsilon^i"'] & \mathbf{Y}' = \sum_{j\geq r} \mathbf{G}_j(\mathbf{Y})\varepsilon^j \arrow[u, "\mathbf{y} = E\mathbf{Y}"']
\end{tikzcd}$$

Figure 2.4: Commutative diagram for $\mathcal{C}^\infty$-equivalence.

From (2.3.11), and effecting a change in the time of the form, $\frac{dt}{dT} =$

$1-\mu_k(\mathbf{x})$, with $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$, we obtain the system

$$\tfrac{d\mathbf{x}}{dT} = \mathbf{x}' = \mathbf{F}_r(\mathbf{x}) + \cdots + \mathbf{F}_{r+k-1}(\mathbf{x}) + (\mathbf{F}_{r+k}(\mathbf{x}) - \mu_k(\mathbf{x})\,\mathbf{F}_r(\mathbf{x})) + \cdots,$$

Lemma 1.2.9 ensures that this system is developed in quasi-homogneous terms of type $\mathbf{t}$ and successive degrees (analogously to system (2.3.11)). Next, to obtain the normal form we apply the transformation $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$, which transforms the above system into,

$$\mathbf{y}' = \mathbf{G}_r(\mathbf{y}) + \cdots + \mathbf{G}_{r+k-1}(\mathbf{y}) + \mathbf{G}_{r+k}(\mathbf{y}) + \cdots. \tag{2.3.14}$$

Using now Propositon 2.3.62, we obtain that

$$\begin{aligned}
\mathbf{G}_r &= \mathbf{F}_r,\\
\mathbf{G}_{r+1} &= \mathbf{F}_{r+1},\\
&\vdots\\
\mathbf{G}_{r+k-1} &= \mathbf{F}_{r+k-1},
\end{aligned}$$

and

$$\mathbf{G}_{r+k} = \mathbf{F}_{r+k} - \mu_k\,\mathbf{F}_r - [\mathbf{P}_k, \mathbf{F}_r] = \mathbf{F}_{r+k} - (\mu_k\,\mathbf{F}_r + \mathbf{L}_{r+k}(\mathbf{P}_k)).$$

This suggests to define the homological operator under $\mathcal{C}^\infty$-equivalence, in the form,

$$\begin{aligned}
\tilde{\mathcal{L}}_{r+k} \quad : \quad & \mathcal{Q}_k^{\mathbf{t}} \times \mathcal{P}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}}\\
& (\mathbf{P}_k, \mu_k) \to \mathcal{L}_{r+k}(\mathbf{P}_k, \mu_k) = \mu_k\,\mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r].
\end{aligned}$$

Now, reasoning as in the classical normal form theory, it is enough to choose $(\mathbf{P}_k, \mu_k)$ adequately in order to simplify the $(r+k)$-degree quasi-homogeneous term in system (3.1.2), by annihilating the part belonging to the range of the linear operator $\tilde{\mathcal{L}}_{r+k}$. In other words, we can achieve that $\mathbf{F}_{r+k}$ belongs to a co-range of $\tilde{\mathcal{L}}_{r+k}$ (a complementary subspace to the Range$(\tilde{\mathcal{L}}_{r+k})$). When this has been done, we say that the corresponding term has been reduced to normal form under $\mathcal{C}^\infty$-equivalence. By performing the procedure for $k=1$, $k=2$, $\cdots$ we obtain a normal form under equivalence.

There are elements of $\mathcal{Q}_{r+k}^{\mathbf{t}}$ in the expression of the transformed vector field which can be annihilated with transformations in the state variables as well as in the time variable . It is important to segregate the simplifications achieved by means of changes in the state variables with those that are consequence of the change in the time. To study this we define the linear operator

$$\begin{aligned} \ell_k \quad : \quad & \mathcal{P}^{\mathbf{t}}_{k-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_k \\ & \mu_{k-r} \to \nabla \mu_{k-r} \cdot \mathbf{F}_r. \end{aligned} \tag{2.3.15}$$

i.e., the Lie derivative respect to $\mathbf{F}_r$ (the lowest degree quasi-homogeneous term of $\mathbf{F}$).

Any function $\mu_k \in \mathcal{P}_k$ can be expressed as $\mu_k = \nu_k + \mu_k^r$ with $\nu_k \in \mathrm{Cor}(\ell_k)$, being $\mathrm{Cor}(\ell_k)$ a complementary subspace to the range of the linear operator ℓ_k, and $\mu_k^r \in \mathrm{Range}(\ell_k)$, i.e., there exists $\mu_{k-r} \in \mathcal{P}^t_{k-r}$ such that $\ell_k(\mu_{k-r}) = \mu_k^r$. So, from (1.2.13), we obtain,

$$\begin{aligned} \tilde{\mathcal{L}}_{r+k}(\mathbf{P}_k, \mu_k) &= [\mathbf{P}_k, \mathbf{F}_r] - \nu_k \mathbf{F}_r - (\nabla \mu_{k-r} \cdot \mathbf{F}_r)\mathbf{F}_r \\ &= [\mathbf{P}_k, \mathbf{F}_r] - \nu_k \mathbf{F}_r - [\mu_{k-r}\mathbf{F}_r, \mathbf{F}_r] + \mu_{k-r}[\mathbf{F}_r, \mathbf{F}_r] \\ &= [\mathbf{P}_k - \mu_{k-r}\mathbf{F}_r, \mathbf{F}_r] - \nu_k \mathbf{F}_r. \end{aligned}$$

Therefore, if we denote the operator $\tilde{\mathcal{L}}_{r+k}$ restricted to $\mathcal{Q}^{\mathbf{t}}_{r+k} \times \mathrm{Cor}(\ell_k)$ as $\mathcal{L}_{r+k}$, both operators have the same range. In this way, due to these considerations we modify the previous definition of the *homological operator under $\mathcal{C}^\infty$-equivalence* as follows,

$$\begin{aligned} \mathcal{L}_{r+k} \quad : \quad & \mathcal{Q}^{\mathbf{t}}_k \times \mathrm{Cor}(\ell_k) \longrightarrow \mathcal{Q}^{\mathbf{t}}_{r+k} \\ & (\mathbf{P}_k, \mu_k) \to \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) = \nu_k\, \mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r]. \end{aligned} \tag{2.3.16}$$

Again, using ideas from the classical theory of normal forms, it is possible to annihilate in each quasi-homogeneous component of $\mathbf{F}$, the part belonging to the range of the linear operator $\mathcal{L}_{r+k}$, by selecting $(\mathbf{P}_k, \mu_k)$ adequately. Therefore, we obtain the following result.

Theorem 2.3.64. *System (2.3.11) is formally equivalent to (2.3.14), where $\mathbf{G}_{r+k}$ belongs to a complementary subspace of the image space of the linear operator $\mathcal{L}_{r+k}$, for all $k \geq 1$.*

2.4 Choice of the type t and normal form to 0-th step.

In the previous sections we have presented the quasi-homogeneous normal forms for vector fields. As we know, the type chosen for developing in quasi-homogeneous terms the vector field, is of great importance, since the quasi-homogeneous type characterizes the first term $\mathbf{F}_r$ of the vector field, whose information is crucial for the normal form. There is, therefore, an obvious question: which is the most suitable type $\mathbf{t}$ in each case? In principle, there is not any criterium for choosing the type $\mathbf{t}$. The most common case is to take $\mathbf{t} = (1, \ldots, 1)$, in this case $\mathcal{Q}_k^{\mathbf{t}}$ coincides with the vector space of homogeneous vector fields of degree $k - 1$. Others criteria may be,

- Take a type that minimizes the co-ranges dimensions of the homological operators.
- Choosing a type associated with a compact faces of the Newton polyhedron of the vector field $\mathbf{F}$. This is, perhaps, the most natural choice, since under certain hypothesis, the first quasi-homogeneous term can determine the topological type of the singularity (see Brunella & Miari [26] and Dumortier [25]).

In our case, we choose the type $\mathbf{t}$, that keeps the topological type of the first quasi-homogeneous component with respect to the complete vector field. That is, we take the type $\mathbf{t}$ that makes, whenever it is possible, the component $\mathbf{F}_r$ structurally stable under perturbations of higher degrees.

Having chosen a suitable type, the next step is to simplify the lower degree quasi-homogeneous component of the vector field. In the classical theory, it is considered a system with non-null linear part, this simplification consist in the transformation of the system by a linear change of variables, to transform the linearization matrix to canonical form (Jordan, of Frobenius, ...). Here we want to generalize this step to the quasi-homogeneous case which will include the case of zero linear part.

For this task, it is fixed a type $\mathbf{t} = (t_1, t_2, \cdots, t_n)$ and the system (2.3.11):

$$\dot{\mathbf{x}} \;=\; \mathbf{F}_r(\mathbf{x}) + \sum_{j\geq 1} \mathbf{F}_{r+j}(\mathbf{x}).$$

By applying, on this system, a change of variables in a neighborhood of a singular point $\mathbf{x} = \mathbf{P}_0(\mathbf{y})$, with $\mathbf{P}_0 \in \tilde{\mathcal{Q}}_0^{\mathbf{t}} = \{\mathbf{P}_0 \in \mathcal{Q}_0^{\mathbf{t}} \,|\; \det(D\mathbf{P}_0(0)) \neq 0\}$, it is obtained a new system:

$$\dot{\mathbf{y}} \;=\; (D\mathbf{P}_0)^{-1}(\mathbf{y})\mathbf{F}_r(\mathbf{P}_0(\mathbf{y})) + (D\mathbf{P}_0)^{-1}(\mathbf{y}) \sum_{j\geq 1} \mathbf{F}_{r+j}(\mathbf{P}_0(\mathbf{y})).$$

Expressing it in quasi-homogeneous terms,

$$\dot{\mathbf{y}} = \mathbf{G}_r(\mathbf{y}) + \mathbf{G}_{r+1}(\mathbf{y}) + \cdots, \tag{2.4.17}$$

we obtain that the transformed first quasi-homogenous component,

$$\mathbf{G}_r(\mathbf{y}) = (D\mathbf{P}_0)^{-1}(\mathbf{y})\mathbf{F}_r(\mathbf{P}_0(\mathbf{y})) \in \mathcal{Q}_r^{\mathbf{t}}$$

At this point, we will use $\mathbf{P}_0$ to try removing parameters in $\mathbf{G}_r$.

The change of variables in the time, that affect to the first quasi-homogeneous term, are reduced to a simple scaling, so they are not useful for removing any term in the first element, $\mathbf{F}_r$. In the same way it occurs with changes in the state variables of the form $\mathbf{x} = D\mathbf{y}$, where D is a diagonal matrix (it is a scaled in the state variables). The zero-degree change of variables, different from the scaling, will be called *non-trivial*.

It should also be noted that, some non-trivial change of variables of degree zero, leave invariant the first quasi-homogeneous term. These changes can be used to eliminate higher order terms.

It would be desirable to choose a type such that the non-trivial change of variables of degree zero that leave invariant the first component do not exist or be minimals. In the example that we have developed at the end of this chapter, the type associated with the compact face of the Newton diagram is also the type that minimizes the non-trivial changes of variables that leave invariant the first quasi-homogeneous component of the vector field.

Next we show these ideas in the following example. In the last section of this chapter is calculated the normal form of this vector field.

Example 1: Consider the system,

$$
\begin{aligned}
\dot{x} &= y + \sum_{i+j\geq 2} a_{ij}x^i y^j, \\
\dot{y} &= \sum_{i+j\geq 2} b_{ij}x^i y^j,
\end{aligned}
\qquad (2.4.18)
$$

with $b_{20} = b_{11} = a_{20} = 0$ and $b_{30} \neq 0$.

This vector field can be written as follows,

$$
\begin{aligned}
\dot{x} &= y + a_{11}xy + a_{30}x^3 + \dots \\
\dot{y} &= b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{40}x^4 + \dots
\end{aligned}
\qquad (2.4.19)
$$

Different types can be considered for this example.

► **Type** $(1,1)$

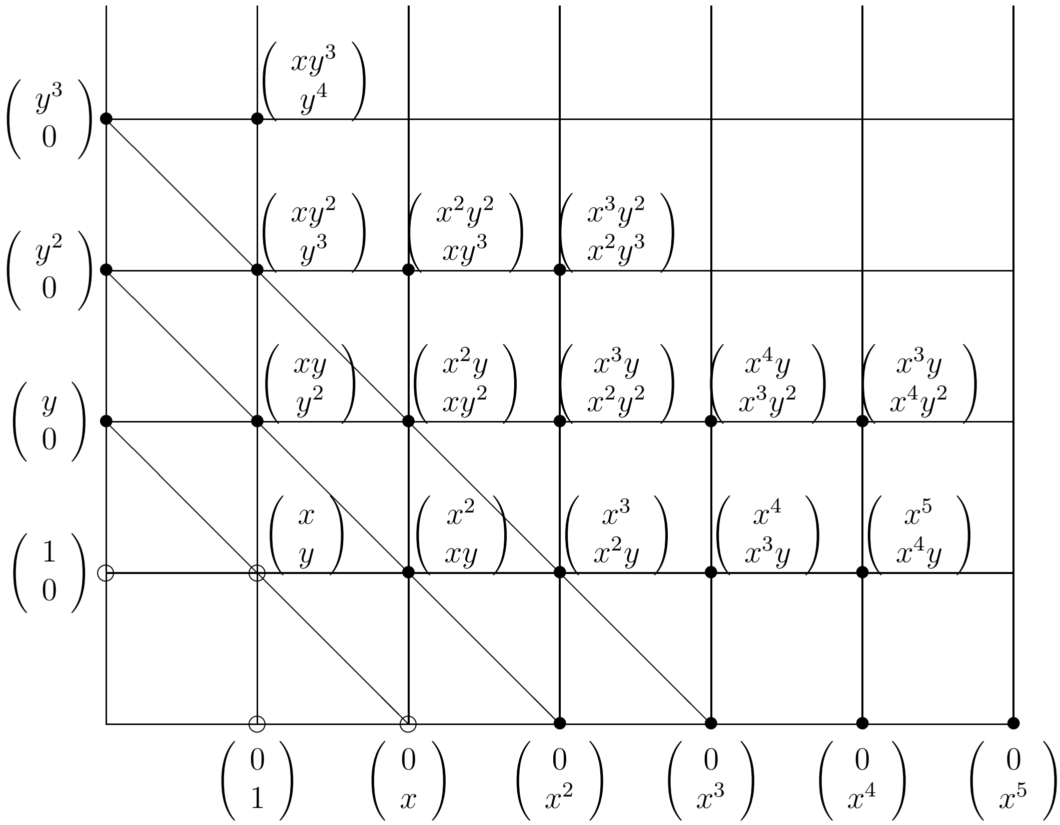

Figure 2.5: Quasi-homogeneous vector fields of type $(1,1)$ and sucessive degrees $0, 1, \dots$

System (2.4.19), respect to the type $(1,1)$, is written of the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ 0 \end{pmatrix} + \begin{pmatrix} a_{11}xy \\ b_{02}y^2 \end{pmatrix} + \begin{pmatrix} a_{30}x^3 \\ b_{30}x^3 + b_{21}x^2y \end{pmatrix} + \begin{pmatrix} 0 \\ b_{40}x^4 \end{pmatrix} + \cdots \tag{2.4.20}$$

In this case the non-trivial change of variables of degree zero are of the form,

$$\begin{aligned} u &= x + \alpha_2 y, \\ v &= \beta_1 x + y, \end{aligned}$$

where $1-\alpha_2\beta_1 \neq 0$, and those that leave invariant the first quasi-homogeneous term $\mathbf{F}_0 = \begin{pmatrix} y \\ 0 \end{pmatrix}$ verify $\beta_1 = 0$. Therefore, the non-trivial change of variables that leave invariant $\mathbf{F}_0$ are of the form,

$$\begin{aligned} u &= x + \alpha_2 y, \\ v &= y. \end{aligned}$$

In the above figure (Figure 2.5) are shown the quasi-homogeneous vector fields of type $(1,1)$ and degrees $0,1,2,\ldots$

► **Type** $(1,3)$

System (2.4.19), respect to the type $(1,3)$, is written of the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 \\ b_{30}x^3 \end{pmatrix} + \begin{pmatrix} 0 \\ b_{40}x^4 \end{pmatrix} + \begin{pmatrix} y + a_{30}x^3 \\ b_{21}x^2y + b_{50}x^5 \end{pmatrix} + \begin{pmatrix} a_{11}xy + b_{40}x^4 \\ b_{02}y^2 + b_{31}x^3y + b_{60}x^6 \end{pmatrix} + \cdots \tag{2.4.21}$$

The change of variables of degree zero are of the form,

$$\begin{aligned} u &= \alpha_1 x, \\ v &= \beta_1 x^3 + \beta_2 y, \end{aligned}$$

and those that leave invariant the first quasi-homogeneous term $\mathbf{F}_0 = \begin{pmatrix} 0 \\ b_{30}x^3 \end{pmatrix}$ verify $\beta_2 = \alpha_1^3$, with $\alpha_1 \neq 0$, i.e., the changes with the form,

$$\begin{aligned} u &= \alpha_1 x, \\ v &= \beta_1 x^3 + \alpha_1^3 y. \end{aligned}$$

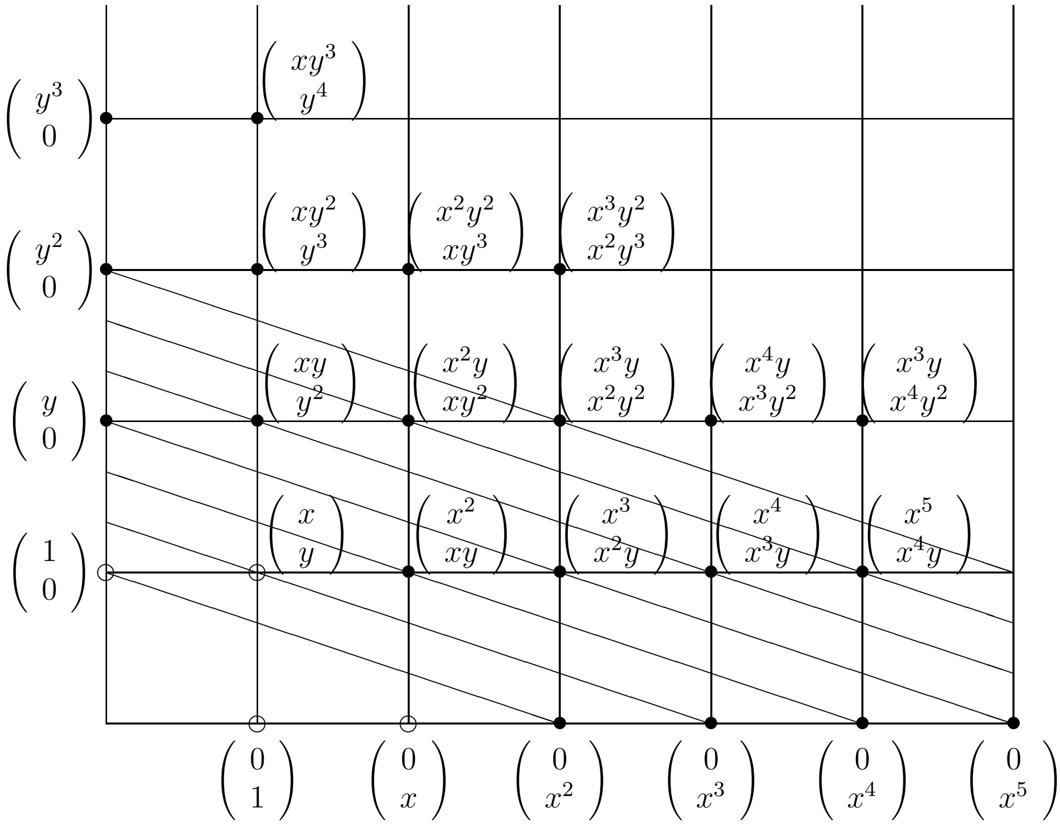

Figure 2.6: Quasi-homogeneous vector fields of type $(1,3)$ and sucessive degrees $-1, 0, 1, \ldots$

The quasi-homogeneous vector fields of type $(1,3)$ and degrees $-1, 0, 1, \ldots$ are shown in Figure (2.6).

► **Type** $(2,3)$

System (2.4.19), respect to the type $(2,3)$, is written of the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ 0 \end{pmatrix} + \begin{pmatrix} a_{11}xy \\ b_{30}x^3 + b_{02}y^2 \end{pmatrix} + \cdots \tag{2.4.22}$$

The change of variables of degree zero are of the form,

$$u = \alpha_1 x,$$
$$v = \beta_2 y.$$

These change of variables are trivial changes.

The quasi-homogeneous vector fields of type $(2,3)$ and degrees $1, 2, \ldots$ are showm in Figure (2.7).

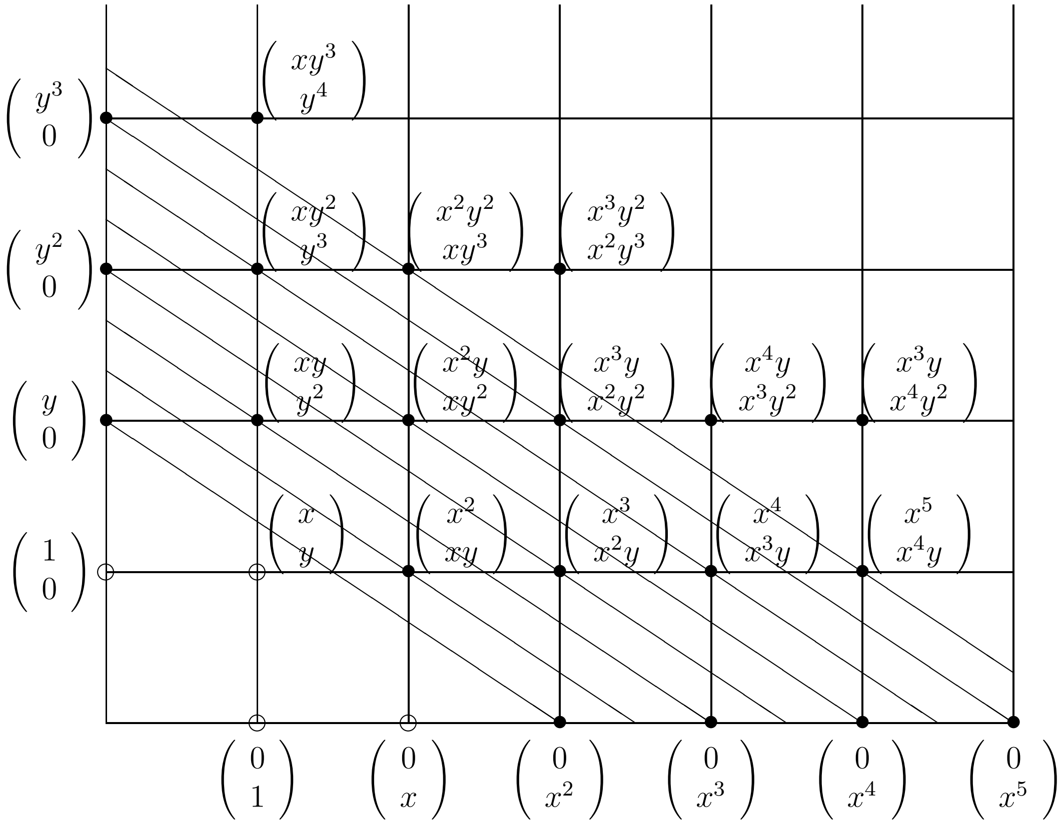

Figure 2.7: Quasi-homogeneous vector fields of type $(2,3)$ and sucessive degrees $1, 2, \ldots$

► **Type** $(1,2)$

System (2.4.19), respect to the type $(1,2)$, is written of the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ b_{30}x^3 \end{pmatrix} + \begin{pmatrix} a_{30}x^3 + a_{11}xy \\ b_{40}x^4 + b_{21}x^2y + b_{02}y^2 \end{pmatrix} + \cdots \tag{2.4.23}$$

In this case, the non-trivial change of variables of zero-degree are of the form,

$$\begin{aligned} u &= x, \\ v &= \beta x^2 + y. \end{aligned}$$

and those leaving the first quasi-homogeneous term invariant verify $\beta = 0$, i.e., there are no non-trivial change of variables that leave invariant the first quasi-homogeneous term.

The quasi-homogeneous vector fields of type $(1,2)$ and degrees $1, 2, \ldots$ are shown in Figure (2.8).

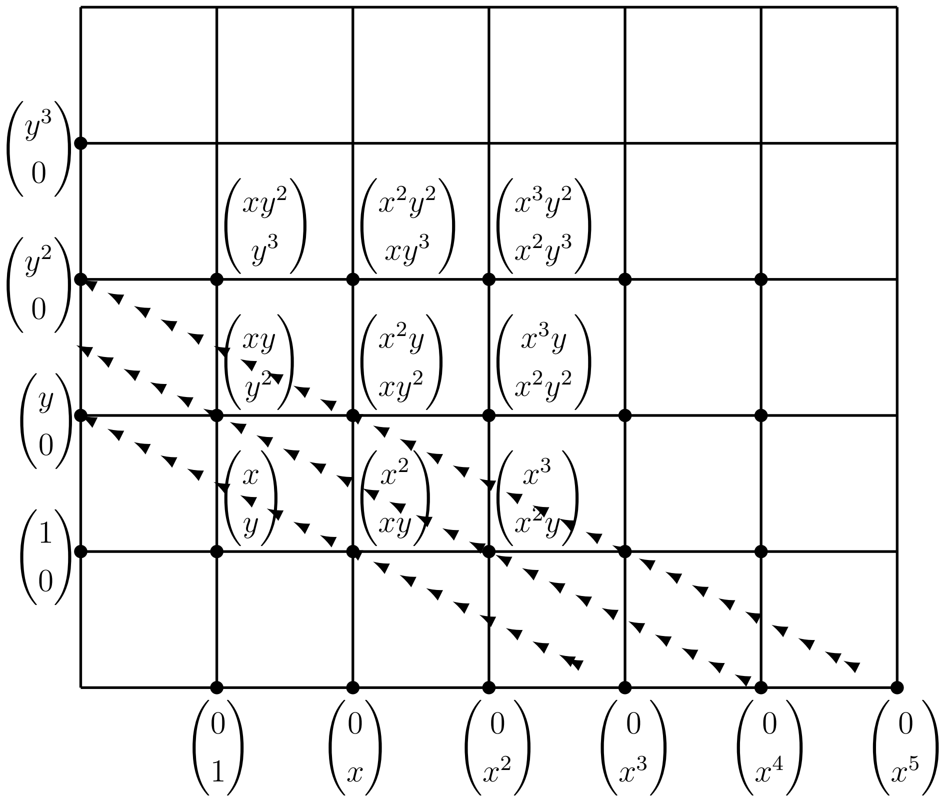

Figure 2.8: Quasi-homogeneous vector fields of type $(1, 2)$ and sucessive degrees $1, 2, \ldots$

2.5 The Lie triangle

Let consider a vector field $\mathbf{U} \in \mathcal{C}^\infty$ in a neighborhood of the origin such that $\mathbf{U}(\mathbf{0}) = \mathbf{0}$. We apply to system 3.1.2 the change of variables $\mathbf{x} = \mathbf{u}(\mathbf{y}, \varepsilon)$, where $\mathbf{u}$ is the only solution of the initial value problem,

$$\frac{\partial}{\partial \varepsilon}\mathbf{u}(\mathbf{y}, \varepsilon) = \mathbf{U}(\mathbf{u}(\mathbf{y}, \varepsilon)), \quad \mathbf{u}(\mathbf{y}, 0) = \mathbf{y}, \tag{2.5.24}$$

i.e., $\mathbf{u}$ is the flow of the autonomous system generated by $\mathbf{U}$. Thus, $\mathbf{U}$ is called generator of the change of varables.

The key idea of this formulation is that is possible to determine the transformed vector field from $\mathbf{F}$ and $\mathbf{U}$, without the need of calculating explicitly $\mathbf{u}$;

i.e., we can determine the transformed vector field directly, without knowing the change of variables made.

The transformed vector field $\mathbf{G}$ depends on the generator $\mathbf{U}$ in the following form, (see Algaba et al. [3]):

$$\mathbf{G}(\mathbf{y}, \varepsilon) = \mathbf{F}(\mathbf{y}) + \sum_{n \geq 1} T^n_{\mathbf{U}}(\mathbf{F}) \frac{\varepsilon^n}{n!}\,, \tag{2.5.25}$$

where $T_{\mathbf{U}}(\mathbf{F}) = [\mathbf{F}, \mathbf{U}]$, and $T^n_{\mathbf{U}}(\mathbf{F}) = T_{\mathbf{U}} \circ \overset{n}{\cdots} \circ T_{\mathbf{U}}(\mathbf{F})$.

In our analysis we use (2.5.25) taking $\varepsilon = 1$. We note that the quasi-homogeneous term of order $r + k$ in the transformed vector field is of the form:

$$\begin{aligned} \mathbf{G}_{r+k} \;\; &= \;\; \mathbf{F}_{r+k} + [\mathbf{F}, \mathbf{U}]_{r+k} + \frac{1}{2!}\,[[\mathbf{F}, \mathbf{U}], \mathbf{U}]_{r+k} \\ &\quad + \frac{1}{3!}\,[[[\mathbf{F}, \mathbf{U}], \mathbf{U}]\,, \mathbf{U}]_{r+k} + \cdots\,, \end{aligned} \tag{2.5.26}$$

where the subscripts indicate the corresponding order of the quasi-homogeneous terms.

Developing $\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2 + \cdots$ in quasi-homogeneous terms, we obtain,

$$\begin{aligned} [\mathbf{F}, \mathbf{U}]_{r+k} &= \sum_{j=1}^{k} [\mathbf{F}_{r+k-j}, \mathbf{U}_j], \\ [[\mathbf{F}, \mathbf{U}], \mathbf{U}]_{r+k} &= \sum_{j=1}^{k-1} [[\mathbf{F}, \mathbf{U}]_{r+k-j}, \mathbf{U}_j]\,, \\ &\;\;\vdots \end{aligned}$$

Note that, in the generator $\mathbf{U}$ we have not included the quasi-homogeneous term of degree zero $\mathbf{U}_0$. This is because we assume that we start from a vector field $\mathbf{F}$ whose first quasi-homogeneous term is already "simplified". This process, of simplifying of the first term, we have called it quasi-homogeneous normal form of zero step and it has been discussed in subsection 2.4.

For obtaining $\mathbf{G}_{r+k}$ we construct the following sequence of functions $\{\mathbf{V}_{r+k,l}\}$ defined in recursive form as is shown below,

$$\begin{aligned} \mathbf{V}_{r+k,0} &= \mathbf{F}_{r+k}, \quad k \geq 0, \\ \mathbf{V}_{r+k,l} &= \sum_{j=1}^{k+1-l} [\mathbf{V}_{r+k-j,l-1}\,, \mathbf{U}_j], \text{ for } l = 1, \cdots, k. \end{aligned} \tag{2.5.27}$$

This sequence can be organized in the following triangular scheme, (called triangle of Lie):

$$
\begin{array}{|c|c|c|c|c|c|}
\mathbf{V}_{r,0} & & & & & \\
\mathbf{V}_{r+1,0} & \mathbf{V}_{r+1,1} & & & & \\
\mathbf{V}_{r+2,0} & \mathbf{V}_{r+2,1} & \mathbf{V}_{r+2,2} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
\mathbf{V}_{r+k,0} & \mathbf{V}_{r+k,1} & \mathbf{V}_{r+k,2} & \cdots & \mathbf{V}_{r+k,k} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\tag{2.5.28}
$$

Figure 2.9: The Lie triangle under $\mathcal{C}^{\infty}$-conjugation.

It is necessary to note that the element in the column i and row $(k+1)$ depends only on the first $k-i+2$ elements of the column $i-1$. Using the equations (2.5.26) is obtained:

$$
\mathbf{G}_{r+k} = \sum_{l=0}^{k} \frac{1}{l!} \mathbf{V}_{r+k,l} = \mathbf{F}_{r+k} + \sum_{l=1}^{k} \frac{1}{l!} \mathbf{V}_{r+k,l}. \tag{2.5.29}
$$

The following result, that is easy to prove, states that the elements of each row of the triangle of Lie are quasi-homogeneous with the same degree.

Proposition 2.5.65. $\mathbf{V}_{r+k,l} \in \mathcal{Q}^{\mathbf{t}}_{r+k}$ *for all* $l = 0, 1, \cdots, k$. *In particular* $\mathbf{G}_{r+k} \in \mathcal{Q}^{\mathbf{t}}_{r+k}$.

In this triangular diagram, the rows are organized according to the quasi-homogeneous degree and columns according to the number of times that the Lie bracket has been applied. The original field is in the first column. The simple Lie brackets are in the second column, the double Lie brackets are in the third column, etc. In this sense, considering that the generators $\mathbf{U}_j$ are free, (i.e., we can express it with undetermined coefficients) we can say that, in the first column are the Lie triangle terms that do not have parameters, in the second column are the parameters with linear dependence, on the third, terms with quadratic dependence, etc. Thus, if we want to define linear operators homological must restrict to the first and second columns.

This triangle of Lie differs from the original proposed by Chow & Hale, [35]; both triangles contain the original system in the first column but are different in the way of building the succession and how to express the transformed system.

The main advantage of this new formulation of the Lie triangle is that it can be adapted to the case of orbital equivalence as we shall see in the next section.

2.6 Lie triangle under $\mathcal{C}^\infty$-equivalence.

To calculate the normal form under $\mathcal{C}^\infty$-equivalence, at first, we apply to the system (2.3.11) a change of variables in the time with the form $\frac{dt}{dT} = 1 - \mu(\mathbf{x})$, where $\mu \in \mathcal{C}^\infty$ in a neighborhood of the origin and $\mu(\mathbf{0}) = 0$. We obtain the system,

$$\tfrac{d\mathbf{x}}{dT} = \mathbf{x}' = \mathbf{F}^*(\mathbf{x}) = \mathbf{F}^*_r(\mathbf{x}) + \cdots + \mathbf{F}^*_{r+k}(\mathbf{x}) + \cdots$$

Writting $\mu = \mu_1 + \mu_2 + \cdots$, with $\mu_j \in \operatorname{Cor}(\ell_j)$ for all $j \geq 1$, (see the theorem 2.3.64), then the terms of the transformed system (2.6.30) are the following:

$$\mathbf{F}^*_r(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}).$$

$$\mathbf{F}^*_{r+k}(\mathbf{x}) = \mathbf{F}_{r+k}(\mathbf{x}) - \sum_{j=1}^{k} \mu_j(\mathbf{x})\mathbf{F}_{r+k-j}(\mathbf{x}), \quad \text{para } k = 1, 2, \cdots$$

Now we apply a change of variables with generator $\mathbf{U}$, and the transformed vector field is given by,

$$\tfrac{d\mathbf{y}}{dT} = \mathbf{y}' = \mathbf{G}(\mathbf{y}) = \mathbf{G}_r(\mathbf{y}) + \cdots + \mathbf{G}_{r+k}(\mathbf{y}) + \cdots \tag{2.6.30}$$

To calculate $\mathbf{G}_{r+k}$, we build the sucession of functions $\left\{\mathbf{V}^*_{r+k,l}\right\}$ defined in recursive form as is shown below,

$$\begin{aligned}
\mathbf{V}^*_{r,0} &= \mathbf{F}^*_r = \mathbf{F}_r, \\
\mathbf{V}^*_{r+k,0} &= \mathbf{F}^*_{r+k} = \mathbf{F}_{r+k} - \sum_{j=1}^{k} \mu_j \mathbf{F}_{r+k-j}, \quad \text{for} \quad k \geq 1, \\
\mathbf{V}^*_{r+k,l} &= \sum_{j=1}^{k+1-l} [\mathbf{V}^*_{r+k-j,l-1}\,, \mathbf{U}_j], \quad \text{for} \quad l = 1, \cdots, k.
\end{aligned}$$

This sequence can be organized similarly to the above triangular scheme. It is easy to deduce that:

$$\mathbf{G}_{r+k} = \sum_{l=0}^{k} \frac{1}{l!} \mathbf{V}^*_{r+k,l}.$$

To find the transformed vector field, we start separating the effects of the change of variables, in the state variables and in the time.

The Lie triangle is determined by the first column, that depend on $\mathbf{F}$ and μ, and the terms of the spatial generators $\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_{k-1}, \cdots$. Therefore, the element $\mathbf{G}_{r+k}$ of the transformed vector field is determined by the row k of the Lie triangle similar to (2.5.28), but it is built with elements of the first column $\mathbf{V}^*_{r+j,0}$. To do this, we define,

$$\mathbf{V}^*_{r+j,0} = \mathbf{V}_{r+j,0} + \mathbf{W}_{r+j,0},\ \ j \geq 0,$$

where

$$\mathbf{V}_{r+j,0} = \mathbf{F}_{r+j}, \quad \mathbf{W}_{r,0} = \mathbf{0}, \quad \mathbf{W}_{r+j,0} = -\sum_{i=1}^{j} \mu_i \mathbf{F}_{r+j-i}.$$

Due to the linearity of the Lie bracket, we can find the transformed vector field to k-th order as the sum of the rows k of two Lie triangles,

► On the one hand, by the triangle defined by the recursive scheme,

$$\begin{aligned} \mathbf{V}_{r+k,0} &= \mathbf{F}_{r+k} \quad k \geq 0, \\ \mathbf{V}_{r+k,l} &= \sum_{j=1}^{k+1-l} [\mathbf{V}_{r+k-j,l-1}, \mathbf{U}_j], \quad 1 \leq l \leq k. \end{aligned} \tag{2.6.31}$$

► Moreover, by the triangle defined by the following sequence of vector fields,

$$\begin{aligned} \mathbf{W}_{r,0} &= \mathbf{0}, \\ \mathbf{W}_{r+k,0} &= -\sum_{j=1}^{k} \mu_j \mathbf{F}_{r+k-j}, \quad k \geq 1, \\ \mathbf{W}_{r+k,l} &= \sum_{j=1}^{k+1-l} [\mathbf{W}_{r+k-j,l-1}, \mathbf{U}_j], \quad 1 \leq l \leq k. \end{aligned} \tag{2.6.32}$$

The Lie triangle (2.6.31) is determined by the quasi-homogeneous terms of the original field, $\mathbf{F}_{r+j}$, and by spatial generators $\mathbf{U}_1, \cdots, \mathbf{U}_{k-1}, \cdots$ This triangle collects the effects of $\mathcal{C}^\infty$-conjugation. Its elements always will be denoted as $\mathbf{V}_{r+j,l}$ and we will refer to it as Lie triangle. Moreover, the second triangle (2.6.32) depends on the spatial generators $\mathbf{U}_1, \cdots, \mathbf{U}_{k-1}, \cdots$, of the quasi-homogeneous terms of the original field, $\mathbf{F}_r, \cdots, \mathbf{F}_{r+k}, \cdots$ and the temporary generators $\mu_1, \cdots, \mu_{k-1}, \cdots$ This triangle introduces the effects of $\mathcal{C}^\infty$-equivalence. Its elements will be denoted as $\mathbf{W}_{r+j,l}$.

In particular,

$$\begin{aligned}
\mathbf{G}_{r+k} &= \sum_{l=0}^{k} \frac{1}{l!}\mathbf{V}^*_{r+k,l} = \mathbf{V}^*_{r+k,0} + \mathbf{V}^*_{r+k,1} + \sum_{l=2}^{k} \frac{1}{l!}\mathbf{V}^*_{r+k,l} \\
&= \mathbf{V}_{r+k,0} + \mathbf{W}_{r+k,0} + \mathbf{V}_{r+k,1} + \mathbf{W}_{r+k,1} + \sum_{l=2}^{k} \frac{1}{l!}\mathbf{V}^*_{r+k,l} \\
&= \mathbf{F}_{r+k} - \sum_{j=1}^{k} \mu_j \mathbf{F}_{r+k-j} + \sum_{j=1}^{k} [\mathbf{V}_{r+k-j,0}\,, \mathbf{U}_j] + \mathbf{W}_{r+k,1} + \sum_{l=1}^{k} \frac{1}{l!}\mathbf{V}^*_{r+k,l} \\
&= \left[\mathbf{F}_{r+k} - \sum_{j=1}^{k-1} \mu_j \mathbf{F}_{r+k-j} + \sum_{j=1}^{k-1} [\mathbf{V}_{r+k-j,0}\,, \mathbf{U}_j] + \mathbf{W}_{r+k,1} + \sum_{l=1}^{k} \frac{1}{l!}\mathbf{V}^*_{r+k,l}\right] \\
&\quad + [\mathbf{F}_r, \mathbf{U}_k] - \mu_k \mathbf{F}_r.
\end{aligned}$$

Thus, denoting,

$$\mathbf{Q}_{r+k} = \mathbf{F}_{r+k} - \sum_{j=1}^{k-1} \mu_j \mathbf{F}_{r+k-j} + \sum_{j=1}^{k-1} [\mathbf{V}_{r+k-j,0}\,, \mathbf{U}_j] + \mathbf{W}_{r+k,1} + \sum_{l=1}^{k} \frac{1}{l!}\mathbf{V}^*_{r+k,l}\,,$$

we obtain,

$$\mathbf{G}_{r+k} = \mathbf{Q}_{r+k} - \mathcal{L}_{r+k}(\mathbf{U}_k, \mu_k),$$

where $\mathbf{Q}_{r+k}$ depends of $\mathbf{F}_r, \cdots, \mathbf{F}_{r+k}$, of the spatial generator $\mathbf{U}_1, \cdots, \mathbf{U}_{k-1}$, and of the temporary generators $\mu_1, \cdots, \mu_{k-1}$. Therefore, using the ideas of the theory normal forms, we can select $\mathbf{U}_k, \mu_k$ in order to simplify the term of degree $r+k$.

These ideas provide an appropriate algorithm for symbolic computation, and consists on building the sequence $\mathbf{V}^*_{r+k,l}$ in each row, then, separate in that row the homological operator under equivalence and the element $\mathbf{Q}_{r+k}$.When we solve the homological equation, we add terms not considered in $\mathbf{V}^*_{r+k,0}$ and $\mathbf{V}^*_{r+k,1}$. The main steps of the algorithm are summarized in the following scheme,

► Let define $\mathbf{V}^*_{r,0} = \mathbf{F}_r$.

► For $k \geq 1$

- Let define,

$$\begin{aligned}
\mathbf{V}^*_{r+k,0} &= \mathbf{F}_{r+k} - \sum_{j=1}^{k-1} \mu_j \mathbf{F}_{r+k-j}, \\
\mathbf{V}^*_{r+k,1} &= \sum_{j=1}^{k-1} [\mathbf{V}^*_{r+k-j,0}\,, \mathbf{U}_j], \\
\mathbf{V}^*_{r+k,l} &= \sum_{j=1}^{k+1-l} [\mathbf{V}^*_{r+k-j,l-1}\,, \mathbf{U}_j], \quad 2 \leq l \leq k.
\end{aligned}$$

► Now, the quasi-homogeneous term of the normal form of degree $r+k$, is computed,

$$\mathbf{G}_{r+k} = \mathrm{Proj}_{\mathrm{Cor}(\mathcal{L}_{r+k})} \left(\sum_{l=0}^{k} \mathbf{V}^*_{r+k,l} \right),$$

where $\mathrm{Cor}\,(\mathcal{L}_{r+k})$ denotes a complementary subspace to the imagen space of $\mathcal{L}_{r+k}$, previously selected.

► Next, to obtain $\mathbf{U}_k$, μ_k, from the homological equation,

$$\mathcal{L}_{r+k}(\mathbf{U}_k, \mu_k) = \mathrm{Proj}_{\mathrm{Im}(\mathcal{L}_{r+k})} \left(\sum_{l=0}^{k} \mathbf{V}^*_{r+k,l} \right). \qquad (2.6.33)$$

► Finally, update

$$\begin{aligned}
\mathbf{V}^*_{r+k,0} &= \mathbf{V}^*_{r+k,0} - \mu_k \mathbf{F}_r, \\
\mathbf{V}^*_{r+k,1} &= \mathbf{V}^*_{r+k,1} + [\mathbf{F}_r, \mathbf{U}_k].
\end{aligned}$$

Similar to the theory of normal forms, sometimes, the process allows additional simplifications making suitable assumptions on quasihomogeneous terms of higher order. In fact, if $\mathrm{Ker}\,(\mathcal{L}_{r+k}) \neq \{0\}$ the homological equation (2.6.33) has not a unique solution, (the solution is determined, except arbitrary terms of the kernel of $\mathcal{L}_{r+k}$.) Thus, in the corresponding row of the Lie triangle can appear arbitrary constants which can be selected to achieve major simplifications in the quasi-homogeneous terms of higher order.

2.7 Study of a case of Takens-Bogdanov singularity with symmetry.

These ideas, about normal form, will be applied to the following system,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ b_{30}x^3 \end{pmatrix} + \begin{pmatrix} a_{30}x^3 + a_{11}xy \\ b_{40}x^4 + b_{21}x^2y + b_{02}y^2 \end{pmatrix} + \cdots \tag{2.7.34}$$

with $b_{30} \neq 0$, which is ordered respect to the type $\mathbf{t} = (t_1, t_2) = (1, 2)$ and the first quasi-homogeneous term is of degree $r = 1$. Using a scaling, it is posible consider $b_{30} = \sigma = \pm 1$. Note that, $\mathbf{F}_1$ is a hamiltonian vector field, with Hamilton function $h = \frac{1}{4}\sigma x^4 - \frac{1}{2}y^2$. It is denoted by $\mathbf{D}_0 = x\,\mathbf{e_1} + 2y\,\mathbf{e}_2$. The aim is to analize the homological operators, under conjugation and equivalence, given in (2.3.13) and (2.3.16), respectively. The homological operators, under conjugancy and equivalence, for a general degree k, are as follows,

$$\begin{aligned} \mathcal{L}_{k+1} \; : \; & \mathcal{Q}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k) \longrightarrow \mathcal{Q}_{k+1}^{\mathbf{t}} \\ & (\mathbf{P}_k, \nu_k) \to \mathcal{L}_{k+1}(\mathbf{P}_k, \nu_k) = \nu_k\,\mathbf{F}_1 + [\mathbf{P}_k, \mathbf{F}_1]. \end{aligned}$$

$$\begin{aligned} \mathbf{L}_{k+1} \; : \; & \mathcal{Q}_k^{\mathbf{t}} \longrightarrow \mathcal{Q}_{k+1}^{\mathbf{t}} \\ & \mathbf{P}_k \to \mathbf{L}_{k+1}(\mathbf{P}_k) = [\mathbf{P}_k, \mathbf{F}_1]. \end{aligned}$$

Now, suitable bases will be chosen for the spaces $\mathcal{Q}_k^{\mathbf{t}}$, $\mathcal{P}_k^{\mathbf{t}}$ and $\mathcal{Q}_{k+1}^{\mathbf{t}}$.

$\mathcal{P}_k^{\mathbf{t}} = span\{x^k, x^{k-2}y, x^{k-4}y^2, \cdots, x^{k-2\lfloor \frac{k}{2}\rfloor}y^{\lfloor \frac{k}{2}\rfloor}\} = span\{x^{k-2i}y^i / i = 0, \cdots, \lfloor \frac{k}{2}\rfloor\}$.

$$\begin{aligned} \mathcal{Q}_k^{\mathbf{t}} \;&=\; span\left\{ \begin{pmatrix} x^{k+1} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{k-1}y \\ 0 \end{pmatrix}, \begin{pmatrix} x^{k-3}y^2 \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} x^{k+1-2\lfloor \frac{k+1}{2}\rfloor}y^{\lfloor \frac{k+1}{2}\rfloor} \\ 0 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 0 \\ x^{k+2} \end{pmatrix}, \begin{pmatrix} 0 \\ x^k y \end{pmatrix}, \begin{pmatrix} 0 \\ x^{k-2}y^2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ x^{k-2\lfloor \frac{k}{2}\rfloor}y^{\lfloor \frac{k}{2}\rfloor+1} \end{pmatrix} \right\} = \\ &=\; span\left\{ \begin{pmatrix} x^{k+1-2i}y^i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x^{k+2-2j}y^j \end{pmatrix} / i = 0, \cdots, \left\lfloor \frac{k+1}{2} \right\rfloor, \; j = 0, \cdots, \left\lfloor \frac{k}{2} \right\rfloor + 1 \right\}. \end{aligned}$$

Another possible basis for $\mathcal{Q}_k^{\mathbf{t}}$ is,

$$\mathcal{Q}_k^{\mathbf{t}} \;=\; span\left\{ \begin{pmatrix} x^{k+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^{k+2} \end{pmatrix}, p_k\mathbf{D}_0, p_{k-1}\mathbf{F}_1 \;/\; p_k \in \mathcal{P}_k^{\mathbf{t}}, p_{k-1} \in \mathcal{P}_{k-1}^{\mathbf{t}} \right\}.$$

Therefore, an element $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$ is of the form,

$$\mathbf{P}_k = \alpha \begin{pmatrix} x^{k+1} \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ x^{k+2} \end{pmatrix} + p_k \mathbf{D}_0 + p_{k-1}\mathbf{F}_1.$$

Now we will operate each of the components of $\mathbf{P}_k$ with $\mathbf{F}_1$ using the Lie bracket:

- $\left[\alpha \begin{pmatrix} x^{k+1} \\ 0 \end{pmatrix}, \mathbf{F}_1\right] = \alpha \begin{pmatrix} (k+1)x^k y \\ -3\,\sigma\, x^{k+3} \end{pmatrix} = -(k+4)\,\sigma\alpha \begin{pmatrix} 0 \\ x^{k+3} \end{pmatrix}$
  $+\,(k+1)\alpha\, x^k \mathbf{F}_1.$
- $\left[\beta \begin{pmatrix} 0 \\ x^{k+2} \end{pmatrix}, \mathbf{F}_1\right] = \beta \begin{pmatrix} -x^{k+2} \\ (k+2)\, x^{k+1} y \end{pmatrix} = -\beta(1+\frac{k+2}{2}) \begin{pmatrix} x^{k+2} \\ 0 \end{pmatrix}$
  $+\,\frac{\beta}{2}(k+2)x^{k+1}\mathbf{D}_0.$
- $[p_k\mathbf{D}_0, \mathbf{F}_1] = (\nabla p_k \mathbf{F}_1)\,\mathbf{D}_0 - p_k\mathbf{F}_1.$
- $[p_{k-1}\mathbf{F}_1, \mathbf{F}_1] = (\nabla p_{k-1}\mathbf{F}_1)\,\mathbf{F}_1.$

Taking into account the previous calculations, the matrix of the homological operator $\mathcal{L}_{k+1}$ is as follows,

$$\left(\begin{array}{c|c|c|c|c||c}
0 & -\beta(1+\frac{k+2}{2})\begin{pmatrix} x^{k+2} \\ 0 \end{pmatrix} & 0 & 0 & 0 & \tilde{\alpha}\begin{pmatrix} x^{k+2} \\ 0 \end{pmatrix} \\
\hline
-(k+4)\,\sigma\alpha\begin{pmatrix} 0 \\ x^{k+3} \end{pmatrix} & 0 & 0 & 0 & 0 & \tilde{\beta}\begin{pmatrix} 0 \\ x^{k+3} \end{pmatrix} \\
\hline
0 & \frac{\beta}{2}(k+2)x^{k+1}\mathbf{D}_0 & (\nabla p_k\mathbf{F}_1)\mathbf{D}_0 & 0 & 0 & \tilde{p}_{k-1}\mathbf{D}_0 \\
\hline
(k+1)\alpha x^k\mathbf{F}_1 & 0 & -p_k\mathbf{F}_1 & (\nabla p_{k-1}\mathbf{F}_1)\mathbf{F}_1 & \nu_k\mathbf{F}_1 & \tilde{p}_k\mathbf{F}_1 \\
\hline\hline
\alpha\begin{pmatrix} x^{k+1} \\ 0 \end{pmatrix} & \beta\begin{pmatrix} 0 \\ x^{k+2} \end{pmatrix} & p_k\mathbf{D}_0 & p_{k-1}\mathbf{F}_1 & \nu_k &
\end{array}\right)$$

From the expression of the above matrix are deduced the following results which show a complementary subspace to the range of the homological operators under $\mathcal{C}^\infty$-equivalence and $\mathcal{C}^\infty$-conjugation. In other words, it is shown a complementary subspace to the range of the operators $\mathbf{L}_{k+1}$ and $\mathcal{L}_{k+1}$ described in (2.3.13) and (2.3.16), respectively.

Proposition 2.7.66. *Consider system (2.7.34). A complementary subspace to the range of the operator given in (2.3.13) is,*

$$\mathrm{Cor}(\mathbf{L}_{1+k}) = \mathrm{Cor}(\ell_{1+k})\mathbf{D}_0 \oplus \mathrm{Cor}(\ell_k)\mathbf{F}_1.$$

Proposition 2.7.67. *Consider system (2.7.34). A complementary subspace of the range to the operator given in (2.3.16) is,*

$$\mathrm{Cor}(\mathcal{L}_{1+k}) = \mathrm{Cor}(\ell_{1+k})\mathbf{D}_0.$$

Moreover, if we consider $k = 4l_1 + l_2$, we obtain the following expression for the kernel of the homological operator,

$$\mathrm{Ker}(\mathcal{L}_{1+k}) = \begin{cases} span\{(h^{l_1}\mathbf{D}_0, h^{l_1})\}, & if \quad l_2 = 0, \\ span\{(h^{l_1}\mathbf{F}_1, 0)\}, & if \quad l_2 = 1, \\ span\{0\}, & otherwise \end{cases}$$

Note that the co-range of the homological operator (under conjugation and equivalence) is given by the co-range of the linear operator, derivative of Lie. For that, we consider the operator ℓ_k defined in (2.3.15). In this example, taking $k = 4m + l$, with $m \in \mathbb{N}$ and $l = \{0, 1, 2, 3\}$, this operator is of the form,

$$\begin{aligned} \ell_{4m+l+1} \quad : \quad & \mathcal{P}^{\mathbf{t}}_{4m+l} \longrightarrow \mathcal{P}^{\mathbf{t}}_{4m+l+1} \\ & \mu_{4m+l} \to \nabla\mu_{4m+l} \cdot \mathbf{F}_1. \end{aligned} \tag{2.7.35}$$

The study of this operator will be decomposed in four cases,

► **Case** $l = 0$.

A basis of $\mathcal{P}^{\mathbf{t}}_{4m}$, is of the form,

$$\mathcal{P}^{\mathbf{t}}_{4m} = span\{x^{4m}, x^{4m-4}h, \ldots, x^4h^{m-1}, h^m, x^{4m-2}y, x^{4m-6}yh, \ldots, x^6yh^{m-2}, x^2yh^{m-1}\}.$$

Therefore, any element $\mu_{4m} \in \mathcal{P}^{\mathbf{t}}_{4m}$ can be written as,

$$\mu_{4m} = \sum_{i=0}^{2m}\alpha_i x^{4m-2i}y^i = \sum_{i=0}^{m}\alpha_i x^{4(m-i)}h^i + \sum_{i=0}^{m-1}\beta_i x^{4(m-i)-2}yh^i.$$

Thus,

$$\begin{aligned} \ell_{4m+1}(\mu_{4m}) \quad &= \quad \nabla\left(\sum_{i=0}^{m}\alpha_i x^{4(m-i)}h^i + \sum_{i=0}^{m-1}\beta_i x^{4(m-i)-2}yh^i\right)\mathbf{F}_1 \\ &= \quad \sum_{i=0}^{m}4(m-i)\alpha_i x^{4(m-i)-1}h^i y + \sum_{i=0}^{m-1}\left((4m-4i-2)\beta_i x^{4m-4i-3}y^2h^i\right. \\ &+ \quad \left.\beta_i\sigma x^{4m-4i+1}h^i\right). \end{aligned}$$

Taking into account that $h = \frac{1}{4}\sigma x^4 - \frac{1}{2}y^2$, i.e., $y^2 = \frac{1}{2}\sigma x^4 - 2h$.

$$\begin{aligned}
\ell_{4m+1}(\mu_{4m}) &= \sum_{i=0}^{m} 4(m-i)\alpha_i x^{4(m-i)-1}h^i y + \sum_{i=0}^{m-1}(2m-2i-1)\sigma\beta_i x^{4m-4i+1}h^i \\
&- \sum_{i=0}^{m-1}(4m-4i-2)2\beta_i x^{4m-4i-3}h^{i+1} + \beta_i\sigma x^{4m-4i+1}h^i \\
&= \sum_{i=0}^{m} 4(m-i)\alpha_i x^{4(m-i)-1}h^i y + \sum_{i=0}^{m-1}\beta_i\sigma(2m-2i)x^{4m-4i+1}h^i \\
&- \sum_{i=1}^{m} 2\beta_{i-1}(4m-4i+2)x^{4m-4i+1}h^i \\
&= \beta_0\sigma 2m x^{4m+1} + \sum_{i=1}^{m-1}\left(\beta_i\sigma(2m-2i) - 2\beta_{i-1}(4m-4i+2)\right)x^{4m-4i+1}h^i \\
&+ 4\beta_{m-1}xh^m + \sum_{i=0}^{m-1} 4(m-i)\alpha_i x^{4(m-i)-1}h^i y\,,
\end{aligned}$$

and taking a basis of $\mathcal{P}^{\mathbf{t}}_{4m+1}$ of the form,

$$\mathcal{P}^{\mathbf{t}}_{4m+1} = span\{x^{4m-1}y, \ldots, x^7yh^{m-2}, x^3yh^{m-1}, x^{4m+1}, x^{4m-3}h, \ldots, x^5h^{m-1}, xh^m\}.$$

The above operator ℓ_{4m+1} has the following matricial expression,

$$\left(\begin{array}{c|c}
\begin{pmatrix}
4m\alpha_0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 4(m-1)\alpha_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 8\alpha_{m-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 4\alpha_{m-1} & 0
\end{pmatrix} & 0 \\
\hline
0 & \begin{pmatrix}
2m\beta_0\sigma & 0 & 0 & 0 & \ldots & 0 & 0 \\
-4(2m-1)\beta_0 & 2\beta_1\sigma(m-1) & 0 & 0 & \ldots & 0 & 0 \\
0 & -4\beta_1(2m-3) & \ddots & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ldots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 4\beta_{m-2}\sigma & 0 \\
0 & 0 & 0 & 0 & \ldots & -12\beta_{m-2} & 2\beta_{m-1}\sigma \\
0 & 0 & 0 & 0 & \ldots & 0 & -4\beta_{m-1}
\end{pmatrix}
\end{array}\right)$$

From the structure of the above matrix, it is possible to deduce that $\mathrm{Ker}(\ell_{4m+1}) = span\{h^m\}$, and using the expression,

$$dim(\mathrm{Cor}(\ell_{4m+1})) = dim(\mathcal{P}^{\mathbf{t}}_{4m+1}) - dim(\mathcal{P}^{\mathbf{t}}_{4m}) + dim(\mathrm{Ker}(\ell_{4m+1})) \quad (2.7.36)$$

it has that $dim(\mathrm{Cor}(\ell_{4m+1}))=1$, therefore $\mathrm{Cor}(\ell_{4m+1})=span\{xh^m\}$.

▶ **Case** $l=1$.

Any element $\mu_{4m+1}\in\mathcal{P}^{\mathbf{t}}_{4m+1}$ can be written as follows,

$$\mu_{4m+1}=\sum_{i=0}^{m}\alpha_i x^{4(m-i)+1}h^i+\sum_{i=0}^{m-1}\beta_i x^{4(m-i)-1}yh^i=x\cdot\mu_{4m},$$

thus,

$$\begin{aligned}
\ell_{4m+2}(\mu_{4m+1}) &= \nabla(x\cdot\mu_{4m})\mathbf{F}_1=\mu_{4m}\cdot y+x\cdot\nabla\mu_{4m}\mathbf{F}_1=\sum_{i=0}^{m}(4(m-i)+1)\alpha_i x^{4(m-i)}yh^i\\
&+ 2\,m\beta_0\sigma x^{4m+2}+\sum_{i=1}^{m-1}\left(2\sigma(m-i)\beta_i-(8m-8i+3)\beta_{i-1}\right)x^{4(m-i)+2}h^i\\
&- 3\beta_{m-1}x^2h^m.
\end{aligned}$$

Taking a basis of $\mathcal{P}^{\mathbf{t}}_{4m+2}$ of the form,

$\mathcal{P}^{\mathbf{t}}_{4m+2}=span\{x^{4m}y,x^{4m-4}yh,x^{4m-8}yh^2,\dots,yh^m,x^{4m+2},x^{4m-2}h,x^{4m-6}h^2,\dots,x^2h^m\}$.

it is obtained the following matrix for the linear operator ℓ_{4m+2},

$$\left(\begin{array}{c|c}
\begin{pmatrix}
(4m+1)\alpha_0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 4(m-3)\alpha_1 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \ddots & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & 4\alpha_{m-1} & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 1
\end{pmatrix} & 0\\
\hline
0 & \begin{pmatrix}
2m\beta_0\sigma & 0 & 0 & 0 & \dots & 0 & 0\\
-8(m-5)\beta_0 & 2\beta_1\sigma(m-1) & 0 & 0 & \dots & 0 & 0\\
0 & -\beta_1(8m-13) & \ddots & 0 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \ddots & \dots & \vdots & \vdots\\
0 & 0 & 0 & \ddots & \ddots & 0 & 0\\
0 & 0 & 0 & 0 & \ddots & 4\beta_{m-2}\sigma & 0\\
0 & 0 & 0 & 0 & \dots & -12\beta_{m-2} & 2\beta_{m-1}\sigma\\
0 & 0 & 0 & 0 & \dots & 0 & -3\beta_{m-1}
\end{pmatrix}
\end{array}\right)$$

From the structure of the above matrix, it is possible to deduce that $\mathrm{Ker}(\ell_{4m+1})=span\{0\}$, and using the expression,

$$dim(\mathrm{Cor}(\ell_{4m+1}))=dim(\mathcal{P}^{\mathbf{t}}_{4m+1})-dim(\mathcal{P}^{\mathbf{t}}_{4m})+dim(\mathrm{Ker}(\ell_{4m+1}))\quad(2.7.37)$$

it has that $dim(\mathrm{Cor}(\ell_{4m+1}))=1$ and $\mathrm{Cor}(\ell_{4m+1})=span\{x^2h^m\}$.

▶ **Case** $l = 2$.

Any element $\mu_{4m+2} \in \mathcal{P}^{\mathbf{t}}_{4m+1}$ can be written in the form,

$$\mu_{4m+2} = \sum_{i=0}^{m} \alpha_i x^{4(m-i)+2} h^i + \sum_{i=0}^{m} \beta_i x^{4(m-i)} y h^i = x^2 \cdot \mu_{4m},$$

thus,

$$\begin{aligned} \ell_{4m+3}(\mu_{4m+2}) &= \sum_{i=0}^{m} (4(m-i)+2)\alpha_i x^{4(m-i)+1} y h^i \\ &+ \beta_0 \sigma (2m+1) x^{4m+3} \\ &+ \sum_{i=1}^{m} \left((2(m-i)+1)\beta_i\, \sigma - 8(m-i+1)\beta_{i-1} \right) x^{4(m-i)+3} h^i. \end{aligned}$$

Taking a basis of $\mathcal{P}^{\mathbf{t}}_{4m+3}$ as follows,

$$\mathcal{P}^{\mathbf{t}}_{4m+3} = span\{x^{4m+3}, x^{4m-1}h, x^{4m-5}h^2, \ldots, x^3 h^m, x^{4m+1}y, x^{4m-3}yh, x^{4m-7}yh^2, \ldots, xyh^m\}.$$

It is obtained the following matrix expression for this operator,

$$\left(\begin{array}{c|c} \begin{pmatrix} (4m+2)\alpha_0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & (4m-2)\alpha_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 6\alpha_{m-1} & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 2\alpha_m \end{pmatrix} & 0 \\ \hline 0 & \begin{pmatrix} (2m+1)\beta_0\sigma & 0 & 0 & 0 & \ldots & 0 & 0 \\ -8m\beta_0 & (2m-1)\beta_1\sigma & 0 & 0 & \ldots & 0 & 0 \\ 0 & -8(m-1)\beta_1 & \ddots & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 4\beta_{m-2}\sigma & 0 \\ 0 & 0 & 0 & 0 & \ldots & -12\beta_{m-2} & 2\beta_{m-1}\sigma \\ 0 & 0 & 0 & 0 & \ldots & 0 & -3\beta_{m-1} \end{pmatrix} \end{array}\right)$$

In this case, using (2.7.37), and from the expression of the above matrix we can conclude that $\mathrm{Ker}(\ell_{4m+3}) = \{0\}$. Therefore $\mathrm{Cor}(\ell_{4m+3}) = \{0\}$.

▶ **Case** $l = 3$.

Any element $\mu_{4m+3} \in \mathcal{P}^{\mathbf{t}}_{4m+3}$ is of the form,

$$\mu_{4m+3} = \sum_{i=0}^{m} \alpha_i x^{4(m-i)+3} h^i + \sum_{i=0}^{m} \beta_i x^{4(m-i)+1} y h^i,$$

hence,

$$\begin{aligned}
\ell_{4m+4}(\mu_{4m+3}) &= \sum_{i=0}^{m}(4(m-i)+3)\alpha_i\, x^{4(m-i)+2}yh^i \\
&+ \sum_{i=1}^{m}\left((2(m-i)+\frac{3}{2})\beta_i\,\sigma - 2(4(m-i)+5)\beta_{i-1}\right)x^{4(m-i)+4}h^i.
\end{aligned}$$

Taking a basis of $\mathcal{P}^{\mathbf{t}}_{4m+4}$ of the form,

$\mathcal{P}^{\mathbf{t}}_{4m+4} = span\{x^{4m+2}y, x^{4m-2}yh, x^{4m-6}yh^2, \ldots, x^2yh^m, x^{4m+4}, x^{4m}h, x^{4m-4}h^2, \ldots, h^{m+1}\}$

It is possible to deduce that ℓ_{4m+4} has the following matrix expression,

$$\left(\begin{array}{c|c}
\begin{pmatrix}
(4m+3)\alpha_0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & (4m-1)\alpha_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ddots & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 7\alpha_{m-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 3\alpha_m
\end{pmatrix} & 0 \\ \hline
0 & \begin{pmatrix}
(2m+\frac{3}{2})\beta_0\sigma & 0 & 0 & 0 & \ldots & 0 & 0 \\
-2(4m+1)\beta_0 & (2m-\frac{1}{2})\beta_1\sigma & 0 & 0 & \ldots & 0 & 0 \\
0 & -2(4m-3)\beta_1 & \ddots & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ldots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & \frac{7}{2}\beta_{m-1}\sigma & 0 \\
0 & 0 & 0 & 0 & \ldots & -10\beta_{m-1} & \frac{3}{2}\beta_m\sigma \\
0 & 0 & 0 & 0 & \ldots & 0 & 2\beta_m
\end{pmatrix}
\end{array}\right)$$

Therefore, from the structure of this matrix, it is possible to conclude that, $\mathrm{Ker}(\ell_{4m+4}) = \{0\}$. Using (2.7.37), we get that $\mathrm{Cor}(\ell_{4m+4}) = h^{m+1}$.

In summary, we have obtained that $\mathrm{Cor}(\ell_{4m+1+l}) = \begin{cases} \{0\} & \text{if } l = 2, \\ x^{l+1}h^m & \text{if } l \in \{0,1\}, \\ h^{m+1} & \text{if } l = 3, \end{cases}$

or equivalently, $\mathrm{Cor}(\ell_{4m+n}) = \begin{cases} \{0\} & \text{if } n = 3, \\ x^n h^m & \text{if } n \in \{0,1,2\}. \end{cases}$

The following theorems show a normal form, under conjugation and equivalence, for system (2.4.19).

Theorem 2.7.68. *A normal form for system (2.4.19), under $\mathcal{C}^\infty$-conjugation, is given by,*

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{pmatrix} y \\ \sigma x^3 \end{pmatrix} + \alpha_0^{(2)}x^2\mathbf{D}_0 + (\beta_0^{(1)}x + \beta_0^{(2)}x^2)\mathbf{F}_1 \\
&+ \sum_{i=1}^{\infty}\sum_{j=0}^{2}(\alpha_i^{(j)}x^jh^i)\mathbf{D}_0 + (\beta_i^{(j)}x^jh^i)\mathbf{F}_1. \qquad (2.7.38)
\end{aligned}$$

Theorem 2.7.69. *A normal form for system (2.4.19), under $\mathcal{C}^\infty$-equivalece, is given by*

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ \sigma x^3 \end{pmatrix} + \alpha_0^{(2)} x^2 \mathbf{D}_0 + \sum_{i=1}^{\infty}\sum_{j=0}^{2}(\alpha_i^{(j)} x^j h^i)\mathbf{D}_0. \qquad (2.7.39)$$

At this point, the question is whether is possible to make more simplifications in the normal form. For obtaining a *reduced normal form* it is necessary to determine the kernel of the homological operator defined in (2.3.16). We describe the process in the next subsection.

2.7.1 A reduced normal form for a nilpotent system.

In order to describe the process, we will consider the normal form given in (2.7.39) and we assume that the coeffcient $\alpha_0^{(2)} \neq 0$. Then we describe the two-step homological operator under equivalence as follows,

$$\begin{aligned} \mathcal{L}_{1+k}^{(2)} \;:\; & \mathcal{Q}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k) \times \mathrm{Ker}(\mathcal{L}_k) \longrightarrow \mathcal{Q}_{1+k}^{\mathbf{t}} \\ & (\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{k-1}, \tilde{\nu}_{k-1})) \to \mathcal{L}_{1+k}^{(2)}(\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{k-1}, \tilde{\nu}_{k-1})) \\ & = [\mathbf{P}_k, \mathbf{F}_1] - \nu_k \mathbf{F}_1 + [\tilde{\mathbf{P}}_{k-1}, \alpha_0^{(2)} x^2 \mathbf{D}_0] - \tilde{\nu}_{k-1}\alpha_0^{(2)} x^2 \mathbf{D}_0 \\ & = \mathcal{L}_{1+k}(\mathbf{P}_k, \nu_k) + [\tilde{\mathbf{P}}_{k-1}, \alpha_0^{(2)} x^2 \mathbf{D}_0] - \tilde{\nu}_{k-1}\alpha_0^{(2)} x^2 \mathbf{D}_0. \qquad (2.7.40) \end{aligned}$$

Taking into account the kernel of the homological operator described in Proposition 2.7.67, we can deduce, if we take $k-1 = 4l_1 + l_2$, with $0 \le l_2 < 4$, the following expression for the homological operator under equivalence,

- For $l_2 = 0$ then $(\tilde{\mathbf{P}}_{k-1}, \tilde{\nu}_{k-1}) = (h^{l_1}\mathbf{D}_0, h^{l_1}) \in \mathrm{Ker}(\mathcal{L}_k)$.

  Thus,

  $$\begin{aligned} [\tilde{\mathbf{P}}_{k-1}, \alpha_0^{(2)} x^2 \mathbf{D}_0] - \tilde{\nu}_{k-1}\alpha_0^{(2)} x^2 \mathbf{D}_0 &= [h^{l_1}\mathbf{D}_0, \alpha_0^{(2)} x^2 \mathbf{D}_0] - h^{l_1}\,\alpha_0^{(2)} x^2 \mathbf{D}_0 \\ &= (4l_1 - 3)\alpha_0^{(2)}\, x^2 h^{l_1}\,\mathbf{D}_0 \;\in \mathrm{Cor}(\mathcal{L}_{4l_1+2}). \end{aligned}$$

- For $l_2 = 1$ then $(\tilde{\mathbf{P}}_{k-1}, \tilde{\nu}_{k-1}) = (h^{l_1}\mathbf{F}_1, 0) \in \mathrm{Ker}(\mathcal{L}_k)$.

Therefore,

$$\begin{aligned}
[\tilde{\mathbf{P}}_{k-1}, \alpha_0^{(2)} x^2 \mathbf{D}_0] - \tilde{\nu}_{k-1}\alpha_0^{(2)} x^2 \mathbf{D}_0 &= [h^{l_1}\mathbf{F}_1, \alpha_0^{(2)} x^2 \mathbf{D}_0] \\
&= -\left(\nabla(\alpha_0^{(2)} x^2 h^{l_1})\mathbf{F}_1\right)\mathbf{D}_0 \\
&\quad + \left((4l_1-1)\alpha_0^{(2)} x^2 h^{l_1}\right)\mathbf{F}_1 \in \mathrm{Im}(\mathcal{L}_{4l_1+3}).
\end{aligned}$$

Note that if $l_2 = 1$, it is not possible to make further simplifications in the normal form given in (2.7.39). In the case $l_2 = 0$ with $\alpha_0^{(2)} \neq 0$, it is possible to eliminate the elements of the form h^j. Therefore, the two-step normal form, under equivalence, is the following:

Theorem 2.7.70. *Considering $\alpha_0^{(2)} \neq 0$, $\alpha_0^{(2)}$ defined in (2.7.39), a normal form for system (2.4.19), under $\mathcal{C}^\infty$-equivalece, is given by,*

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ \sigma x^3 \end{pmatrix} + \alpha_0^{(2)} x^2 \mathbf{D}_0 + \sum_{i=1}^{\infty}\sum_{j=1}^{2}(\alpha_i^{(j)} x^j h^i)\mathbf{D}_0 \qquad (2.7.41)$$

Remark 11. *The normal form, described in (2.7.41), is the unique normal form for system (2.7.39). It is possible to prove that, the kernels of the two-step homological operators do not get to obtain more simplifications in the normal form given in (2.7.41).*

CHAPTER 3

Planar quasi-homogeneous normal forms.

3.1 Introduction.

In the previous chapter we have done a reminder of the classical theory of normal forms and an introduction to the quasi-homogeneous normal form. Also, we have calculated the normal form for a particular case of the Takens-Bogdanov degeneration. With this example, we have wanted to show the large number of calculations necessary to obtain this normal form to infinite order. In this chapter, our goal is to develop a theory that allows us a remarkable simplification of the calculations for obtaining the normal form. Let us consider the following system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \text{with} \;\; \mathbf{x} \in \mathbb{R}^2. \tag{3.1.1}$$

The vector field (3.1.1) can be always written as the sum of quasi-homogeneous terms of type $\mathbf{t}$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots , \tag{3.1.2}$$

where $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ for all k, and we assume that $\mathbf{F}_r \neq \mathbf{0}$ being $r \in \mathbb{Z}$. If we select the type $\mathbf{t} = (1,1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees. The main tools we use in this chapter are two types of decompositions for vector fields. These decompositions provide notable simplifications in the computation of the normal form. The first decomposition is the called *conservative-dissipative decomposition*, and was showed in Chapter 1. Fixed a type $\mathbf{t}$, any quasi-homogeneous vector field of a given degree k can be decomposed in a unique way as a sum of two quasi-homogeneous vector fields with the same type and degree that the original vector field: one with null divergence and other with divergence equal to the original vector field. This decomposition, generalizes those given, for the homogeneous case, by Baider [21] and Collins [39]. The second one, is derived from the above mentioned but it introduces a new component. This new component is a multiple of the quasi-homogeneous term of the vector field with lower degree; i.e., given a type $\mathbf{t}$, any quasi-homogeneous vector field of a given degree k can be decomposed, in a unique way, as a sum of three quasi-homogeneous vector fields.

In summary, this chapter is structured as follows: in the next section we present the decompositions above described. In section 3, we apply these decompositions to obtain a normal form, under equivalence, for vector fields whose quasi-homogeneous lower degree term is a Hamiltonian, i.e., $\mathbf{F}_r = \mathbf{X}_h$ and h only has simple factors in its factorization on $\mathbb{C}[x, y]$. We show this normal form in the main result of this chapter, Theorem 3.3.93. In section 4 we calculate, using this theorem, the normal form of some families of degenerate vector fields.

3.2 Decompositions of a quasi-homogeneous vector field.

The following proposition provides the decomposition of any quasi-homogeneous vector field. This decomposition was showed in Chapter 1, but we show it because it will be used along this and subsequent chapters.

Proposition 3.2.71. *(*Conservative-dissipative decomposition*)*

Assume that $\mathbf{P}_k \in \mathcal{Q}_k^{\mathbf{t}}$, *then there exist unique polynomials* $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$ *and* $h_{k+|\mathbf{t}|} \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$ *such that:*

$$\mathbf{P}_k = \mathbf{X}_{h_{k+|\mathbf{t}|}} + \mu_k \mathbf{D}_0,$$

where $h_{k+|\mathbf{t}|} = \frac{1}{k+|\mathbf{t}|}\left(\mathbf{D}_0 \wedge \mathbf{P}_k\right)$ *and* $\mu_k = \frac{1}{k+|\mathbf{t}|} div(\mathbf{P}_k)$.

Next, we show some technical lemmas that will allow us to describe the new decomposition.

Lemma 3.2.72. *Given $p \in \mathcal{P}_k^t$, it is verified*

a) $[\mathbf{X}_p, \mathbf{X}_h] = \mathbf{X}_f$ *with* $f = \nabla p \cdot \mathbf{X}_h \in \mathcal{P}_{r+k}^t$.

b) $p\mathbf{X}_h = \mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0$ *with* $\tilde{h} = \frac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|}ph$ *and* $\tilde{\mu} = \frac{1}{r+k+|\mathbf{t}|}\nabla p\mathbf{X}_h$.

Proof.

a) $[\mathbf{X}_p, \mathbf{X}_h] = \begin{pmatrix} p_{yx}h_y - p_{yy}h_x - h_{yx}p_y + h_{yy}p_x \\ -p_{xx}h_y + p_{xy}h_x + h_{xx}p_y - h_{xy}p_x \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}(p_x h_y - p_y h_x) \\ -\frac{\partial}{\partial x}(p_x h_y - p_y h_x) \end{pmatrix} = -\mathbf{X}_{p_x h_y - p_y h_x} = \mathbf{X}_{\nabla p \cdot \mathbf{X}_h}$.

b) The conservative part of $p\mathbf{X}_h$ is

$$\tilde{h} = \frac{1}{r+k+|\mathbf{t}|}[\mathbf{D}_0 \wedge (\mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0)] = \frac{1}{r+k+|\mathbf{t}|}[\mathbf{D}_0 \wedge (p\mathbf{X}_h)] = \frac{r+|\mathbf{t}|}{r+k+|\mathbf{t}|}ph,$$

and the dissipative part is

$$\tilde{\mu} = \frac{1}{r+k+|\mathbf{t}|}div(\mathbf{X}_{\tilde{h}} + \tilde{\mu}\mathbf{D}_0) = \frac{1}{r+k+|\mathbf{t}|}div(p\mathbf{X}_h) = \frac{1}{r+k+|\mathbf{t}|}\nabla p\mathbf{X}_h.$$

■

In order to give a second decomposition for quasi-homogeneous vector fields, we will show that the space $\mathcal{Q}_k^{\mathbf{t}}$ can be decomposed as a direct sum of three subspaces. To this end, let us define,

$$h\mathcal{P}_{k-r}^{\mathbf{t}} = \{h(x,y)\gamma(x,y) \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}} : \gamma \in \mathcal{P}_{k-r}^{\mathbf{t}}\}, \tag{3.2.3}$$

and we denote by $\Delta_{k+|\mathbf{t}|}$ a complementary subespace of $h\mathcal{P}_{k-r}^{\mathbf{t}}$, i.e., $\mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}} = \Delta_{k+|\mathbf{t}|} \oplus h\mathcal{P}_{k-r}^{\mathbf{t}}$.

Also, let us define the following subspaces,

- $\mathcal{F}_k^t := \left\{\lambda \cdot \mathbf{F}_r \in \mathcal{Q}_k^{\mathbf{t}} / \lambda \in \mathcal{P}_{k-r}^t\right\}$.
- $\mathcal{D}_k^t := \{\eta \cdot \mathbf{D}_0 \in \mathcal{Q}_k^{\mathbf{t}} / \eta \in \mathcal{P}_k^t\}$.
- $\mathcal{C}_k^t := \{\mathbf{X}_g \in \mathcal{Q}_k^{\mathbf{t}} / g \in \Delta_{k+|\mathbf{t}|}\}$.

Proposition 3.2.73. *Assume that* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ *and* $h \in \mathcal{P}^t_{r+|\mathbf{t}|}\setminus\{0\}$. *Then* $\mathcal{Q}^t_k = \mathcal{C}^t_k \oplus \mathcal{D}^t_k \oplus \mathcal{F}^t_k$

Proof. Obviously $\mathcal{C}^t_k + \mathcal{D}^t_k + \mathcal{F}^t_k \subset \mathcal{Q}^t_k$. Moreover

- $\mathcal{D}^t_k \cap \mathcal{F}^t_k = \{0\}$. In fact, if $\mathbf{P}_k \in \mathcal{D}^t_k \cap \mathcal{F}^t_k$, then there exist $\lambda \in \mathcal{P}^t_{k-r}$ and $\eta \in \mathcal{P}^t_k$ verifying $\mathbf{P}_k = \lambda\mathbf{F}_r = \eta\mathbf{D}_0$. Therefore $0 = (\eta\mathbf{D}_0) \wedge \mathbf{D}_0 = \mathbf{P}_k \wedge \mathbf{D}_0 = \lambda\mathbf{F}_r \wedge \mathbf{D}_0 = \frac{r+|\mathbf{t}|}{k+|\mathbf{t}|}\lambda h$ and, as $h \neq 0$, $\lambda = 0$ and consequently $\mathbf{P}_k = \mathbf{0}$.

- $(\mathcal{D}^t_k + \mathcal{F}^t_k) \cap \mathcal{C}^t_k = \{0\}$, otherwise, let $\mathbf{P}_k \in \mathcal{D}^t_k + \mathcal{F}^t_k$, then there exist $\lambda \in \mathcal{P}^t_{k-r}$ and $\eta \in \mathcal{P}^t_k$ verifying $\mathbf{P}_k = \eta\mathbf{D}_0 + \lambda\mathbf{F}_r$. By other hand, $\mathbf{P}_k \in \mathcal{C}^t_k$, then there exists $g \in \Delta_{k+|\mathbf{t}|}$, such that $\mathbf{P}_k = \mathbf{X}_g$. Therefore

  $(k+|\mathbf{t}|)g = \mathbf{D}_0 \wedge \mathbf{X}_g = \mathbf{D}_0 \wedge \mathbf{P}_k = \mathbf{D}_0 \wedge (\eta\mathbf{D}_0 + \lambda\mathbf{F}_r) = (r+|\mathbf{t}|)\lambda h.$

  Hence $g \in \Delta_{k+|\mathbf{t}|} \cap h\mathcal{P}^{\mathbf{t}}_{k-r}$ and we can conclude $\lambda = 0$. In consequence $g = 0$, and $\mathbf{P}_k = \mathbf{0}$.

Now only remains to prove $\mathcal{Q}^t_k \subset \mathcal{C}^t_k + \mathcal{D}^t_k + \mathcal{F}^t_k$.
We consider $\mathbf{P}_k \in \mathcal{Q}^t_k$, from Proposition 1.2.18, $\mathbf{P}_k = \mathbf{X}_{h_{k+|\mathbf{t}|}} + \mu_k\mathbf{D}_0$ with $\mu_k \in \mathcal{P}^t_k$ and $h_{k+|\mathbf{t}|} \in \mathcal{P}^t_{k+|\mathbf{t}|}$. Since $\mathcal{P}^t_{k+|\mathbf{t}|} = \Delta_{k+|\mathbf{t}|} \oplus h\mathcal{P}^{\mathbf{t}}_{k-r}$, we can express $h_{k+|\mathbf{t}|} = g + \lambda h$ with $g \in \Delta_{k+|\mathbf{t}|}$ and $\lambda \in \mathcal{P}^t_{k-r}$. Therefore $\mathbf{P}_k = \mathbf{X}_g + \mathbf{X}_{\lambda h} + \mu_k\mathbf{D}_0$. By other hand, from Lemma 3.2.72 b) we know that $\lambda\mathbf{X}_h = \mathbf{X}_{\frac{r+|\mathbf{t}|}{k+|\mathbf{t}|}\lambda h} + \frac{1}{k+|\mathbf{t}|}\nabla(\lambda\mathbf{X}_h)\mathbf{D}_0$, that is, $\mathbf{X}_{\lambda h} = \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mathbf{X}_h - \frac{1}{r+|\mathbf{t}|}(\nabla\lambda\mathbf{X}_h)\mathbf{D}_0$. Hence

$$\begin{aligned}\mathbf{P}_k &= \mathbf{X}_g + \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mathbf{X}_h + \left(\mu_k - \frac{1}{r+|\mathbf{t}|}(\nabla\lambda\mathbf{X}_h)\right)\mathbf{D}_0 = \\ & \mathbf{X}_g + \left(\mu_k - \frac{1}{r+|\mathbf{t}|}(\nabla\lambda\mathbf{X}_h) - \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mu\right)\mathbf{D}_0 + \frac{k+|\mathbf{t}|}{r+|\mathbf{t}|}\lambda\mathbf{F}_r.\end{aligned} \tag{3.2.4}$$

■

This proposition allows us to enunciate the following result.

Proposition 3.2.74. *Assume that* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ *and* $h \neq 0$. *For any* $\mathbf{P}_k \in \mathcal{Q}^{\mathbf{t}}_k$, *there exist unique polynomials* $g \in \Delta_{k+|\mathbf{t}|}$, $\eta \in \mathcal{P}^{\mathbf{t}}_k$ *and* $\lambda \in \mathcal{P}^{\mathbf{t}}_{k-r}$, *such that*

$$\mathbf{P}_k = \mathbf{X}_g + \eta\mathbf{D}_0 + \lambda\mathbf{F}_r, \tag{3.2.5}$$

where $g = \frac{Proy_{\Delta_{k+|\mathbf{t}|}}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{k+|\mathbf{t}|}$, $\lambda = \frac{Proy_{h\mathcal{P}^{\mathbf{t}}_{k-r}}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{(r+|\mathbf{t}|)h}$, *and* $\eta = \frac{\mathrm{div}(\mathbf{P}_k) - \nabla\lambda\mathbf{F}_r - \lambda\mathrm{div}(\mathbf{F}_r)}{k+|\mathbf{t}|}$.

Proof. The existence and uniqueness is proved in Lemma 3.2.73, only remains to find the expressions of g, η and λ.

$$\mathbf{D}_0 \wedge \mathbf{P}_k = \mathbf{D}_0 \wedge (\mathbf{X}_g + \eta\mathbf{D}_0 + \lambda\mathbf{F}_r) = (k + |\mathbf{t}|)g + (r + |\mathbf{t}|)\lambda h$$

Therefore $g = \frac{Proy_{\Delta_{k+|\mathbf{t}|}}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{k+|\mathbf{t}|}$ and $\lambda = \frac{Proy_{h\mathcal{P}^{\mathbf{t}}_{k-r}}(\mathbf{D}_0 \wedge \mathbf{P}_k)}{(r+|\mathbf{t}|)h}$. From (3.2.5), $\mathrm{div}(\mathbf{P}_k) = (k + |\mathbf{t}|)\eta + \nabla\lambda\mathbf{F}_r + \lambda\mathrm{div}(\mathbf{F}_r)$, that is, $\eta = \frac{\mathrm{div}(\mathbf{P}_k) - \nabla\lambda\mathbf{F}_r - \lambda\mathrm{div}(\mathbf{F}_r)}{k+|\mathbf{t}|}$ ■

Remark 12. *Notice that λ is polynomial because the numerator of λ is the projection of $\mathbf{D}_0 \wedge \mathbf{P}_k$ over $h\mathcal{P}^{\mathbf{t}}_{k-r}$ and consequently, it is multiple of h.*

3.3 Normal Form under $\mathcal{C}^\infty$-equivalence.

In the applications, when one tries of determining a normal form of system (3.1.2), it is very important to reduce the lowest-order term $\mathbf{F}_r$ to an adequate form. When this has been done, system (3.1.2) is called a 0-th order normal form. The calculation of such 0-th order normal form is not a trivial task since this involves an adequate selection of the type $\mathbf{t}$. This selection is very important because the lowest-order quasi-homogeneous term $\mathbf{F}_r$ defines the homological operator, and it determines the further simplifications that can be reached in the normal form. Nevertheless, this nontrivial question has not effect from a theoretical perspective. In this section we provides a normal form under conjugantion and equivalence, by making simplifications in the quasi-homogeneous terms with degree greater than r. The 0-th order normal form was treated in the section 2.4 of the previous chapter.

To describe the normal form it is necessary to focus on the homological operators described in (2.3.13) and (2.3.16),

$$\begin{aligned} \mathbf{L}_{r+k} \;:\; & \mathcal{Q}^{\mathbf{t}}_k \longrightarrow \mathcal{Q}^{\mathbf{t}}_{r+k} \quad \text{(homological operator for conjugation)} \\ & \mathbf{P}_k \to \mathbf{L}_{r+k}(\mathbf{P}_k) = [\mathbf{P}_k, \mathbf{F}_r] \\ \mathcal{L}_{r+k} \;:\; & \mathcal{Q}^{\mathbf{t}}_k \times \mathrm{Cor}(\ell_k) \longrightarrow \mathcal{Q}^{\mathbf{t}}_{r+k} \quad \text{(homological operator under equivalence)} \\ & (\mathbf{P}_k, \mu_k) \to \mathcal{L}_{r+k}(\mathbf{P}_k, \mu_k) = \mu_k\,\mathbf{F}_r + [\mathbf{P}_k, \mathbf{F}_r] \end{aligned}$$

The following theorem describes a normal form, under conjugation and equivalence, for system (3.1.2).

Theorem 3.3.75. *A formal normal form for system (3.1.2) is given by*

$$\dot{\mathbf{x}} = \sum_{j\geq 1} \mathbf{G}_{r+j}(\mathbf{x}), \ \ \textit{with } \mathbf{G}_r = \mathbf{F}_r,$$

with $\mathbf{G}_{r+j} \in \mathrm{Cor}(\mathbf{L}_{r+j})$ *in the case of* $\mathcal{C}^\infty$*-conjugation, and* $\mathbf{G}_{r+j} \in \mathrm{Cor}(\mathcal{L}_{r+j})$ *in the case of* $\mathcal{C}^\infty$*-equivalence, for* $j \geq 1$.

At this point, we want to provide a theory that allows us to reduce the calculations of the co-ranges of the homological operators and giving an expression of them in a simpler way. Consider the system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) := \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \tag{3.3.6}$$

where $\mathbf{F}_r(\mathbf{x}) = \mathbf{X}_h + \mu\mathbf{D}_0$ with $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ and $\mu \in \mathcal{P}^{\mathbf{t}}_r$.

For calculating a normal form of system (3.3.6) under equivalence, it is necessary to calculate a complementary subspace of the range of the homological operator $\mathcal{L}_{r+k}$ given in (2.3.16). For this task, we need to define the following linear operators,

$$\begin{aligned} \ell_k^{(c)} \ &: \ \mathcal{P}^{\mathbf{t}}_{k-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_k \\ & \quad \mu_{k-r} \longrightarrow \nabla\mu_{k-r} \cdot \left(\mathbf{X}_h + (1 - \frac{r+|\mathbf{t}|}{k})\mu\mathbf{D}_0 \right). \end{aligned}$$

Remark 13. *Observe that the operators* $\ell_k^{(c)}$ *agree with the linear operator given in (2.3.15) in the case* $\mu \equiv 0$.

To achieve this goal we need the following lemma, which will be useful later to characterize the homological operator.

Lemma 3.3.76. *Assume that* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$ *with* $h \neq 0$. *Given* $p \in \mathcal{P}^{\mathbf{t}}_k$, *it is verified*

a) $[\mathbf{X}_p, \mathbf{F}_r] = \mathbf{X}_g + \eta\mathbf{D}_0 + \lambda \cdot \mathbf{F}_r$,

with $g = Proy_{\Delta_{r+k}}(\ell^{(c)}_{r+k}(p))$, $\quad \lambda = \frac{r+k}{r+|\mathbf{t}|}\frac{Proy_{h\mathcal{P}^{\mathbf{t}}_{k-r}}(\ell^{(c)}_{r+k}(p))}{h}$ *and*

$\eta = -\frac{(r+2k-|\mathbf{t}|)\nabla\mu\mathbf{X}_p + \nabla\lambda\cdot\mathbf{X}_h + (r+k)\lambda\mu}{r+k}$.

b) $[p\mathbf{D}_0, \mathbf{F}_r] = \eta\mathbf{D}_0 + \lambda \cdot \mathbf{F}_r$, *where* $\eta = \nabla p \cdot \mathbf{F}_r = \ell_{k+r}(p)$ *and* $\lambda = -rp$.

c) $[p\mathbf{F}_r, \mathbf{F}_r] = \lambda \cdot \mathbf{F}_r$, *where* $\lambda = \nabla p \cdot \mathbf{F}_r = \ell_{k+r}(p)$.

Proof.

a) To show the expressions of g and λ we will use Lemma 3.2.72. In first time, we can develop $[\mathbf{X}_p, \mathbf{F}_r]$ as follows,

$$\begin{aligned} [\mathbf{X}_p, \mathbf{F}_r] &= [\mathbf{X}_p, \mathbf{X}_h + \mu\mathbf{D}_0] = [\mathbf{X}_p, \mathbf{X}_h] + [\mathbf{X}_p, \mu\mathbf{D}_0] \\ &= \mathbf{X}_{\nabla p \mathbf{X}_h} - [\mu\mathbf{D}_0, \mathbf{X}_p] \\ &= \mathbf{X}_{\nabla p \mathbf{X}_h} - (\nabla\mu\mathbf{X}_p)\mathbf{D}_0 + \mu[\mathbf{X}_p, \mathbf{D}_0] \\ &= \mathbf{X}_{\nabla p \mathbf{X}_h} - (\nabla\mu\mathbf{X}_p)\mathbf{D}_0 + (k - |\mathbf{t}|)\mu\mathbf{X}_p. \end{aligned} \tag{3.3.7}$$

Therefore, by one hand, using (3.3.7), we obtain,

$$\begin{aligned} \mathbf{D}_0 \wedge [\mathbf{X}_p, \mathbf{F}_r] &= (r+k)\nabla p \cdot \mathbf{X}_h + k(k - |\mathbf{t}|)\mu p \\ &= (r+k)\ell^{(c)}_{r+k}(p). \end{aligned} \tag{3.3.8}$$

By other hand,

$$\mathbf{D}_0 \wedge (\mathbf{X}_g + \eta\mathbf{D}_0 + \lambda \cdot \mathbf{F}_r) = (r+k)g + (r + |\mathbf{t}|)\lambda h. \tag{3.3.9}$$

From (3.3.8) and (3.3.9) we get,

$$g = Proy_{\Delta_{r+k}}\left(\ell^{(c)}_{r+k}(p) - \frac{r + |\mathbf{t}|}{r+k}\lambda h\right) = Proy_{\Delta_{r+k}}\left(\ell^{(c)}_{r+k}(p)\right).$$

From (3.3.8) and (3.3.9) we have,

$$\lambda = \frac{r+k}{(r+|\mathbf{t}|)h} Proy_{h\mathcal{P}^{\mathbf{t}}_{k-r}}\left(\ell^{(c)}_{r+k}(p) - g\right) = \frac{r+k}{(r+|\mathbf{t}|)h} Proy_{h\mathcal{P}^{\mathbf{t}}_{k-r}}\left(\ell^{(c)}_{r+k}(p)\right).$$

To prove the expression of η, we consider (3.3.7), then

By one hand,

$$\begin{aligned} \operatorname{div}([\mathbf{X}_p, \mathbf{F}_r]) &= -(r+k)\nabla\mu \cdot \mathbf{X}_p - (k - |\mathbf{t}|)\nabla\mu \cdot \mathbf{X}_p \\ &= -(r + 2k - |\mathbf{t}|)\nabla\mu \cdot \mathbf{X}_p. \end{aligned} \tag{3.3.10}$$

By other hand,

$$\mathrm{div}(\mathbf{X}_g + \eta \mathbf{D}_0 + \lambda \cdot \mathbf{F}_r) = (r+k)\eta + \nabla\lambda \cdot \mathbf{X}_h + (r+k)\lambda\mu. \quad (3.3.11)$$

From (3.3.10) and (3.3.11), we obtain,

$$\eta = -\frac{(r+2k-|\mathbf{t}|)\nabla\mu\mathbf{X}_p + \nabla\lambda \cdot \mathbf{X}_h + (r+k)\lambda\mu}{r+k}.$$

b) $[p\mathbf{D}_0, \mathbf{F}_r] = (\nabla p \cdot \mathbf{F}_r)\mathbf{D}_0 + p[\mathbf{D}_0, \mathbf{F}_r] = \ell_{r+k}(p)\mathbf{D}_0 - rp\mathbf{F}_r$.

c) $[p\mathbf{X}_h, \mathbf{F}_r] = (\nabla p \cdot \mathbf{F}_r)\mathbf{F}_r + p[\mathbf{F}_r, \mathbf{F}_r] = \ell_{r+k}(p)\mathbf{F}_r$.

■

The following proposition shows a complementary subspace to the range of the homological operator, under equivalence, described above. In its proof, it is possible see that, the matrix of the homological operator has a triangular structure generated by the decomposition presented in the Proposition 3.2.74. To prove this proposition, we need the following two technical lemmas.

Lemma 3.3.77. *Consider $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ irreducible and $f \in \mathbb{C}[x,y]$ an irreducible invariant curve of $\mathbf{F}_r$. If $L_{\mathbf{F}_r}(p) \in \langle f \rangle$ then $p \in \langle f \rangle$.*

Proof. If $L_{\mathbf{F}_r}(p) = 0$ then p is a first integral of $\dot{\mathbf{x}} = \mathbf{F}_r$, and a first integral of $\mathbf{F}_r$ is vanished on any invariant curve of it, i.e., $p(\mathbf{x}) = 0$ when $f(\mathbf{x}) = 0$. Therefore, by Hilbert‘s Nullstellensatz $p \in rad\,\langle f \rangle$. Since $\langle f \rangle$ is a prime ideal, then $\langle f \rangle = rad\,\langle f \rangle$, in consequence $p \in \langle f \rangle$.

If $L_{\mathbf{F}_r}(p) \neq 0$, let $\nu \in \mathbb{C}[x,y] \setminus \{0\}$ such that $f\nu = L_{\mathbf{F}_r}(p)$. Consider $\gamma(t)$, real or complex, a solution curve of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ which is a parametrization of $f(\mathbf{x}) = 0$. Suppose also that $\lim_{t\to-\infty}\gamma(t) = 0$, (the other case $\lim_{t\to+\infty}\gamma(t) = 0$ is proved analogously). Since $p(\mathbf{0}) = 0$ then

$$\begin{aligned} p(\gamma(t)) &= p(\gamma(t)) - p(\mathbf{0}) = \int_{-\infty}^{t} \tfrac{dp(\gamma(s))ds}{ds} = \int_{-\infty}^{t} \nabla_{\mathbf{x}}\, p \cdot \mathbf{F}_r(\gamma(s))ds \\ &= \int_{-\infty}^{t} \ell_{r+k}(p)(\gamma(s))ds = \int_{-\infty}^{t} f(\gamma(s))\nu(\gamma(s))ds = 0. \end{aligned}$$

Since $f(\mathbf{x}) = 0$ is union of orbits we have $p(\mathbf{x}) = 0$ when $f(\mathbf{x}) = 0$. Therefore, by Hilbert‘s Nullstellensatz $p \in rad\,\langle f\rangle$. Since $\langle f\rangle$ is a prime ideal, then $\langle f\rangle = rad\,\langle f\rangle$, in consequence $p \in \langle f\rangle$. ∎

Lemma 3.3.78. *Consider* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ *irreducible and* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *only has simple factors in its factorization on* $\mathbb{C}[x, y]$*. If* $L_{\mathbf{F}_r}(p) \in \langle h\rangle$ *then* $p \in \langle h\rangle$*.*

Proof. Let $h = \prod_{i=1}^{s} f_i$ the decomposition in irreducible factors of h on $\mathbb{C}[x, y]$. If $L_{\mathbf{F}_r}(p) \in \langle h\rangle$ then $L_{\mathbf{F}_r}(p) \in \langle f_i\rangle$ for all $i = 1\cdots s$. As f_i is an irreducible curve of $\mathbf{F}_r$ for all $i = 1\cdots s$, by Proposition 3.3.77, $p \in \langle f_i\rangle$ for all $i = 1\cdots s$. Therefore $p \in \langle h\rangle$. ∎

Remark 14. *Observe that the hypothesis,* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ *is* irreducible, *is necessary. Let consider,*

$$\mathbf{F}_r = \begin{pmatrix} -y^2 \\ xy \end{pmatrix} = y\begin{pmatrix} -y \\ x \end{pmatrix}.$$

$\mathbf{F}_r$ *is reducible,* y *is an irreducible invariant curve of* $\mathbf{F}_r$*, and* $\nabla x \cdot \mathbf{F}_r = -y^2 \in \langle y\rangle$*. Nevertheless,* $x \notin \langle y\rangle$*.*

Proposition 3.3.79. *Assume* $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$ *irreducible. Consider* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *only has simple factors in its factorization on* $\mathbb{C}[x, y]$*. Then,*

$$\mathrm{Cor}(\mathbf{L}_{r+k}) = \mathbf{X}_{\mathrm{Cor}(\ell^{(c)}_{r+k+|\mathbf{t}|})\cap\Delta_{r+k+|\mathbf{t}|}} \oplus \mathrm{Cor}(\ell_{r+k})\mathbf{D}_0 \oplus \mathrm{Cor}(\ell_k)\mathbf{F}_r.$$

$$\mathrm{Cor}(\mathcal{L}_{r+k}) = \mathbf{X}_{\mathrm{Cor}(\ell^{(c)}_{r+k+|\mathbf{t}|})\cap\Delta_{r+k+|\mathbf{t}|}} \oplus \mathrm{Cor}(\ell_{r+k})\mathbf{D}_0.$$

Proof. We prove the case of equivalence, for the proof of the another case is sufficient consider $\nu \equiv 0$ in that follows. From Lemma 3.2.73, we know that $\mathcal{Q}_k^t = \mathcal{C}_k^t \oplus \mathcal{D}_k^t \oplus \mathcal{F}_k^t$. Therefore, the homological operator given in (2.3.16) has, taking into account the decomposition given in Proposition 3.2.74 and Lemma 3.3.76, the following form

$$\mathcal{L}_{r+k} \;:\; \left(\mathcal{C}_k^t \oplus \mathcal{D}_k^t \oplus \mathcal{F}_k^t\right) \times \mathrm{Cor}(\ell_k) \longrightarrow \mathcal{C}_{r+k}^t \oplus \mathcal{D}_{r+k}^t \oplus \mathcal{F}_{r+k}^t$$

defined as follows $\mathcal{L}_{r+k}(\mathbf{X}_g + \eta\mathbf{D}_0 + \lambda\mathbf{X}_h, \nu) =$

$$\left(\mathbf{X}_{g_{r+k+|\mathbf{t}|}} + (\eta_{r+k} + \ell_{r+k}(\eta))\mathbf{D}_0 + (\lambda_k - r\eta + \ell_k(\lambda) + \nu)\mathbf{F}_r\right),$$

where

$g_{r+k+|\mathbf{t}|} = Proy_{\Delta_{k+r+|\mathbf{t}|}}\left(\ell^{(c)}_{k+r+|\mathbf{t}|}(g)\right)$, $\lambda_k = \frac{k+r+|\mathbf{t}|}{r+|\mathbf{t}|}\frac{Proy_{h\mathcal{P}^{\mathbf{t}}_k}\left(\ell^{(c)}_{k+r+|\mathbf{t}|}(g)\right)}{h}$

and $\eta_{r+k} = -\frac{(r+2(r+k)-|\mathbf{t}|)\nabla\mu\mathbf{X}_\eta + \nabla\lambda_k\cdot\mathbf{X}_h + (2r+k)\lambda_k\mu}{2r+k}$.

Therefore, taking a suitable basis, we obtain a triangular-block matrix,

$\mathbf{X}_{g_{k+r+\lvert\mathbf{t}\rvert}}$	0	0	0	$\mathcal{C}^t_{r+k}$
$\eta_{r+k}\mathbf{D}_0$	$\ell_{r+k}(\eta)\mathbf{D}_0$	0	0	$\mathcal{D}^t_{r+k}$
$\lambda_k\mathbf{F}_r$	$-r\eta\mathbf{F}_r$	$\ell_k(\lambda)\mathbf{F}_r$	$\nu\mathbf{F}_r$	$\mathcal{F}^t_{r+k}$
$\mathbf{X}_g \in \mathcal{C}^t_k$	$\eta\mathbf{D}_0 \in \mathcal{D}^t_k$	$\lambda\mathbf{F}_r \in \mathcal{F}^t_k$	$\nu \in \mathrm{Cor}(\ell_k)$	

From Proposition 3.3.78, we can deduce that the upper left block diagonal of the above matrix has maximum range. Taking into account the structure of the above matrix we can derive the result. ■

The following theorem shows a formal normal form, under equivalence, for system (3.3.6).

Theorem 3.3.80. *Consider $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0 \in \mathcal{Q}^{\mathbf{t}}_r$ irreducible and assume that $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ only has simple factors in its factorization on $\mathbb{C}[x,y]$.*

A formal normal form, under conjugation, for system (3.3.6) is,

$$\dot{\mathbf{x}} = \mathbf{F}_r + \sum_{k\geq 1}\left(\mathbf{X}_{g_{r+k+|\mathbf{t}|}} + \eta_{r+k}\mathbf{D}_0 + \lambda_k\mathbf{F}_r\right). \tag{3.3.12}$$

where $g_{r+k+|\mathbf{t}|} \in \mathrm{Cor}(\ell^{(c)}_{r+k+|\mathbf{t}|}) \cap \Delta_{r+k+|\mathbf{t}|}$, $\eta_{r+k} \in \mathrm{Cor}(\ell_{r+k})$ and $\lambda_k \in \mathrm{Cor}(\ell_k)$.

A formal normal form, under equivalence, for system (3.3.6) is,

$$\dot{\mathbf{x}} = \mathbf{F}_r + \sum_{k\geq 1}\left(\mathbf{X}_{g_{r+k+|\mathbf{t}|}} + \eta_{r+k}\mathbf{D}_0\right). \tag{3.3.13}$$

where $g_{r+k+|\mathbf{t}|} \in \mathrm{Cor}(\ell^{(c)}_{r+k+|\mathbf{t}|}) \cap \Delta_{r+k+|\mathbf{t}|}$ and $\eta_{r+k} \in \mathrm{Cor}(\ell_{r+k})$.

3.3.1 Calculation of the normal form for a Takens-Bogdanov singularity.

In this section we obtain the ∞-order normal form under conjugation and equivalence for higher-order perturbations of non-integrable quasi-homogeneous Takens-Bogdanov singularities. With this example, we show how difficult, in terms of the calculations, is to obtain a normal form to infinite order. A Takens-Bogdanov singularity can be written as:

$$\begin{aligned} \dot{x} &= y + x f_1(x) + y f(x,y), \\ \dot{y} &= g_1(x) + y g_2(x) + y^2 g(x,y). \end{aligned} \tag{3.3.14}$$

Let us denote by:

- m, the lowest-degree in the Taylor expansion of $g_1(x)$.
- n, the minimum of the lowest-degrees of the Taylor expansions for $f_1(x)$ and $g_2(x)$.

Hence, $m = \infty$ arises if $g_1(x) \equiv 0$ and $n = \infty$ corresponds to $f_1(x) \equiv g_2(x) \equiv 0$. Then, we can write system (3.3.14) as

$$\begin{aligned} \dot{x} &= y + x^{n+1}\Psi_1(x) + y f(x,y), \\ \dot{y} &= x^n y \Phi_1(x) + x^m \Phi_3(x) + y^2 g(x,y), \end{aligned} \tag{3.3.15}$$

where $m \in \mathbf{N} \cup \{\infty\}$, $n \in \mathbb{N}_0 \cup \{\infty\}$, and $\Psi_1(x) = a + \mathcal{O}(x)$, $\Phi_1(x) = b + \mathcal{O}(x)$, $\Phi_3(x) = c + \mathcal{O}(x)$, $f(x,y) = \mathcal{O}(x,y)$, $g(x,y) = \mathcal{O}(x,y)$.

To exclude some cases corresponding to Hamiltonian lowest-degree quasi-homogeneous term, we will asume $n \neq \infty$ and $m \geq 2n+1$.

By selecting the type $\mathbf{t} = (1, n+1)$, the quoted lowest-degree quasi-homogeneous term is of degree equal to n and it is of the form,

$$\mathbf{F}_n(x,y) = \begin{cases} \begin{pmatrix} y + a x^{n+1} \\ b x^n y \end{pmatrix}, & \text{if } m \neq 2n+1; \\ \begin{pmatrix} y + a x^{n+1} \\ b x^n y + c x^{2n+1} \end{pmatrix}, & \text{if } m = 2n+1. \end{cases}$$

We can deal with both cases at once by writing

$$\mathbf{F}_n(x,y) = \begin{pmatrix} y + ax^{n+1} \\ bx^n y + Ax^{2n+1} \end{pmatrix},$$

where $A = 0$ if $m \neq 2n+1$, and $A = c$ if $m = 2n+1$.

Next, we show the decomposition (1.2.5) (conservative-dissipative sppliting), for the lowest-degree quasi-homogeneous term, $\mathbf{F}_n$,

$$\begin{aligned} h(x,y) &= \tfrac{A}{2(n+1)}x^{2(n+1)} + \left(\tfrac{b}{2(n+1)} - \tfrac{1}{2}a\right)x^{n+1}y - \tfrac{1}{2}y^2 \\ &= -\tfrac{1}{2}\left(y - \left(\tfrac{b}{2(n+1)} - \tfrac{a}{2}\right)x^{n+1}\right)^2 + \left(\tfrac{A}{2(n+1)} + \tfrac{1}{2}\left(\tfrac{b}{2(n+1)} - \tfrac{a}{2}\right)^2\right)x^{2(n+1)}, \\ \mu(x,y) &= \left(\tfrac{b}{2(n+1)} + \tfrac{1}{2}a\right)x^n. \end{aligned}$$

Again to exclude cases where $\mathbf{F}_n$ is Hamiltonian, we take $\frac{b}{2(n+1)} + \frac{1}{2}a \neq 0$.

The change of variables $u = x$, $v = y - \left(\frac{b}{2(n+1)} - \frac{a}{2}\right)x^{n+1}$ and an adequate rescaling, transform system (3.3.15) into a higher-order quasi-homogeneous perturbation of $\mathbf{G}_n := \mathbf{X}_h + \mu\mathbf{D}_0$, where

$$\begin{aligned} h(x,y) &= -\tfrac{1}{2}y^2 + \tfrac{\sigma}{2}x^{2(n+1)}, \text{ with } \sigma \in \{-1,0,1\}. \\ \mu(x,y) &= dx^n, \text{ with } d \in \mathbb{R}\setminus\{0\}, \end{aligned}$$

Hence:

$$\mathbf{G}_n(x,y) = \begin{pmatrix} y + dx^{n+1} \\ \sigma(n+1)x^{2n+1} + (n+1)dx^n y \end{pmatrix}. \tag{3.3.16}$$

Remark 15. *The case $n = 0$ corresponds to perturbations of linear systems with nonzero trace: linear focus, node and saddle with nonzero divergence.*

Firstly, we build an adequate basis for the spaces $\mathcal{P}_k^{\mathbf{t}}$. To this end, we require the following technical result:

Lemma 3.3.81. *Let us consider $k \in \mathbf{N}$. Then, there exist $k_1, k_2 \in \mathbf{N}, k_3 \in \mathbb{N}_0$, such that $k = k_1 + (n+1)k_2 + 2(n+1)k_3$, where $0 \leq k_1 \leq n$, $k_2 = 0,1$, and $k_3 \in \mathbb{N}_0$. Moreover, k_1, k_2, k_3 are unique.*

Proof. It is a simple matter to show that $k_3 = \left\lfloor \frac{k}{2(n+1)} \right\rfloor$ (the quotient of the division $\frac{k}{2(n+1)}$), where $\lfloor\cdot\rfloor$ denotes the *floor function*. Then, $2k_3 + k_2$ and

k_1 are, respectively, the quotient and the rest of the division $\frac{k}{n+1}$. Hence, $k_2 = \left\lfloor \frac{k}{n+1} \right\rfloor - 2\left\lfloor \frac{k}{2(n+1)} \right\rfloor$ and $k_1 = k - \left\lfloor \frac{k}{n+1} \right\rfloor (n+1)$. ∎

Lemma 3.3.82. *Let $k \in \mathbf{N}$, and write $k = k_1 + (n+1)k_2 + 2(n+1)k_3$ as in Lemma 3.3.81. Then:*

- *If $k_2 = 1$, a basis of $\mathcal{P}_k^{\mathbf{t}}$ is,*

$$\mathfrak{B} = \left\{ x^{k-2i(n+1)}h^i, \left(y - x^{n+1}\right) x^{k-(2i+1)(n+1)}h^i : i = 0, \ldots, k_3 \right\}.$$

- *If $k_2 = 0$, a basis of $\mathcal{P}_k^{\mathbf{t}}$ is,*

$$\mathfrak{B} \setminus \left\{ \left(y - x^{n+1}\right) x^{k-(2k_3+1)(n+1)}h^{k_3} \right\}$$

.

Proof. Let us take $p_k \in \mathcal{P}_k^{\mathbf{t}}$. Then, we can write $p_k(x,y)$ as follows,

$$p_k(x,y) = x^{k_1} \sum_{j=0}^{2k_3+k_2} \alpha_j x^{(n+1)(2k_3+k_2-j)} y^j.$$

In the above sum, we will consider separately the terms with even/odd powers of y. So:

$$\begin{aligned}
p_k(x,y) &= x^{k_1} \sum_{i=0}^{k_3} \alpha_{2i} x^{(n+1)(2k_3+k_2-2i)} y^{2i} + x^{k_1} y \sum_{i=0}^{k_3+k_2-1} \alpha_{2i+1} x^{(n+1)(2k_3+k_2-(2i+1))} y^{2i} \\
&= \sum_{i=0}^{k_3} \alpha_{2i} x^{k-2i(n+1)} y^{2i} + y \sum_{i=0}^{k_3+k_2-1} \alpha_{2i+1} x^{k-(2i+1)(n+1)} y^{2i} \\
&= \sum_{i=0}^{k_3+k_2-1} \left(\alpha_{2i} + \alpha_{2i+1}\right) x^{k-2i(n+1)} h^i + \left(y - x^{n+1}\right) \sum_{i=0}^{k_3+k_2-1} \alpha_{2i+1} x^{k-(2i+1)(n+1)} h^i \\
&\quad + \alpha_{2k_3} x^{k-2k_3(n+1)} h^{k_3},
\end{aligned}$$

where the last term: $\alpha_{2k_3} x^{k-2k_3(n+1)} h^{k_3}$ must be dropped if $k_2 = 1$. The result follows by substituting $y^2 = \sigma x^{2(n+1)} - 2h$. ∎

Lemma 3.3.83. *Let us consider $k \in \mathbf{N}$, and write $k = k_1+(n+1)k_2+2(n+1)k_3$ as in Lemma 3.3.81. Then, the matrix of the linear operator ℓ_k, associated to the bases given in Lemma 3.3.82 for $\mathcal{P}_k^{\mathbf{t}}$ and $\mathcal{P}_{k+n}^{\mathbf{t}}$, is*

$$\begin{array}{|c|c|c|c|c|}
\hline
A_0 & & & & \\
\hline
B_0 & A_1 & & & \\
\hline
 & B_1 & \ddots & & \\
\hline
 & & \ddots & A_{k_3-1} & \\
\hline
 & & & B_{k_3-1} & A_{k_3} \\
\hline
 & & & & B_{k_3} \\
\hline
\end{array},$$

where

- $A_i = \left(\begin{array}{cc} (d+1)k - 2i(n+1) & (\sigma-1)(k-2i(n+1)) \\ k-2i(n+1) & (d-1)k+2i(n+1) \end{array} \right)$ *and*
  $B_i = \left(\begin{array}{cc} 0 & -2\left(k-(2i+1)(n+1)\right)_{2\times 2} \\ 0 & 0 \end{array} \right)_{2\times 2}$, *for $i = 0, \ldots, k_3-1$,*
- ▶ $A_{k_3} = (dk)_{1\times 1}$ *and $B_{k_3}$is the empty matrix, if $k_1 = k_2 = 0$,*
  ▶ $A_{k_3} = \left(\begin{array}{c} (d+1)k - 2k_3(n+1) \\ k_1 \end{array} \right)_{2\times 1}$ *and B_{k_3} is the empty matrix if $k_2 = 0$, $k_1 > 0$,*
  ▶ $A_{k_3} = \left(\begin{array}{cc} (d+1)k - 2k_3(n+1) & (\sigma-1)(k-2k_3(n+1)) \\ k-2k_3(n+1) & (d-1)k+2k_3(n+1) \end{array} \right)_{2\times 2}$, *if $k_2 = 1$; and B_{k_3} is the empty matrix if $k_1 = 0$ and $k_2 = 1$, or $B_{k_3} = (0 \quad -2k_1)_{1\times 2}$ if $k_1 > 0$ and $k_2 = 1$.*

Proof. It is a simple matter to show that:

$$\begin{aligned}
\nabla x \cdot \mathbf{G}_n &= y + dx^{n+1} = \left(y - x^{n+1}\right) + (d+1)x^{n+1}, \\
\nabla h \cdot \mathbf{G}_n &= 2(n+1)dx^n h, \\
\nabla\left(y - x^{n+1}\right) \cdot \mathbf{G}_n &= (n+1)(d-1)x^n\left(y - x^{n+1}\right) + (n+1)(\sigma-1)x^{2n+1} \\
\left(y - x^{n+1}\right)\nabla x \cdot \mathbf{G}_n &= (d-1)x^{n+1}\left(y - x^{n+1}\right) + (\sigma-1)x^{2(n+1)} - 2h.
\end{aligned}$$

Let us assume $k_2 = 1$. According to Lemma 3.3.82, we can write the elements of $\mathcal{P}_k^{\mathbf{t}}$ as:

$$p_k(x,y) = \sum_{i=0}^{k_3} a_i x^{k-2i(n+1)} h^i + \sum_{i=0}^{k_3} b_i \left(y - x^{n+1}\right) x^{k-(2i+1)(n+1)} h^i.$$

After some operations we get:

$$
\begin{aligned}
\ell_k\left(p_k\right) &= k\left((d+1)a_0+(\sigma-1)b_0\right)x^{k+n}+k\left(a_0+(d-1)b_0\right)\left(y-x^{n+1}\right)x^{k-1}\\
&+ \sum_{i=1}^{k_3-1}\left(-2\left(k-(2i-1)(n+1)\right)b_{i-1}+\left((d+1)k-2i(n+1)\right)a_i\right.\\
&+ \left.(\sigma-1)(k-2i(n+1))b_i\right)\times x^{k-2i(n+1)+n}h^i\\
&+ \left(y-x^{n+1}\right)\sum_{i=1}^{k_3-1}\left(\left(k-2i(n+1)\right)a_i\right.\\
&+ \left.\left((d-1)k+2i(n+1)\right)b_i\right)x^{k-2i(n+1)-1}h^i\\
&+ \left(-2\left(k-(2k_3-1)(n+1)\right)b_{k_3-1}+\left((d+1)k-2k_3(n+1)\right)a_{k_3}\right.\\
&+ \left.(\sigma-1)(k-2k_3(n+1))b_{k_3}\right)x^{k-2k_3(n+1)+n}h^{k_3}\\
&+ \left(\left(k-2k_3(n+1)\right)a_{k_3}+\left((d-1)k\right.\right.\\
&+ \left.\left.2k_3(n+1)\right)b_{k_3}\right)\left(y-x^{n+1}\right)x^{k-2k_3(n+1)-1}h^{k_3}\\
&- 2k_1b_{k_3}x^{k-2(k_3+1)(n+1)+n}h^{k_3+1}.
\end{aligned}
$$

The proof in this case is completed by looking at the basis of $\mathcal{P}^{\mathbf{t}}_{k+n}$, described in Lemma 3.3.82.

In the case $k_2=0$, the term $b_{k_3}\left(y-x^{n+1}\right)x^{k-(2k_3+1)(n+1)}h^{k_3}$ in $p_k(x,y)$ must be omitted, and we can exploit the above computations by putting $b_{k_3}=0$. ∎

In the following two proposition we describe the corange of ℓ_k and $\ell_k^{(c)}$.

Proposition 3.3.84. *Let $k\in\mathbf{N}$, and write $k=k_1+(n+1)k_2+2(n+1)k_3$ as in Lemma 3.3.81. Let us assume that $\sigma\in\{-1,0\}$; or $\sigma=1$, $|d|\neq 1-\frac{2i(n+1)}{k}$, for $i=0,\ldots,k_3+k_2-1$. Then,* $\operatorname{Ker}\left(\ell_{k+n}\right)=\{0\}$. *Moreover, a complementary subspace to* $\operatorname{Range}\left(\ell_{k+n}\right)$ *is:*

$$
\operatorname{Cor}\left(\ell_{k+n}\right) = \begin{cases} \{0\}, & \textit{if} \quad k_1=0,\\ \operatorname{Span}\left\{x^{k_1+n-k_2(n+1)}h^{k_3+k_2}\right\}, & \textit{if} \quad k_1>0. \end{cases}
$$

Proof. From Lemma 3.3.83, we have $\det\left(A_i\right)=k^2d^2-\sigma\left(k-2i(n+1)\right)^2$, for $i=0,\ldots,k_3+k_2-1$. Hence, $\det\left(A_i\right)\neq 0$, if $\sigma=-1$; and also if $\sigma=1$, $|d|\neq 1-\frac{2i(n+1)}{k}$. Consequently:

- If $k_2=1$, $k_1>0$, then $\operatorname{Cor}\left(\ell_{k+n}\right)=\operatorname{Span}\left\{x^{k_1-1}h^{k_3+1}\right\}$.

- If $k_2 = 1$, $k_1 = 0$, then B_3 is the empty matrix and so $\text{Cor}\,(\ell_{k+n}) = \{0\}$.
- If $k_2 = 0$, $k_1 > 0$, then B_3 is the empty matrix, $A_{k_3} = \begin{pmatrix} (d+1)k - 2k_3(n+1) \\ k_1 \end{pmatrix}_{2\times 1}$, and $\text{Cor}\,(\ell_{k+n}) = \text{Span}\left\{x^{k_1+n}h^{k_3}\right\}$ is a complementary subspace to $\text{Range}\,(\ell_{k+n})$.
- If $k_1 = k_2 = 0$, then B_3 is the empty matrix, and $A_{k_3} = [dk]_{1\times 1}$. As $d \neq 0$, we find $\text{Range}\,(\ell_{k+n}) = \mathcal{P}^{\mathbf{t}}_{k+n}$ and $\text{Cor}\,(\ell_{k+n}) = \{0\}$.

■

Proposition 3.3.85. *Let us consider $k \in \mathbf{N}$, and assume that $\sigma \in \{-1, 0\}$; or $\sigma = 1$, $|d| \neq 1 + \frac{2(n+1)}{k}$. Then*

$$\text{Ker}\left(\ell^{(c)}_{k+n+2}\right) = \text{Cor}\left(\ell^{(c)}_{k+n+2}\right) = \{0\}\,.$$

Proof. Let us take the following bases for the complementary subspaces to Δ_{k+n+2} and Δ_{k+2n+2}:

$$\begin{aligned} \Delta_{k+n+2} &= \text{Span}\left\{x^{k+n+2}, x^{k+1}\left(y - x^{n+1}\right)\right\}, \\ \Delta_{k+2n+2} &= \text{Span}\left\{x^{k+2(n+1)}, x^{k+n+1}\left(y - x^{n+1}\right)\right\}. \end{aligned}$$

Then, we write the elements of Δ_{k+n+2} as:

$$g = a_0 x^{k+n+2} + a_1 x^{k+1}\left(y - x^{n+1}\right) = (a_0 - a_1)x^{k+n+2} + a_1 x^{k+1} y,$$

and, after some computations, we get

$$\begin{aligned} \ell^{(c)}_{k+2n+2}(g) = \ & \text{Proj}_{\Delta_{k+2n+2}}\Big((k+n+2)\left(\left(d\tfrac{k}{k+2(n+1)} + 1\right)a_0 + (\sigma - 1)a_1\right)x^{k+2(n+1)} \\ & +(k+n+2)\left(a_0 + \left(d\tfrac{k}{k+2(n+1)} - 1\right)a_1\right)x^{k+n+1}\left(y - x^{n+1}\right) \\ & -2(k+1)a_1 x^k h\Big). \end{aligned}$$

In this way, the matrix for the linear transformation $\ell^{(c)}_{r+2n+2}$ is:

$$(k+n+2)\begin{pmatrix} d\frac{k}{k+2(n+1)} + 1 & \sigma - 1 \\ 1 & d\frac{k}{k+2(n+1)} - 1 \end{pmatrix}.$$

To obtain the result, it is enough to observe that its determinant: $d^2\frac{k^2}{(k+2(n+1))^2} - \sigma$ is nonzero if $\sigma \in \{-1, 0\}$; or $\sigma = 1$, $|d| \neq 1 + \frac{2(n+1)}{k}$. ■

Theorem 3.3.86. *Let us consider the system:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{G}(x,y) = \mathbf{G}_n(x,y) + H.O.T., \tag{3.3.17}$$

where $\mathbf{G}_n$ is given in (3.3.16), and H.O.T. denotes quasi-homogeneous terms of type $\mathbf{t} = (1, n+1)$ with degree higher than n. Let us assume that $d \in \mathbb{R}\setminus\{0\}$, $n \in \mathbb{N}_0$, and also that $\sigma \in \{-1, 0\}$; or $\sigma = 1$, $|d| \neq 1 + \frac{2(n+1)}{k}$, $|d| \neq 1 - \frac{2i(n+1)}{k}$, for all $i = 0, \ldots, \lfloor \frac{k}{n+1} \rfloor - \left\lfloor \frac{k}{2(n+1)} \right\rfloor - 1$, and $k \in \mathbf{N}$.

Then, the unique normal form under conjugation for system (3.3.17) is:

$$\begin{aligned} \dot{\mathbf{x}} = \mathbf{G}_n(\mathbf{x}) &+ \sum_{j=n+1}^{2n} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j\neq n}}^{2n} \alpha_j^{(l)} x^j h^l \mathbf{D}_0 + \sum_{j=1}^{2n} \beta_j^{(0)} x^j \mathbf{G}_n \\ &+ \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j\neq n}}^{2n} \beta_j^{(l)} x^j h^l \mathbf{G}_n, \end{aligned} \tag{3.3.18}$$

and the unique normal form under equivalence is:

$$\dot{\mathbf{x}} = \mathbf{G}_n + \sum_{j=n+1}^{2n} \gamma_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{\substack{j=0 \\ j\neq n}}^{2n} \gamma_j^{(l)} x^j h^l \mathbf{D}_0. \tag{3.3.19}$$

Proof. From Proposition 3.3.85 (recall that we assume $\sigma \in \{-1, 0\}$; or $\sigma = 1$, $|d| \neq 1 + \frac{2(n+1)}{k}$ for all $k \in \mathbf{N}$), we get $\mathrm{Cor}\left(\ell^{(c)}_{k+2n+2}\right) = \{0\}$. Moreover, if $\sigma \in \{-1, 0\}$; or $\sigma = 1$, $|d| \neq 1 - \frac{2i(n+1)}{k}$, for $i = 0, \ldots, \lfloor \frac{k}{n+1} \rfloor - \left\lfloor \frac{k}{2(n+1)} \right\rfloor - 1$, for all $k \in \mathbf{N}$; from Proposition 3.3.84 we get $\mathrm{Ker}\,(\ell_k) = \{0\}$ for all $k \in \mathbf{N}$, and $\mathrm{Cor}\,(\ell_{k+n}) = \mathrm{Span}\left\{x^{k_1+n-k_2(n+1)} h^{k_3+k_2}\right\}$ if $k_1 > 0$, and zero otherwise.

The complementary subspaces $\mathrm{Cor}\,(\ell_{k+n})$ are spanned by the monomials shown in the following table.:

k	1	$\cdots$	n	$n+1$	$n+2$	$\cdots$	$2n+1$	$2(n+1)$	$2n+3$	$\cdots$
$\mathrm{Cor}\,(\ell_{k+n})$	x^{n+1}	$\cdots$	x^{2n}	0	h	$\cdots$	$x^{n-1}h$	0	$x^{n+1}h$	$\cdots$

Table 3.1: Table of the co-ranges of ℓ_{k+n}, $n \in \mathbb{N}_0$.

Moreover, if $1 \leq k \leq n$, then $\mathcal{P}^{\mathbf{t}}_{k-n} = \{0\}$, and so $\mathrm{Cor}\,(\ell_k) = \mathrm{Span}\left\{x^k\right\}$. For $k > n$, the subspaces $\mathrm{Cor}\,(\ell_k)$ can be obtained from the above table (with an adequate displacement):

k	1	$\cdots$	n	$n+1$	$\cdots$	$2n$	$2n+1$	$2(n+1)$	$\cdots$	$3n+1$	$2(n+1)+n$	$\cdots$
$\mathrm{Cor}\,(\ell_k)$	x	$\cdots$	x^n	x^{n+1}	$\cdots$	x^{2n}	0	h	$\cdots$	$x^{n-1}h$	0	$\cdots$

Table 3.2: Table of the co-ranges of ℓ_k.

The result follows from Proposition 3.3.79. ∎

This example illustrates the difficulty in the calculations for obtaining a normal form when $\mathbf{F}_r = \mathbf{X}_h + \mu\mathbf{D}_0$. Our aim, in what follows, will be to reduce this difficulty considering that the principal quasi-homogeneous component, $\mathbf{F}_r$, is Hamiltonian. Therefore, we will work with system (3.3.6) where $\mathbf{F}_r = \mathbf{X}_h$ and $h \in \mathcal{P}^{\mathbf{t}}_{k+|\mathbf{t}|}$, i.e.,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) := \mathbf{X}_h + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots \tag{3.3.20}$$

where $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$.

Considering system (3.3.20) and Proposition 3.3.79, Theorem 3.3.80 can be enunciated in the form,

Proposition 3.3.87. *Assume $\mathbf{F}_r = \mathbf{X}_h$, $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|} \setminus \{0\}$. A complementary subspace of the range of the operators given in 2.3.13 and 2.3.16 is,*

$$\mathrm{Cor}(\mathbf{L}_{r+k}) = \mathbf{X}_{\mathrm{Cor}(\ell_{r+k+|\mathbf{t}|})\cap\Delta_{r+k+|\mathbf{t}|}} \oplus \mathrm{Cor}(\ell_{r+k})\mathbf{D}_0 \oplus \mathrm{Cor}(\ell_k)\mathbf{F}_r.$$

$$\mathrm{Cor}(\mathcal{L}_{r+k}) = \mathbf{X}_{\mathrm{Cor}(\ell_{k+r+|\mathbf{t}|})\cap\Delta_{k+r+|\mathbf{t}|}} \oplus \mathrm{Cor}(\ell_{r+k})\mathbf{D}_0$$

where ℓ_k is the Lie derivative operator respect to $\mathbf{F}_r$ defined in (2.3.15).

Theorem 3.3.88. *A formal normal form for system (3.3.20), under $\mathcal{C}^\infty$-conjugation, is given by*

$$\dot{\mathbf{x}} = \mathbf{F}_r + \sum_{k\geq 1}\Big(\mathbf{X}_{g_{r+k+|\mathbf{t}|}} + \eta_{r+k}\mathbf{D}_0 + \lambda_k\mathbf{F}_r\Big). \tag{3.3.21}$$

where $g_{r+k+|\mathbf{t}|} \in \mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}) \cap \Delta_{r+k+|\mathbf{t}|}$, $\eta_{r+k} \in \mathrm{Cor}(\ell_{r+k})$ and $\lambda_k \in \mathrm{Cor}(\ell_k)$, for $k \geq 1$.

A formal normal form for system (3.3.20), under $\mathcal{C}^\infty$-equivalence, is given by

$$\dot{\mathbf{x}} = \mathbf{X}_h + \sum_{k\geq 1}(\mathbf{X}_{g_{r+k+|\mathbf{t}|}} + \eta_{r+k}\mathbf{D}_0). \tag{3.3.22}$$

where $g_{r+k+|\mathbf{t}|} \in \mathrm{Cor}(\ell_{k+r+|\mathbf{t}|}) \cap \Delta_{k+r+|\mathbf{t}|}$ *and* $\eta_{r+k} \in \mathrm{Cor}(\ell_{r+k})$*, for* $k \geq 1$.

As follows from Theorem 3.3.88, the co-ranges of the homological operators are determined by the co-ranges of the linear operators ℓ_{r+k}, for $k \geq 1$. Next, we show that, if h only has simple factors in its factorization on $\mathbb{C}[x,y]$, there exists cyclicity in the expression of these co-ranges. Specifically, we will see that it is only necessary to calculate some initial co-ranges, the rest of them are derived from knowing the previous ones. We study it in the next section.

3.3.2 Computing $\mathrm{Cor}(\ell_k)$, $k \geq 1$.

The following two lemmas are necessary to prove the existence of the cyclicity of the co-ranges of the linear operators ℓ_k. The first one shows a relationship between the dimensions of the spaces $\mathcal{P}_k^t$ with different degrees, the second one gives an expression of the kernel of ℓ_k. Their proofs can be seen in [9].

Lemma 3.3.89. *Let* $k \in \mathbb{N}$, $r \in \mathbb{N}\cup\{0\}$ *such that* $r+|\mathbf{t}| \geq t_1t_2$*. If* $\mathcal{P}_{k-r} \neq \{0\}$*, then*

$$\dim\left(\mathcal{P}^{\mathbf{t}}_{k+r+|\mathbf{t}|}\right) - \dim\left(\mathcal{P}^{\mathbf{t}}_{k+|\mathbf{t}|}\right) = \dim\left(\mathcal{P}^{\mathbf{t}}_{k}\right) - \dim\left(\mathcal{P}^{\mathbf{t}}_{k-r}\right).$$

Lemma 3.3.90. *Assume that* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *only has simple factors in its factorization on* $\mathbb{C}[x,y]$*, then*

$$Ker(\ell_k) = \begin{cases} \langle h^l \rangle & if \quad k-r = l(r+|\mathbf{t}|), \\ 0 & if \quad k-r \neq l(r+|\mathbf{t}|). \end{cases}$$

The below lemma relates the co-ranges of ℓ_k and $\ell_{k+r+|\mathbf{t}|}$.

Lemma 3.3.91. *Suppose that* h *only has simple factors in its factorization on* $\mathbb{C}[x,y]$*, then we can choose a complementary subspace to* $\mathrm{Im}(\ell_{k+r+|\mathbf{t}|})$ *such that*

$$h\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+r+|\mathbf{t}|}).$$

Proof. We prove that $h\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+r+|\mathbf{t}|})$ or equivalently that $h\mathrm{Cor}(\ell_k))\cap \mathrm{Im}(\ell_{k+r+|\mathbf{t}|}) = \{0\}$. By *reductio ad absurdum*, let $\mu_k \in \mathrm{Cor}(\ell_k) \setminus \{0\}$ such that $h\mu_k \in \mathrm{Im}(\ell_{k+r+|\mathbf{t}|})$. Then there exists $\nu_{k+|\mathbf{t}|} \in \mathcal{P}^{\mathbf{t}}_{k+|\mathbf{t}|}$ such that $h\mu_k = \ell_{k+r+|\mathbf{t}|}(\nu_{k+|\mathbf{t}|}) = \nabla\nu_{k+|\mathbf{t}|} \cdot \mathbf{X}_h$, i.e., $\ell_{k+r+|\mathbf{t}|}(\nu_{k+|\mathbf{t}|}) \in \langle h\rangle$, by using Proposition 3.3.77, $\nu_{k+|\mathbf{t}|} \in \langle h\rangle$ and we can consider $\nu_{k+|\mathbf{t}|} = \lambda_{k-r}h$. Hence

$$h\mu_k = \ell_{k+r+|\mathbf{t}|}(\nu_{k+|\mathbf{t}|}) = \ell_{k+r+|\mathbf{t}|}(\lambda_{k-r}h) = h\ell_k(\lambda_{k-r}),$$

Consequently, as $h \neq 0$, $\mu_k \in \mathrm{Im}(\ell_k)$ and this is a contradiction. Therefore $h\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+r+|\mathbf{t}|})$. ■

Next statement establishes the cyclicity relation between the co-ranges of the operators ℓ_k.

Theorem 3.3.92. *Assume that h only has simple factors in its factorization on $\mathbb{C}[x,y]$ and $\mathcal{P}^{\mathbf{t}}_{k-r} \neq \{0\}$, then we can choose a complementary subspace to $\mathrm{Im}(\ell_{r+k+|\mathbf{t}|})$ such that*

$$\mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}) = h\mathrm{Cor}(\ell_k)$$

Proof. From Lemma 3.3.91 we know that $h\mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{r+k+|\mathbf{t}|})$. Therefore it is enough to prove that $\dim(h\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}))$.

Since ℓ_k and $\ell_{k+r+|\mathbf{t}|}$ are linear operators, we get

$$\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathcal{P}^{\mathbf{t}}_k) - \dim(\mathcal{P}^{\mathbf{t}}_{k-r}) + \dim(\mathrm{Ker}(\ell_k)). \tag{3.3.23}$$

$$\dim(\mathrm{Cor}(\ell_{r+k+|\mathbf{t}|})) = \dim(\mathcal{P}^{\mathbf{t}}_{r+k+|\mathbf{t}|}) - \dim(\mathcal{P}^{\mathbf{t}}_{k+|\mathbf{t}|}) + \dim(\mathrm{Ker}(\ell_{r+k+|\mathbf{t}|})). \tag{3.3.24}$$

From (3.3.23), (3.3.24) and using Lemma 3.3.89,

$$\dim(\mathrm{Cor}(\ell_{r+k+|\mathbf{t}|})) = \dim(\mathrm{Cor}(\ell_k)) - \dim(\mathrm{Ker}(\ell_k)) + \dim(\mathrm{Ker}(\ell_{r+k+|\mathbf{t}|})). \tag{3.3.25}$$

Using Lemma 3.3.90 we obtain $\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{r+k+|\mathbf{t}|}))$ and this complete the proof. ■

The next result provides a formal normal form of (3.3.20) in the case that h only has simple factors in its factorization on $\mathbb{C}[x,y]$.

Theorem 3.3.93. *Consider* $\mathbf{F}_r = \mathbf{X}_h$*,* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *only has simple factors in its factorization on* $\mathbb{C}[x,y]$*. A formal normal form under* $\mathcal{C}^\infty$*-equivalence for system (3.3.20) is*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \sum_{j=r+|\mathbf{t}|+1}^{n_0+r+|\mathbf{t}|-1} \mathbf{X}_{g_j} + \sum_{j=r+1}^{n_0-1} \eta_j^{(0)} \mathbf{D}_0 + \sum_{i=0}^{\infty} \sum_{j=n_0}^{n_0+r+|\mathbf{t}|-1} \eta_j^{(i)} h^i \mathbf{D}_0,$$

with $\eta_j^{(i)} \in \mathrm{Cor}(\ell_j)$*,* $g_j \in \mathrm{Cor}(\ell_j) \cap \Delta_j$ *and* $n_0 := 1 + max\,\{k \in \mathbb{N}/\mathcal{P}_{k-r} = \{0\}\}$*.*

Proof. First we have to specify that it is always possible to achieve the value n_0. In [9] is proved that, if $k - r > t_1 t_2 - t_1 - t_2$ then $\mathcal{P}^{\mathbf{t}}_{k-r} \neq \{0\}$. From it is easy to prove the existence of n_0 and verify that $n_0 \leq t_1 t_2 - t_1 - t_2 + 1 + r$. By Theorem 3.3.88, the co-range of the homological operator is determined by the co-range of the linear operator ℓ_k. From Theorem 3.3.92, if $k \geq n_0 + r + |\mathbf{t}|$ then $\mathrm{Cor}(\ell_k) = h\mathrm{Cor}(\ell_{k-r-|\mathbf{t}|})$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k = \{0\}$.

Let $i = \lfloor \frac{k-n_0}{r+|\mathbf{t}|} \rfloor$ and $j = k - (r + |\mathbf{t}|)i$, therefore $n_0 \leq j \leq n_0 + r + |\mathbf{t}1$. By applying, repeatedly, Theorem 3.3.92, we obtain $\mathrm{Cor}(\ell_k) = h^i \mathrm{Cor}(\ell_j)$. This complete de proof. ∎

Remark: In the particular case that h only has simple factors in its factorization on $\mathbb{C}[x,y]$, we only need the computation of a certain number of these co-ranges. Particularly, from $r+1$ to $n_0 + r + |\mathbf{t}| - 1$.

3.4 Calculation of the normal form of some families of planar vector fields.

In this section, we give a formal normal form of several families of nilpotent and degenerate vector fields.

3.4.1 Normal Form of Nilpotent Systems.

Here we show a formal normal form of nilpotent vector fields whose first quasi-homogeneous term is $\mathbf{F}_r = \begin{pmatrix} y \\ a\,x^n \end{pmatrix}$, with $a \in \mathbb{N}$ and $n \geq 2$. For that, we decompose the natural number n in the form $n = 2l + m$ with $m \in \{0, 1\}$ and $l \in \mathbb{N} \cup \{0\}$ and we distinguish two different cases:

► Case $m = 0$. In this case we choose the type $\mathbf{t} = (2, 2l+1)$ hence $r = 2l-1$. Making a scaling in the time, it is possible to consider $a = 1$, therefore, the vector field can be written in the form

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ x^{2l} \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N}. \tag{3.4.26}$$

Theorem 3.4.94. *System (3.4.26) is formally equivalent to*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ x^{2l} \end{pmatrix} + \sum_{j=l}^{2l-1} \alpha_{2j} x^j \mathbf{D}_0 + \sum_{j=0}^{2l-1} x^j f_j(h) \mathbf{D}_0, \tag{3.4.27}$$

where $f_j(h) = \alpha_{0,j} + \alpha_{1,j} h + \cdots \in \mathbb{C}[[h]]$, and $\mathbb{C}[[h]]$ is the vector space of the power series in the h variable with coefficients in $\mathbb{C}$.

Proof. To calculate the normal form, it is sufficient to apply Theorem 3.3.93 to system (3.4.26). In this particular case, $n_0 = 4l-1$, $r = 2l-1$ and $|\mathbf{t}| = 2l+3$ and it is sufficient to compute $\mathrm{Cor}(\ell_k)$ for $k =$ 2l, $\cdots$, $8l$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k$ for $k = 4l + 3, \cdots, 8l$ with $l \in \mathbb{N}$.

In order to obtain $\mathrm{Cor}(\ell_k)$ for $k =$ 2l, $\cdots$, $8l$, we distinguish two cases:

a) k odd, i.e., $k = 2j + 1$ with $l \leq j \leq 4l - 1$ and $j \in \mathbb{N}$.

- Case $l \leq j \leq 3l - 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j+1-l}\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y\}$. If we take $\mu_{k-r} = \alpha x^{j+1-l} \in \mathcal{P}^t_{k-r}$ then, $\ell_k(\mu_{k-r}) = \alpha(j + 1 - l)x^{j-l}y$. Therefore $\mathrm{Range}(\ell_k) = span\{x^{j-l}y\}=\mathcal{P}^t_k$. In consequence, $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $j = 3l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{2l+1}, h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{2l}y\}$. Taking $\mu_{k-r} = \alpha x^{2l+1} + \beta h \in \mathcal{P}^t_{k-r}$ we get, $\ell_k(\mu_{k-r}) = \alpha(2l + 1)x^{2l}y$. Consequently, it is deduced that $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $3l + 1 \leq j \leq 4l - 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j+1-l}, x^{j-3l}h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y, x^{j-3l-1}yh\}$. Proceeding analogously to the previous cases, it is easy to prove that $\mathrm{Cor}(\ell_k) = \{0\}$.

b) k even, i.e., $k = 2j$ with $j \leq m \leq 4l$. and $j \in \mathbb{N}$.

- Case $l \leq j \leq 2l-1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{0\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^j\}$. Thus $\mathrm{Cor}(\ell_k) = \{x^j\}$.

- Case $j = 2l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{y\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{2l}\}$. If we take $\mu_{k-r} = \alpha y \in \mathcal{P}^{t}_{k-r}$ then, $\ell_k(\mu_{k-r}) = \alpha x^{2l}$. Therefore, we obtain $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $2l+1 \leq j \leq 4l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-2l}y\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^j, x^{j-2l-1}h\}$. Proceeding analogously to the previous cases, it is easy to prove that $\mathrm{Cor}(\ell_k) = \{x^{j-2l-1}h\}$.

By applying Theorem 3.3.93 we get a formal normal form, under equivalence, for system (3.4.26) given in (3.4.27). ■

► Case $m = 1$ In this case we choose the type $\mathbf{t} = (1, l+1)$, hence $r = l$. Making a scaling in the time, it is possible to consider $a = \pm 1$, therefore, the vector field can be written in the form

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ ax^{2l+1} \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N} \cup \{0\} \text{ and } a = \pm 1. \tag{3.4.28}$$

Theorem 3.4.95. *System (3.4.28) is formally equivalent to:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ ax^{2l+1} \end{pmatrix} + \sum_{j=l+1}^{2l} \alpha_j x^j \mathbf{D}_0 + \sum_{\substack{j=0 \\ j \neq l}}^{2l} x^j f_j(h) \mathbf{D}_0, \tag{3.4.29}$$

where $f_j(h) = \alpha_{0,j} + \alpha_{1,j} h + \cdots \in \mathbf{C}[[h]]$.

Proof. In order to calculate the normal form, it is sufficient to apply Theorem 3.3.93 to system (3.4.28). Concretely for this case, $n_0 = l$, $r = l$ and $|\mathbf{t}| = l+2$ and it is enough to compute $\mathrm{Cor}(\ell_k)$ for $k = \mathrm{l}{+}1, \cdots, 3l+1$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k$ for $k = 2l+3, \cdots, 3l+1$ with $l \in \mathbb{N}$.

- Case $l+1 \leq k < 2l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-l} = span\{x^{k-l}\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^k, x^{k-(l+1)}y\}$. If we take $\mu_{k-l} = \alpha x^{k-l} \in \mathcal{P}^{t}_{k-l}$ then, $\ell_k(\mu_{k-l}) = \alpha(k-l)x^{k-(l+1)}y$. Therefore, $\mathrm{Range}(\ell_k) = span\{x^{k-(l+1)}y\}$ and $\mathrm{Cor}(\ell_k) = span\{x^k\}$.

- Case $k = 2l + 1$. In this case $\mathcal{P}^{\mathbf{t}}_{l+1} = span\{x^{l+1}, y\}$ and $\mathcal{P}^{\mathbf{t}}_{2l+1} = span\{x^{2l+1}, x^l y\}$. Considering $\mu_{l+1} = \alpha x^{l+1} + \beta y \in \mathcal{P}^{t}_{l+1}$ we get $\ell_{2l+1}(\mu_{l+1}) = \beta a x^{2l+1} - \alpha(l+1)x^l y$. Hence we obtain $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $2l + 2 \leq k \leq 3l + 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-l} = span\{x^{k-l}, x^{k-(2l+1)}y\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}h\}$. Taking $\mu_{k-l} = \alpha x^{k-l} + \beta x^{k-(2l+1)} y \in \mathcal{P}^{t}_{k-l}$ then, $\ell_k(\mu_{k-l}) = \beta a x^k + \alpha(k-l)x^{k-(l+1)}y + \beta(k-2l-1)x^{k-2(l+1)}h$. Thus we can choose $\mathrm{Cor}(\ell_k) = span\{x^{k-2(l+1)}h\}$.

Consequently, for $k = 2l + 3, \cdots, 3l + 1$, it is satisfied that, $\mathrm{Cor}(\ell_k) =< x^{k-2(l+1)}h >$, therefore $\mathrm{Cor}(\ell_k) \subset h\mathrm{Cor}(\ell_{k-2l-2})$, for $k = 2l+3, \cdots, 3l+1$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k = \{0\}$ for $k = 2l + 3, \cdots, 3l + 1$.

By applying Theorem 3.3.93 we get a normal form, under equivalence, for system (3.4.28) given in (3.4.29). ∎

3.4.2 Normal form of some generalized nilpotent vector fields

In this subsection we show a normal form of vector fields whose first quasi-homogeneous term is $\mathbf{F}_r = \mathbf{X}_h$ with $h(0, y) = -\frac{1}{3}y^3$ and $h(x, 0) = \frac{1}{n+1}x^{n+1}$. So we decompose the natural number n in the form $n = 3l + m$ with $m \in \{0, 1, 2\}$ and $l \in \mathbb{N} \cup \{0\}$ and we distinguish three different cases:

► <u>Case $m = 0$</u> In this case $\mathbf{t} = (3, 3l + 1)$, $r = 6l - 1$ and the vector field is of the form

$$\dot{\mathbf{x}} = \begin{pmatrix} y^2 \\ x^{3l} \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N}. \tag{3.4.30}$$

Theorem 3.4.96. *System (3.4.30) is formally equivalent to:*

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} y^2 \\ x^{3l} \end{pmatrix} + \sum_{j=2l+1}^{3l-1} \beta_{3(j-1)} \mathbf{X}_{x^j y} + \sum_{j=2l}^{3l-1} \alpha_{3j} x^j \mathbf{D}_0 + \sum_{j=l}^{3l-1} \alpha_{3(j+l)+1} x^j y \mathbf{D}_0 \\ &+ \sum_{j=0}^{3l-1} x^j h f^{(0)}_j(h) \mathbf{D}_0 + \sum_{j=0}^{3l-1} x^j y h f^{(1)}_j(h) \mathbf{D}_0.\end{aligned} \tag{3.4.31}$$

where $f^{(i)}_j = \alpha^{(i)}_{0,j} + \alpha^{(i)}_{1,j} h + \cdots \in \mathbf{C}[[h]]$, $i = 0, 1$.

Proof. To obtain the normal form, it is sufficient to apply Theorem 3.3.93 to system (3.4.30). In this particular case, $n_0 = 12l-1$, $r = 6l-1$, $|\mathbf{t}| = 3l+4$ and it is only necessary to calculate $\mathrm{Cor}(\ell_k)$ for $k = 6\mathrm{l}, \cdots, 21l+1$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k$ for $k = 9l+4, \cdots, 21l+1$ with $l \in \mathbb{N}$.

In order to calculate $\mathrm{Cor}(\ell_k)$ for $k = 6\mathrm{l}, \cdots, 21l+1$, we distinguish three cases: $k = 3j+i$ with $j = 0, 1, 2$ and $j \in \mathbb{N} \cup \{0\}$. Therefore we must consider $2l \leq j \leq 7l - 1 + \frac{4-i}{3}$ for calculating $\mathrm{Cor}(\ell_k)$ and $3l + \frac{4-i}{3} \leq j \leq 7l - 1 + \frac{4-i}{3}$ for calculating $\mathrm{Cor}(\ell_k) \cap \Delta_k$.

a) $k = 3j$ $(i = 0)$.

- Case $2l \leq j \leq 3l-1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^j\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^j\}$.

- Case $j = 3l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{y\}$ and $\mathcal{P}^{\mathbf{t}}_{3m+6l} = span\{x^{3l}\}$. It is easy to prove that $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $3l+1 \leq j \leq 6l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-3l}y\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^j, x^{j-3l-1}h\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-3l}y$, then $\ell_k(\mu_{k-r}) = -3\alpha(j-3l)x^{j-3l-1}h + \frac{\alpha(3j-6l+1)}{3l+1}x^j$, consequently $\mathrm{Cor}(\ell_k) = span\{x^{j-3l-1}h\}$.

- Case $j = 6l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{3l+1}y, yh\}$ and $\mathcal{P}^{\mathbf{t}}_{6l+1} = span\{x^{6l+1}, x^{3l}h\}$. Hence $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $6l+2 \leq j \leq 7l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-3l}y, x^{j-6l-1)}yh\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^j, x^{j-3l-1}h, x^{j-6l-2)}h^2\}$. If we take $\mu_{k-r} = \alpha x^{j-3l}y + \beta x^{j-6l-1)}yh$, then $\ell_k(\mu_{k-r}) = -3\beta(j-6l-1)x^{j-6l-2}h^2 - \left[3(j-3l)\alpha + \frac{3(j-5l-1)+1}{3l+1}\beta\right] x^{j-3l-1}h + \frac{3(j-2l)+1}{3l+1}\alpha x^j$, and we can choose $\mathrm{Cor}(\ell_k) = span\{x^{j-6l-2}h^2\}$.

  In these cases, it is obvious that $\mathrm{Cor}(\ell_k) \cap \Delta_k = \{0\}$.

b) $k = 3j+1$ $(i = 1)$.

- Case $2l \leq j \leq 4l-1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^{j-l}y\}$.

- Case $j = 4l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{y^2\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{3l}y\}$. Considering $\mu_{k-r} = \alpha y^2$, then $\ell_k(\mu_{k-r}) = 2\alpha x^{3l}y$ and consequently $\mathrm{Cor}(\ell_k) = span\{0\}$.

- Case $4l+1 \leq j \leq 7l$. In this case $\mathcal{P}^t_{k-r} = \{x^{j-4l}y^2\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y, x^{j-4l-1}yh\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-4l}y^2$, then $\ell_k(\mu_{k-r}) = -3\alpha(j-4l)x^{j-4l-1}yh + \frac{3(j-2l)+2}{3l+1}\alpha x^{j-l}y$ thus we can choose $\mathrm{Cor}(\ell_k) = span\{x^{j-4l-1}yh\}$.

  In these cases, it is easy to prove that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{x^{j-l}y\}$ for $j = 3l+1, \cdots, 4l-1$.

c) $k = 3j+2$ $(i=2)$.

- Case $2l \leq j < 5l-1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j+1-2l}\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-2l}y^2\}$. If we take $\mu_{k-r} = \alpha x^{j+1-2l} \in \mathcal{P}^t_{k-r}$ then, $\ell_k(\mu_{k-r}) = \alpha(j+1-2l)x^{j-2l}y^2$. Therefore $\mathrm{Range}(\ell_k) = span\{x^{j-2l}y^2\} = \mathcal{P}^t_k$ and, in consequence, $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $m = 5l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{3l+1}, h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{3l}y^2\}$. Considering $\mu_{k-r} = \alpha x^{3l+1} + \beta h \in \mathcal{P}^t_{k-r}$ then, $\ell_k(\mu_{k-r}) = \alpha(3l+1)x^{3l}y^2$. Therefore we obtain $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $5l+1 \leq m \leq 7l-1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j+1-2l}, x^{j-5l)}h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-2l}y^2, x^{j-5l-1}y^2h\}$. Proceeding analogously to the previous cases $\mathrm{Cor}(\ell_k) = \{0\}$.

  In these cases, it is obvious that $\mathrm{Cor}(\ell_k) \cap \Delta_k = \{0\}$.

Applying Theorem 3.3.93 we complete the proof. ∎

▶ Case $m = 1$ In this case $\mathbf{t} = (3, 3l+2)$, $r = 6l+1$ and the vector field is of the form

$$\dot{\mathbf{x}} = \begin{pmatrix} y^2 \\ x^{3l+1} \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N}. \tag{3.4.32}$$

Theorem 3.4.97. *System (3.4.32) is formally equivalent to:*

$$\begin{aligned}\dot{\mathbf{x}} \;=\; & \begin{pmatrix} y^2 \\ x^{3l+1} \end{pmatrix} + \sum_{j=2l+2}^{3l} \beta_{3(j-l)}\mathbf{X}_{x^jy} + \sum_{j=2l+1}^{3l} \alpha_{3j}x^j\mathbf{D}_0 + \sum_{j=l}^{3l} \alpha_{3(j+l)+2}x^jy\mathbf{D}_0 \\ & + \sum_{j=0}^{3l} x^jhf_j^{(0)}(h)\mathbf{D}_0 + \sum_{j=0}^{3l} x^jyhf_j^{(1)}(h)\mathbf{D}_0. \end{aligned} \tag{3.4.33}$$

where $f_j^{(i)} = \alpha_{0,j}^{(i)} + \alpha_{1,j}^{(i)}h + \cdots \in \mathbf{C}[[h]]$, $i = 0, 1$.

Proof. With the purpose of calculating the normal form, it is sufficient to apply Theorem 3.3.93 to system (3.4.32). In this particular case, $n_0 = 12l+3$, $r = 6l+1$ and $|\mathbf{t}| = 3l+5$ and it is only necessary to calculate $\mathrm{Cor}(\ell_k)$ for $k = 6l+2, \cdots, 21l+8$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k$ for $k = 9l+7, \cdots, 21l+8$ with $l \in \mathbb{N}$.

In order to calculate $\mathrm{Cor}(\ell_k)$ for $k = 6l+2, \cdots, 21l+8$, we distinguish three cases: $k = 3j+i$ with $i = 0, 1, 2$ and $j \in \mathbb{N} \cup \{0\}$. Therefore we must consider $\frac{2-i}{3} + 2l \leq j \leq 7l + 2 + \frac{2-i}{3}$ for calculating $\mathrm{Cor}(\ell_k)$ and $3l + 1 + \frac{4-i}{3} \leq j \leq 7l + 2 + \frac{2-i}{3}$ for calculating $\mathrm{Cor}(\ell_k) \cap \Delta_k$.

a) $k = 3j$ $(i = 0)$, $k - r = 3(j - 2l) - 1$.

- Case $2l+1 \leq j \leq 3l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_k = \mathcal{P}^{\mathbf{t}}_{3j} = span\{x^j\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^j\}$.
- Case $j = 3l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{y\}$ and $\mathcal{P}^{t}_k = span\{x^{3l+1}\}$. If we take $\mu_{k-r} = \alpha y$, then $\ell_k(\mu_{k-r}) = \alpha x^{3l+1}$, consequently, $\mathrm{Cor}(\ell_k) = \{0\}$.
- Case $3l + 2 \leq j \leq 6l + 2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-(3l+1)}y\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^j, x^{j-(3l+2)}h\}$ and we can choose $\mathrm{Cor}(\ell_k) = span\{x^{j-(3l+2)}h\}$.
- Case $6l + 3 \leq j \leq 7l + 2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r-r-|\mathbf{t}|} = \mathcal{P}^{\mathbf{t}}_{3(j-6l-3)-3l+2} = span\{x^{j-6l-3}y\}$. From Theorem 3.3.92 we get $\mathrm{Cor}(\ell_k) = h\mathrm{Cor}(\ell_{k-r-|\mathbf{t}|}) = h\mathrm{Cor}(\ell_{3(j-3l-2)})$ so:
  - If $j = 6l + 3$, $\mathrm{Cor}(\ell_{3(3l+1)}) = \{0\}$ and then $\mathrm{Cor}(\ell_k) = \{0\}$
  - If $6l + 4 \leq j \leq 7l + 2$, $\mathrm{Cor}(\ell_{3(j-3l-2)}) = \{x^{j-6l-4}h\}$ and then $\mathrm{Cor}(\ell_k) = \{x^{j-6l-4}h^2\}$

b) $k = 3j + 1$ $(i = 1)$, $k - r = 3(j - 2l)$.

- Case $2l + 1 \leq j < 5l + 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-2l}\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-(2l+1)y^2}\}$. Therefore $\mathrm{Cor}(\ell_k) = \{0\}$.
- Case $5l + 2 \leq j \leq 7l + 2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r-r-|\mathbf{t}|} = \mathcal{P}^{\mathbf{t}}_{3(j-5l-2)} = span\{x^{j-5l-2}\} \neq \{0\}$. From Theorem 3.3.92 we get $\mathrm{Cor}(\ell_k) = h\mathrm{Cor}(\ell_{k-r-|\mathbf{t}|}) = h\mathrm{Cor}(\ell_{3(j-3l-2)+1}) = \{0\}$.

c) $k = 3j + 2$ $(i = 2)$, $k - r = 3(j - 2l) + 1$.

- Case $2l \leq j \leq 4l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^{j-l}y\}$.

- Case $j = 4l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{y^2\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{3l+1}y\}$. If we take $\mu_{k-r} = \alpha y^2$, then $\ell_k(\mu_{k-r}) = 2\alpha x^{3l+1}y$, consequently, $\mathrm{Cor}(\ell_k) = \{0\}$.
- Case $4l+2 \leq j \leq 7l+2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-(4l+1)}y^2\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y, x^{j-(4l+2)}yh\}$. Hence, it is easy to prove $\mathrm{Cor}(\ell_k) = span\{x^{j-(4l+2)}yh\}$.

Applying Theorem 3.3.93, we complete the proof. ■

► Case $m=2$ In this case $\mathbf{t} = (1, l+1)$, $r = 2l+1$ and the vector field is of the form

$$\dot{\mathbf{x}} = \mathbf{X}_h = \begin{pmatrix} y^2 + 2\tilde{a}x^{l+1}y + \tilde{b}x^{2(l+1)} \\ -\tilde{a}(l+1)x^l y^2 - 2\tilde{b}(l+1)x^{2l+1}y + \tilde{c}x^{3l+2} \end{pmatrix} + \cdots,. \tag{3.4.34}$$

with $l \in \mathbb{N} \cup \{0\}$, $\tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{R}$ and $h := -\frac{1}{3}y^3 - \tilde{a}x^{l+1}y^2 - \tilde{b}x^{2(l+1)}y + \frac{\tilde{c}}{3(l+1)}x^{3(l+1)}$ has only simple factors.

Lemma 3.4.98. *System (3.4.34) is analytically equivalent to system*

$$\dot{\mathbf{x}} = \mathbf{X}_h = \begin{pmatrix} y^2 + 2ax^{l+1}y + \sigma x^{2(l+1)} \\ -a(l+1)x^l y^2 - 2\sigma(l+1)x^{2l+1}y \end{pmatrix} + \cdots, \text{ with } l \in \mathbb{N} \cup \{0\},\ \sigma = \pm 1. \tag{3.4.35}$$

and $a^2 \neq \frac{4\sigma}{3}$, i.e. h has only simple factors. Moreover $a = 0$ if $l = 0$.

Proof. Using the change of variables $x = u$, $y = v - \alpha_0 u^{l+1}$, where α_0 is a real roots of $(l+1)\alpha^3 + 3\tilde{a}(l+1)\alpha^2 + 3\tilde{b}(l+1)\alpha - \tilde{c} = 0$, system (3.4.40) is transformed into

$$\dot{\mathbf{x}} = \mathbf{X}_h = \begin{pmatrix} y^2 + 2ax^{l+1}y + bx^{2(l+1)} \\ -a(l+1)x^l y^2 - 2b(l+1)x^{2l+1}y \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N} \cup \{0\}$$

with $h := y[-\frac{1}{3}y^2 - ax^{l+1}y - bx^{2(l+1)}]$, $a^2 \neq \frac{4b}{3}$ and $b \neq 0$ i.e. h has only simple factors in its decomposition on $\mathbb{C}[x,y]$. Using an scaling, we can get $b = \pm 1$.

In the particular case $l = 0$, using the change of variables $x = u + \frac{-a}{2}v$, $y = v$ and a scaling in the variables of state, we can get $a = 0$. ■

Theorem 3.4.99. *System (3.4.35) is formally equivalent to:*

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{pmatrix} y^2 + 2ax^{l+1}y + \sigma x^{2(l+1)} \\ -a(l+1)x^l y^2 - 2\sigma(l+1)x^{2l+1}y \end{pmatrix} + \sum_{j=l+2}^{2l} \beta_{j+l}\mathbf{X}_{x^j y^2} + \sum_{j=0}^{l-1}[x^{j+l+1}yf_j^{(0)}(h)\mathbf{D}_0 \\
&\quad + x^j y^2 f_j^{(1)}(h)\mathbf{D}_0] + x^l y^2 f_l^{(2)}(h)\mathbf{D}_0 + \sum_{j=0}^{l-1}[x^{j+l+1}y^2 f_j^{(3)}(h)\mathbf{D}_0 + x^j h f_j^{(4)}(h)\mathbf{D}_0] \\
&\quad + x^l h f_l^{(5)}(h)\mathbf{D}_0 + \sum_{j=0}^{l}[x^{j+l+1}hf_j^{(6)}(h)\mathbf{D}_0 + x^j y h f_j^{(7)}(h)\mathbf{D}_0] \qquad (3.4.36)
\end{aligned}
$$

where $f_j^{(i)} = \alpha_{0,j}^{(i)} + \alpha_{1,j}^{(i)}h + \cdots \in \mathbf{C}[[h]]$, $i = 0, 1, 2, 3, 4, 5, 6, 7$.

Proof. In order to calculate a normal form, it is sufficient to apply Theorem 3.3.93 to the system (3.4.35). In this particular case, $n_0 = 2l+2$, $r = 2l+1$ and $|\mathbf{t}| = l+2$ and it is only necessary to calculate $\mathrm{Cor}(\ell_k)$ for $k = 2l+2, \cdots, 5l+4$.

a) $2l + 2 \le k \le 3l + 1$. In this case, $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{k-r}\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}y^2\}$. If we take $\mu_{k-r} = \alpha x^{k-r}$ then $\ell_k(\mu_{k-r}) = (k - 2l - 1)\alpha[x^k + 2ax^{k-l-1}y + x^{k-2(l+1)}y^2]$. Therefore, we can choose $\mathrm{Cor}(\ell_k) = span\{x^{k-(l+1)}y, x^{k-2(l+1)}y^2\}$.

b) $k = 3l + 2$. In this case, $\mathcal{P}^{\mathbf{t}}_{k-r} = \mathcal{P}^{\mathbf{t}}_{l+1} = span\{x^{l+1}, y\}$ and $\mathcal{P}^{\mathbf{t}}_k = \mathcal{P}^{\mathbf{t}}_{3l+2} = span\{x^{3l+2}, x^{2l+1}y, x^l y^2\}$. Considering $\mu_{l+1} = \alpha x^{l+1} + \beta y$ then $\ell_k(\mu_{l+1}) = (l+1)[\alpha x^{3l+2} + 2(\alpha a - \sigma\beta)x^{2l+1}y + (\alpha - \beta a)x^l y^2]$. It is possible to choose $\mathrm{Cor}(\ell_k) = span\{x^l y^2\}$.

c) $3l + 3 \le k \le 4l + 2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{k-r}, x^{k-r-(l+1)}y\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}y^2, x^{k-3(l+1)}h\}$. Taking $\mu_{k-r} = \alpha x^{k-r} + \beta x^{k-r-(l+1)}y$ then $\ell_k(\mu_{k-r}) = \alpha(k - 2l - 1)x^k + 2(k - 2l - 1)(\alpha a - \sigma\beta)x^{k-(l+1)}y + (k - 2l - 1)(\alpha - \beta a)x^{k-2(l+1)}y^2 - 3\beta(k - 3l - 2)x^{k-3(l+1)}h$. Therefore we can choose $\mathrm{Cor}(\ell_k) = span\{x^{k-2(l+1)}y^2, x^{k-3(l+1)}h\}$.

d) $k = 4l + 3$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \mathcal{P}^{\mathbf{t}}_{2(l+1)} = span\{x^{2(l+1)}, x^{l+1}y, y^2\}$ and $\mathcal{P}^{\mathbf{t}}_k = \mathcal{P}^{\mathbf{t}}_{4l+3} = span\{x^{4l+3}, x^{3l+2}y, x^{2l+1}y^2, x^l h\}$. If we take $\mu_{2(l+1)} = \alpha x^{2(l+1)} + \beta x^{(l+1)}y + \gamma y^2$ then $\ell_k(\mu_{2(l+1)}) = +2(l+1)\alpha x^{4l+3} + 2(l+1)(2a\alpha - 2\sigma\beta + a(1+2\sigma)\gamma)x^{3l+2}y + 2(l+1)(\alpha - a\beta - (2\sigma - 3a^2)\gamma)x^{2l+1}y^2 - 3(l+1)(\beta - 2a\gamma)x^l h$. Thus we can choose $\mathrm{Cor}(\ell_k) = span\{x^l h\}$.

e) $4l+4 \leq k \leq 5l+3$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{k-r}, x^{k-r-(l+1)}y, x^{k-r-2(l+1)}y^2\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}y^2, x^{k-3(l+1)}h, x^{k-4(l+1)}yh\}$. Then $\ell_k(\mu_{k-r}) = \alpha(k-2l-1)x^k + (k-2l-1)(2\alpha a - 2\sigma\beta + 3\gamma a)x^{k-l-1}y + (k-2l-1)(\alpha - \beta a + \gamma(3a^2 - 2\sigma))x^{k-2l-2}y^2 + 3(-\beta(k-3l-2) + a\gamma(k-2l-1))x^{k-3l-3}h - 3\gamma(k-4l-3)x^{k-4l-4}yh$ and we can choose $\mathrm{Cor}(\ell_k) = span\{x^{k-3(l+1)}h, x^{k-4(l+1)}yh\}$

f) $k = 5l+4$. In this case $\mathcal{P}^{\mathbf{t}}_{k-5l-4} =< 1 >\neq \{0\}$ and, from Theorem 3.3.92, $\mathrm{Cor}(\ell_{5l+4}) = h\mathrm{Cor}(\ell_{k-3(l+1)}) = h\mathrm{Cor}(\ell_{2l+1}) = span\{x^{2l+1}h, x^l yh\}$.

From the above calculations and Theorem 3.3.93 we get a formal normal form.

■

3.4.3 Normal form of some degenerate vector fields.

In this section we present a formal normal form of vector fields whose first quasi-homogeneous term is $\mathbf{F}_r = \mathbf{X}_h$ with $h(0,y) = -\frac{1}{4}y^4$ and $h(x,0) = \frac{b}{n+1}x^{n+1}$. For that, we decompose the natural number n in the form $n = 4l+m$ with $m \in \{0,1,2,3\}$ and $l \in \mathbb{N}\cup\{0\}$ and we distinguish four different cases.

► <u>Case $m=0$</u> In this case $\mathbf{t} = (4, 4l+1)$, $r = 12l-1$ and the vector field is of the form

$$\dot{\mathbf{x}} = \begin{pmatrix} y^3 \\ b\,x^{4l} \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N}.$$

Using a scaling in the variables of state, we can get $b=1$. Therefore the vector field is of the form:

$$\dot{\mathbf{x}} = \begin{pmatrix} y^3 \\ x^{4l} \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N}. \tag{3.4.37}$$

Theorem 3.4.100. *System (3.4.37) is formally equivalent to:*

$$
\begin{aligned}
\dot{\mathbf{x}} = \begin{pmatrix} y^3 \\ x^{4l} \end{pmatrix} \quad &+ \sum_{j=4l+1}^{5l-1} \mathbf{X}_{x^{j-l}y} + \sum_{j=4l+1}^{6l-1} \mathbf{X}_{x^{j-2l}y^2} + \sum_{j=3l}^{4l-1} \alpha_{4j} x^j \mathbf{D}_0 \\
&+ \sum_{j=4l+1}^{6l-1} \alpha_{4j} x^{j-(4l+1)} h \mathbf{D}_0 + \sum_{j=3l}^{5l-1} \alpha_{4j+1} x^{j-l} y \mathbf{D}_0 \\
&+ \sum_{j=5l+1}^{6l-1} \alpha_{4j+1} x^{j-(5l+1)} y h \mathbf{D}_0 \\
&+ \sum_{j=3l}^{6l-1} \alpha_{4j+2} x^{j-2l} y^2 \mathbf{D}_0 + \sum_{i=0}^{\infty} \left[\sum_{j=6l}^{8l} \alpha_{4j+i(16l+4)} x^{j-(4l+1)} h^{i+1} \right. \\
&+ \sum_{j=8l+2}^{10l} \alpha_{4j+i(16l+4)} x^{j-2(4l+1)} h^{i+2} + \sum_{j=6l}^{9l} \alpha_{4j+1+i(16l+4)} x^{j-(5l+1)} y h^{i+1} \\
&+ \sum_{j=9l+2}^{10l} \alpha_{4j+1+i(16l+4)} x^{j-(9l+2)} y h^{i+2} + \sum_{j=6l}^{9l} \alpha_{4j+1+i(16l+4)} x^{j-(5l+1)} y h^{i+1} \\
&+ \sum_{j=9l+2}^{10l} \alpha_{4j+1+i(16l+4)} x^{j-(9l+2)} y h^{i+2} \\
&+ \left. \sum_{j=6l+1}^{10l} \alpha_{4j+2+i(16l+4)} x^{j-(6l+1)} y^2 h^{i+1} \right] \mathbf{D}_0 .
\end{aligned}
$$

Proof. We apply Theorem 3.3.93 to system (3.4.37). In this particular case, $n_0 = 24l - 1$, $r = 12l - 1$, $|\mathbf{t}| = 4l + 5$ and it is necessary to calculate $\mathrm{Cor}(\ell_k)$ for $k = 12l, \cdots, 40l + 2$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k$ for $k = 16l + 5, \cdots, 40l + 2$ with $l \in \mathbb{N}$.

In order to calculate $\mathrm{Cor}(\ell_k)$ we distinguish four cases: $k = 4j + i$ with $j = 0, 1, 2, 3$ and $j \in \mathbb{N} \cup \{0\}$.

a) $k = 4j$ $(i = 0)$.

- Case $3l \leq j \leq 4l - 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^j\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^j\}$.
- Case $j = 4l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{y\}$ and $\mathcal{P}^{\mathbf{t}}_{4(4l)} = span\{x^{4l}\}$ and it is easy to prove that $\mathrm{Cor}(\ell_k) = \{0\}$.
- Case $4l + 1 \leq j \leq 8l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-4l}y\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^j, x^{j-(4l+1)}h\}$. Considering $\mu_{k-r} = \alpha x^{j-4l}y$, then $\ell_k(\mu_{k-r}) =$

$-\frac{(12l-1-4j)\alpha}{4l+1}x^j + 4(-j+4l)\alpha x^{j-4l-1}h$ and we can choose $\mathrm{Cor}(\ell_k) = span\{x^{j-4l-1}h\}$.

• Case $j = 8l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{4l+1}y, yh\}$ and $\mathcal{P}^{\mathbf{t}}_{4(8l+1)} = span\{x^{8l+1}, x^{4l}h\}$. Therefore $\mathrm{Cor}(\ell_k) = \{0\}$.

• Case $8l+2 \le j \le 10l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-4l}y, x^{j-8l-)}yh\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^j, x^{j-4l-1}h, x^{j-8l-2}h^2\}$. Taking $\mu_{k-r} = \alpha x^{j-4l}y + \beta x^{j-8l-1}yh$, then $\ell_k(\mu_{k-r}) = -\frac{\alpha(12l-1-4j)}{4l+1}x^j + \frac{4(4l+1)(4l-j)\alpha-(28l+3-4j)\beta}{4l+1}x^{j-4l-1}h + 4(-j+8l+1)\beta x^{j-8l-2}h^2$, and we can choose $\mathrm{Cor}(\ell_k) = span\{x^{j-8l-2}h^2\}$.

In these cases, it is easy to prove that $\mathrm{Cor}(\ell_k) \cap \Delta_k = \{0\}$.

b) $k = 4j+1$ $(i=1)$.

• Case $3l \le j \le 5l-1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^{j-l}y\}$.

• Case $j = 5l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{y^2\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{4l}y\}$. Hence, if we take $\mu_{k-r} = \alpha y^2$, then $\ell_k(\mu_{k-r}) = 2\alpha x^{4l}y$ and consequently, $\mathrm{Cor}(\ell_k) = span\{0\}$.

• Case $5l+1 \le j \le 9l$. In this case $\mathcal{P}^{t}_{k-r} = \{x^{j-5l}y^2\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y, x^{j-5l-1}yh\}$. Considering $\mu_{k-r} = \alpha x^{j-5l}y^2$, then $\ell_k(\mu_{k-r}) = 2\frac{\alpha(2j-6l+1)}{4l+1}x^{j-l}y - 4\alpha(j-5l)x^{j-5l-1}yh$ and we can choose $\mathrm{Cor}(\ell_k) = span\{x^{j-5l-1}yh\}$.

• Case $j = 9l+1$. In this case $\mathcal{P}^{t}_{k-r} = \{x^{4l+1}y^2, y^2h\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y, x^{j-5l-1}yh\}$. It is easy to prove that $\mathrm{Cor}(\ell_k) = span\{0\}$.

• Case $9l+2 \le j \le 10l$. In this case $\mathcal{P}^{t}_{k-r} = \{x^{j-5l}y^2, x^{j-9l-1}y^2h\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y, x^{j-5l-1}yh, x^{j-9l-2}yh^2\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-5l}y^2 + \beta x^{j-9l-1}y^2h$, then $\ell_k(\mu_{k-r}) = 2\frac{\alpha(-6l+1+2j)}{4l+1}x^{j-l}y - 2\frac{(14l+1-2j)\beta+2(4l+1)(j-5l)\alpha}{4l+1}x^{j-5l-1}yh - 4(j-9l-1)\beta x^{j-9l-2}yh^2$ consequently, $\mathrm{Cor}(\ell_k) = span\{x^{j-9l-2}yh^2\}$.

In these cases, we can see that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{x^{j-l}y\}$ for $j = 4l+1, \cdots, 5l-1$.

c) $k = 4j+2$ $(i=2)$.

• Case $3l \le j < 6l-1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{0\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-2l}y^2\}$. In consequence $\mathrm{Cor}(\ell_k) = \{x^{j-2l}y^2\}$.

- Case $j = 6l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{y^3\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-2l}y^2\}$. It is easy to prove that $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $6l + 1 \leq j \leq 10l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-6l}y^3\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-2l}y^2,$
$x^{j-6l-1}y^2h\}$. If we take $\mu_{k-r} = \alpha x^{j-6l}y^3$, then $\ell_k(\mu_{k-r}) = -\frac{\alpha(-4j+12l-3)}{4l+1}x^{j-2l}y^2+$ $4\alpha(-j+6l)x^{j-6l-1}y^2h$ consequently, $\mathrm{Cor}(\ell_k) = span\{x^{j-6l-1}y^2h\}$.

In these cases we can see that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{x^{j-2l}y^2\}$ for $j = 4l+1, \cdots, 6l-1$.

d) $k = 4j + 3$ $(i = 3)$.

- Case $3l \leq j < 7l - 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-(3l-1)}\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-3l}y^3\}$. In consequence $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $j = 7l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{4l+1}, h\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{4l}y^3\}$. It is easy to prove that $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $7l + 1 \leq j \leq 10l - 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-(3l-1)}, x^{j-7l}h\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-3l}y^3, x^{j-(7l+1)}y^3h\}$. If we take $\mu_{k-r} = \alpha x^{j-3l+1} + \beta x^{j-7l}h$, then $\ell_k(\mu_{k-r}) = \alpha(j - 3l + 1)x^{j-3l}y^3 + \beta(j - 7l)x^{j-7l-1}y^3h$. Consequently, $\mathrm{Cor}(\ell_k) = \{0\}$.

In these cases we can see that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{0\}$.

■

► <u>Case $m = 1$</u> In this case $\mathbf{t} = (2, 2l+1)$, $r = 6l+1$ and the vector field is of the form

$$\dot{\mathbf{x}} = \begin{pmatrix} y^3 + 2\tilde{a}x^{2l+1}y \\ bx^{4l+1} - (2l+1)\tilde{a}x^{2l}y^2 \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N}.$$

Using the scaling in the variables of state, $x = \frac{1}{b^{\frac{3}{12l+2}}}u$, $y = \frac{1}{b^{\frac{1}{12l+2}}}v$, this system is transformed into,

$$\dot{\mathbf{x}} = \begin{pmatrix} y^3 + 2ax^{2l+1}y \\ x^{4l+1} - (2l+1)ax^{2l}y^2 \end{pmatrix} + \cdots, \text{ with l } \in \mathbb{N}. \tag{3.4.38}$$

Theorem 3.4.101. *System (3.4.38) is formally equivalent to:*

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{pmatrix} y^3 + 2ax^{2l+1}y \\ x^{4l+1} - (2l+1)ax^{2l}y^2 \end{pmatrix} \\
&+ \sum_{j=4l+2}^{8l+2} X_{x^{j-(2l+1)}y^2} + \sum_{j=4l+2}^{8l+1} X_{x^{j-l}y} + \sum_{j=3l+1}^{4l} (\alpha_{2j}^{(0)} x^j + \alpha_{2j}^{(1)} x^{j-(2l+1)} y^2)\mathbf{D}_0 \\
&+ \sum_{j=3l+1}^{4l-1} \alpha_{2j+1} x^{j-l} y \mathbf{D}_0 + \sum_{i=0}^{\infty} [\alpha_{8l+2} x^{2l} y^2 h^i + \sum_{j=4l+2}^{6l+1} (\alpha_{2j+i(14l+5)}^{(0)} x^{j-(2l+1)} y^2 h^i \\
&+ \alpha_{2j+i(14l+5)}^{(1)} x^{j-(4l+2)} h^{i+1}) + \alpha_{12l+4} x^{2l} h^{i+1} + \sum_{j=6l+3}^{8l+2} (\alpha_{2j+i(14l+5)}^{(0)} x^{j-(4l+2)} h^{i+1} \\
&+ \alpha_{2j+i(14l+5)}^{(1)} x^{j-(6l+3)} y^2 h^{i+1}) + \sum_{j=4l}^{5l} \alpha_{(2j+1)+i(14l+5)} x^{j-l} y h^i \\
&+ \sum_{j=5l+2}^{8l+1} \alpha_{(2j+1)+i(14l+5)} x^{j-(5l+2)} y h^{i+1}]\mathbf{D}_0.
\end{aligned}$$

Proof. We apply Theorem 3.3.93 to system (3.4.38). In this particular case, $n_0 = 8l+1$, $r = 6l+1$, $|\mathbf{t}| = 8l+4$ and it is necessary to calculate $\mathrm{Cor}(\ell_k)$ for $k = 6l+2, \cdots, 16l+4$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k$ for $k = 8l+4, \cdots, 16l+4$ with $l \in \mathbb{N}$.

In order to calculate $\mathrm{Cor}(\ell_k)$ we distinguish two cases: $k = 2j+i$ with $i = 0, 1$ and $j \in \mathbb{N} \cup \{0\}$.

a) $k = 2j$ ($i = 0$).

- Case $3l+1 \le j \le 4l$. In this case $\mathcal{P}_{k-r}^{\mathbf{t}} = \{0\}$ and $\mathcal{P}_k^{\mathbf{t}} = span\{x^j, x^{j-(2l+1)}y^2\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^j, x^{j-(2l+1)}y^2\}$.

- Case $j = 4l+1$. In this case $\mathcal{P}_{k-r}^{\mathbf{t}} = span\{y\}$ and $\mathcal{P}_{2(4l+1)}^{\mathbf{t}} = span\{x^{4l+1}, x^{2l}y^2\}$. Hence, if we take $\mu_{2l+1} = \alpha y$ then $\ell_{2(4l+1)}(\mu_{2l+1}) = \alpha x^{4l+1} - \alpha(2l+1)ax^{2l}y^2$ and we can choose $\mathrm{Cor}(\ell_k) = \{x^{2l}y^2\}$.

- Case $4l+2 \le j \le 6l+1$. In this case $\mathcal{P}_{k-r}^{\mathbf{t}} = \{x^{j-(4l+1)}y\}$ and $\mathcal{P}_k^{\mathbf{t}} = span\{x^j, x^{j-(2l+1)}y^2, x^{j-(4l+2)}h\}$. Considering $\mu_{k-r} = \alpha x^{j-(4l+1)}y$, then $\ell_k(\mu_{k-r}) = -\frac{(2j-(6l+1))\alpha}{2l+1}x^j - (2j-(6l+1))\alpha a x^{j-(2l+1)}y^2 - 4(j-(4l+1))\alpha x^{j-(4l+2)}h$ and it is possible to choose $\mathrm{Cor}(\ell_k) = span\{x^{j-(2l+1)}y^2, x^{j-(4l+2)}h\}$.

- Case $j = 6l+2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{2l+1}y, y^3\}$ and $\mathcal{P}^{\mathbf{t}}_{2(4l+1)} = span\{x^{6l+2}, x^{4l+1}y^2, x^{2l}h\}$. Taking $\mu_{6l+2} = \alpha y x^{2l+1} + \beta y^3$ then $\ell_{2(6l+2)}(\mu_{2l+1}) = -3(2a\beta - \alpha)x^{6l+2} + 3(-2\alpha al - \alpha a + \beta + 8a^2\beta l + 4a^2\beta)x^{4l+1}y^2 + 4(2l+1)(-\alpha + 3a\beta)x^{2l}h$ and we can choose $\mathrm{Cor}(\ell_k) = \{x^{2l}h\}$.

- Case $6l+3 \leq j \leq 8l+2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-(4l+1)}y, x^{j-(6l+2)}y^3\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^j, x^{j-(2l+1)}y^2, x^{j-(4l+2)}h, x^{j-(6l+3)}y^2h\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-(4l+1)}y + \beta x^{j-6l+2y^3}$, then $\ell_k(\mu_{k-r}) = \frac{(2j-1-6l)(2\beta a-\alpha)}{2l+1}x^j + \frac{(8a^2\beta l-\alpha a+4a^2\beta+\beta-2\alpha la)(2j-1-6l)}{2l+1}x^{j-2l-1}y^2 + 4(-6\beta la + 4\alpha l + \alpha - \alpha j - \beta a + 2\beta ja)x^{j-4l-2}h - 4\beta(j-6l-2)y^2x^{j-6l-3}h$, consequently $\mathrm{Cor}(\ell_k) = span\{x^{j-(4l+2)}h, x^{j-(6l+3)}y^2h\}$.

  In these cases we can see that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{x^{j-(2l+1)}y^2\}$ for $j = 4l+2, \cdots, 8l+2$.

b) $k = 2j+1$ $(i = 1)$.

- Case $3l+1 \leq j \leq 5l$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-3l}\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y, x^{j-(3l+1)}y^3\}$. If we take $\mu_{k-r} = \alpha x^{j-3l}$ then $\ell_k(\mu_{k-r}) = 2\alpha(j-3l)ax^{j-l}y - \alpha(j-3l)x^{j-(3l+1)}y^3$ and we can choose $\mathrm{Cor}(\ell_k) = \{x^{j-l}y\}$.

- Case $j = 5l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{2l+1}, y^2\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{4l+1}y, x^{2l}y^3\}$. It is easy to prove that $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $5l+2 \leq j \leq 7l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-3l}, x^{j-(5l+1)}y^2\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y, x^{j-(3l+1)}y^3, x^{j-(5l+2)}yh\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-3l} + \beta x^{j-(5l+1)l}y^2$ then $\ell_k(\mu_{k-r}) = \frac{2(j-3l)(2\alpha la+\alpha a+\beta)}{2l+1}x^{j-l}y - (j-3l)(-\alpha+2\beta a)x^{j-(3l+1)}y^3 - 4\beta(j-(5l+1))x^{j-(5l+2)}yh$ and we can choose $\mathrm{Cor}(\ell_k) = \{x^{j-(5l+2)}yh\}$.

- Case $j = 7l+2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{4l+2}, x^{2l+1}y^2, h\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{6l+2}y, x^{4l+1}y^3,$
  $x^{2l}yh\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{4l+2} + \beta x^{2l+1}y^2 + \gamma h$ then $\ell_k(\mu_{k-r}) = 4(2\alpha al + \alpha a + \beta)x^{6l+2}y - 2(2l+1)(-\alpha+2\beta a)x^{4l+1}y^3 - 4\beta(2l+1)x^{2l}yh$ and we can choose $\mathrm{Cor}(\ell_k) = \{x^{j-(5l+2)}yh\}$.

- Case $7l+3 \leq j \leq 8l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-3l}, x^{j-(5l+1)}y^2, x^{j-(7l+2)}h\}$ and $\mathcal{P}^{\mathbf{t}}_{k} = span\{x^{j-l}y, x^{j-(3l+1)}y^3, x^{j-(5l+2)}yh, x^{j-(7l+3)}y^3h\}$. Hence, if we take $\mu_{k-r} = \alpha x^{j-3l} + \beta x^{j-(5l+1)l}y^2 + \gamma x^{j-(7l+2)}h$ then $\ell_k(\mu_{k-r}) =$

$\frac{2(j-3l)(\alpha a+\beta+2\alpha la)}{2l+1}x^{j-l}y-(j-3l)(2\beta a-\alpha)x^{j-(3l+1)}y^3+2(10\beta l-7\gamma la+2\beta-2\gamma a-2\beta j+\gamma ja)x^{j-(5l+2)}yh+\gamma(j-7l-2)x^{j-(7l+3))}y^3h$ and we can choose $\mathrm{Cor}(\ell_k)=\{x^{j-(5l+2)}yh\}$.

In these cases we can see that $\mathrm{Cor}(\ell_k)\cap\Delta_k = span\{x^{j-l}y\}$ for $j = 4l+2,\cdots,8l+1$.

■

► Case $m=2$ In this case $\mathbf{t}=(4,4l+3)$, $r=12l+5$ and the vector field is of the form

$$\dot{\mathbf{x}}=\begin{pmatrix} y^3\\ bx^{4l+2}\end{pmatrix}+\cdots,\ \text{with l }\in\mathbb{N}.$$

Using the scaling in the variables of state, $x=\frac{1}{b^{\frac{3}{12l+2}}}u$, $y=\frac{1}{b^{\frac{1}{12l+2}}}v$, this system is transformed into,

$$\dot{\mathbf{x}}=\begin{pmatrix} y^3\\ x^{4l+2}\end{pmatrix}+\cdots,\ \text{with l }\in\mathbb{N}. \tag{3.4.39}$$

Theorem 3.4.102. *System (3.4.39) is formally equivalent to:*

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{pmatrix} y^3\\ x^{4l+2}\end{pmatrix}\\
&+ \sum_{j=4l+1}^{6l+2} X_{x^{j-(2l+1)}y^2}+\sum_{j=4l+3}^{5l+1} X_{x^{j-l}y}+\sum_{j=3l+2}^{4l+1}\alpha_{4j}x^j\mathbf{D}_0+\sum_{j=4l+3}^{6l+2}\alpha_{4j}x^{j-(4l+3)}h\mathbf{D}_0\\
&+ \sum_{j=3l+2}^{6l+2}\alpha_{4j+2}x^{j-(2l+1)}y^2\mathbf{D}_0+\sum_{j=3l+1}^{5l+1}\alpha_{4j+2}x^{j-l}y\mathbf{D}_0+\sum_{j=5l+3}^{6l+1}\alpha_{4j+2}x^{j-(5l+3)}yh\mathbf{D}_0\\
&+ \sum_{i=0}^{\infty}[\alpha_{8l+2}x^{2l}y^2h^i+\sum_{j=4l+2}^{6l+1}(\alpha^{(0)}_{2j+i(14l+5)}x^{j-(2l+1)}y^2h^i\\
&+ \alpha^{(1)}_{2j+i(14l+5)}x^{j-(4l+2)}h^{i+1})+\alpha_{12l+4}x^{2l}h^{i+1}+\sum_{j=6l+3}^{8l+2}(\alpha^{(0)}_{2j+i(14l+5)}x^{j-(4l+2)}h^{i+1}\\
&+ \alpha^{(1)}_{2j+i(14l+5)}x^{j-(6l+3)}y^2h^{i+1})+\sum_{j=4l}^{5l}\alpha_{(2j+1)+i(14l+5)}x^{j-l}yh^i\\
&+ \sum_{j=5l+2}^{8l+1}\alpha_{(2j+1)+i(14l+5)}x^{j-(5l+2)}yh^{i+1}]\mathbf{D}_0.
\end{aligned}$$

Proof. We apply Theorem 3.3.93 to the system (3.4.39). In this particular case, $n_0 = 24l + 11$, $r = 12l + 5$, $|\mathbf{t}| = 4l + 7$ and it is necessary to calculate $\mathrm{Cor}(\ell_k)$ for $k = 12l+6, \cdots, 40l+22$ and $\mathrm{Cor}(\ell_k) \cap \Delta_k$ for $k = 16l+13, \cdots, 40l+22$ with $l \in \mathbb{N}$.

In order to calculate $\mathrm{Cor}(\ell_k)$ we distinguish four cases: $k = 4j + i$ with $j = 0, 1, 2, 3$ and $j \in \mathbb{N} \cup \{0\}$.

a) $k = 4j$ $(i = 0)$.

- Case $3l + 2 \leq j \leq 4l + 1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^j\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^j\}$.

- Case $j = 4l + 2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{y\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{4l+2}\}$. Therefore $\mathrm{Cor}(\ell_k) = \{0\}$.

- Case $4l + 3 \leq j \leq 8l + 4$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-(4l+2)}y\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^j, x^{j-(4l+3)}h\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-(4l+2)}y$, then $\ell_k(\mu_{k-r}) = \frac{\alpha(4j-5-12l)}{4l+3)} x^j - 4\alpha(j - 4l - 2)x^{j-(4l+3)}h$, consequently $\mathrm{Cor}(\ell_k) = span\{x^{j-(4l+3)}h\}$.

- Case $j = 8l + 5$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{4l+3}y, yh\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{8l+5}, x^{4l+2}h\}$. Hence, if we take $\mu_{k-r} = \alpha x^{4l+3}y + \beta yh$, then $\ell_k(\mu_{k-r}) = 5\alpha x^(8l + 5 - (16\alpha l + 12\alpha - \beta)x^{4l+2}h$ and consequently $\mathrm{Cor}(\ell_k) = span\{0\}$.

- Case $8l+6 \leq j \leq 10l+5$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{j-(4l+2)}y, x^{j-(8l+5)}yh\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^j, x^{j-(4l+3)}h, x^{j-(8l+6)}h^2\}$. If we take $\mu_{k-r} = \alpha x^{j-(4l+2)}y + \beta x^{j-(8l+5)}yh$, then $\ell_k(\mu_{k-r}) = \frac{\alpha(-12l-5+4j)}{4l+3} x^j - \frac{(17+28l-4j)\beta+(4(4l+3))(j-4l-2)\alpha}{4l+3} x^{j-(4l+3)}h - 4\beta(j - 8l - 5)x^{j-(8l+6)}h^2$, and we can choose $\mathrm{Cor}(\ell_k) = span\{x^{j-(8l+6)}h^2\}$.

In these cases, it is easy to prove that $\mathrm{Cor}(\ell_k) \cap \Delta_k = \{0\}$.

b) $k = 4j + 1$ $(i = 1)$.

- Case $3l + 2 \leq j \leq 7l + 3$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-(3l+1)}\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-(3l+2)}y^3\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{0\}$.

- Case $j = 7l+4$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{4l+3}, h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{4l+2}y^3\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{4l+3} + \beta h$, then $\ell_k(\mu_{k-r}) = \alpha(4l + 3)x^{4l+2}y^3$. Consequently, $\mathrm{Cor}(\ell_k) = span\{0\}$.

- Case $7l+5 \le j \le 10l+5$. In this case $\mathcal{P}^t_{k-r} = \{x^{j-(3l+1)}, x^{j-(7l+4)}h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-(3l+2)}y^3, x^{j-(7l+5)}y^3h\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-(3l+1)} + \beta x^{j-(7l+4)}h$, then $\ell_k(\mu_{k-r}) = \alpha(j-3l-1)x^{j-(3l+2)}y^3$. Consequently, $\mathrm{Cor}(\ell_k) = span\{0\}$.

  Obviously, in these cases we can see that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{0\}$ for $j = 4l+1, \cdots, 5l-1$.

c) $k = 4j+2$ $(i = 2)$.

- Case $3l+1 \le j \le 6l+2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-(2l+1)}y^2\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^{j-(2l+1)}y^2\}$.

- Case $j = 6l+3$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{y^3\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-(2l+1)}y^2\}$. Therefore, if we take $\mu_{k-r} = \alpha y^3$, then $\ell_k(\mu_{k-r}) = 3\alpha y^2 x^{4l+2}$. Consequently, $\mathrm{Cor}(\ell_k) = span\{0\}$.

- Case $6l+4 \le j \le 10l+5$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{x^{j-(6l+3)}y^3\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-(2l+1)}y^2,$
  $x^{j-(6l+4)}y^2h\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-(6l+3)}y^3$, then $\ell_k(\mu_{k-r}) = \frac{\alpha(4j-3-12l)}{4l+3}x^{j-(2l+1)}y^2 - 4\alpha(j-6l-3)x^{j-(6l+4)}y^2h$. Consequently, $\mathrm{Cor}(\ell_k) = span\{x^{j-(6l+4)}y^2h\}$.

  Obviously, in these cases we can see that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{x^{j-(2l+1)}y^2\}$ for $j = 4l+1, \cdots, 6l+2$.

d) $k = 4j+3$ $(i = 3)$.

- Case $3l+1 \le j \le 5l+1$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{0\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^{j-l}y\}$.

- Case $j = 5l+2$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \{y^2\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{4l+2}y\}$. Therefore, if we take $\mu_{k-r} = \alpha y^2$, then $\ell_k(\mu_{k-r}) = 2\alpha x^{4l+2}y$ consequently, $\mathrm{Cor}(\ell_k) = span\{0\}$.

- Case $5l+3 \le j \le 9l+4$. In this case $\mathcal{P}^t_{k-r} = \{x^{j-(5l+2)}y^2\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y, x^{j-(5l+3)}yh\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-(5l+2)}y^2$, then $\ell_k(\mu_{k-r}) = 2\frac{\alpha(2j-6l-1)}{4l+3}x^{j-l}y - 4\alpha(j-5l-2)x^{j-(5l+3)}yh$ consequently, $\mathrm{Cor}(\ell_k) = span\{x^{j-(5l+3)}yh\}$.

- Case $j = 9l+5$. In this case $\mathcal{P}^t_{k-r} = \{x^{4l+3}y^2, y^2h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{8l+5}y, x^{4l+2}yh\}$. Therefore $\mathrm{Cor}(\ell_k) = span\{0\}$.

• Case $9l+6 \leq j \leq 10l+4$. In this case $\mathcal{P}^t_{k-r} = \{x^{j-(5l+2)}y^2, x^{j-(9l+5)}y^2h\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^{j-l}y, x^{j-(5l+3)}yh, x^{j-(9l+6)}yh^2\}$. Therefore, if we take $\mu_{k-r} = \alpha x^{j-(5l+2)}y^2 + \beta x^{j-(9l+5)}y^2h$, then $\ell_k(\mu_{k-r}) = 2\frac{\alpha(2j-6l-1)}{4l+3}x^{j-l}y - 2\frac{(14l+7-2j)\beta+2(4l+3)(j-5l-2)\alpha}{4l+3}x^{j-(5l+3)}yh - 4(j-9l-5)\beta x^{j-(9l+6)}yh^2$ consequently, $\mathrm{Cor}(\ell_k) = span\{x^{j-(9l+6)}yh^2\}$.

In these cases we can see that $\mathrm{Cor}(\ell_k) \cap \Delta_k = span\{x^{j-l}y\}$ for $j = 4l+3, \cdots, 5l+1$.

■

► <u>Case $m = 3$</u> In this case $\mathbf{t} = (1, l+1)$, $r = 3l+2$ and the vector field is of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{X}_h + \cdots \\ &= \begin{pmatrix} y^3 + 3\tilde{a}x^{l+1}y^2 + 2\tilde{b}x^{2(l+1)}y + \tilde{c}x^{3(l+1)} \\ -\tilde{a}(l+1)x^ly^3 - 2\tilde{b}(l+1)x^{2l+1}y^2 - 3(l+1)\tilde{c}x^{3l+2}y + \tilde{d}x^{4l+3}\end{pmatrix} + \cdots,\end{aligned} \tag{3.4.40}$$

with $l \in \mathbb{N}\cup\{0\}$, $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}$ and $h := -\frac{1}{4}y^4 - \tilde{a}x^{l+1}y^3 - \tilde{b}x^{2(l+1)}y^2 - \tilde{c}x^{3(l+1)}y + \frac{\tilde{d}}{4(l+1)}x^{4(l+1)}$ has only simple factors.

Lemma 3.4.103. *System (3.4.40) is analytically equivalent to the system,*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \cdots \tag{3.4.41}$$

where $h = -\frac{1}{4}\left[y^2 + \sigma_1 x^{2(l+1)}\right]\left[(y - ax^{l+1})^2 + \sigma_2 b^2 x^{2(l+1)}\right]$, *with* $l \in \mathbf{N} \cup \{0\}$, $\sigma_1, \sigma_2 = \pm 1$, *and* $4a^2\sigma_1 + (a^2 + b^2\sigma_2 - \sigma_1)^2 \neq 0$, *i.e., h has only simple factors. Moreover if* $l = 0$, $(2 + \sigma_1 + \sigma_2) > 0$ *or* $l = 0$, $\sigma_1 = \sigma_2 = -1$ *and* $|a| \notin (|b-1|, b+1)$ *then*

$$h = -\frac{1}{4}\left[y^2 + \sigma_1 x^{2(l+1)}\right]\left[y^2 + \sigma_2 \tilde{b}^2 x^{2(l+1)}\right] \quad \text{with } \tilde{b} \neq \sigma_1\sigma_2$$

and if $l = 0$, $\sigma_1 = \sigma_2 = -1$ *and* $|a| \in (|b-1|, b+1)$, *then*

$$h = -\frac{1}{3}xy\left[y^2 + 2Axy - x^2\right] \quad \text{with } A \in \mathbb{R}$$

Proof. Since h only has simple factors, it is posible to expres it in the form:

$$h = -\frac{1}{4}\left[(y-\tilde{a}_1x^{l+1})^2+\sigma_1\tilde{b}_1^2x^{2(l+1)}\right]\left[(y-\tilde{a}_2x^{l+1})^2+\sigma_2\tilde{b}_2^2x^{2(l+1)}\right],$$

with $\tilde{a}_1,\tilde{a}_2,\tilde{b}_1,\tilde{b}_2\in\mathbb{R}$, $\tilde{b}_1>0$, $\tilde{b}_2>0$, $\sigma_1,\sigma_2\in\{-1,1\}$ such that using the change of variables $u=\sqrt[l+1]{\tilde{b}_1}x$, $v=y-\tilde{a}_1x^{l+1}$, system (3.4.40) is transformed into $\dot{\mathbf{u}}=\mathbf{X}_h+\cdots$, being $\mathbf{X}_h$ the following vector field,

$$\begin{pmatrix} v^3-\frac{3}{2}a_2u^{l+1}v^2+\frac{1}{2}(\sigma_1+a_2^2+\sigma_2b_2^2)u^{2(l+1)}v-\frac{1}{2}\sigma_1a_2u^{3(l+1)} \\ \frac{(l+1)a_2}{2}u^lv^3-\frac{(l+1)(\sigma_1+a_2^2+\sigma_2b_2^2)}{2}u^{2l+1}v^2+\frac{3(l+1)\sigma_1a_2}{2}u^{3l+2}v-\sigma_1(l+1)(a_2^2+\sigma_2b_2^2)u^{4l+3}\end{pmatrix}$$

with $l\in\mathbb{N}\cup\{0\}$, $h:=-\frac{1}{4}\left[v^2+\sigma_1u^{2(l+1)}\right]\left[(v-a_2u^{l+1})^2+\sigma_2b_2^2u^{2(l+1)}\right]$, $4a_2^2\sigma_1+(a_2^2+\sigma_2b_2^2-\sigma_1)^2\neq 0$, i.e., h has only simple factors in its decomposition on $\mathbb{C}[x,y]$.

In the particular case $l=0$, using the change of variables $x=u-\sigma_1\alpha v$, $y=\alpha u+v$, we obtain the vector field $\mathbf{X}_{\tilde{h}}$ where:

$$\begin{aligned}\tilde{h} &= -\frac{1}{4}\left[v^2+\sigma_1u^{2(l+1)}\right]\left[((a_2^2+\sigma_2b_2^2)\alpha^2+2\sigma_1a_2\alpha+1)v^2+\right.\\ &\quad \left.(\alpha^2-2a_2\alpha+a_2^2+\sigma_2b_2^2)u^2+2(a_2\sigma_1\alpha^2+(1-\sigma_1(a_2^2+\sigma_2b_2^2))\alpha-a_2)uv\right]\end{aligned}$$

at this point, we intend to eliminate any parameter to achieve the reversibility criterion, in this sense we choose α so that the coefficient of uv is zero. For this task, it is sufficient to require that the discriminant is non-negative, i.e., $\Delta=(1-\sigma_1(a_2^2+\sigma_2b_2^2))^2+4\sigma_1a_2^2>0$, (otherwise, if $\Delta=0$ then h not have simple factors).

- If $a_2=0$ then we have just achieved our aim.
- If $a_2\neq 0$ and $\sigma_1=1$, we also reached our aim.
- If $a_2\neq 0$ y $\sigma_1=-1$ then

  $$\Delta = (1+a_2^2+\sigma_2b_2^2)^2-4a_2^2=\left[(1+a_2)^2+\sigma_2b_2^2\right]\left[(1-a_2)^2+\sigma_2b_2^2\right]$$

  - If $\sigma_2=1$ we have achieved our objective.
  - If $\sigma_2=-1$ then,

    $$\begin{aligned}\Delta &= (1+a_2+b_2)(1+a_2-b_2)(1-a_2+b_2)(1-a_2-b_2)=\\ &\quad \left[(b_2+1)^2-a_2^2\right]\left[(b_2-1)^2-a_2^2\right].\end{aligned}$$

In summary, the condition $\Delta > 0$ is equivalent to $(2+\sigma_1+\sigma_2) > 0$ or $\sigma_1 = \sigma_2 = -1$ and $b_2+1 < |a_2|$ or $|a_2| < |b_2-1|$, e.i., $\sigma_1 = \sigma_2 = -1$ and $|a_2| \notin [|b_2-1|, b_2+1]$.

Finally, if $\sigma_1 = \sigma_2 = -1$ and $|a_2| \in (|b_2-1|, b_2+1)$, then $\Delta < 0$ and there is no change of variables to annul a_2, i.e., X_h is not orbitally R_x-reversible. In this case, $h := -\frac{1}{4}\left[y^2-x^2\right]\left[(y-a_2x)^2-b_2^2x^2\right]$. Applying the change of variables $u = y-x$, $v = y+x$, we obtain a new field $\dot{x} = \mathbf{X}_{\tilde{h}}$ where:

$$\tilde{h} = -\tfrac{1}{8}uv\left[((a_2-1)^2-b_2^2)v^2+2(b_2^2+1-a_2^2)uv+((a_2+1)^2-b_2^2)u^2\right]$$

since $\Delta = ((a_2-1)^2-b_2^2)((a_2+1)^2-b_2^2) < 0$, we can express it in the form,

$$\tilde{h} = -\tfrac{(a_2-1)^2-b_2^2}{8}uv\left[v^2+2\tfrac{b_2^2+1-a_2^2}{(a_2-1)^2-b_2^2}uv+\tfrac{(a_2+1)^2-b_2^2}{(a_2-1)^2-b_2^2}u^2\right]$$

with a scaling in the state variables and time, we can obtain,

$$h = -\tfrac{1}{3}uv[v^2+2Auv-u^2]$$

■

Theorem 3.4.104. *System (3.4.41) is formally equivalent to:*

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{X}_h + \sum_{j=2l+2}^{3l+1}\alpha_j x^j\mathbf{D}_0 + \sum_{j=l+1}^{3l+2}\alpha_{l+1+j}x^jy\mathbf{D}_0 \\ &\quad + \sum_{\substack{j=0\\ j\neq 2l+1}}^{3l+1} x^j f_j^{(0)}(h)\mathbf{D}_0 + \sum_{\substack{j=0\\ j\neq l}}^{3l+2} x^j f_j^{(1)}(h)\mathbf{D}_0 \end{aligned} \tag{3.4.42}$$

where $h = -\frac{1}{4}\left[y^2+\sigma_1x^{2(l+1)}\right]\left[(y-ax^{l+1})^2+\sigma_2b^2x^{2(l+1)}\right]$ *and* $f_j^{(i)} = \alpha_{0,j}^{(i)} + \alpha_{1,j}^{(i)}h+\cdots \in \mathbf{C}[[h]]$, $i = 0,1$.

Proof. To calculate a normal form, it is sufficient to apply Theorem 3.3.93 to system (3.4.35). In this particular case, $n_0 = 3l+3$, $r = 3l+2$ and $|\mathbf{t}| = l+2$ and it is only necessary to calculate $\mathrm{Cor}(\ell_k)$ for $k = 3l+3,\cdots,7l+6$.

a) $3l+3 \leq k \leq 4l+2$. In this case, $\mathcal{P}_{k-r}^{\mathbf{t}} = span\{x^{k-r}\}$ and $\mathcal{P}_k^{\mathbf{t}} = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}(y^2+\sigma_1x^{2(l+1)}), x^{k-3(l+1)}y(y^2+\sigma_1x^{2(l+1)})\}$. If we take $\mu_{k-r} =$

αx^{k-r} then $\ell_k(\mu_{k-r}) = \frac{1}{2}(k-3l-2)\alpha x^{k-3l-3}[\sigma_1 a x^{3(l+1)} + (a^2+\sigma_2 b^2 - \sigma_1)x^{2(l+1)}y - 3ax^{l+1}(y^2+\sigma_1 x^{2(l+1)}) + 2y(y^2+\sigma_1 x^{2(l+1)})]$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}(y^2+\sigma_1 x^{2(l+1)})\}$.

b) $k = 4l+3$. In this case, $\mathcal{P}^{\mathbf{t}}_{k-r} = \mathcal{P}^{\mathbf{t}}_{l+1} = span\{x^{l+1}, y\}$ and $\mathcal{P}^{\mathbf{t}}_k = \mathcal{P}^{\mathbf{t}}_{4l+3} = span\{x^{4l+3}, x^{3l+2}y, x^{2l+1}(y^2+\sigma_1 x^{2(l+1)}), x^l y(y^2+\sigma_1 x^{2(l+1)})\}$. If we take $\mu_{l+1} = \alpha x^{l+1} + \beta y$ then

$$\begin{array}{rcl} \ell_k(\mu_{l+1}) & = & \frac{1}{2}(l+1)[\sigma_1(2a\alpha - (a^2+\sigma_2 b^2 - \sigma_1)\beta)x^{4l+3} \\ & & +((a^2+\sigma_2 b^2-\sigma_1)\alpha + 2a\sigma_1\beta)x^{3l+2}y \\ & & -(3a\alpha - (a^2+\sigma_2 b^2+\sigma_1)\beta)x^{2l+1}(y^2+\sigma_1 x^{2(l+1)}) \\ & & +(2\alpha + a\beta)x^l y(y^2+\sigma_1 x^{2(l+1)})]. \end{array}$$

Taking into account that $4a^2\sigma_1 + (a^2+\sigma_2 b^2 - \sigma_1)^2 \neq 0$, we get $\mathrm{Cor}(\ell_k) = span\{x^{2l+1}(y^2+\sigma_1 x^{2(l+1)}), x^l y(y^2+\sigma_1 x^{2(l+1)})\}$.

For the case $l=0$, $\sigma_1=\sigma_2=-1$, $|a| \in (|b-1|, b+1)$ we obtain that $\ell_k(\mu_{l+1}) = -\frac{1}{3}(\alpha x^3 - (4A\alpha+3\beta)x^2y - (3\alpha - 4A\beta)xy^2 + \beta y^3)$. Therefore $\mathrm{Cor}(\ell_k) = span\{x^2y, xy^2\}$.

c) $4l+4 \leq k \leq 5l+3$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{k-r}, x^{k-r-(l+1)}y\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}(y^2+\sigma_1 x^{2(l+1)}), x^{k-3(l+1)}y(y^2+\sigma_1 x^{2(l+1)}), x^{k-4(l+1)}h\}$. If we take $\mu_{k-r} = \alpha x^{k-r} + \beta x^{k-r-(l+1)}y$ then

$$\begin{array}{rcl} \ell_k(\mu_{k-r}) & = & -\frac{k-3l-2}{2}\left[\sigma_1(2a\alpha - (a^2+\sigma_2 b^2-\sigma_1)\beta)x^k \right. \\ & & -((a^2+\sigma_2 b^2-\sigma_1)\alpha + 2a\sigma_1\beta)x^{k-l-1}y + \\ & & (3a\alpha + (a^2+\sigma_2 b^2+\sigma_1)\beta)x^{k-2(l+1)}(y^2+\sigma_1 x^{2(l+1)}) \\ & & -(2\alpha + a\beta)x^{k-3(l+1)}y(y^2+\sigma_1 x^{2(l+1)}) \\ & & \left. +\frac{8(k-4l-3)}{k-3l-2}\beta x^{k-4(l+1)}h\right]. \end{array}$$

Taking into account that $4a^2\sigma_1 + (a^2+\sigma_2 b^2 - \sigma_1)^2 \neq 0$, we can choose $\mathrm{Cor}(\ell_k) = span\{x^{k-2(l+1)}(y^2+\sigma_1 x^{2(l+1)}), x^{k-3(l+1)}y(y^2+\sigma_1 x^{2(l+1)}), x^{k-4(l+1)}h\}$.

d) $k = 5l+4$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = \mathcal{P}^{\mathbf{t}}_{2(l+1)} = span\{x^{2(l+1)}, x^{l+1}y, y^2\}$ and $\mathcal{P}^{\mathbf{t}}_k = \mathcal{P}^{\mathbf{t}}_{5l+4} = span\{x^{5l+4}, x^{4l+3}y, x^{3l+2}(y^2+\sigma_1 x^{2(l+1)}), x^{2l+1}y(y^2+\sigma_1 x^{2(l+1)}), x^l h\}$.

If we take $\mu_{2(l+1)} = \alpha x^{2(l+1)} + \beta x^{(l+1)}y + \gamma y^2$ then

$$\begin{aligned}\ell_k(\mu_{2(l+1)}) \;=\; & (l+1)\left[-\sigma_1(\gamma M_2\sigma_2N_2^2 + M_2\alpha + 2\beta N_2^2\sigma_2 + 2\beta M_2^2 + \gamma M_2^3)x^{5l+4}\right.\\ & -(-\alpha N_2^2\sigma_2 - \alpha M_2^2 - \alpha\sigma_1 - 3\beta\sigma_1M_2 + 2\gamma\sigma_1N_2^2\sigma_2)x^{4l+3}y\\ & -(-M_2\beta - \gamma M_2^2 + \gamma N_2^2\sigma_2 + \gamma\sigma_1 - 2\alpha)x^{2l+1}y(y^2+\sigma_1x^{2l+2})\\ & -4(\beta+\gamma M_2)x^lh + (-\beta\sigma_1 - \beta M_2^2 - \beta N_2^2\sigma_2 + 2\gamma\sigma_1M_2 - \gamma M_2^3\\ & \left.-\gamma M_2\sigma_2N_2^2 - 3M_2\alpha)x^{3l+2}(y^2+\sigma_1x^{2l+2})\right].\end{aligned}$$

Therefore we can choose $\text{Cor}(\ell_k) = span\{x^lh\}$

e) $5l{+}5 \leq k \leq 6l{+}4$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{k-r}, x^{k-r-(l+1)}y, x^{k-r-2(l+1)}y^2,\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}y^2, x^{k-3(l+1)}y^3, x^{k-4(l+1)}h, x^{k-5(l+1)}yh\}$. Then $\ell_k(\mu_{k-r}) = \alpha(k-2l-1)x^k + (k-2l-1)(2\alpha a - 2\sigma\beta + 3\gamma a)x^{k-l-1}y + (k-2l-1)(\alpha - \beta a + \gamma(3a^2-2\sigma))x^{k-2l-2}y^2 + 3(-\beta(k-3l-2) + a\gamma(k-2l-1))x^{k-3l-3}h - 3\gamma(k-4l-3)x^{k-4l-4}yh$ and we can choose $\text{Cor}(\ell_k) = span\{x^{k-3(l+1)}h, x^{k-4(l+1)}yh\}$

f) $k = 6l+5$. $\mathcal{P}^{\mathbf{t}}_{k-r} = \mathcal{P}^{\mathbf{t}}_{3(l+1)} = span\{x^{3(l+1)}, x^{2(l+1)}y, x^{l+1}y^2, y^3\}$ and $\mathcal{P}^{\mathbf{t}}_k = \mathcal{P}^{\mathbf{t}}_{6l+5} = span\{x^{6l+5}, x^{5l+4}y, x^{4l+3}y^2, x^{3l+2}y^3, x^{2l+1}h, x^{l+1}yh, y^2h\}$. If we take $\mu_{3(l+1)} = \alpha x^{2(l+1)} + \beta x^{(l+1)}y + \gamma y^2$ then $\ell_k(\mu_{2(l+1)}) = +2(l+1)\alpha x^{4l+3} + 2(l+1)(2a\alpha - 2\sigma\beta + a(1+2\sigma)\gamma)x^{3l+2}y + 2(l+1)(\alpha - a\beta - (2\sigma - 3a^2)\gamma)x^{2l+1}y^2 - 3(l+1)(\beta - 2a\gamma)x^lh$. Therefore we can choose $\text{Cor}(\ell_k) = span\{x^lh\}$.

g) $6l{+}6 \leq k \leq 7l{+}5$. In this case $\mathcal{P}^{\mathbf{t}}_{k-r} = span\{x^{k-r}, x^{k-r-(l+1)}y, x^{k-r-2(l+1)}y^2, x^{k-r-3(l+1)}y^3\}$ and $\mathcal{P}^{\mathbf{t}}_k = span\{x^k, x^{k-(l+1)}y, x^{k-2(l+1)}y^2, x^{k-3(l+1)}y^3, x^{k-4(l+1)}h, x^{k-5(l+1)}yh, x^{k-6(l+1)}y^2h\}$. Then $\ell_k(\mu_{k-r}) = \alpha(k-2l-1)x^k + (k-2l-1)(2\alpha a - 2\sigma\beta + 3\gamma a)x^{k-l-1}y + (k-2l-1)(\alpha - \beta a + \gamma(3a^2 - 2\sigma))x^{k-2l-2}y^2 + 3(-\beta(k-3l-2) + a\gamma(k-2l-1))x^{k-3l-3}h - 3\gamma(k-4l-3)x^{k-4l-4}yh$ and we can choose $\text{Cor}(\ell_k) = span\{x^{k-3(l+1)}h, x^{k-4(l+1)}yh\}$

h) $k = 7l+6$. In this case $\mathcal{P}^{\mathbf{t}}_{k-7l-6} =< 1 >\neq \{0\}$ and, from Theorem 3.3.92, $\text{Cor}(\ell_{7l+6}) = h\text{Cor}(\ell_{k-4(l+1)}) = h\text{Cor}(\ell_{3l+2}) = span\{x^{3l+2}h, x^{2l+1}yh, x^ly^2h\}$.

From the above calculations and Theorem 3.3.93 we get a formal normal form.

■

CHAPTER 4

Applications: Integrability and inverse integrating factor.

4.1 Introduction

We consider an autonomous system with the form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T, \ \mathbf{x} \in \mathbb{C}^2, \tag{4.1.1}$$

where $\mathbf{F}$ is a formal planar vector field defined in a neighborhood of the origin $U \subset \mathbb{C}^2$ having an equilibrium point at the origin, i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $P, Q \in \mathbb{C}[[x, y]]$ (algebra of the power series in x and y with coefficients in $\mathbb{C}$, convergent or not).

In this chapter is treated the study of the integrability of $\mathbf{F}$. The integrability problem in $\mathbb{R}^2$, consists in determining when a planar vector field, in our case, system (4.1.1), has a first integral, i.e., a non-constant function h which is constant on each solution curve of (4.1.1). The importance of first integrals is in its level sets. The existence of a first integral h on U determines the phase portrait of system (4.1.1) in U. This is because the level sets $\{h(x, y) = c\} \subseteq U$, contain the orbits of the system (4.1.1) in U. Among other applications, the existence of an analytic first integral defined in a neighborhood of the origin can be use to characterize when a monodromic singular point is a center or a focus, see [9]. For these reasons, the integrability problem is an important question in the qualitative theory of dynamical systems. Necessary conditions for the characterization of the existence of a first intergral are given in [57]. In [9, Theorem 3.19] are given necessary and sufficient conditions for the existence of an analytic first integral in a neighborhood of the origin for systems, where the first quasi-homogeneous term is Hamiltonian and its Hamiltonian function does not have multiple factors. In Algaba *et al.* [11] is characterized the analytic integrability, around the origin, of a family of degenerate differential systems. In [10] is studied the analytic integrability problem through the formal integrability problem and we show its connection, in some cases, with the existence of invariant analytic (sometimes algebraic) curves.

The second problem that is treated in this chapter is the existence of a inverse integrating factor. A non-null $\mathcal{C}^1$ class function V is an inverse integrating factor of system (4.1.1) on U if it satisfies the linear partial differential equation $L_{\mathbf{F}}V = \mathrm{div}(\mathbf{F})V$, being $\mathrm{div}(\mathbf{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ the divergence of $\mathbf{F}$. We will say that V is a *formal inverse integrating factor* of system (4.1.1) if $V \in \mathbb{C}[[x, y]]$. Also we will say than V is an *algebraic inverse integrating factor* for system (4.1.1), if $V \in \mathbb{C}((x, y))$, where $\mathbb{C}((x, y))$ denotes the quotient field of the algebra of the power series $\mathbb{C}[[x, y]]$. For more detail see [79, 84]. The only results that we know in this sense are due to *Walcher* [84] for non-degenerate cusp nilpotent singularity and Algaba *et. al.* [15], for the nilpotent systems in general.

The study of the inverse integrating factor is an essential tool for studying other problems, such as the integrability problem. The existence of inverse integrating factor is also strongly associated with the problem of integrability. It is known that, *if system (4.1.1) has a formal inverse integrating factor non-zero at origin then the system (4.1.1) is formally integrable*. Therefore, *if system (4.1.1) is not formally integrable and it has an inverse integrating factor V, then $V(\mathbf{0}) = 0$*. For more details about the relation between the integrability and the inverse integrating factor see [6, 54]. In addition, the expressions of V usually are simpler than the expressions of the first integrals, see [29, 31]. The domain of definition and the regularity of V usually are larger than the domain and the regularity of the first integral, see [32, 47, 74, 79].

Another reason for studying the existence of inverse integrating factors is its relation with existence of limit cycles, because the zero-set of V, $\{V = 0\}$, is formed by orbit of system (4.1.1) and it contains the limit cycles of system (4.1.1) which are in U, whenever they exist. This interesting result is due to Giacomini *et al.* [56]. Moreover, the cyclicity of a limit cycle is related with the vanishing order of V, see [49], therefore the existence of an inverse integrating factor is very important in the algebraic Hilbert problem.

This chapter is structured as follows. In the next section we give some previous concepts. In section 4.3, we study the existence of an inverse integrating factor and give necessary and sufficient conditions for the existence of a formal/algebraic inverse integranting factor. Finally, in section 4.4, we apply the results obtained for studying several families of polynomial vectors fields.

4.2 Preliminaries and previous concepts.

As we said in the introduction, a non-constant function h defined in a neighborhood $U \subseteq \mathbb{R}^2$ is a first integral for system (4.1.1), if h is constant on each solution of this system contained in $\mathbf{U}$. Obviously, in the case that $h \in \mathcal{C}^1$, then the definition is equivalent to the equality,

$$\dot{h} = P\frac{\partial h}{\partial x} + Q\frac{\partial h}{\partial y} \equiv 0, \text{ in } U.$$

This expression is equivalent to the equality $\nabla h{\cdot}\mathbf{F} = 0$, that we call *integrability equation*.

Once we know the concept of first integral for a system, we can remember the well-knowed concept of *Hamiltonian system* that already have been used in this memory. System (4.1.1) is a Hamiltonian system if there exists a function h such that,

$$P = -\frac{\partial h}{\partial y} \text{ and } Q = \frac{\partial h}{\partial x}$$

Another important concept related to the integrability problem and the existence of an inverse integrating factor is the concept of invariant curve. Let $f \in \mathbb{C}[[x, y]]$ be, we say that $f(x, y) = 0$, or simply f, is invariant curve for system (4.1.1) if,

$$\nabla f \cdot \mathbf{F} = K \cdot \mathbf{F},$$

where $K \in \mathbb{C}[[x, y]]$ is called the cofactor of f.

An irreducible invariant curve $f(x, y) = 0$ is an invariant curve such that f is irreducible on $\mathbb{C}[[x, y]]$, except multiplication by a unit element.

Remark 16. *As regards the concept of irreducibility, it worth clarifying that, if f is an irreducible invariante curve of a vector field $\mathbf{F}$, i.e., $\nabla f \cdot \mathbf{F} = K \cdot f$, and we consider an unit element u, then $f \cdot u$ is also an irreducible invariant curve of $\mathbf{F}$, because $\nabla g \cdot \mathbf{F} = f \cdot u(K + \frac{\nabla u \cdot \mathbf{F}}{u})$ and $K + \frac{\nabla u \cdot \mathbf{F}}{u} \in \mathbb{C}[[x, y]]$. Moreover in a neightborhood of the origin, both curves agree.*

Since the gradient of f at the points (x, y) such that $f(x, y) = 0$, is orthogonal to the vector field X (considering X the vector field associated with the system (4.1.1) i.e., $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ or simply $X = (P, Q)$), then this vector field is tangent to the curve $f = 0$. Hence, the curve $f = 0$ is formed by trajectories of X. A solution of (4.1.1) either has empty intersection with the zero set of f or is contained in it.

Finally, as we advance in the introduction, we treat with the concept of *inverse integrating factor*, a non-null $\mathcal{C}^1$ class function V that satisfies the linear partial differential equation

$$\nabla V \cdot \mathbf{F} = \operatorname{div}(\mathbf{F}) \cdot V.$$

In summary, $V = 0$ is an invariant curve of system (4.1.1) whose cofactor is the divergence of $\mathbf{F}$.

In this chapter we consider the following systems

$$\dot{\mathbf{x}} = \mathbf{X}_h + \text{q-h.h.o.t.}, \tag{4.2.2}$$

where $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ only has simple factors in its factorization on $\mathbb{C}[x, y]$, i.e., systems which can be considered as perturbations of a Hamiltonian system whose Hamiltonian function h only has simple factors in its factorization on $\mathbb{C}[x, y]$.

Making a reminder, in Charpter 3, we have calculated a formal orbital equivalent normal form of system (4.2.2), i.e., an expression of this system after a change of state variables and a reparametrization of the time, Theorem 3.3.93. This normal form is given by

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mathbf{X}_g + \mu \mathbf{D}_0, \tag{4.2.3}$$

with g a polynomial and $\mu = \sum_{j>r} \mu_j$, $\mu_j \in \mathrm{Cor}(\ell_j)$, being $\mathrm{Cor}(\ell_j)$ a complementary subspace to the range of the linear operator

$$\begin{aligned} \ell_j \ : \ & \mathcal{P}^{t}_{j-r} \longrightarrow \mathcal{P}^{\mathbf{t}}_{j} \\ & \mu_{j-r} \longrightarrow \ell_j(\mu_{j-r}) := L_{\mathbf{F}_r}\mu_{j-r}. \end{aligned} \tag{4.2.4}$$

We focus on the systems with $g \equiv 0$, i.e., the formally orbital equivalent systems to

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu \mathbf{D}_0, \text{ with } \mu = \sum_{j>r} \mu_j, \ \mu_j \in \mathrm{Cor}(\ell_j) \tag{4.2.5}$$

where $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ only has simple factors in its factorization on $\mathbb{C}[x, y]$. They are a wide class of systems. For instance, the systems with linear part non-null are included in this class, among others. The following theorem characterizes these systems. Before showing it, we present two technical lemmas that we will use in the proof of the theorem.

Lemma 4.2.105. *Let $\mathbf{F} = \sum_{j\geq r} \mathbf{F}_j$ be, with $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, $\mathbf{F}_j \in \mathcal{Q}^{\mathbf{t}}_j$ and $\mathbf{F}_r = \mathbf{X}_h$. If $h = 0$ is an invariant curve of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ then $\mathbf{F} = (1 + \lambda)\mathbf{X}_h + \eta \mathbf{D}_0$, where $\lambda = \sum_{j\geq 1} \lambda_j$, $\lambda_j \in \mathcal{P}^{\mathbf{t}}_j$ and $\eta = \sum_{j>r} \eta_j$, $\eta_j \in \mathcal{P}^{\mathbf{t}}_j$.*

Proof. If h is an invariant curve of $\mathbf{F}$, as h is a quasi-homogeneous polynomial, then h is an invariant curve of each $\mathbf{F}_j$, that is $L_{\mathbf{F}_j} h := \nabla h \cdot \mathbf{F}_j = K_j h$ with $K_j \in \mathcal{P}^{\mathbf{t}}_j$. Using Lemma 3.2.74, $\mathbf{F}_j = \mathbf{X}_{g_{j+|\mathbf{t}|}} + \eta_j \mathbf{D}_0 + \lambda_{j-r}\mathbf{X}_h$, $\lambda_{j-r} \in \mathcal{P}^{\mathbf{t}}_{j-r}$, $\eta_j \in \mathcal{P}^{\mathbf{t}}_j$ y $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$. Thus,

$$K_j h = \nabla h \cdot \mathbf{F}_j = \nabla h \cdot \left(\mathbf{X}_{g_{j+|\mathbf{t}|}} + \eta_j \mathbf{D}_0 + \lambda_{j-r}\mathbf{X}_h \right) = \nabla h \cdot \mathbf{X}_{g_{j+|\mathbf{t}|}} + (r + |\mathbf{t}|)\eta_j h,$$

i.e., h is an invariant curve of $\mathbf{X}_{g_{j+|\mathbf{t}|}}$. Therefore, $g_{j+|\mathbf{t}|}$ belongs to the ideal generated by h and as $g_{j+|\mathbf{t}|} \in \Delta_{j+|\mathbf{t}|}$ it has that $g_{j+|\mathbf{t}|} = 0$.

So, for each $j > r$, $\mathbf{F}_j = \eta_j \mathbf{D}_0 + \lambda_j \mathbf{X}_h$, hence $\mathbf{F} = (1 + \lambda)\mathbf{X}_h + \eta \mathbf{D}_0$ with $\lambda = \sum_{j>r} \lambda_{j-r}$ and $\eta = \sum_{j>r} \eta_j$.

■

Lemma 4.2.106. *Let Φ be a near identity diffeomorphism on $U \subset \mathbb{C}^2$. If $f(\mathbf{x}) = 0$ is an invariant curve of the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ with cofactor K, then*

$f(\Phi(\mathbf{y})) = 0$ is an invariant curve of the system $\dot{\mathbf{y}} = (\Phi_*((1+\alpha)\mathbf{F}))(\mathbf{y})$ with cofactor $(1+\alpha\circ\Phi)(K\circ\Phi)$, for any α a $\mathcal{C}^\infty$-class scalar function with $\alpha(\mathbf{0}) = 0$.

Proof. Indeed, let $\mathbf{G} = \Phi_*((1+\alpha)\mathbf{F})$, it has that

$$\begin{aligned} L_{\mathbf{G}}(f\circ\Phi)(\mathbf{y}) &= \nabla f(\Phi(\mathbf{y}))\cdot D^{-1}\Phi(\mathbf{y})(1+\alpha(\Phi(\mathbf{y})))\mathbf{F}(\Phi(\mathbf{y})) \\ &= (1+\alpha(\mathbf{x}))\nabla f(\mathbf{x})D\Phi(\mathbf{y})D^{-1}\Phi(\mathbf{y})\mathbf{F}(\mathbf{x}) \\ &= (1+\alpha(\mathbf{x}))\nabla f(\mathbf{x})\cdot\mathbf{F}(\mathbf{x}) = (1+\alpha(\mathbf{x}))K(\mathbf{x})f(\mathbf{x}) \\ &= (1+\alpha(\Phi(\mathbf{y})))K(\Phi(\mathbf{y}))f(\Phi(\mathbf{y})). \end{aligned}$$

■

Theorem 4.2.107. *Let $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ be. A system $\dot{\mathbf{x}} = \mathbf{X}_h +$ q-h.h.o.t. is formally orbital equivalent to $\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0$ with $\mu = \sum_{j>r}\mu_j$, $\mu_j \in \mathcal{P}^{\mathbf{t}}_j$, if and only if it has an invariant curve $f = 0$ of the form $f = h +$ q-h.h.o.t. with f a function conjugated to h (i.e., there exists a formal diffeomorphism Φ such that $h = f\circ\Phi$).*

Proof. We prove the necessity. We assume that there is a $\mu = \sum_{j>r}\mu_j$, $\mu_j \in \mathcal{P}^{\mathbf{t}}_j$ such that $\mathbf{F}$ and $\mathbf{G} := \mathbf{X}_h + \mu\mathbf{D}_0$ are orbitally equivalent. That is, there is a Ψ near-identity diffeomorphism and α scalar function with $\alpha(\mathbf{0}) = 0$ such that $\Psi_*(1+\alpha)(\mathbf{X}_h + \mu\mathbf{D}_0) = \mathbf{F}$. We note that h is an invariant curve of $\mathbf{X}_h + \mu\mathbf{D}_0$ with cofactor $(r+|\mathbf{t}|)\mu$ since $L_{\mathbf{G}}h = \mu\, L_{\mathbf{D}_0}h = (r+|\mathbf{t}|)\mu h$.

So, from Lemma 4.2.106, $f = h(\Psi) = h +$ q-h.h.o.t. is an invariant curve of $\mathbf{F}$, that is, f is conjugated to h.

Now, we prove the sufficient condition. We suppose that there exists an invariant curve f of $\mathbf{F}$ such that $f = h +$ q-h.h.o.t. and it is conjugated to h, thus there is a formal diffeomorphism Φ such that $h = f(\Phi)$. From Lemma 4.2.106 with $\alpha \equiv 0$ it has that h is an invariant curve of $\tilde{\mathbf{G}} := \Phi_*\mathbf{F}$. By applying Lemma 4.2.105, we get $\tilde{\mathbf{G}} = (1+\lambda)\mathbf{X}_h + \eta\mathbf{D}_0$. Therefore,

$$\mathbf{F} \;=\; \Phi^*\tilde{\mathbf{G}} = \Phi^*((1+\lambda)(\mathbf{X}_h + \frac{\eta}{1+\lambda}\mathbf{D}_0) = \Phi^*((1+\lambda)(\mathbf{X}_h + \mu\mathbf{D}_0)$$

with $\mu = \frac{\eta}{1+\lambda}$. So, $\mathbf{F}$ is orbitally equivalent to $\mathbf{X}_h + \mu\mathbf{D}_0$. ■

From Algaba et al. [9], the systems formally orbital equivalent to systems (4.2.5) are integrable if and only if $\mu \equiv 0$ and, in such a case, they have a first integral of the form $h+$ q-h.h.o.t. Therefore, for $\mu \not\equiv 0$, the systems (4.2.5) do not have any formal first integral. We focus our study about the existence of an inverse integrating factor for systems (4.2.5).

Next we present a result that we will use to study the integrability of different families of generalized nilpotent polynomial vector fields. It is a adapted version of Theorem 3.19 proved in [9].

Theorem 4.2.108. *System*

$$\dot{\mathbf{x}} = \mathbf{X}_H + \nu \mathbf{D}_0 \tag{4.2.6}$$

with $H \in \bigoplus_{j \geq r+|\mathbf{t}|} \mathcal{P}_j^{\mathbf{t}}$ *and* $\nu \in \bigoplus_{j>r} \mathrm{Cor}(\ell_j)$ *is formally integrable if and only if* $\nu \equiv 0$.

4.3 Characterization of the existence of an inverse integrating factor.

In Chapter 2 of this memory, we have presented a normal form for some degenerate vector fields. Among them, we have described the normal form for system (4.1.1) based on the lowest-order quasi-homogeneous term $\mathbf{F}_r$. This term determines the homological operator and, consequently, the simplifications can be reached in the normal form. In this subsection we want to obtain necessary and sufficient conditions for the existence of an inverse integrating factor for systems with the following form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) := \mathbf{X}_h + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots$$

where $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ only has simple factors in its factorization on $\mathbb{C}[x,y]$.

The normal form for this system was given in Theorem 3.3.88, with the expression (3.3.22). If we assume that $g \equiv 0$ we obtain

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{X}_h + \mu \mathbf{D}_0, \text{ being } \mu = \sum_{j=N}^{\infty} \mu_j, \text{ with } \mu_j \in \mathrm{Cor}(\ell_j) \tag{4.3.7}$$

Attending to this system we can write that,

- If $\mu \equiv 0$, then, from Theorem 4.2.108, the above system is integrable. Therefore h is an inverse integrating factor for system (4.3.7), among others.

- If $\mu \not\equiv 0$, then system (4.3.7) is non-integrable. In this case, we consider $N = min\{j/\mu_j \neq 0\}$. This first non-zero element will be considered to calculate a reduced normal form. This system can be written as follows,

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_N \mathbf{D}_0 + \mu \mathbf{D}_0, \tag{4.3.8}$$

being $\mu_N \in \mathrm{Cor}(\ell_N) \setminus \{0\}$ and $\mu = \sum_{j=N}^{\infty} \mu_j$ with $\mu_j \in \mathrm{Cor}(\ell_j)$

4.3.1 The two-step normal form under $\mathcal{C}^\infty$-equivalence

We consider the above system (4.3.8). The procedure for obtaining the two-step normal form of this system is as follows: Firstly, we make a reparametrization of the time $dt/dT = 1 - \nu_k(\mathbf{x})$ and a change of variables $\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y})$ that transforms the k-degree quasi-homogeneous terms into a normal form as it was explained in section 2.3 of Chapter 2. For the two-step normal form, we consider another reparametrization of the time $dt/dT = 1 - \tilde{\nu}_{k+r-N}(\mathbf{x})$ and a near-identity transformation $\mathbf{x} = \mathbf{y} + \tilde{\mathbf{P}}_{k+r-N}(\mathbf{y})$ with $(\tilde{\mathbf{P}}_{k+r-N}, \tilde{\nu}_{k+r-N}) \in \mathrm{Ker}(\mathcal{L}_{2r+k-N})$. This process describes the two-step homological operator, for every $k \geq 1$, as follows

$$\begin{aligned}
\mathcal{L}^{(2)}_{r+k} \quad : \quad & \mathcal{Q}^{\mathbf{t}}_k \times \mathrm{Cor}(\ell_k) \times \mathrm{Ker}(\mathcal{L}_{2r+k-N}) \longrightarrow \mathcal{Q}^{\mathbf{t}}_{r+k} \\
& \Big(\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{k+r-N}, \tilde{\nu}_{k+r-N})\Big) \to \mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, (\tilde{\mathbf{P}}_{k+r-N}, \tilde{\nu}_{k+r-N})) \\
& = [\mathbf{P}_k, \mathbf{F}_r] + \nu_k \mathbf{F}_r + [\tilde{\mathbf{P}}_{k+r-N}, \mathbf{F}_N] + \tilde{\nu}_{k+r-N} \mathbf{F}_N,
\end{aligned} \tag{4.3.9}$$

being $\mathbf{F}_N = \mu_N \mathbf{D}_0$ and, recalling the hypothesis assumed in the development of this work for the lowest-order quasi-homogeneous term $\mathbf{F}_r = \mathbf{X}_h$.

For determining the homologic operator $\mathcal{L}^{(2)}_{r+k}$ it is necessary to compute $\mathrm{Ker}(\mathcal{L}_{r+k})$. Next lemma shows its expression.

Lemma 4.3.109. *Let $k = s(r + |\mathbf{t}|) + m$, with $0 \leq m < r + |\mathbf{t}|$. It has that:*

$$\mathrm{Ker}(\mathcal{L}_{r+k}) = \begin{cases} span\{(\mathbf{0}, h^s\mathbf{D}_0, \mathbf{0}, r\, h^s)\} \text{ if } m = 0,\ r \neq 0, \\ span\{(\mathbf{0}, \mathbf{0}, h^s\mathbf{X}_h, \mathbf{0})\} \text{ if } m = r,\ r \neq 0, \\ span\{(\mathbf{0}, \mathbf{0}, h^s\mathbf{X}_h, \mathbf{0}), (\mathbf{0}, h^s\mathbf{D}_0, \mathbf{0}, \mathbf{0})\} \text{ if } m = r = 0, \\ span\{\mathbf{0}\} \quad \text{in other case.} \end{cases}$$

Proof.

It is enough to consider the expression of the homological operator given in (2.3.16) and Lemma 3.3.90. ∎

Remark: Obviously, the expression of the two-step homological operator $\mathcal{L}^{(2)}_{r+k}$, in the case that $\mathrm{Ker}(\mathcal{L}_{r+k})$ is spanned by 0 ($k \neq s(r + |\mathbf{t}|)$), is the same that the expression of $\mathcal{L}_{r+k}$.

Our goal is to show the expression of a co-range of the homological operator $\mathcal{L}^{(2)}_{r+k}$. For this task, analogously to the description of the previous homological operator described in section 3, we need define the following linear operator.

Definition 4.3.110. *Consider $r + k = N + s(r + |\mathbf{t}|) + m$,*

$$\begin{aligned} \ell^{(2)}_{r+k} \ &: \ \mathcal{P}^t_k \times \mathrm{Ker}(\ell_{2r+k-N}) \longrightarrow \mathcal{P}^{\mathbf{t}}_{r+k} \\ &(\mu_k, \alpha\, h^s) \longrightarrow \ell^{(2)}_{r+k}(\mu_k, \alpha\, h^s) = \begin{cases} \nabla\mu_k \cdot \mathbf{X}_h + \alpha\mu_N h^s \ \ if \ \ m = 0, \\ \ell_{r+k}(\mu_k) \qquad otherwise. \end{cases} \end{aligned}$$

This linear operator provides an expression of $\mathrm{Cor}(\mathcal{L}^{(2)}_{r+k})$, and consequently, an expression of a reduced normal form. We show it in the following proposition.

Proposition 4.3.111. *Let system (4.3.8) be. It has that:*

1. *$Cor(\mathcal{L}^{(2)}_{r+k}) = Cor(\ell^{(2)}_{r+k})\mathbf{D}_0$, if $N \neq r+s(r+|\mathbf{t}|)$, for any s natural number.*
2. *For $N = r + s(r + |\mathbf{t}|)$, it holds:*
   - *(a) $Cor(\mathcal{L}^{(2)}_{r+k}) = Cor(\ell^{(2)}_{r+k})\mathbf{D}_0$, if $k \neq 2s(r + |\mathbf{t}|)$.*
   - *(b) Otherwise, $Cor(\mathcal{L}^{(2)}_{r+2s(r+|\mathbf{t}|)}) = Cor(\ell_{r+2s(r+|\mathbf{t}|)})\mathbf{D}_0$.*

Proof. Let consider the elements of $\mathrm{Ker}(\mathcal{L}_{r+k})$ calculated in lemma 4.3.109, for $r \neq 0$:

1. $(\mathbf{0}, \mathbf{0}, \alpha\; h^s \mathbf{X}_h, \mathbf{0})$. Therefore, using (4.3.9)

$$\begin{aligned}
\mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, (\mathbf{0}, \mathbf{0}, \alpha\, h^s \mathbf{X}_h, \mathbf{0})\,) &= [\mathbf{P}_k, \mathbf{F}_r] + \nu_k \mathbf{F}_r + [(\mathbf{0}, \mathbf{0}, \alpha\, h^s \mathbf{X}_h), \mu_N \mathbf{D}_0] \\
&= [\mathbf{P}_k, \mathbf{F}_r] + \nu_k \mathbf{F}_r + [\alpha\, h^s \mathbf{X}_h, \mu_N \mathbf{D}_0] \\
&= \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha\; (\nabla h^s (\mu_N \mathbf{D}_0) \mathbf{X}_h \\
&\quad + h^s [\mathbf{X}_h, \mu_N \mathbf{D}_0]) = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) \\
&\quad + \alpha\; ((\nabla h^s \mu_N) \mathbf{D}_0) \mathbf{X}_h - h^s (\nabla \mu_N \mathbf{X}_h) \mathbf{D}_0 \\
&\quad - r\; h^s \mu_N \mathbf{X}_h) = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) \\
&\quad + \alpha\; ((r + s(r + |\mathbf{t}|) h^s \mu_N \mathbf{X}_h - h^s (\nabla \mu_N \mathbf{X}_h) \mathbf{D}_0)\,.
\end{aligned}$$

Taking into accout that, $h^s (\nabla \mu_N \mathbf{X}_h) \mathbf{D}_0 = [h^s \mu_N \mathbf{D}_0, \mathbf{X}_h] + r\, h^s \mu_N \mathbf{X}_h$, we obtain that,

$$\begin{aligned}
\mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, (\mathbf{0}, \mathbf{0}, \alpha\, h^s \mathbf{X}_h, \mathbf{0})\,) &= \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha\; ((r + s(r + |\mathbf{t}|) h^s \mu_N \mathbf{X}_h \\
&\quad - [h^s \mu_N \mathbf{D}_0, \mathbf{X}_h] - r\; h^s \mu_N \mathbf{X}_h) \\
&= \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) - \alpha\; ([h^s \mu_N \mathbf{D}_0, \mathbf{X}_h] \\
&\quad - s(r + |\mathbf{t}|) h^s \mu_N \mathbf{X}_h) = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) \\
&\quad - \alpha\, \mathcal{L}_{r+k}(h^s \mu_N \mathbf{D}_0, -(k - N) h^s \mu_N)
\end{aligned}$$

Therefore, this element does not provide the possibility of more simplifications in the normal form (4.3.8).

2. $(\mathbf{0}, \alpha\, h^s \mathbf{D}_0, \mathbf{0}, \alpha\, r\, h^s)$. Thus, for (4.3.9)

$$\begin{aligned}
\mathcal{L}^{(2)}_{r+k}(\mathbf{P}_k, \nu_k, (\mathbf{0}, \alpha\, h^s \mathbf{D}_0, \mathbf{0}, \alpha\, r\, h^s)\,) &= [\mathbf{P}_k, \mathbf{F}_r] + \nu_k \mathbf{F}_r + [(\mathbf{0}, \alpha\, h^s \mathbf{D}_0, \mathbf{0}), \mu_N \mathbf{D}_0] \\
&\quad + \alpha\, r\, h^s \mu_N \mathbf{D}_0 = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) \\
&\quad + [\alpha\, h^s \mathbf{D}_0, \mu_N \mathbf{D}_0] + \alpha\, r\, h^s \mu_N \mathbf{D}_0 \\
&= \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \nabla \alpha\, h^s (\mu_N \mathbf{D}_0) \mathbf{D}_0 \\
&\quad + \alpha\, h^s [\mathbf{D}_0, \mu_N \mathbf{D}_0] + \alpha\, r\, h^s \mu_N \mathbf{D}_0 \\
&= \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) + \alpha\, s\, (r + |\mathbf{t}|) h^s \mu_N \mathbf{D}_0 \\
&\quad - \alpha\, (N - r) h^s \mu_N \mathbf{D}_0 = \mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) \\
&\quad + \alpha\; (s\, (r + |\mathbf{t}|) - (N - r))\, h^s \mu_N \mathbf{D}_0
\end{aligned}$$

At this point we can afirm:

- If $N \neq r + s\,(r + |\mathbf{t}|)$ then $\mathrm{Im}(\mathcal{L}^{(2)}_{r+k}) = \mathrm{Im}(\mathcal{L}_{r+k}) \oplus \langle h^s \mu_N \mathbf{D}_0 \rangle$. Consequently $\mathrm{Cor}(\mathcal{L}^{(2)}_{r+k}) = \mathrm{Cor}(\ell^{(2)}_{r+k})\mathbf{D}_0$ and we are in the situation **a)**.
- If $N = r + s\,(r + |\mathbf{t}|)$ it is possible two option:
  - There exist a natural number $s := s_0$ such that $N = r + s_0(r + |\mathbf{t}|)$, then $k = 2s_0(r + |\mathbf{t}|) + m$, and, if $m = 0$ it holds that, $\mathrm{Im}(\mathcal{L}^{(2)}_{r+s_0(r+|\mathbf{t}|)}) = \mathrm{Im}(\mathcal{L}_{r+s_0(r+|\mathbf{t}|)})$ and we can conclude that $\mathrm{Cor}(\mathcal{L}^{(2)}_{r+s_0(r+|\mathbf{t}|)}) = \mathrm{Cor}(\ell_{r+s_0(r+|\mathbf{t}|)})\mathbf{D}_0$
  - Otherwise,

    $$\mathrm{Im}(\mathcal{L}^{(2)}_{r+k}) = \mathrm{Im}(\mathcal{L}_{r+k}) + \mathrm{span}\{h^{l_1} \mu_N \mathbf{D}_0\}.$$

    Moreover, as the system (4.3.8) is a normal form, then

    $$\mathrm{Range}(\mathcal{L}_{r+k}) = \mathbf{X}_{\mathcal{P}^{\mathbf{t}}_{r+k+|\mathbf{t}|}} + \mathrm{Range}(\ell_{r+k})\mathbf{D}_0.$$

    Thus, it concludes that $\mathrm{Cor}(\mathcal{L}^{(2)}_{r+k}) = \mathrm{Cor}(\ell^{(2)}_{r+k})\mathbf{D}_0$.

For $r = 0$, this result can be similarly proved. ■

Proposition 4.3.111 yields the next theorem, which determines a reduced normal form of non-integrable systems.

Theorem 4.3.112. *Let $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ be a polynomial whose factorization on $\mathbb{C}[x, y]$) only has simple factors. We consider the system,*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_N \mathbf{D}_0 + \mu \mathbf{D}_0,$$

being $\mu_N \in \mathrm{Cor}(\ell_N) \setminus \{0\}$ and $\mu = \sum_{j>N}^{\infty} \mu_j$ with $\mu_j \in \mathrm{Cor}(\ell_j)$. It has that:

1. *If $N \neq r + s(r + |\mathbf{t}|)$ with s a natural number, then a formal normal form under $\mathcal{C}^\infty$-equivalence for system (4.3.7) is*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_N + \sum_{j>N} \tilde{\mu}_j \mathbf{D}_0. \tag{4.3.10}$$

   where $\tilde{\mu}_j \in \mathrm{Cor}(\ell^{(2)}_j)$.

2. *If there exists* s_0 *a natural number, such that,* $N = r + s_0(r + |\mathbf{t}|)$*, then a formal normal form under* $\mathcal{C}^\infty$*-equivalence for system (4.3.7) is*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_N + \sum_{j>N} \tilde{\mu}_j \mathbf{D}_0. \tag{4.3.11}$$

where $\tilde{\mu}_{N+s_0(r+|\mathbf{t}|)} \in \mathrm{Cor}(\ell_{N+s_0(r+|\mathbf{t}|)})$ *and* $\tilde{\mu}_j \in \mathrm{Cor}(\ell_j^{(2)})$ *with* $j \neq N + s_0(r + |\mathbf{t}|)$

4.3.2 Existence of a formal inverse integrating factor.

The following proposition gives a necessary condition for the existence of an inverse integrating factor.

Proposition 4.3.113. *Consider the system*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \lambda g(h)\mathbf{D}_0, \tag{4.3.12}$$

with $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$*,* $\lambda \in \mathcal{P}^{\mathbf{t}}_{r+s(r+|\mathbf{t}|)}$ *and* $g(h) = 1 + \sum_{j\geq 1} a_j h^j$*, being* s *a natural number. Then, the function* $V = h^{s+1}g(h)$ *is an inverse integrating factor of the system.*

Proof. Applying the Euler theorem for quasi-homogeneous functions, it has that,

$$L_{\mathbf{F}}V = V'(h)L_{\mathbf{F}}h = (r + |\mathbf{t}|)\lambda h g(h)V'(h),$$

and

$$\begin{aligned}
\mathrm{div}(\mathbf{F}) &= \mathrm{div}(\lambda g(h)\mathbf{D}_0) \\
&= g(h)L_{\mathbf{D}_0}\lambda + \lambda g'(h)L_{\mathbf{D}_0}h + |\mathbf{t}|\lambda g(h) \\
&= (s+1)(r+|\mathbf{t}|)\lambda g(h) + (r+|\mathbf{t}|)\lambda g'(h)h \\
&= (r+|\mathbf{t}|)((s+1)\lambda g(h) + \lambda g'(h)h).
\end{aligned}$$

Therefore, $V\mathrm{div}(\mathbf{F}) = (r + |\mathbf{t}|)((s + 1)h^s\lambda g(h) + \lambda g'(h)h^{s+1})hg(h) = (r + |\mathbf{t}|)\lambda hg(h)V'(h)$. So, $L_{\mathbf{F}}V - V\mathrm{div}(\mathbf{F}) = 0$. This completes the proof. ∎

Next we relate the terms of an inverse integrating factor and give the shape of its lowest-degree term.

Proposition 4.3.114. *Consider system* $\dot{\mathbf{x}} = \mathbf{X}_h + \mu\mathbf{D}_0$ *where* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *only has simple factors in its factorization on* $\mathbb{C}[x,y]$ *and* $\mu \not\equiv 0$. *Assume that the system has an inverse integrating factor* V *with the form* $V = \sum_{i\geq i_0} V_i \in \mathcal{P}^{\mathbf{t}}_i$. *Then, for each* $k \in \mathbb{N}$, $k \geq i_0$ *it holds*

$$L_{\mathbf{X}_h} V_k = -\sum_{l=N}^{k+r-i_0} (k + r - |\mathbf{t}| - 2l) V_{k+r-l}\, \mu_l. \qquad (4.3.13)$$

Moreover,

a) $i_0 = m(r + |\mathbf{t}|)$, *and* V *is of the form* $V = h^m + \sum_{j>i_0} V_j$.

b) If $\mu_j \in \mathrm{Cor}(\ell_j)$, *for all* j, *then* $V = h^m + \sum_{i>m} b_i h^i$.

c) $\mu = \mu_N + \sum_{j>N} \mu_j$ *with* $N = r + s(r + |\mathbf{t}|)$ *and* $s = m - 1 \in \mathbb{N}$.

Proof. If V is a formal inverse integrating factor of system (4.2.2) it holds the equation $L_{\mathbf{F}}V - V\mathrm{div}(\mathbf{F}) = 0$, therefore,

$$L_{\mathbf{F}}V - V\mathrm{div}(\mathbf{F}) = L_{\mathbf{X}_h}V + \mu L_{\mathbf{D}_0}V - V\mathrm{div}(\mu\mathbf{D}_0) = 0$$

Thus, considering $i_0 = min\{i \in \mathbb{N}/V_i \neq 0\}$,

$$\begin{aligned}
L_{\mathbf{X}_h}V &= V\mathrm{div}(\mu\mathbf{D}_0) - \mu L_{\mathbf{D}_0}V \\
&= \left(\sum_{i\geq i_0} V_i\right)\left(\sum_{j\geq N}(j + |\mathbf{t}|)\mu_j\right) - \left(\sum_{j\geq N}\mu_j\right)\left(\sum_{i\geq i_0} iV_i\right) \\
&= \sum_{k=i_0+N}^{\infty}\sum_{l=N}^{k-i_0}(l + |\mathbf{t}|)V_{k-l}\,\mu_l - \sum_{k=i_0+N}^{\infty}\sum_{l=N}^{k-i_0}(k - l)V_{k-l}\,\mu_l \\
&= -\sum_{k=i_0+N}^{\infty}\sum_{l=N}^{k-i_0}(k - |\mathbf{t}| - 2l)V_{k-l}\,\mu_l.
\end{aligned}$$

Therefore, for each $k \in \mathbb{N}$, $k \geq i_0$, we obtain,

$$L_{\mathbf{X}_h} V_k = -\sum_{l=N}^{k+r-i_0} (k + r - |\mathbf{t}| - 2l) V_{k+r-l}\, \mu_l$$

We prove the second part,

a) If $V = \sum_{i \geq i_0} V_i$ is an inverse integrating factor of system (4.2.2), then the equation (4.3.13) for $k = i_0$ is $L_{\mathbf{X}_h} V_{i_0} = 0$. So, V_{i_0} is a formal first integral of $\mathbf{X}_h$, therefore $V_{i_0} = h^m$ with $i_0 = m(r + |\mathbf{t}|)$.

b) Consider now $\mu_j \in \mathrm{Cor}(\ell_j)$. We know that $V = h^m + \sum_{i > m(r+|\mathbf{t}|)}^{\infty} V_i$, for some $m \in \mathbb{N}$. Assume $j_0 = min\{i \in \mathbb{N}/V_i$ is not a function of $h\}$ then $m(r+|\mathbf{t}|) < j_0$ and $V = h^m + \sum_{l=m+1}^{\lfloor \frac{j_0}{r+|\mathbf{t}|} \rfloor} b_l h^l + V_{j_0} + \cdots$. The equation (4.3.13), for $k = j_0$ is

$$\nabla V_{j_0} \mathbf{X}_h = - \sum_{l=0}^{j_0+r-i_0} (r + j_0 - |\mathbf{t}| - 2l) V_{j_0+r-l}\, \mu_l.$$

We can see the left side of the above equation is in the $\mathrm{Im}(\ell_{r+j_0})$ and the right side is in the $\mathrm{Cor}(\ell_{r+j_0})$ because V_{j_0+r-l} are function of h (observe that $r - l < 0$). Therefore, $\nabla V_{j_0} \mathbf{X}_h = 0$ and we conclude that $V_{j_0} = 0$ or $V_{j_0} = h^n$ for some $n \in \mathbb{N}$, $n > m$. Moreover, the above equality can be expressed as,

$$0 = - \sum_{l=0}^{k+r-i_0} (r + k - |\mathbf{t}| - 2l) V_{k+r-l}\, \mu_l. \tag{4.3.14}$$

c) For $k = m(r + |\mathbf{t}|) + N - r$, the equation (4.3.14) is of the form,

$$\begin{aligned} 0 &= (r + m(r + |\mathbf{t}|) + N - r - |\mathbf{t}| - 2N) V_{m(r+|\mathbf{t}|)}\, \mu_N \\ &= ((m-1)(r + |\mathbf{t}|) - N + r) h^m\, \mu_N. \end{aligned}$$

It follows that $N = r + (m-1)(r + |\mathbf{t}|) > r$, hence $m \geq 2$.

Taking $s = m - 1$ we get $N = r + s(r + |\mathbf{t}|)$ with $s \in \mathbb{N}$.

■

The following theorem is the main result of this subsection. This theorem provides necessary and sufficient conditions for the existence of a formal inverse integrating factor.

Theorem 4.3.115. *System (4.3.7) has a formal inverse integrating factor if, and only if, it is formally orbital equivalent either to*

$$\dot{\mathbf{x}} = \mathbf{X}_h \quad \textit{(integrable system)}$$

or to system (4.3.11) with $\mu_{r+s_0(r+|\mathbf{t}|)} = \mu_{r+2s(r+|\mathbf{t}|)} = \alpha h^s \mu_{r+s(r+|\mathbf{t}|)}$ *and* $\tilde{\mu}_j = 0,\ j \neq N + s(r+|\mathbf{t}|)$*, being* s *a natural number,* α *real and* $\mu_{r+s(r+|\mathbf{t}|)} \in Cor(\ell_{r+s(r+|\mathbf{t}|)}) \setminus \{0\}$ *(non-integrable system).*

Proof.

Sufficient condition: It is follows from Proposition 4.3.113.

Necessary condition: If $\mu_j = 0$ for all j, the system is a Hamiltonian system whose first integral is h. In such a case, h is also an inverse integrating factor. Otherwise, let $N = \min\{j,\ \mu_j \neq 0\}$. Hence, by Proposition 4.3.114, if $\mu = \mu_N +$ q-h.h.o.t. with $\mu_N \neq 0$, then $N = r + s(r+|\mathbf{t}|)$ and $V = h^{s+1} + \sum_{j>s} b_j h^j$. From Theorem 4.3.112, system (4.3.7) can be transformed into system (4.3.11).
We will prove that $V = h^{s+1} + b_{2s+1}h^{2s+1}$ and $\mu = \mu_{r+s(r+|\mathbf{t}|)} + b_{2s+1}h^s\mu_{r+s(r+|\mathbf{t}|)}$.
We do the proof in several steps:
Step 1. We see that $b_j = 0$ and $\tilde{\mu}_{r+(j-1)(r+|\mathbf{t}|)} = 0$, for $j = s+2, ..., 2s$.
Indeed, we assume the contrary, i.e., there exists $j_0 = min\{j,\ b_j \neq 0,\ s+2 \leq j \leq 2s\}$.

The equality (4.3.14), for $k = (j_0+s)(r+|\mathbf{t}|)$, has only two components $V_{r+k-l} \neq 0$ with $N \leq l \leq r + (j_0-1)(r+|\mathbf{t}|)$. In particular, for $l = r + (j_0-1)(r+|\mathbf{t}|)$ we have $V_{r+k-l} = V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$ and for $l = N$, $V_{r+k-l} = V_{j_0(r+|\mathbf{t}|)} = b_{j_0}h^{j_0}$. Then, the equality (4.3.14), for $k = (j_0+s)(r+|\mathbf{t}|)$, becomes

$$\begin{aligned} 0 &= (j_0-s-1)(r+|\mathbf{t}|)b_{j_0}h^{j_0}\mu_N - (j_0-s-1)(r+|\mathbf{t}|)h^{s+1}\tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} \\ &= (j_0-s-1)(r+|\mathbf{t}|)h^{s+1}\left[b_{j_0}h^{j_0-s-1}\mu_N - \tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)}\right]. \end{aligned}$$

Consequently, by Theorem **??**,

$$\tilde{\mu}_{r+(j_0-1)(r+|\mathbf{t}|)} = b_{j_0}h^{j_0-s-1}\mu_N \in \mathrm{Cor}(\ell^{(2)}_{r+(j_0-1)(r+|\mathbf{t}|)}) \setminus \{0\}, \qquad (4.3.15)$$

but also $b_{j_0}h^{j_0-s-1}\mu_N = \ell^{(2)}_{r+(j_0-1)(r+|\mathbf{t}|)}(0, b_{j_0}h^{j_0-s-1})$, which is a contradiction.

Step 2. We see that $\tilde{\mu}_{r+2s(r+|\mathbf{t}|)} = b_{2s+1}h^s\mu_{r+s(r+|\mathbf{t}|)}$.
From expression (4.3.15), for $j_0 = 2s+1$ it has that $\tilde{\mu}_{r+2s(r+|\mathbf{t}|)} = b_{2s+1}h^s\mu_{r+s(r+|\mathbf{t}|)}$.

Step 3. We see that $b_j = 0$, and $\tilde{\mu}_{r+(j-1)(r+|\mathbf{t}|)} = 0$, for all $j \geq 2s+2$.
Indeed, we assume the contrary, i.e., there exists $j_0 = min\{j,\ b_j \neq 0,\ j \geq 2s+2\}$. Thus, there exists $m_0 \geq 2$ such that $j_0 \in \{m_0 s+2, ..., (m_0+1)s+1\}$.

The equality (4.3.14), for $k = (j_0+s)(r+|\mathbf{t}|)$, has only three components $V_{r+k-l} \neq 0$ with $N \leq l \leq r+(j_0-1)(r+|\mathbf{t}|)$. In particular, for $l = r+(j_0-1)(r+|\mathbf{t}|)$ we have $V_{r+k-l} = V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$, for $l = r+(j_0-s-1)(r+|\mathbf{t}|)$, $V_{r+k-l} = V_{(2s+1)(r+|\mathbf{t}|)} = b_{2s+1}h^{2s+1}$ and for $l = N$, $V_{r+k-l} = V_{j_0(r+|\mathbf{t}|)} = b_{j_0}h^{j_0}$. The term $V_{r+k-l} = V_{(2s+1)(r+|\mathbf{t}|)}$ is multiplied by $\mu_{r+(j_0-s-1)(r+|\mathbf{t}|)}$ which is zero. So, the equality (4.3.14) gets (4.3.15) for $j_0 \in \{m_0 s+2, ..., (m_0+1)s+1\}$. Thus, by Theorem 4.3.112, it arrives to contradiction.

Step 4. We prove that $\tilde{\mu}_j = 0$, for all $j > N+s(r+|\mathbf{t}|)$ and $j \neq r+n(r+|\mathbf{t}|)$, for any n. We use *reductio ad absurdum.* Let $j_0 = min\{j > N+s(r+|\mathbf{t}|),\ \tilde{\mu}_j \neq 0\}$ be. There exists $m_0 \geq 2$ such that $j_0 \in \{r+m_0 s(r+|\mathbf{t}|)+1, ..., (m_0+1)s(r+|\mathbf{t}|)-1\}$.

For $k = j_0 - r + (s+1)(r+|\mathbf{t}|)$, the equality (4.3.14) has two factors $V_{(s+1)(r+|\mathbf{t}|)}\tilde{\mu}_{j_0}$ and $V_{(2s+1)(r+|\mathbf{t}|)}\tilde{\mu}_N$. This second term is zero. Thus, (4.3.14) becomes

$$
\begin{aligned}
0 &= \sum_{l=N}^{j_0}(j_0 - r + s(r+|\mathbf{t}|) - 2l)V_{j_0+(s+1)(r+|\mathbf{t}|)-l}\,\mu_l \\
&= [s(r+|\mathbf{t}|) - (j_0 - r)]V_{(s+1)(r+|\mathbf{t}|)}\,\mu_{j_0},
\end{aligned}
$$

and as $V_{(s+1)(r+|\mathbf{t}|)} = h^{s+1}$ and $j_0 - r \neq s(r+|\mathbf{t}|)$, we obtain $\mu_{j_0} = 0$, a contradiction. ■

4.3.3 Results for nilpotent and generalized nilpotent vector fields

The following propositions give a relationship between inverse integrating factors of conjugated vector fields (the first one) and orbitally equivalent vector fields (the second one).

A proof of the following propositions can be seen in Enciso and Peralta-Salas [46].

Proposition 4.3.116. *Let Φ be a diffeomorphism on $U \subset \mathbb{R}^2$. If V is an inverse integrating factor of system (4.1.1), then det$(D\Phi)^{-1}V \circ \Phi$ is an inverse integrating factor of $\dot{\mathbf{y}} = \Phi_*\mathbf{F}(\mathbf{y}) := D\Phi(\mathbf{y})^{-1}\mathbf{F}(\Phi(\mathbf{y}))$.*

Proposition 4.3.117. *Let Φ be a diffeomorphism and η a functions on $U \subset \mathbb{R}^2$ such that detDΦ has no zero on U and $\eta(\mathbf{0}) \neq 0$. If $V(\mathbf{x})$ is an inverse integrating factor of system (4.1.1), then $\eta(\mathbf{y}det(D\Phi))^{-1}V(\Phi(\mathbf{y}))$ is an inverse integrating factor of $\dot{\mathbf{y}} = \Phi_*(\eta\mathbf{F})(\mathbf{y}) := D\Phi(\mathbf{y})^{-1}\eta(\mathbf{y})\mathbf{F}(\Phi(\mathbf{y}))$.*

The following three theorems give conditions for the existence of a formal inverse integrating factor.

Theorem 4.3.118. *If system (3.4.26) is non-integrable then it has not an inverse integrating factor.*

Proof. A formal normal form for system (3.4.26) is expressed by (3.4.27) where there exists some $\alpha_{k,j}^{(i)}$ non-null (in other case the system is integrable). Let N be the least degree of the monomials of the normal form (3.4.27). Observe that N is always a natural even number, therefore, $N \neq r + s(r + |\mathbf{t}|) = 2l - 1 + 2s(2l + 1)$ for all $s \in \mathbb{N}$. In conclusion, from Proposition 4.3.115, system (3.4.26) has not a formal inverse integrating factor. ■

Theorem 4.3.119. *System (3.4.28) is non-integrable and has a formal inverse integrating factor V if and only if it is formal orbitally equivalent to $(\dot{x}, \dot{y}) = (y,\ \sigma x^{2l+1}) + \alpha_{l+s(r+|\mathbf{t}|)} h^s \mathbf{D}_0$ for some $s \in \mathbb{N}$ and a formal inverse integrating factor is of the form $h^{s+1} + \cdots$, where $h = \frac{1}{2}(\sigma x^2 - (l+1)y^2)$.*

Proof. *Necessaty condition* From Proposition (**??**), a formal normal form of system (3.4.28) is

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ \sigma x^{2l+1} \end{pmatrix} + \alpha_{l+s(2l+2)}(x^l h^s)\mathbf{D}_0, \tag{4.3.16}$$

where $\alpha_{l+s(2l+2)} \neq 0$ for $s \in \mathbb{N}$ (in other case system (3.4.28) is integrable).

We prove that the polynomial h^{s+1} is an inverse integrating factor of (4.3.16), being $h = \frac{1}{2}(\sigma x^2 - (l+1)y^2)$. For that we verify that que $L_{\mathbf{F}}V = V\mathrm{div}\mathbf{F}$.

- $L_{\mathbf{F}}V = \nabla V(\alpha_{l+s(r+|\mathbf{t}|)}x^l h^s\,\mathbf{D}_0) = \alpha_{l+s(r+|\mathbf{t}|)}x^l h^s\,\nabla V\cdot\mathbf{D}_0 = \alpha_{l+s(r+|\mathbf{t}|)}(s+1)(r+|\mathbf{t}|)x^l h^{2s+1}$.
- $V\mathrm{div}\mathbf{F} = V\mathrm{div}(\alpha_{l+s(r+|\mathbf{t}|)}x^l h^s\,\mathbf{D}_0) = V[\alpha_{l+s(r+|\mathbf{t}|)}(l+s(r+|\mathbf{t}|))h^s + \alpha_{l+s(r+|\mathbf{t}|)}|\mathbf{t}|h^s] = \alpha_{l+s(r+|\mathbf{t}|)}(s+1)(r+|\mathbf{t}|)h^{2s+1}$.

Suffcient condition From Proposition 4.3.117, $h^{s+1}+\cdots$ is an inverse integrating factor of system (3.4.95). ■

Theorem 4.3.120. *If system (3.4.28) with $l=0$ is non-integrable then it has a formal inverse integrating factor.*

Theorem 4.3.121. *System (3.4.35), for $l=0$ or $l=1$, is non-integrable and has a formal inverse integrating factor V if and only if it is formal orbitally equivalent to $(\dot{x},\dot{y}) = (y^2+2ax^{l+1}y+\sigma x^{2(l+1)}, -a(l+1)x^l y^2 - 2\sigma(l+1)x^{2l+1}y) + (\alpha^{(1)}x^l y h^s + \alpha^{(2)}x^{2l+1}h^s)\mathbf{D}_0$ for some $s\in\mathbb{N}$ and a formal inverse integrating factor is of the form $h^{s+1}+\cdots$, where $h := y[-\frac{1}{3}y^2 - ax^{l+1}y - \sigma x^{2(l+1)}]$ with $a^2 \neq \frac{4\sigma}{3}$ and $a=0$ when $l=0$.*

Proof. *Necessary condition:* From Theorem 4.3.112, a formal normal form of system (3.4.40) is

$$\dot{\mathbf{x}} = \begin{pmatrix} y^2+2ax^{l+1}y+x^{2(l+1)} \\ -a(l+1)x^l y^2 - 2\sigma(l+1)x^{2l+1}y \end{pmatrix} + (\alpha_N^{(1)}x^l y h^s + \alpha_N^{(2)}x^{2l+1}h^s)\mathbf{D}_0, \tag{4.3.17}$$

where $\alpha_N^{(1)} + \alpha_N^{(2)} \neq 0$ for some $s\in\mathbb{N}$ (in other case system (3.4.40) is integrable).

We prove that the polynomial h^{s+1} is an inverse integrating factor of (4.3.17). For that we verify that $L_{\mathbf{F}}V = V\mathrm{div}\mathbf{F}$.

- $L_{\mathbf{F}}V = \nabla V(\alpha_N^{(1)}x^l y h^s + \alpha_N^{(2)}x^{2l+1}h^s)\mathbf{D}_0 = [\alpha_N^{(1)}x^l y + \alpha_N^{(2)}x^{2l+1}](s+1)(3l+3)h^{2s+1}$.

- $V\text{div}\mathbf{F} = V\text{div}[\alpha_N^{(1)}x^lyh^s+\alpha_N^{(2)}x^{2l+1}h^s)\mathbf{D}_0] = V[(2l+1+s(3l+3))\alpha_N^{(1)}x^lyh^s+\alpha_N^{(1)}(l+2)x^lyh^s + (2l+1+s(3l+3))\alpha_N^{(2)}x^{2l+1}h^s + \alpha_N^{(2)}(l+2)x^{2l+1}h^s] = [\alpha_N^{(1)}x^ly + \alpha_N^{(2)}x^{2l+1}](s+1)(3l+3)h^{2s+1}$.

Suffcient condition From Proposition 4.3.117, $h^{s+1} + \cdots$ is an inverse integrating factor of system (3.4.40).

∎

4.3.4 Existence of an algebraic inverse integrating factor.

In this subsection we want to characterize the existence of algebraic inverse integrating factor. First we introduce some results that relate the formal inverse integrating factor with the algebraic inverse integrating factor. The following proposition provides an expression of an algebraic inverse integrating factor for system (4.1.1).

Proposition 4.3.122. *If system (4.1.1) has an algebraic inverse integrating factor, then it also admits an inverse integrating factor of the form $V = (W_1/W_2)^d$ with W_1 and W_2 formal series and d a positive rational number.*

Proof. From [74, Propositions 1 and 2] and particularizing in our context, if system (4.1.1) has an algebraic inverse integrating factor, then it also admits an inverse integrating factor V of the specific form $V = \phi_1^{d_1} \cdots \phi_s^{d_s}$, with $\phi_i \in \mathbb{C}[[x,y]]$, non-invertible, irreducible invariant curves and d_i a non-zero rational numbers . (The possibility $s = 0$ is included, with inverse integrating factor 1). So, if we write $d_i = m_i/n_i$ and denote $N = lcm(\{|n_1|, \ldots, |n_s|\})$ and $M = gcd(\{N|\frac{m_1}{n_1}|, \ldots, N|\frac{m_s}{n_s}|\})$,then $V = (\prod_{i=1}^s \phi_i^{k_i})^{\frac{M}{N}}$, with $\{k_1, \ldots, k_s\}$ integer. ∎

The two following results provide some properties of the inverse integrating factors which are powers of quotient of formal series.

Lemma 4.3.123. *We assume that $V = (W_1/W_2)^d$ is an algebraic inverse integrating factor of $\mathbf{F} = \mathbf{F}_r + q-h.h.o.t.$, with $W_1 = W_{1,m} + q-h.h.o.t.$ and $W_2 = W_{2,n} + q-h.h.o.t.$ where $W_{1,m} \in \mathcal{P}_m^{\mathbf{t}}$ and $W_{2,n} \in \mathcal{P}_n^{\mathbf{t}}$ and d a positive* *rational number. Then,* $(W_{1,m}/W_{2,n})^d$ *is an algebraic inverse integrating factor of* $\mathbf{F}_r$*. Moreover, in the case of* $\mathbf{F}_r = \mathbf{X}_h$ *with* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *,it has that* $W_{1,m}/W_{2,n}$ *is a rational first integral of* $\mathbf{X}_h$*. In addition, if the factorization of* $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ *on* $\mathbb{C}[x,y]$ *only has simple factors then, there exists an integer number non-zero* k *such that* $W_{1,m}/W_{2,n} = h^k$*.*

Proof. If V is an algebraic inverse integrating factor of $\mathbf{F}$, V satisfies equation $L_F V - V \cdot \mathrm{div}(\mathbf{F}) = 0$. From Proposition 4.3.122, if we replace V by $(W_1/W_2)^d$ then

$$(W_1/W_2)^{d-1} W_2^{-2}[dW_2 \cdot L_F W_1 - dW_1 \cdot L_F W_2 - W_1 W_2 \,\mathrm{div}(\mathbf{F})] = 0.$$

Multiplying by $(W_{1,m}/W_{2,n})^{d-1} W_{2,n}^{-2}$ it follows easily that $(W_{1,m}/W_{2,n})^d$ verifies

$$L_{\mathbf{F}_r}(W_{1,m}/W_{2,n})^d - (W_{1,m}/W_{2,n})^d \mathrm{div}(\mathbf{F}_r) = 0.$$

Hence, it is an inverse integrating factor of $\mathbf{F}_r$, which is algebraic over $\mathbb{C}((x,y))$. We prove the second part. Obviously, if $\mathbf{F}_r = \mathbf{X}_h$, then $\mathrm{div}(\mathbf{F}_r) = 0$ and by (4.3.18), $W_{1,m}/W_{2,n}$ is a rational first integral of $\mathbf{F}_r$.

We see that it is a power of the polynomial h, when h only has simple factors in its factorization on $\mathbb{C}[[x,y]]$. We note that the quotient $W_{1,m}/W_{2,n}$ can not be irreducible. In such case, there exist two quasi-homogeneous coprime polynomials $W^*_{1,\tilde{m}}$ and $W^*_{2,\tilde{n}}$ such that $W_{1,m}/W_{2,n} = W^*_{1,\tilde{m}}/W^*_{2,\tilde{n}}$. Since $W^*_{1,\tilde{m}}/W^*_{2,\tilde{n}}$ is a rational first integral of $\mathbf{X}_h$, then $L_{\mathbf{X}_h}(W^*_{1,\tilde{m}}/W^*_{2,\tilde{n}}) = 0$, that is,

$$\left(\nabla W^*_{1,\tilde{m}} \cdot \mathbf{X}_h\right) W^*_{2,\tilde{n}} = \left(\nabla W^*_{2,\tilde{n}} \cdot \mathbf{X}_h\right) W^*_{1,\tilde{m}}$$

Consequently, as $W^*_{1,\tilde{m}}$ and $W^*_{2,\tilde{n}}$ are coprime, there exists $K \in \mathbb{C}[x,y]$, a quasi-homogeneous polynomial, such that $\nabla W^*_{1,\tilde{m}} \cdot \mathbf{X}_h = K W^*_{1,m}$ and $\nabla W^*_{2,\tilde{n}} \cdot \mathbf{X}_h = K W^*_{2,n}$, i.e., $W^*_{1,\tilde{m}}$ and $W^*_{2,\tilde{n}}$ are algebraic invariant curves of $\mathbf{X}_h$ which arrives at the origin. So, if $h = f_1 \cdots f_s$, with f_i irreducible factors on $\mathbb{C}[x,y]$, the unique irreducible invariant curves of $\mathbf{X}_h$ that arrives at the origin are $f_1 = 0, \cdots, f_s = 0$.

Therefore, $W^*_{1,\tilde{m}} = f_1^{n_1} \cdots f_s^{n_s}$ and $W^*_{1,\tilde{m}} = f_1^{n_1} \cdots f_s^{n_s}$, that is $W^*_{1,\tilde{m}}/W^*_{2,\tilde{n}} = f_1^{k_1} \cdots f_s^{k_s}$ with k_i integer numbers. So, if $M = lcm\{|k_i|\}$ with

$k_i < 0$, $i = 1, ..., s$, the function $W^*_{1,\tilde{m}}/W^*_{2,\tilde{n}}h^M$ is a quasi-homogeneous first integral of the system $\dot{\mathbf{x}} = \mathbf{X}_h$ since it is a product of two first integrals. As h only has simple factors in its factorization on $\mathbb{C}[x, y]$, the quasi-homogeneous first integrals of $\mathbf{X}_h$ are h^l, with l a natural number. Therefore, $W_{1,m}/W_{2,n} = W^*_{1,\tilde{m}}/W^*_{2,\tilde{n}} = h^k$ with k integer number non-zero. ■

Lemma 4.3.124. *Assume that $V_1 = (W_1/W_2)^{\frac{p_1}{q_1}}$ and $V_2 = (\tilde{W}_1/\tilde{W}_2)^{\frac{p_2}{q_2}}$ are algebraic inverse integrating factor of $\mathbf{F}_r$, with $W_1, W_2, \tilde{W}_1$ and $\tilde{W}_2$ formal series and $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ rational numbers. Then there exists a natural number l such that $(V_1/V_2)^l$ is a first integral of $\mathbf{F}$ and belongs to $\mathbb{C}((x, y))$.*

Proof. Taking $l = lcm\{|q|, |q|\}$, then $(V_1/V_2)^l = \frac{W_1^{lp_1/q_1}\tilde{W}_2^{lp_2/q_2}}{W_2^{lp_1/q_1}\tilde{W}_1^{lp_2/q_2}}$, i.e., it is a quotient of formal series. To prove that $(V_1/V_2)^l$ is a first integral of $\mathbf{F}$, it is enough to prove that V_1/V_2 is a first integral. Indeed,

$$L_{\mathbf{F}}\frac{V_1}{V_2} = \frac{1}{V_2}L_{\mathbf{F}}V_1 - \frac{V_1}{V_2^2}L_{\mathbf{F}}V_2 = \frac{1}{V_2}V_1\mathrm{div}(\mathbf{F}) - \frac{V_1}{V_2^2}V_2\mathrm{div}(\mathbf{F}) = 0$$

Consequently, V_1/V_2 is a first integral of $\mathbf{F}$. ■

The following result is key in our study.

Proposition 4.3.125. *Let system (4.1.1) be with $\mathbf{F} = \mathbf{X}_h$ + q-h.h.o.t. and $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$, where the factorization of h on $\mathbb{C}[x, y]$ only has simple factors. We assume that system (4.1.1) has an algebraic inverse integrating factor.*
Then, system (4.1.1) admits an algebraic inverse integrating factor of the form $V = W^q$ being W a formal series $W = h$ + q-h.h.o.t. and q a positive rational number.

Moreover, if system (4.1.1) is not formally integrable, then the algebraic inverse integrating factor is unique, up to a multiplicative constant.

Proof. Given a number non-zero λ such that the quasi-homogeneous polynomial $H(x, y) = x^{2t_2} + \lambda y^{2t_1} \in \mathcal{P}^{\mathbf{t}}_{2t_1t_2}$ is not factor of h, we consider the unique solution $(Cs(\theta), Sn(\theta))$ of the initial value problem

$$\frac{d\mathbf{x}}{d\theta} = \mathbf{X}_H(\mathbf{x}), \qquad \mathbf{x}(\mathbf{0}) = (1, 0)^T.$$

These functions, $Cs(\theta)$ and $Sn(\theta)$, named *generalized trigonometric functions,* are periodic and T will denote their minimal period. For more details, see Dumortier [43]. System (4.1.1) by means of the change

$$x = u^{t_1} Cs(\theta), \qquad y = u^{t_2} Sn(\theta), \tag{4.3.18}$$

with $u \geq 0$ and $\theta \in [0, T)$, and rescaling the time by $dt = (2t_1t_2/u^r)d\tau$, is transformed into

$$u' = \tfrac{du}{d\tau} = -h'(\theta)u + O(u^2), \qquad \theta' = \tfrac{d\theta}{d\tau} = (r + |\mathbf{t}|)h(\theta) + O(u). \tag{4.3.19}$$

where $h(\theta) := h(Cs(\theta), Sn(\theta))$.

The equilibria of (4.3.19) on $u = 0$ are $(u, \theta) = (0, \theta_j)$, $j = 1 \ldots s$, where θ_j are all roots of $h(\theta)$ since we have chosen λ such that H is not a factor of h (otherwise, this factor would not be in the expression of the system (4.3.19) since $H(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta)) = 1$).

The linearization of the system (4.3.19) about the fixed points has eigenvalues non-zero (since the factors of h are simple) with different sign; thus, by applying a result of Seidenberg [76], we can ensure the existence of an unique solution different from $u = 0$ of the form $\theta - \theta_j + \phi^{(j)}(u, \theta) = 0$, with $\phi^{(j)}(u, \theta) = \mathcal{O}\left(|u, \theta - \theta_j|^2\right)$. We note that such solution curves are invariant curves of the system (4.3.19).

From Proposition 4.3.122, if system (4.1.1) has an algebraic inverse integrating factor, then it admits an inverse integrating factor of the form $V = (W_1/W_2)^d$. The irreducible factors of W_1 and of W_2 are invariant curves which arrive at the origin, since they are non-invertible, see proof of Proposition 4.3.122. So, it has that

$$\tfrac{W_1(u^{t_1}Cs(\theta), u^{t_2}Sn(\theta))}{W_2(u^{t_1}Cs(\theta), u^{t_2}Sn(\theta))} = \psi(u, \theta) u^m \prod_{j=1}^{s} \left(\theta - \theta_j + \phi^{(j)}(u, \theta)\right)^{n_j},$$

with m and n_j integer numbers and $\psi(0, \theta_j) \neq 0$ for each $j = 1, \ldots, s$. Undoing the change (4.3.18), we obtain that

$$\tfrac{W_1(x,y)}{W_2(x,y)} = \psi(x, y) \prod_{j=1}^{s} \left[f_j(x, y) + \text{q-h.h.o.t.}\right]^{n_j},$$

with $\psi(0, 0) \neq 0$. Moreover, we can assume that $\psi(0, 0) > 0$.

From Lemma 4.3.123, we have that $\prod_{j=1}^{s}(f_j)^{n_j}$ is an algebraic inverse integrating factor of $\mathbf{X}_h$. In fact, it is a rational first integral of $\mathbf{X}_h$. As h only has simple factors on $\mathbb{C}[x,y]$, the first integrals, which are quotient of quasi-homogeneous polynomials, are h^n with n a integer number non-zero. If we write $\psi(x,y)=\Psi(x,y)^n$ (it holds $\Psi(0,0)\neq 0$ and it can be expanded as a series of quasi-homogeneous terms), it has that $W_1/W_2=(\Psi\,(h+\phi))^n$, with $h+\phi$ unique. Thus, $V=(h+\text{q-h.h.o.t.})^q$ with $q=dn$.

To prove the second part, we see that if there were two distinct algebraic inverse integrating factors, then the system would be formally integrable. Indeed, let consider $V_1=(\Psi_1\,(h+\phi))^{q_1}$, $V_2=(\Psi_2\,(h+\phi))^{q_2}$, $q_1,q_2\in\mathbb{Q}\setminus\{0\}$, $\Psi_1(0,0)\Psi_2(0,0)\neq 0$, two algebraic inverse integrating factors, we can suppose that $q_2\geq q_1$. From Lemma 4.3.124, there is a natural number l such that $(V_2/V_1)^l=(\Psi_2^{lq_2})/(\Psi_1^{lq_1})\,(h+\phi)^{l(q_2-q_1)}$ is a first integral of $\mathbf{F}$, and is formal since $q_2\geq q_1$ and Ψ_1, Ψ_2 are invertible series. ∎

Now, we provide a series of properties of the inverse integrating factor of the system (4.1.1) in order to give conditions that ensure its existence.

Next, we show a class of systems (4.1.1) having an algebraic inverse integrating factor.

Proposition 4.3.126. *Let system (4.1.1) be, with $\mathbf{F}=\mathbf{X}_h+\mu_N\mathbf{D}_0$, $h\in\mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ and $\mu_N\in\mathcal{P}^{\mathbf{t}}_N$. The function $V(h)=h^{\frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}}$ is an algebraic inverse integrating factor of (4.1.1).*

Proof. Applying Euler theorem for quasi-homogeneous function, i.e. $L_{\mathbf{D}_0}f=sf$ with $f\in\mathcal{P}^{\mathbf{t}}_s$, then

$$\begin{aligned} L_{\mathbf{F}}V &= \nabla V\cdot\mathbf{F}=\nabla V\cdot(\mathbf{X}_h+\mu_N\mathbf{D}_0)=\mu_N\,\nabla V\cdot\mathbf{D}_0\\ &= (N+|\mathbf{t}|)\mu_N\,h^{\frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}} \end{aligned}$$

and

$$V\cdot\operatorname{div}(\mathbf{F}) = h^{\frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}}\,(N+|\mathbf{t}|)\mu_N.$$

So, $L_{\mathbf{F}}V-V\operatorname{div}(\mathbf{F})=0$. This completes the proof. ∎

The following proposition gives a necesary condition for the existence of an algebraic inverse integrating factor.

Proposition 4.3.127. *Consider system (4.3.10) where $h \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ only has simple factors in its factorization on $\mathbb{C}[x, y]$. Then, if W^q is an algebraic inverse integrating factor of the system (4.3.10), with q an rational number non-zero and $W = \sum_{j \geq r+|\mathbf{t}|} W_j \in \mathbb{C}[[x, y]]$, $W_j \in \mathcal{P}^{\mathbf{t}}_j$, being $W_{r+|\mathbf{t}|} = h$, for each positive integer k it holds:*

$$L_{\mathbf{X}_h} W_k = \sum_{l=N}^{k-|\mathbf{t}|} \left(\frac{l + |\mathbf{t}|}{q} - (r + k - l) \right) W_{r+k-l}\mu_l. \tag{4.3.20}$$

Moreover,

a) $q = \frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}$

b) $W = h + \sum_{i>1} b_i h^i$, *for all j*

c) $W = h$.

Proof.

By one hand, if W^q is an algebraic inverse integrating factor of system (4.3.11), then, by definition, $L_{\mathbf{F}} W^q - W^q \mathrm{div}(\mathbf{F}) = q W^{q-1} \left[L_{\mathbf{F}} W - \frac{1}{q} W \mathrm{div}(\mathbf{F}) \right] = 0$, thus

$$L_{\mathbf{X}_h + \mu \mathbf{D}_0} W - \tfrac{1}{q} W \mathrm{div}(\mathbf{X}_h + \mu \mathbf{D}_0) = L_{\mathbf{X}_h} W + \mu L_{\mathbf{D}_0} W - \tfrac{1}{q} W \mathrm{div}(\mu \mathbf{D}_0) = 0,$$

Therefore, $\quad L_{\mathbf{X}_h} W = \frac{1}{q} W \mathrm{div}(\mu \mathbf{D}_0) - \mu L_{\mathbf{D}_0} W$

Considering now $W = \sum_{j \geq r+|\mathbf{t}|} W_j$ and $\mu = \sum_{i \geq N} \mu_i$, we obtain,

$$\begin{aligned} L_{\mathbf{X}_h} W &= \tfrac{1}{q} \left(\sum_{j \geq r+|\mathbf{t}|} W_j \right) \left(\sum_{i \geq N} (i + |\mathbf{t}|) \mu_i \right) - \left(\sum_{i \geq N} \mu_i \right) \left(\sum_{j \geq r+|\mathbf{t}|} j W_j \right) \\ &= \sum_{k=r+|\mathbf{t}|}^{\infty} \sum_{l=N}^{k-|\mathbf{t}|} \frac{l + |\mathbf{t}|}{q} W_{r+k-l} \mu_l - \sum_{k=r+|\mathbf{t}|}^{\infty} \sum_{l=N}^{k-|\mathbf{t}|} (r + k - l) W_{r+k-l} \mu_l \\ &= \sum_{k=r+|\mathbf{t}|}^{\infty} \sum_{l=N}^{k-|\mathbf{t}|} \left(\frac{l + |\mathbf{t}|}{q} - (r + k - l) \right) W_{r+k-l} \mu_l \end{aligned}$$

We prove the second part,

a) Considering $k = N + |\mathbf{t}|$ in the equality (4.3.20), we obtain,

$$L_{\mathbf{X}_h} W_{N+|\mathbf{t}|} = \left(\frac{N + |\mathbf{t}|}{q} - (r + |\mathbf{t}|) \right) W_{r+|\mathbf{t}|} \mu_N = \left(\frac{N + |\mathbf{t}|}{q} - (r + |\mathbf{t}|) \right) h\mu_N$$

Taking into account that $L_{\mathbf{X}_h} W_{N+|\mathbf{t}|} \in \mathrm{Range}(\ell_{r+N+|\mathbf{t}|})$, $h\mu_N \neq 0$ and $h\mu_N \in \mathrm{Cor}(\ell_{r+N+|\mathbf{t}|})$ then $\frac{N+|\mathbf{t}|}{q} - (r + |\mathbf{t}|) = 0$ and we conclude that $q = \frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}$.

b) We know that $W = h + \sum_{j>r+|\mathbf{t}|}^{\infty} W_j$.

Consider now $j_0 = min\{j \in \mathbb{N}/W_j$ is not a function of $h\}$, then $W = h + \sum_{l\geq 1}^{\lfloor \frac{j_0}{r+|\mathbf{t}|} \rfloor} b_l h^l + W_{j_0} + \cdots$. The equation (4.3.20), for $k = j_0$ is

$$L_{\mathbf{X}_h} W_{j_0} = \sum_{l=N}^{j_0-|\mathbf{t}|} \left(\frac{l + |\mathbf{t}|}{q} - (r + j_0 - l) \right) W_{r+j_0-l} \mu_l$$

Observe that $r + j_0 - l < j_0$ puesto que $r - l < 0$. Therefore W_{r+j_0-l} are function of h and $W_{r+j_0-l}\mu_l \in \mathrm{Cor}(\ell_{r+j_0})$. By other hand $L_{\mathbf{X}_h} W_{j_0} \in \mathrm{Range}(\ell_{r+j_0})$ and we can conclude that $W_{j_0} = 0$ or $W_{j_0} = h^n$, for some $n \in \mathbf{N}$, $n > 1$

c) We know that, $W = h + \sum_{j>1}^{\infty} b_j W_{j(r+|\mathbf{t}|)}$, being $W_{j(r+|\mathbf{t}|)} = h^j$. If $b_j = 0$ for all j, item c) is verified. In other case, let $s_0 = min\{j \in \mathbb{N}/b_j \neq 0\}$. Then, for $k = s_0(r + |\mathbf{t}|) + N - r$, the equality (4.3.20) has only two components $W_{r+k-l} \neq 0$, these are the values corresponding to $l = N$ and $l = N + (s_0 - 1)(r + |\mathbf{t}|)$. Therefore the above equality has the form,

$$\begin{aligned} 0 &= \left(\frac{N + |\mathbf{t}|}{q} - (s_0 - 1)(r + |\mathbf{t}|) \right) W_{s_0(r+|\mathbf{t}|)} \mu_N + \left(\frac{N - r + s_0(r + |\mathbf{t}|)}{q} \right. \\ &- \left. (r + |\mathbf{t}|) \right) W_{(r+|\mathbf{t}|)} \mu_{N+(s_0-1)(r+|\mathbf{t}|)} \end{aligned}$$

Taking into account that,

$$W_{s_0(r+|\mathbf{t}|)} \mu_N = b_{s_0} h^{s_0} \mu_N$$

$$W_{(r+|\mathbf{t}|)} \mu_{N+(s_0-1)(r+|\mathbf{t}|)} = h\mu_{N+(s_0-1)(r+|\mathbf{t}|)}$$

and

$q = \frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}$, the above expression is the following,

$$\begin{aligned} 0 &= -(s_0-1)(r+|\mathbf{t}|)h^{s_0}\mu_N + \frac{(s_0-1)(r+|\mathbf{t}|)^2}{(N+|\mathbf{t}|)} h\mu_{N+(s_0-1)(r+|\mathbf{t}|)} \\ &= (s_0-1)(r+|\mathbf{t}|)h\left(-h^{s_0-1}\mu_N + \frac{(r+|\mathbf{t}|)}{(N+|\mathbf{t}|)}\mu_{N+(s_0-1)(r+|\mathbf{t}|)}\right). \end{aligned}$$

Therefore $\mu_{N+(s_0-1)(r+|\mathbf{t}|)} = \frac{(N+|\mathbf{t}|)}{(r+|\mathbf{t}|)} h^{s_0-1}\mu_N \in \mathrm{Range}(\ell^{(2)}_{N+(s_0-1)(r+|\mathbf{t}|)}) \setminus \{0\}$, which is a contradiction because we know that $\mu_{N+(s_0-1)(r+|\mathbf{t}|)} \in \mathrm{Cor}(\ell^{(2)}_{N+(s_0-1)(r+|\mathbf{t}|)})$. Thus, we can conclude that $W = h$.

■

The following theorem provides a necessary and sufficient condition for the existence of an algebraic inverse integrating factor.

Theorem 4.3.128. *Consider the system $\dot{\mathbf{x}} = \mathbf{X}_h + \cdots$ then, it has an algebraic inverse integrating factor if and only if it is orbitally equivalent to*

$$\dot{\mathbf{x}} = \mathbf{X}_h + \mu_N \mathbf{D}_0.$$

Moreover $V = (h+\cdots)^{\frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}}$.

Proof.

The sufficient condition is proved in Proposition 4.3.126. Next we prove the necessary condition.

A formal normal form for this system is of the form (4.3.8) and a reduced normal form is given in (4.3.10). From Proposition 4.3.127 c), we know that $V = h^{\frac{N+|\mathbf{t}|}{r+|\mathbf{t}|}}$ is an algebraic inverse integrating factor of (4.3.8). Consider $j_0 = min\{j > N / \mu_j \neq 0\}$, from the equality (4.3.20) and taking $k = j_0 + |\mathbf{t}|$, we get

$$\begin{aligned} 0 &= \sum_{l=N}^{j_0} \left(\frac{l+|\mathbf{t}|}{q} - (r+j_0+|\mathbf{t}|-l)\right) W_{r+j_0+|\mathbf{t}|-l}\,\mu_l \\ &= \left(\frac{j_0+|\mathbf{t}|}{q} - (r+|\mathbf{t}|)\right) W_{r+|\mathbf{t}|}\,\mu_{j_0} \\ &= \frac{j_0-N}{q} h\mu_{j_0} \end{aligned}$$

and, as $W_{r+|\mathbf{t}|} = h$ and $\frac{j_0-N}{q}$ because $j_0 > N$ and q is a possitive racional number, we obtain $\mu_{j_0} = 0$ which is a contradiction. ■

4.4 Applications.

4.4.1 Study of the integrability for a case of generalized nilpotent systems with quasi-homogeneous first component $\mathbf{F}_r = (y^2, x^3)^T$.

Consider the following family,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, \qquad (4.4.21)$$

with $b_{30} \neq 0$.

Theorem 4.4.129. *System (4.4.21) is analytically integrable if and only if it is verified one of the following conditions:*

a) $3a_{30} + b_{21} = a_{21} + b_{12} = a_{12} + 3b_{03} = 0$

b) $3a_{30} + b_{21} = a_{21} + b_{12} = b_{30}^2(a_{30}a_{12} + a_{03}b_{30} - a_{30}b_{03}) - 3a_{30}^4 = \frac{6a_{32}a_{50}}{b_{50}} + \frac{135a_{50}^4}{b_{50}^3} + \frac{3a_{41}^2}{b_{50}} + 2a_{23} + \frac{54a_{41}a_{50}^2}{b_{50}^2} - \frac{2b_{23}a_{50}}{b_{50}} = 0$

Proof.

Using a scaling in the state of variables, system (4.4.21) can be transformed into a system of the form (3.4.30) with $l = 1$. From Theorem 3.4.96, a formal normal form of system (4.4.21) can be expressed by (3.4.31) that, in the particular case $l = 1$, can be written in the form (4.2.6) with $H = -\frac{y^3}{3} + b_{30}\frac{x^4}{4} \in \mathcal{P}_{12}^{(3,4)}$ and $\nu = \alpha_{02}^{(0)}x^2 + \alpha_{01}^{(1)}xy + \alpha_{02}^{(1)}x^2y + \alpha_{00}^{(2)}h \cdots \in \bigoplus_{j>r} \mathrm{Cor}(\ell_j)$, where $\cdots$ stands for quasi-homogeneous terms of order higher than 12.

Necessary condition. If system (4.4.21) is analytically integrable, then system (4.2.6) is formally integrable, therefore, from Theorem 4.2.108, $\nu \equiv 0$, in particular $\alpha_{02}^{(0)} = \alpha_{01}^{(1)} = \alpha_{02}^{(1)} = \alpha_{00}^{(2)} = 0$.

For the calculation of the values $\alpha_{ij}^{(k)}$, we apply the algorithm given in [7]. For system (4.4.21), the first coefficient $\alpha_{02}^{(0)}$ is,

$$\alpha_{02}^{(0)} = \frac{a_{30}}{3} + \frac{b_{21}}{9} = 0, \text{ if and only if } 3a_{30} + b_{21} = 0$$

Under this hypothesis, the next coefficient $\alpha_{01}^{(1)}$ is,

$$\alpha_{01}^{(1)} = \frac{a_{21} + b_{12}}{3} = 0, \text{ if and only if } a_{21} + b_{12} = 0$$

Following the same process, the next coefficient $\alpha_{02}^{(1)}$ is,

$$\alpha_{02}^{(1)} = (a_{21}b_{30} + 3a_{30}^2)(a_{12} + 3b_{03}) = 0,$$

- If $a_{12} + 3b_{03} = 0$, then we are in the situation a)
- If $a_{21}b_{30} + 3a_{30}^2 = 0$ and $a_{12} + 3b_{03} \neq 0$, the next coefficient $\alpha_{00}^{(2)}$ is,

$$\alpha_{00}^{(2)} = b_{30}^2(a_{30}a_{12} + a_{03}b_{30} - a_{30}b_{03}) - 3a_{30}^4 = 0$$

and from here we obtain the condition b).

Sufficient condition If condition a) is verified, then system (4.4.21) is a Hamiltonian system and therefore it is analytically integrable.

If condition b) is verified we find an inverse integrating factor of the form $W = V^A$, with

$$V = 1 + \frac{a_{12} + 3b_{03}}{Ab_{30}}(b_{30}x + a_{30}y) \text{ and } A = \frac{b_{30}^2(a_{12} + 3b_{03})}{b_{30}^2 a_{12} - 3a_{30}^3}$$

for $b_{30}^2 a_{12} - 3a_{30}^3 \neq 0$. In the particular case $b_{30}^2 a_{12} - 3a_{30}^3 = 0$, there exists an integrating factor e^V, where $V = \frac{a_{12}+3b_{03}}{b_{30}}(a_{30}y - b_{30}x)$. Since that, in both cases, there exists an analytically inverse integrating factor W such that $W(0,0) = 1$, then system (4.4.21) is C^ω-integrable. ∎

4.4.2 Study of the integrability for a case of generalized nilpotent systems with quasi-homogeneous first component $\mathbf{F}_r = (y^2, x^4)^T$.

Consider the following family,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y^2 + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 \\ b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4 \end{pmatrix} \quad (4.4.22)$$

with $b_{40} \neq 0$.

Theorem 4.4.130. *System (4.4.22) is analytically integrable if and only if it is verified one of the following conditions:*

a) $4a_{40} + b_{31} = 3a_{31} + 2b_{22} = a_{13} + 4b_{04} = 2a_{22} + 3b_{13} = 0$

b) $4a_{40} + b_{31} = 3a_{31} + 2b_{22} = b_{04}b_{40}^3 + 3a_{40}^4 + a_{40}b_{13}b_{40}^2 = a_{13}b_{40}^3 + b_{40}^2(2a_{22} - b_{13})a_{40} - 12a_{40}^4 = a_{04}b_{40}^4 - b_{40}^2(a_{22} - b_{13})a_{40}^2 + 9a_{40}^5 = a_{31}b_{40} + 4a_{40}^2 = 0$ *and* $2a_{22} + 3b_{13} \neq 0$

Proof. Using a scaling in the state of variables, system (4.4.22) can be transformed into a system of the form (3.4.32) with $l = 1$. From Theorem 3.4.97, a formal normal form of system (4.4.22) can be expressed by (3.4.33) that, in the particular case $l = 1$, can be written in the form (4.2.6), with $H = -\frac{y^3}{3} + b_{40}\frac{x^5}{5} \in \mathcal{P}_{15}^{(3,5)}$ and $\nu = \alpha_{01}^{(0)}xy + \alpha_{03}^{(1)}x^3 + \alpha_{02}^{(0)}x^2y + \alpha_{03}^{(0)}x^3y + \alpha_{00}^{(3)}h + \alpha_{01}^{(3)}xh + \cdots \in \bigoplus_{j>7}\operatorname{Cor}(\ell_j)$.

Necessary condition. If system (4.4.22) is analytically integrable, then system (4.2.6) is formally integrable, therefore from Theorem 4.2.108 $\nu \equiv 0$, in particular $\alpha_{01}^{(0)} = \alpha_{03}^{(1)} = \alpha_{02}^{(0)} = \alpha_{03}^{(0)} = \alpha_{00}^{(3)} = \alpha_{01}^{(3)} = 0$. Again for the calculation of the values $\alpha_{ij}^{(k)}$, we apply the algorithm given in [7]. For system (4.4.22), the first coefficient $\alpha_{01}^{(0)}$ is $\alpha_{01}^{(0)} = 4a_{40} + b_{31} = 0$.

Under this hypothesis, the next coefficient $\alpha_{03}^{(1)}$ is, $3a_{31} + 2b_{22} = 0$.

Again, under these hypotheses, the next coefficient $\alpha_{02}^{(0)}$ is $b_{40}(a_{13}+4b_{04}) + a_{40}(2a_{22} + 3b_{13}) = 0$.

Following the same process, the next coefficient is $\alpha_{03}^{(0)}$, i.e. $\alpha_{14} = [b_{40}(36a_{31}{}^2_{40} + 3a_{31}^2b_{40} - 8b_{40}^2b_{04}) - a_{40}(8b_{13}b_{40}^2 + 72a_{40}^3)](2a_{22} + 3b_{13}) = 0$.

- If $2a_{22} + 3b_{13} = 0$, then we are in the situation a).

- If $[b_{40}(36a_{31}{}^2_{40} + 3a_{31}^2b_{40} - 8b_{40}^2b_{04}) - a_{40}(8b_{13}b_{40}^2 + 72a_{40}^3)](2a_{22} + 3b_{13}) = 0$ and $2a_{22} + 3b_{13} \neq 0$, the next coefficient is $\alpha_{00}^{(3)}$, i.e., $\alpha_{00}^{(3)} = b_{40}(24a_{22}a_{40}^2b_{40} + 8a_{22}a_{31}b_{40}^2 + 8a_{40}^2b_{40}^3 + 240a_{40}^3a_{31} - 8a_{31}b_{13}b_{40}^3 - 24a_{40}^2b_{13}b_{40} + 45a_{40}a_{31}^2b_{40}) = 0$. In these conditions, the next coefficient is $\alpha_{01}^{(3)} = a_{31}b_{40} + 4a_{40}^2 = 0$ and, from here we obtain the condition b).

Sufficient condition If condition a) is verified system (3.4.32) is a Hamiltonian system and consequently, analytically integrable.

If condition b) is verified it is possible to find an inverse integrating factor of the form $W = V^A$, with

$$V = 1 + \frac{2a_{22} + 3b_{13}}{2Ab_{40}^2}(b_{40}x - a_{40}y)^2 \text{ and } A = \frac{b_{40}^2(2a_{22} + 3b_{13})}{b_{40}^2 a_{22} - 6a_{40}^3}$$

for $b_{40}^2 a_{22} - 6a_{40}^3 \neq 0$. In the particular case $b_{40}^2 a_{22} - 6a_{40}^3 = 0$, there exists an inverse integrating factor e^V, where $V = \frac{3(4a_{40}^3 + b_{13}b_{40}^2)}{2b_{40}^4}(b_{40}x - a_{40}y)^2$. Since that, in both cases, there exist an analytically inverse integrating factor W such that $W(0,0) = 1$, then system (4.4.21) is C^ω-integrable. ■

4.4.3 Study of the integrability for a case of generalized nilpotent systems with quasi-homogeneous first component $\mathbf{F}_r = (y^2, x^5)^T$.

Consider the following family,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y^2 + a_{50}x^5 + a_{41}x^4y + a_{32}x^3y^2 + a_{23}x^2y^3 + a_{14}xy^4 + a_{05}y^5 \\ b_{50}x^5 + b_{41}x^4y + b_{32}x^3y^2 + b_{23}x^2y^3 + b_{14}xy^4 + b_{05}y^5 \end{pmatrix}, \tag{4.4.23}$$

with $b_{50} \neq 0$.

Theorem 4.4.131. *System (4.4.23) is analytically integrable if and only if it is verified one of the following conditions*

a) $5a_{50} + b_{41} = 2a_{41} + b_{32} = \frac{6a_{50}}{b_{50}}(b_{23} + a_{32}) + 4b_{14} + 2a_{23} = 2a_{14} + 10b_{05} + \frac{3b_{23}a_{41}}{b_{50}} + \frac{9a_{50}^2 b_{23}}{b_{50}^2} + \frac{9a_{50}^2 a_{32}}{b_{50}^2} + \frac{3a_{32}a_{41}}{b_{50}} = a_{32} + b_{23} = 0$

b) $5a_{50} + b_{41} = 2a_{41} + b_{32} = \frac{6a_{50}}{b_{50}}(b_{23} + a_{32}) + 4b_{14} + 2a_{23} = 2a_{14} + 10b_{05} + \frac{3b_{23}a_{41}}{b_{50}} + \frac{9a_{50}^2 b_{23}}{b_{50}^2} + \frac{9a_{50}^2 a_{32}}{b_{50}^2} + \frac{3a_{32}a_{41}}{b_{50}} = \frac{6a_{32}a_{50}}{b_{50}} + \frac{135a_{50}^4}{b_{50}^3} + \frac{3a_{41}^2}{b_{50}} + 2a_{23} + \frac{54a_{41}a_{50}^2}{b_{50}^2} - \frac{2b_{23}a_{50}}{b_{50}} = \frac{18a_{50}^2 a_{32}}{b_{50}^2} + \frac{6a_{32}a_{41}}{b_{50}} - \frac{12a_{50}^2 b_{23}}{b_{50}^2} + 4a_{14} + \frac{270a_{50}^3 a_{41}}{b_{50}^3} + \frac{405a_{50}^5}{b_{50}^4} + \frac{45a_{41}^2 a_{50}}{b_{50}^2} - \frac{4b_{23}a_{41}}{b_{50}} = 0$

Proof. Using the scaling in the state of variables $u = (\frac{b_{50}}{2})^{\frac{2}{9}}x$, $v = (\frac{b_{50}}{2})^{\frac{1}{9}}y$ system (4.4.23) is transformed into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y^2 + A_{50}x^5 + A_{41}x^4y + A_{32}x^3y^2 + A_{23}x^2y^3 + A_{14}xy^4 + A_{05}y^5 \\ 2x^5 + B_{41}x^4y + B_{32}x^3y^2 + B_{23}x^2y^3 + B_{14}xy^4 + B_{05}y^5 \end{pmatrix}, \tag{4.4.24}$$

where $A_{50} = (\frac{2}{b_{50}})^{\frac{8}{9}} a_{50}$, $A_{41} = (\frac{2}{b_{50}})^{\frac{7}{9}} a_{41}$, $A_{32} = (\frac{2}{b_{50}})^{\frac{2}{3}} a_{32}$, $A_{23} = (\frac{2}{b_{50}})^{\frac{5}{9}} a_{23}$, $A_{14} = (\frac{2}{b_{50}})^{\frac{4}{9}} a_{14}$, $A_{05} = (\frac{2}{b_{50}})^{\frac{1}{3}} a_{05}$, $B_{41} = (\frac{2}{b_{50}})^{\frac{8}{9}} b_{41}$, $B_{32} = (\frac{2}{b_{50}})^{\frac{7}{9}} b_{32}$, $B_{23} = (\frac{2}{b_{50}})^{\frac{2}{3}} b_{23}$, $B_{14} = (\frac{2}{b_{50}})^{\frac{5}{9}} b_{14}$, $B_{05} = (\frac{2}{b_{50}})^{\frac{4}{9}} b_{05}$.

Next, applying the change of variables $x = u$, $y = u^2 - v$, system (4.4.24) is transformed into the form of system (3.4.42) with $a = -1$ and $\sigma = 1$. It can be expressed in the form (4.2.6), with $H = -\frac{y^3}{3} + x^2y^2 - x^4y \in \mathcal{P}_6^{(1,2)}$ and $\nu = \alpha_{00}^{(0)} x^2y + \alpha_{00}^{(1)} y^2 + \alpha_{01}^{(2)} xy^2 + \alpha_{00}^{(3)} x^2y^2 + \alpha_{00}^{(4)} h + \alpha_{01}^{(5)} xh + \alpha_{00}^{(6)} x^2h + \alpha_{00}^{(7)} yh + \alpha_{01}^{(6)} x^3h + \alpha_{01}^{(7)} xyh + \cdots$.

Necessary condition. If system (4.4.23) is analytically integrable, system (4.2.6) is formally integrable. Therefore, from Theorem 4.2.108, $\nu \equiv 0$, in particular $\alpha_{00}^{(0)} = \alpha_{00}^{(1)} = \alpha_{01}^{(2)} = \alpha_{00}^{(3)} = \alpha_{00}^{(4)} = \alpha_{01}^{(5)} = \alpha_{00}^{(6)} = \alpha_{00}^{(7)} = \alpha_{01}^{(6)} = \alpha_{01}^{(7)} = 0$. Analogously with the above cases, we use the algorithm described in [7] for the calculation of the values $\alpha_{ij}^{(k)}$. For system (4.4.23), the first coefficient $\alpha_{00}^{(0)} = 5A_{50} + B_{41} = 0$, and the second one is $\alpha_{00}^{(1)} = 0$.

Under this hypothesis, the next coefficient is $\alpha_{01}^{(2)} = 2A_{41} + B_{32} = 0$.

Again, under these hypotheses, the next two coefficients are $\alpha_{00}^{(3)} = \alpha_{00}^{(4)} = 0$.

Analogously the next coefficient is

$$\alpha_{01}^{(5)} = 3A_{50}(B_{23} + A_{32}) + 4B_{14} + 2A_{23} = 0.$$

Imposing this new condition, we obtain,

$$\alpha_{00}^{(6)} = -\frac{12}{5}A_{14} - 12B_{05} - \frac{9}{5}B_{23}A_{41} - \frac{27}{10}A_{50}^2B_{23} - \frac{27}{10}A_{50}^2A_{32} - \frac{9}{5}A_{32}A_{41} = 0.$$

Again, under these hypotheses, the next three coefficient are,

$$\alpha_{00}^{(7)} = \alpha_{01}^{(6)} = \alpha_{01}^{(7)} = 0.$$

and, following with the process, the next coefficient is,

$$\alpha_{11}^{(1)} = (A_{32}+B_{23})(24A_{32}A_{50}+135A_{50}^4+12A_{41}^2+16A_{23}+108A_{41}A_{50}^2-8B_{23}A_{50}).$$

At this point, we distinguish two cases,

- If $A_{32} + B_{23} = 0$ and we are in situation **a)**.
- If $A_{32} + B_{23} \neq 0$ and considering

$$24A_{32}A_{50} + 135A_{50}^4 + 12A_{41}^2 + 16A_{23} + 108A_{41}A_{50}^2 - 8B_{23}A_{50} = 0,$$

we calculate the next coefficient,

$$\begin{aligned}\alpha_{11}^{(2)} &= (B_{23}+A_{32})(72A_{50}^2A_{32}+48A_{32}A_{41}48A_{50}^2B_{23}+64A_{14}+540A_{50}^3A_{41}+\\ &+ 405A_{50}^5+180A_{41}^2A_{50}-32B_{23}A_{41})\end{aligned}$$

Imposing $72A_{50}^2A_{32}+48A_{32}A_{41}48A_{50}^2B_{23}+64A_{14}+540A_{50}^3A_{41}+405A_{50}^5+180A_{41}^2A_{50}-32B_{23}A_{41}=0$, we are in the situation **b)**.

Sufficient condition. If condition a) is verified, then system (4.4.23) is a Hamiltonian system and, consequently, analytically integrable.

If condition b) is verified, we find an inverse integrating factor of the form W^A with $W = 1-\frac{1}{16}(A_{32}+B_{23})(-24A_{50}Ax^2y-12A(2A_{41}+3A_{50}^2)x^3y+6A(2A_{41}+3A_{50}^2)xy^2+24A_{50}Ax^4+(-8B_{23}-8A_{32}+27AA_{50}^3+18A_{50}AA_{41}+8A_{32}A)x^6+(-27AA_{50}^3-18A_{50}AA_{41}+8B_{23}-8A_{32}A+8A_{32})y^3+(-54A_{50}AA_{41}+24B_{23}-81AA_{50}^3+24A_{32}-24A_{32}A)x^4y+(24A_{32}A-24A_{32}+54A_{50}AA_{41}+81AA_{50}^3-24B_{23})x^2y^2-16Ax^3+6A(2A_{41}+3A_{50}^2)x^5)/A^2$,

where A is a solution of the following second degree equation,

$-64(A_{32}+B_{23})^2+64(A_{32}+B_{23})^2\,A+(144A_{32}A_{41}A_{50}-64A_{32}B_{23}+216A_{50}^3A_{32}+324A_{41}^2A_{50}^2-144B_{23}A_{41}A_{50}+972A_{50}^4A_{41}-216A_{50}^3B_{23}+128A_{05}+729A_{50}^6)A^2=0$.

Therefore,

. If the above equation has only real solutions, then system (4.4.23) has a real inverse integrating factor W such that $W(0,0)=1$, then system (4.4.23) is C^ω-integrable.

. If the above equation has complex solutions, then the real and the imaginary part of it are real inverse integrating factors such that do not vanish at the origin. Therefore system (4.4.23) is C^ω-integrable. ■

4.4.4 Study of the existence of a formal inverse integrating factor for some families of generalized nilpotent systems.

▶ Example 1:

$$\dot{\mathbf{x}} = \begin{pmatrix} y^2 + \sigma x^2 \\ -2\sigma xy \end{pmatrix} + \begin{pmatrix} a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ b_{30}x^3 + b_{12}xy^2 + b_{03}y^3 \end{pmatrix} \tag{4.4.25}$$

Theorem 4.4.132. *System (4.4.25) has a formal inverse integrating factor if and only if it is verified one of the following conditions:*

a) $a_{12} + 3b_{03} = b_{12} + a_{21} = 0$

b) $a_{12} = b_{03} = b_{30} = 0$ *and* $b_{12} + a_{21} \neq 0$.

c) $a_{12} = b_{03} = b_{30} - \sigma(b_{12} - 2a_{21}) = 0$ *and* $b_{30}(b_{12} + a_{21}) \neq 0$.

d) $a_{12} = b_{03} = 0$, $b_{12} + a_{21} \neq 0$ *and* $b_{30}(b_{30} - \sigma(b_{12} - 2a_{21})) \neq 0$

Moreover, in the cases b), c) and d), systems (4.4.25) are integrable and, in the case a) systems (4.4.25) are non-integrable.

Proof.

Necessary condition: A formal normal form for system (4.4.25) is given in (3.4.35) for $l = 0$ and $a = 0$. In this particular case, $h = -\frac{1}{3}y^3 - \sigma x^2 y \in \mathcal{P}_3^{(1,1)}$ and $\mu = \sum_{j\geq 2} \mu_j$ with $\mu_j \in \mathrm{Cor}(\ell_j)$. From Proposition 4.3.115 $\mu_j = 0$ for $2 \leq j < 1+3s$, for some $s \in \mathbb{N}$. Therefore, the coefficients $\alpha_{00}^{(2)}$ and $\alpha_{00}^{(5)}$ must be null. So, by applying the algorithm given in [7], we obtain $\alpha_{00}^{(2)} = 3b_{03}+a_{12} = 0$ and, imposing this conditions we obtain, $\alpha_{00}^{(5)} = b_{03}(b_{12}+a_{21}) = 0$. At this point we have two options:

1. $b_{12} + a_{21} = 0$. We are in the situation a).

2. $b_{03} = 0$ and $b_{12} + a_{21} \neq 0$. The next coefficient is $\alpha_{00}^{(6)} = b_{30}(b_{12} + a_{21})(2a_{21}\sigma + b_{30} - b_{12}\sigma) = 0$. At this point we have again two options:

   2.a) $b_{30} = 0$. In this case, we are in the situation b).

2.b) $b_{30} = \sigma(b_{12} - 2a_{21}) \neq 0$. In this case, we are in the situation c).

2.c) $b_{30}(b_{30} - \sigma(b_{12} - 2a_{21})) \neq 0$. In this case, we are in the situation d).

Sufficient condition: If condition a) is satisfied, then system (4.4.25) is a Hamiltonian system, therefore the Hamiltonian function h and $f(h)$, being f any scalar function, are inverse integrating factors of system (4.4.25).

If condition b) is satisfied, then we obtain the following situations:

1. If $b_{12} \neq 0$, we find an inverse integrating factor W^A with $W = 1 - \frac{b_{12}}{2\sigma}y$ and $A = \frac{2(a_{21}+b_{12})}{b_{12}}$.
2. If $b_{12} = 0$, we find an exponential inverse integrating factor e^W with $W = \frac{a_{21}+b_{12}}{\sigma}y$.

If condition c) is verified, we find a complex inverse integrating factor W^A, with W the following expression:

$$W = 1 - \frac{a_{21}+b_{12}}{A\sigma}y + \frac{2a_{21}^2 + a_{21}b_{12} - b_{12}^2}{2A\sigma}x^2 - \frac{(A-1)a_{21}^2 + (A-2)a_{21}b_{12} - b_{12}^2}{2A^2\sigma^2},$$

and A is the solution of the second degree equation $(-\sigma a_{03}b_{12} + 2\sigma a_{03}a_{21} + a_{21}b_{12})A^2 - (a_{21}+b_{12})A + (a_{21}+b_{12})^2 = 0$ with discriminant $\Delta = -(a_{21} + b_{12})^2[4\sigma(2a_{21}+b_{12})a_{03} - (a_{21}+b_{12})^2] < 0$

As the field is real, both, the real and the imaginary part of the complex inverse integrating factor of this field are also real inverse integrating factors.

If condition d) is verified system (4.4.25) is of the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} y^2 + \sigma x^2 \\ -2\sigma xy \end{pmatrix} + \begin{pmatrix} a_{21}x^2y + a_{03}y^3 \\ b_{30}x^3 + b_{12}xy^2 \end{pmatrix}$$

which is a $\mathbb{R}_x$-reversible system. If we apply the change of variables $x^2 = \frac{2\sigma}{b_{30}}(u+v)$, $y = v$ and the scaling in the time $dT = 2\sigma x dt$, the above system is transformed into

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} \frac{b_{12}}{2\sigma}u^2 \\ \frac{a_{21}}{\sigma}uv + \frac{-b_{12}\sigma + b_{30} + 2a_{21}\sigma}{2}u^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{b_{30}a_{03}}{2}u^3 \end{pmatrix}$$

The result follows by Theorem 4.3.120. ■

▶ Example 2.

Consider the following family,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y^2 + 2ax^2y + \sigma x^4 \\ -2axy^2 - 4\sigma x^3 y \end{pmatrix} + \begin{pmatrix} a_{50}x^5 + a_{31}x^3y + a_{12}xy^2 \\ b_{60}x^6 + b_{41}x^4y + b_{22}x^2y^2 + b_{03}y^3 \end{pmatrix}, \tag{4.4.26}$$

Theorem 4.4.133. *System (4.4.26) has a formal inverse integrating factor if and only if it is verified one of the following conditions:*

a) $5a_{50} + b_{41} = 3a_{31} + 2b_{22} = a_{12} + 3b_{03} = 0$

b) $-2a(5a_{50}+b_{41})+\sigma(3a_{31}+2b_{22}) = \sigma(a_{12}+3b_{03})-(5a_{50}+b_{41}) = \sigma(-6a_{12}+3b_{03})+(2a_{50}-b_{41}) = a(-2a_{50}+b_{41})+\sigma(2a_{31}-b_{22}) = 0$ *and* $5a_{50}+b_{41} \neq 0$

c) $-2a(5a_{50} + b_{41}) + \sigma(3a_{31} + 2b_{22}) = 0$
$\sigma(a_{12} + 3b_{03}) - (5a_{50} + b_{41}) = 0$
$3(-2a_{12} + b_{03})\sigma^2 - \sigma a(-2a_{31} + b_{22}) + (a^2 - \sigma)(-2a_{50} + b_{41}) = 0$
$-\sigma(a^2 - 6\sigma)\mathrm{C} + 2(a^2 - \sigma)b_{60} = 0$
$2(a^2 - \sigma)b_{41} + 5a\mathrm{C}\sigma = 0$
$-4(a^2 - \sigma)^2(5a_{50} + b_{41})^2 - 5a\mathrm{C}(2a^2 - 3\sigma)(a^2 - \sigma)(5a_{50} + b_{41}) + 50\mathrm{C}^2(a^6 - 3a^4\sigma - 18a^2\sigma^2 + 27\sigma^3) = 0$ *with* $5a_{50} + b_{41} \neq 0$ *and* $\mathrm{C} = \frac{a(2a_{50}-b_{41})-\sigma(2a_{31}+b_{22})}{7\sigma} \neq 0$.

Moreover, in all cases system (4.4.26) is analitically integrable.

Proof. Using the conservative-dissipative decomposition system (4.4.26) can be write in the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y^2 + 2ax^2y + \sigma x^4 \\ -2axy^2 - 4\sigma x^3 y \end{pmatrix} + \mathrm{X}_{c_{70}x^7+c_{51}x^5y+c_{32}x^3y^2+c_{13}xy^3} + (d_{40}x^4 + d_{21}x^2y + d_{02}y^2)\mathbf{D}_0 \tag{4.4.27}$$

with

$$c_{70} = (1/6)b_{60} \quad , \quad d_{40} = (5/7)a_{50} + (1/7)b_{41}$$

$$c_{51} = -(2/7)a_{50} + (1/7)b_{41} \quad , \quad d_{21} = (3/7)a_{31} + (2/7)b_{22}$$
$$c_{32} = -(2/7)a_{31} + (1/7)b_{22} \quad , \quad d_{02} = (1/7)a_{12} + (3/7)b_{03}$$
$$c_{13} = -(2/7)a_{12} + (1/7)b_{03}$$

From Theorem 3.4.104, a formal normal form of system (4.4.26) is given in (3.4.42) for the particular case $l = 1$, with $h = -\frac{1}{3}y(y^2 + 3ax^2y + 3\sigma x^4) \in \mathcal{P}_6^{(1,2)}$ and $\mu = \alpha_{00}^{(0)}x^2y + \alpha_{00}^{(1)}y^2 + \alpha_{01}^{(2)}xy^2 + \alpha_{00}^{(3)}x^2y^2 + \alpha_{00}^{(4)}h + \alpha_{01}^{(5)}xh + \cdots \in \bigoplus_{j>3} \mathrm{Cor}(\ell_j)$.

Necessary condition. From Proposition 4.3.115 $\mu_j = 0$ for $3 \leq j < 3+6s$, for some $s \in \mathbb{N}$. In particular $\alpha_{00}^{(0)} = \alpha_{00}^{(1)} = \alpha_{01}^{(2)} = \alpha_{00}^{(3)} = \alpha_{00}^{(4)} = \alpha_{01}^{(5)} = 0$. By applying the algorithm given in [7] for the calculation of the values $\alpha_{ij}^{(k)}$, we obtain for system (4.4.26), the first coefficients are $\alpha_{00}^{(0)} = -2ad_{40} + d_{21}\sigma$ and $\alpha_{00}^{(1)} = d_{02}\sigma - d_{40}$. Taking $d_{02} = \frac{d_{40}}{\sigma}$ and $d_{21} = \frac{2a}{\sigma d_{40}}$, the next coefficient $\alpha_{01}^{(2)}$ is $\alpha_{01}^{(2)} = d_{40}\left(3c_{13}\sigma^2 - \sigma a c_{32} + (a^2 - \sigma)c_{51}\right)$.

Then, we obtain,

1. If $d_{40} = 0$, then we are in the situation **a)**.
2. If $3c_{13}\sigma^2 - \sigma a c_{32} + (a^2 - \sigma)c_{51} = 0$ and $d_{40} \neq 0$, taking $c_{13} = -\frac{1}{3}\frac{a^2c_{51} - c_{32}\sigma a - c_{51}\sigma}{\sigma^2}$, the following coefficients are

$$\begin{aligned} \alpha_{00}^{(3)} &= d_{40}(ac_{51} - c_{32}\sigma)(a^3c_{51} - c_{32}\sigma a^2 + 14a^2c_{70} - 6ac_{51}\sigma + 6\sigma^2c_{32} \\ &\quad -14\sigma c_{70}) = 0 \\ \alpha_{00}^{(4)} &= d_{40}(ac_{51} - c_{32}\sigma)(a^2c_{51} - c_{32}\sigma a + 14ac_{70} - 10c_{51}\sigma - 4d_{40}\sigma) = 0 \end{aligned}$$

Focusing on $\alpha_{00}^{(3)}$ we can distinguish the following cases:

2.a) If $ac_{51} - c_{32}\sigma = 0$, then we are in the condition **b)**.

2.b) If $ac_{51} - c_{32}\sigma \neq 0$ and $a^3c_{51} - c_{32}\sigma a^2 + 14a^2c_{70} - 6ac_{51}\sigma + 6\sigma^2c_{32} - 14\sigma c_{70} = 0$ or equivalently $(a^2 - 6\sigma)(ac_{51} - c_{32}\sigma) + 14(a^2 - \sigma)c_{70} = 0$ (observe that, $ac_{51} - c_{32}\sigma \neq 0$ therefore, in this case $a^2 - \sigma \neq 0$ otherwise $a^2 = \sigma$ and, consequently $a^2 - 6\sigma = -5\sigma = 0$ and this implies that $\sigma = 0$ which is impossible).

At this point, to facilitate expression of the coefficients $\alpha_{ij}^{(k)}$, we make a renaming of parameters as follows:

$$ac_{51} - c_{32}\sigma = -\sigma \mathrm{C}, \text{ i.e. } c_{32} = \mathrm{C} + \frac{a}{\sigma}c_{51}.$$

(It should be noted that, from now on, C is nonzero, since otherwise we have already previously studied).
With this change, the above coefficients $\alpha_{00}^{(3)}$ and $\alpha_{00}^{(4)}$ are of the form:

$$\alpha_{00}^{(3)} = -d_{40}\sigma\mathrm{C}(-a^2\mathrm{C}\sigma + 14a^2c_{70} + 6\mathrm{C}\sigma^2 - 14\sigma c_{70}),$$

and we take $c_{70} = \frac{1}{14}\frac{\sigma(a^2-6\sigma)\mathrm{C}}{a^2-\sigma}$,

$$\alpha_{00}^{(4)} = d_{40}\sigma\mathrm{C}(a\sigma\mathrm{C} - 14ac_{70} + 10c_{51}\sigma + 4d_{40}\sigma),$$

and we consider $c_{51} = -\frac{1}{10}\frac{4d_{40}(a^2-\sigma)+5a\sigma\mathrm{C}}{a^2-\sigma}$.
Following the same procedure, under these hypotheses, the next coefficient is

$$\begin{aligned}\alpha_{01}^{(5)} &= 50a^6C^2 - 70a^5Cd_{40} - 196d_{40}^2a^4 - 150a^4C^2\sigma + 175d_{40}a^3C\sigma \\ &- 900a^2\sigma^2C^2 + 392d_{40}^2a^2\sigma - 105d_{40}\sigma^2aC - 196d_{40}^2\sigma^2 \\ &+ 1350C^2\sigma^3 = 0,\end{aligned}$$

and, from here, we obtain the condition **c)**.

Sufficient condition: If condition **a)** is verified, system (3.4.40) is a Hamiltonian system and consequently, analytically integrable.

If condition **b)** is verified it is possible to find an inverse integrating factor of the form $W = V^A$, with

$$V = 1 + \frac{a_{50}}{\sigma}x \quad \text{and} \quad A = \frac{5a_{50} + b_{41}}{a_{50}},$$

for $a_{50} \neq 0$. In the particular case $a_{50} = 0$, there exists an inverse integrating factor e^V, where $V = \frac{2a_{50}-b_{41}}{\sigma}x$.

If condition **c)** is verified it is possible to find an inverse integrating factor of the form $W = V^A$, with V and A having the following form,

$$\begin{aligned}V &= 1 + \alpha_{10}x + \alpha_{20}x^2 + \alpha_{01}y + \alpha_{30}x^3 + \alpha_{11}xy + \alpha_{40}x^4 + \alpha_{21}x^2y + \alpha_{02}y^2, \\ A &= \frac{15(5a_{50} + b_{41})(a^2 - \sigma)}{5\mathrm{C}a(2a^2 - 3\sigma) + 8(a^2 - \sigma)(5a_{50} + b_{41})},\end{aligned}$$

(we omit the expressions of α_{ij} because these expressions are too long), under the condition $5\mathrm{C}a(2a^2 - 3\sigma) + 8(a^2 - \sigma)(5a_{50} + b_{41}) \neq 0$. In the particular case

$5\mathrm{C}a(2a^2-3\sigma)+8(a^2-\sigma)(5a_{50}+b_{41})=0$ we find and exponential integrating factor e^V, where V is of the form

$$V=\frac{5}{128}\frac{(2a^2-3\sigma)(2\mathrm{C}a^3x^2+16\sigma a^2x-8a^2\mathrm{C}y-7\mathrm{C}\sigma ax^2-16\sigma^2x+8\mathrm{C}\sigma y)a\mathrm{C}}{(a^2-\sigma)^2\sigma^2}.$$

■

4.4.5 Study of the existence of an algebraic inverse integrating factor for some families of generalized nilpotent systems.

In this subsection we obtain necessary conditions for the existance of an algebraic inverse integrating factor.

▶ Example 1

$$\dot{\mathbf{x}} = \begin{pmatrix} y^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3 \\ b_{21}x^2y+b_{12}xy^2+b_{03}y^3 \end{pmatrix} \tag{4.4.28}$$

with $a_{30},\ a_{21},\ a_{12},\ a_{03},\ b_{12},\ b_{21}$ and b_{03} real numbers. These systems consist in a Hamiltonian system, whose Hamiltonian function is $h=\frac{1}{4}x^4-\frac{1}{3}y^3\in\mathcal{P}_{12}^{(3,4)}$, perturbed by cubic terms.

It has the following result,

Proposition 4.4.134. *System (4.4.28) with* $3a_{30}+b_{21}\neq 0$ *is not formally integrable. Moreover, if it has an algebraic inverse integrating factor, then* $13(a_{21}+b_{12})+(3a_{30}+b_{21})(4a_{30}-3b_{21})=0$ *and the algebraic inverse integrating factor is equal to* $(4y^3-3y^4+h.o.t.)^{13/12}$.

Proof. The normal form for system (4.4.28) is given in (3.4.31) for $l=1$. In this particular case, the Hamiltonian function is $h=\frac{1}{4}x^4-\frac{1}{3}y^3\in\mathcal{P}_{12}^{(3,4)}$ and $\mu=\alpha_6x^2+\alpha_7xy+\alpha_{10}x^2y+\cdots\in\bigoplus_{j>5}\mathrm{Cor}(\ell_j)$. From Proposition (4.3.128), $\mu_j=0$ for all j. Therefore the coeficients of μ must be zero. By appliying the algorithm given in [7], we obtain that, the first coefficient $\alpha_6=3a_{30}+b_{21}$ and the following coefficient $\alpha_7=13(a_{21}+b_{12})+(3a_{30}+b_{21})(4a_{30}-3b_{21})$. At this point we can conlude that, if $3a_{30}+b_{21}\neq 0$ then, from Theorem (4.2.108), system (4.4.28) is a non-integrable system. In this case, from theorem (**??**), we can conclude that, system (4.4.28) has an algebraic inverse integrating factor, then must be verified that $13(a_{21}+b_{12})+(3a_{30}+b_{21})(4a_{30}-3b_{21})=0$ and this algebraic inverse integrating factor is of the form $(\frac{1}{4}x^4-\frac{1}{3}y^3+h.o.t.)^{13/12}$.

$\blacksquare$

Moreover, we have studied this family in two cases described below.

A) Case $a_{30}=0$, $a_{03}=0$. The family has the form,

$$\begin{aligned}(\dot{x},\dot{y})^T &= (y^2,x^3)^T+(a_{21}x^2y+a_{12}xy^2,\\ & \quad b_{21}x^2y+b_{12}xy^2+b_{03}y^3)^T.\end{aligned} \tag{4.4.29}$$

with $a_{21},\ a_{12},\ b_{12},\ b_{21}$ and b_{03} real numbers.

We obtain the following result,

Theorem 4.4.135. *We assume that system (4.4.29) has an algebraic inverse integrating factor. It has that:*

a) If $b_{21}\neq 0$, then $13(a_{21}+b_{12})=3b_{21}^2$, (non-integrable case),

b) If $b_{21}=0$, then system (4.4.29) has a formal inverse integrating factor (integrable case).

Moreover, in such a case, system (4.4.29) is one of the following systems

1. *$b_{21}=a_{21}+b_{12}=a_{12}+3b_{03}=0$, (Hamiltonian case).*
2. *$a_{21}=a_{03}=b_{21}=b_{12}=0, a_{12}+3b_{03}\neq 0$, (non-Hamiltonian case).*

Proof. The proof of item a) is followed from Proposition (4.4.134). To prove item b) we assume that $b_{21}=0$ (i.e., $\alpha_6=0$). If $a_{21}+b_{12}\neq 0$ (i.e., $\alpha_7\neq 0$), it is easy to check that $\alpha_{10},\alpha_{12},\alpha_{15}$ and α_{16} are not zero simultaneouly. Therefore, by Theorem (4.3.128), system (4.4.28) does not have an algebraic inverse integrating factor. Otherwise, $a_{21}+b_{12}=0$ (i.e., $\alpha_6=\alpha_7=0$), then the following coefficient α_{10} is $\alpha_{10} = (3b_{12} - 4a_{21})(a_{12} + 3b_{03})$. If it is not zero, the following coefficients under the cancellation of the above ones are

$$\begin{aligned}
\alpha_{12} &= (a_{12} + 3b_{03})(98a_{03} + (3b_{12} - 4a_{21})^2),\\
\alpha_{15} &= (3b_{12} - 4a_{21})^2(a_{12} + 3b_{03})(5b_{03} - 4a_{12}),\\
\alpha_{16} &= (3b_{12} - 4a_{21})(a_{12} + 3b_{03})((289/1372)(3b_{12} - 4a_{21})^3 + (11/25)(a_{12} + 3b_{03})^2),\\
\alpha_{18} &= (3b_{12} - 4a_{21})^2(a_{12} + 3b_{03})^3.
\end{aligned}$$

Thus, α_{18} is different from zero and therefore system (4.4.28) does not have an algebraic inverse integrating factor.
Otherwise, $(3b_{12} - 4a_{21})(a_{12} + 3b_{03}) = 0$ (i.e., $\alpha_6 = \alpha_7 = \alpha_{10} = 0$). If $a_{12} + 3b_{03} = 0$, system (4.4.28) is a Hamiltonian system whose Hamiltonian is a polynomial inverse integrating factor and a first integral. So, the system is formally integrable and it has a formal inverse integrating factor. In this case we are in the situaction b) 1.

If $a_{12} + 3b_{03} \neq 0$ and $3b_{12} - 4a_{21} = 0$, it has that

$$\begin{aligned}
\alpha_{12} &= (a_{12} + 3b_{03})a_{03},\\
\alpha_{15} &= a_{03}(a_{12} + 3b_{03})(11b_{03} - 8a_{12}),\\
\alpha_{16} &= a_{03}^2(a_{12} + 3b_{03}).
\end{aligned}$$

If $a_{03} \neq 0$, then α_{12} and α_{16} are different from zero. So, the existence of an algebraic inverse integrating factor arrives to $a_{03} = 0$. (We are in the situation b) 2.). It is easy to check that

$$\begin{aligned}
V = \;& 1 + (a_{12} + 3b_{03})x + (3/2)b_{03}(a_{12} + 3b_{03})x^2\\
& -(1/2)b_{03}(a_{12} - 3b_{03})(a_{12} + 3b_{03})x^3\\
& +(1/2)a_{12}b_{03}(-3b_{03} + 2a_{12})(a_{12} - 3b_{03})x^4\\
& -(1/2)b_{03}(-b_{03} + a_{12})(a_{12} - 3b_{03})(-3b_{03} + 2a_{12})y^3\\
& -(1/2)b_{03}(-b_{03} + a_{12})(a_{12} - 3b_{03})(-3b_{03} + 2a_{12})a_{12}xy^3,
\end{aligned}$$

is a polynomial inverse integrating factor for family b) 2. with $V(0,0) = 1$. Thus,

$$H = -\int P/V\,dy + \int \left(Q/V + \frac{\partial}{\partial x}\int P/V\,dy\right)dx$$

is a formal first integral defined in a neighborhood of the origin. Therefore, the system is formally integrable. ■

B) Case $a_{30} \neq 0,\ a_{21} = a_{12} = a_{03} = b_{12} = 0$. The family has the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} y^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} a_{30}x^3 \\ b_{21}x^2y + b_{03}y^3 \end{pmatrix} \tag{4.4.30}$$

with $a_{30} \neq 0$. It has the following result.

Theorem 4.4.136. *System (4.4.30), with* $a_{30} \neq 0$, *has an algebraic inverse integrating factor if, and only if, it satisfies one of the following conditions:*

1. $3a_{30} + b_{21} = b_{03} = 0$, *(Hamiltonian system)*
2. $3b_{21} - 4a_{30} = 0$ *and* $b_{03} = 0$, *(non-integrable system).*
   Moreover, in this case, an algebraic inverse integrating factor is $V = (4y^3 - 3x^4)^{13/12}$.

Proof. We assume that system (4.4.30) with $a_{30} \neq 0$, has an algebraic inverse integrating factor. The first coefficient of the quasi-homogeneous normal form of (4.4.30), is $\alpha_6 = 3a_{30} + b_{21}$. Therefore, there is two options,

- If $3a_{30} + b_{21} = 0$, then α_6 and α_7 are zero and $\alpha_{10} = a_{30}^2 b_{03}$. This arrives to $b_{03} = 0$, i.e., it is a Hamiltonian system.
- If $3a_{30} + b_{21} \neq 0$, it arrives to $4a_{30} - 3b_{21} = 0$. In such case, the following coefficient of the normal form is $\alpha_{10} = a_{30}^2 b_{03}$. So, $b_{03} = 0$. It is easy to check that $V = (4y^3 - 3x^4)^{13/12}$ is an algebraic inverse integrating factor of the system.

■

CHAPTER 5

Quasi-homogeneous tridimensional normal forms.

5.1 Introduction

In the context of dynamical systems modeled by systems of nonlinear differential equations, the theory of normal forms focuses on the identification the simplest expressions. This theory is a basic tool for the study of various problems in differential equations, such as: study of bifurcations, stability analysis, among others. The main idea of this theory is the use of near-identity changes of variables to eliminate non-essential terms, from the dynamic point of view, in the analytical expression of the vector field. In this memory, we have generalized these ideas using quasi-homogeneous vector fields and we have used this generalization for calculating, to infinite order, a normal form of some planar vectors fields. In this chapter, we apply the quasi-homogenous normal forms to tridimensional vector fields. In the first part of this chapter we develop the theory of normal forms for vector field with the form $\dot{\mathbf{x}} = (\mathbf{X}_h, f(x,y))^T + \cdots$ being $\mathbf{x} = (x,y,z)^T$ i.e., we consider systems,

$$\dot{\mathbf{x}} = \sum_{j \geq r} \mathbf{F}_j, \tag{5.1.1}$$

being $\mathbf{F}_r$ a quasi-homogeneous vector field in $\mathbb{R}^3$, $\mathbf{F}_r$ is independent on z and $\mathrm{div}(\mathbf{F}_r) = 0$. We present two results where the normal form of these vectors fields is described under $\mathcal{C}^\infty$-conjugation (Theorem 5.3.140) and $\mathcal{C}^\infty$-equivalence (Theorem 5.3.141), respectively. This theory is applicable to a wide class of systems, one of these systems is a particular case of triple-zero singularity with geometric multiplicity equals to two, namely,

$$\dot{\mathbf{x}} = \mathbf{F}_r + \cdots, \quad \text{where } \mathbf{F}_r = \begin{pmatrix} y \\ x^2 \\ \hline 2x^3 - 3y^2 \end{pmatrix},$$

or the Hopf-Zero singularity, described by the following system,

$$\dot{\mathbf{x}} = \mathbf{F}_r + \cdots, \quad \text{where } \mathbf{F}_r = \mathbf{F}_0 = \begin{pmatrix} -y \\ x \\ \hline x^2 + y^2 \end{pmatrix}.$$

The first works published about this last singularity, were obtained by Ushiki [82]. Later Algaba *et al.* [2] described the hypernormal form of vector fields having a Hopf-zero singularity. More recently Chen *et al.* [33, 34], Gazor *et al.* [53] and Gazor & Mokhtari [52] provide different normal forms of Hopf-zero singularity. In the cited works of Chen *et al.* [33,34], a normal form of the Hopf-Zero singularity is considered. Moreover, they give the unique normal forms, under conjugancy and orbital equivalence for this singularity. In others words, they describe the unique normal form for system (5.1.1) considering $\mathbf{F}_0 = (y, -x, 0)^T$ and $\mathbf{t} = (1, 1, 1)$, more specifically, vector fields of the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + \begin{pmatrix} b_0 z \begin{pmatrix} x \\ y \end{pmatrix} \\ z^2 \pm (x^2 + y^2) \end{pmatrix} + \sum_{k \geq 3} \mathbf{F}_k.$$

In this case, they obtain, under $\mathcal{C}^\infty$-equivalence, the following normal form,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + \begin{pmatrix} b_0 z \begin{pmatrix} x \\ y \end{pmatrix} \\ z^2 \pm (x^2 + y^2) \end{pmatrix} + \begin{pmatrix} b_1 h \begin{pmatrix} x \\ y \end{pmatrix} \\ a_1 z^3 \end{pmatrix} + \sum_{m \geq 2} \begin{pmatrix} b_m h^m \begin{pmatrix} x \\ y \end{pmatrix} \\ a_m h^m \end{pmatrix}.$$

This normal form is not correct since, for $m \geq 1$, it has,

$$\left[\begin{pmatrix} -y \\ x \\ x^2+y^2 \end{pmatrix}, \begin{pmatrix} zh^{m-1}\begin{pmatrix} x \\ y \end{pmatrix} \\ z^2h^{m-1} \end{pmatrix}\right] = \begin{pmatrix} h^m\begin{pmatrix} x \\ y \end{pmatrix} \\ 0 \end{pmatrix},$$

$$\left[\begin{pmatrix} -y \\ x \\ x^2+y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ zh^{m-1} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ -h^m \end{pmatrix}.$$

Therefore, $\begin{pmatrix} h^m\begin{pmatrix} x \\ y \end{pmatrix} \\ 0 \end{pmatrix} \in \mathrm{Im}(\mathcal{L}_{2m})$ and $\begin{pmatrix} 0 \\ 0 \\ h^m \end{pmatrix} \in \mathrm{Im}(\mathcal{L}_{2(m-1)})$.

Gazor & Mokhtari [52], compute the simplest normal form for free divergence systems with Hopf-Zero singularity, i.e., systems with the form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sum\limits_{k\geq 0}\sum\limits_{-1\leq l\leq k} \dfrac{l+1}{2} a_k^l h^{k-l} z^l \begin{pmatrix} x \\ y \end{pmatrix} - \sum\limits_{n\geq 0}\sum\limits_{0\leq m\leq n} b_n^m h^{n-m} z^m \begin{pmatrix} -y \\ x \end{pmatrix} \\ \sum\limits_{k\geq 0}\sum\limits_{-1\leq l\leq k} (k-l+1) a_k^l h^{k-l} z^{l+1} \end{pmatrix},$$

where $a_k^l,\ b_n^m \in \mathbb{R}$, being $a_0^{-1} \neq 0$, $a_0^0 = 0$ and $b_0^0 = 1$.

They obtain, under equivalence, the following normal form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y\, g(z) \mp \frac{p+1}{2} z^p x - z^p x \sum_{k=1}^{\infty} \frac{(k+p+1)\alpha_{k+p}}{2} z^k \\ -x\, g(z) \mp \frac{p+1}{2} z^p y - z^p y \sum_{k=1}^{\infty} \frac{(k+p+1)\alpha_{k+p}}{2} z^k \\ (x^2+y^2) \pm z^{p+1} + z^{p+1} \sum_{k=1}^{\infty} \alpha_{k+p} z^k \end{pmatrix}.$$

This normal form is equal to one obtained by us in Theorem 5.4.145, no more than impose divergence equal to zero. More specifically, the condition of free divergence for system (5.4.10) is given by the expression $2g_1(z) + g_2'(z) = 0$. Imposing this condition, we obtain the system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -y\, g(z) - \sum\limits_{k=1}^{\infty} \dfrac{(k+1)\alpha_{k+1}}{2} z^k x \\ x\, g(z) - \sum\limits_{k=1}^{\infty} \dfrac{(k+1)\alpha_{k+1}}{2} z^k y \\ (x^2+y^2) + \sum\limits_{k=1}^{\infty} \alpha_{k+1} z^{k+1} \end{pmatrix},$$

and considering $p = min\{k \in \mathbb{N}/a_k \neq 0\}$ and making a scaling, we obtain the above normal form.

5.2 Preliminaries and some key lemmas

Our aim is to calculate the normal form, under equivalence, for system (5.1.1). More specifically, we consider

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) := \mathbf{F}_r(x,y) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \quad \text{being } \mathbf{F}_r = \begin{pmatrix} \mathbf{X}_h \\ \hline f \end{pmatrix}, \tag{5.2.2}$$

where $\mathbf{x} = (x,y,z)$, is the lowest degree quasi-homogeneous term respect the type $\mathbf{t} = (t_1,t_2,t_3)$, $\mathbf{X}_h \in \mathcal{Q}_r^{\hat{\mathbf{t}}}$, $h \in \mathcal{P}_{r+|\hat{\mathbf{t}}|}^{\hat{\mathbf{t}}}$ and $f \in \mathcal{P}_{r+t_3}^{\hat{\mathbf{t}}}$.

Remark: We denote by $\mathbf{t} = (t_1,t_2,\cdots,t_n)$ the type of a vector field $\mathbf{F} = (F_1,F_2,\cdots,\ F_n)$ with degree equal to k, i.e., with $F_j \in \mathcal{P}_{k+j}^{\mathbf{t}}$. In this chapter we will work with two and three-dimensional vector fields so we need to distinguish their types. In what follows, we denote by $\hat{\mathbf{t}} = (t_1,t_2)$ the type of a two-dimensional quasi-homogeneous vector field and by $\mathbf{t} = (t_1,t_2,t_3)$ the type of a three-dimensional quasi-homogeneous vector field.

Next, we present a decomposition similar to (3.2.74), for this purpose we define a similar set to which was defined in (3.2.3),

$$h\mathcal{P}_{k-r}^{\mathbf{t}} = \{h(x,y)\gamma(x,y,z) \in \mathcal{P}_{k+|\hat{\mathbf{t}}|}^{\mathbf{t}} : \gamma \in \mathcal{P}_{k-r}^{\mathbf{t}}\},$$

and denote by $\Delta_{k+|\hat{\mathbf{t}}|}$ a complementary subspace, of $h\mathcal{P}_{k-r}^{\mathbf{t}}$ in $\mathcal{P}_{k+|\hat{\mathbf{t}}|}^{\mathbf{t}}$.

Now we define the following subspaces,

- $\mathcal{C}_k^{\mathbf{t}} = \left\{ \begin{pmatrix} \mathbf{X}_g \\ \hline 0 \end{pmatrix} \in \mathcal{Q}_k^{\mathbf{t}} \ / \ g \in \Delta_{k+|\hat{\mathbf{t}}|},\ g(0,0,z) = 0 \right\}$, where $\mathbf{X}_g = \begin{pmatrix} -\frac{\partial g(x,y,z)}{\partial y} \\ \frac{\partial g(x,y,z)}{\partial x} \end{pmatrix}$.

- $\mathcal{D}_k^{\mathbf{t}} = \left\{ \begin{pmatrix} \mu \mathbf{D}_0 \\ \hline 0 \end{pmatrix} \in \mathcal{Q}_k^{\mathbf{t}} \ / \ \mu \in \mathcal{P}_k^{\mathbf{t}} \right\}$.

- $\mathcal{F}_k^{\mathbf{t}} = \left\{ \begin{pmatrix} \lambda \mathbf{X}_h \\ \hline 0 \end{pmatrix} \in \mathcal{Q}_k^{\mathbf{t}} \ / \ \lambda \in \mathcal{P}_{k-r}^{\mathbf{t}} \right\}$.

- $\mathcal{G}_k^{\mathbf{t}} = \left\{ \begin{pmatrix} 0 \\ \hline \varsigma \end{pmatrix} \in \mathcal{Q}_k^{\mathbf{t}} \ / \ \varsigma \in \mathcal{P}_{k+t_3}^{\mathbf{t}} \right\}$.

The following proposition is a generalization of the decomposition given in Proposition 3.2.73 for three-dimensional vector fields.

Proposition 5.2.137. *Consider* $\mathbf{F}_r = \begin{pmatrix} \mathbf{X}_h \\ \hline f \end{pmatrix}$ *with* $h \in \mathcal{P}^{\hat{\mathbf{t}}}_{r+|\hat{\mathbf{t}}|} \setminus \{0\}$. *Then*

$$\mathcal{Q}^{\mathbf{t}}_k = \mathcal{C}^{\mathbf{t}}_k \oplus \mathcal{D}^{\mathbf{t}}_k \oplus \mathcal{F}^{\mathbf{t}}_k \oplus \mathcal{G}^{\mathbf{t}}_k.$$

Proof. We first show that $\mathcal{Q}^{\mathbf{t}}_k = \mathcal{C}^{\mathbf{t}}_k + \mathcal{D}^{\mathbf{t}}_k + \mathcal{F}^{\mathbf{t}}_k + \mathcal{G}^{\mathbf{t}}_k$. Obviously $\mathcal{C}^{\mathbf{t}}_k + \mathcal{D}^{\mathbf{t}}_k + \mathcal{F}^{\mathbf{t}}_k + \mathcal{G}^{\mathbf{t}}_k \subset \mathcal{Q}^{\mathbf{t}}_k$.

To prove the converse inclusion, let us to consider $\mathbf{F}_k \in \mathcal{Q}^{\mathbf{t}}_k$. We can write it as follows,

$$\mathbf{F}_k = \begin{pmatrix} \mathbf{G}_k \\ \hline 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \hline \varsigma_{k+t_3} \end{pmatrix},$$

where $\mathbf{G}_k$ depends on the variables x, y, z and can be expressed in the form,

$$\mathbf{G}_k = \sum_{i=0}^{s} z^i \widehat{\mathbf{G}}_{k-it_3}(x,y).$$

where $s = \lfloor \frac{k}{t_3} \rfloor$ and $\widehat{\mathbf{G}}_{k-it_3} \in \mathcal{Q}^{\hat{\mathbf{t}}}_{k-it_3}$. Using Propositon 3.2.74, we can write,

$$\mathbf{G}_k = \sum_{i=0}^{s} z^i \widehat{\mathbf{G}}_{k-it_3}(x,y) = \sum_{i=0}^{s} z^i \left(\mathbf{X}_{g_{k-it_3+|\hat{\mathbf{t}}|}(x,y)} + \mu_{k-it_3}(x,y)\mathbf{D}_0 + \lambda_{k-r-it_3}(x,y)\mathbf{X}_h \right),$$

where $g_{k-it_3+|\hat{\mathbf{t}}|} = \frac{Proy_{\Delta_{k-it_3+|\hat{\mathbf{t}}|}}(\mathbf{D}_0 \wedge \widehat{\mathbf{G}}_{k-it_3})}{k-it_3+|\hat{\mathbf{t}}|}$, $\mu_{k-it_3} = \frac{\operatorname{div}(\widehat{\mathbf{G}}_{k-it_3}) - \nabla\lambda_{k-r-it_3}\mathbf{X}_h}{k-it_3+|\hat{\mathbf{t}}|}$ and $\lambda_{k-r-it_3} = \frac{\operatorname{Proj}_{h\mathcal{P}^{\mathbf{t}}_{k-r-it_3}}(\mathbf{D}_0 \wedge \widehat{\mathbf{G}}_{k-it_3})}{(r+|\hat{\mathbf{t}}|)h}$.

Therefore

$$\mathbf{G}_k = \mathbf{X}_g + \mu\mathbf{D}_0 + \lambda\mathbf{X}_h,$$

where $g = \sum_{i=0}^{s} z^i g_{k-it_3+|\hat{\mathbf{t}}|}$, $\lambda = \sum_{i=0}^{s} z^i \lambda_{k-r-it_3}$ and $\mu = \sum_{i=0}^{s} z^i \mu_{k-it_3}$.

Finally only remains to prove the uniqueness. It is clear that, $\mathcal{C}^{\mathbf{t}}_k \cap \mathcal{G}^{\mathbf{t}}_k = \{0\}$, $\mathcal{D}^{\mathbf{t}}_k \cap \mathcal{G}^{\mathbf{t}}_k = \{0\}$ and $\mathcal{F}^{\mathbf{t}}_k \cap \mathcal{G}^{\mathbf{t}}_k = \{0\}$. Therefore, to complete the proof, we indicate that,

- $\mathcal{D}^{\mathbf{t}}_k \cap \mathcal{F}^{\mathbf{t}}_k = \{0\}$. It is a consequence of the planar decomposition given in Proposition 3.2.73.

- $(\mathcal{D}_k^{\mathbf{t}} + \mathcal{F}_k^{\mathbf{t}}) \cap \mathcal{C}_k^{\mathbf{t}} = \{0\}$. Anagously to above item, it is a consequence of the planar decomposition given in Proposition 3.2.73.

$\blacksquare$

Remark 17. *A consequence of this proposition is that, for any $P_k \in \mathcal{Q}_k^{\mathbf{t}}$, there exist unique polynomials $g \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$, $\mu \in \mathcal{P}_k^{\mathbf{t}}$, $\lambda \in \mathcal{P}_{k-r}^{\mathbf{t}}$ and $\varsigma \in \mathcal{P}_{k+t_3}^{\mathbf{t}}$ such that,*

$$\mathbf{P}_k = \left(\frac{X_g}{0}\right) + \left(\frac{\mu\mathbf{D}_0}{0}\right) + \left(\frac{\lambda\mathbf{X}_h}{0}\right) + \left(\frac{0}{\varsigma}\right).$$

The following lemma will be use, later, to simplify the matrix expression of the homological operator.

Lemma 5.2.138. *It holds the following properties:*

1. *If $g \in \Delta_{k+|\hat{\mathbf{t}}|}$, with $g(0,0,z) = 0$, then*

$$\left[\left(\frac{\mathbf{X}_g}{0}\right), \mathbf{F}_r\right] = \left(\frac{\mathbf{X}_{\tilde{g}}}{0}\right) + \left(\frac{\tilde{\mu}\mathbf{D}_0}{0}\right) + \left(\frac{\tilde{\lambda}\mathbf{X}_h}{0}\right) + \left(\frac{\mathbf{0}}{\nabla f \cdot \mathbf{X}_g}\right),$$

where $\tilde{g} = \mathrm{Proj}_{\Delta_{r+k+|\hat{\mathbf{t}}|}}\left(\nabla_{(x,y)} g \cdot \mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|} f \frac{\partial g}{\partial z}\right)$,

$\tilde{\eta} = \frac{1}{r+k+|\hat{\mathbf{t}}|}(\nabla f \cdot \mathbf{X}_{\frac{\partial g}{\partial z}} - \nabla\tilde{\lambda}\mathbf{X}_h)$,

$\tilde{\lambda} = \frac{1}{(r+|\hat{\mathbf{t}}|)h}\mathrm{Proj}_{h\cdot\mathcal{P}_k^{\mathbf{t}}}(\nabla_{(x,y)} g \cdot \mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|} f \frac{\partial g}{\partial z})$.

2. $$\left[\left(\frac{\mu\mathbf{D}_0}{0}\right), \mathbf{F}_r\right] = \left(\frac{(\nabla\mu \cdot \mathbf{F}_r)\mathbf{D}_0}{0}\right) - \left(\frac{r\mu\mathbf{X}_h}{0}\right) + \left(\frac{\mathbf{0}}{(r+t_3)\mu f}\right).$$

3. $$\left[\left(\frac{\lambda\mathbf{X}_h}{0}\right), \mathbf{F}_r\right] = \left(\frac{(\nabla\lambda \cdot \mathbf{F}_r)\mathbf{X}_h}{0}\right) + \left(\frac{\mathbf{0}}{\lambda \cdot \nabla f \cdot \mathbf{X}_h}\right).$$

4. $$\left[\left(\frac{\mathbf{0}}{\varsigma}\right), \mathbf{F}_r\right] = \left(\frac{\mathbf{0}}{\nabla\varsigma \cdot \mathbf{F}_r}\right).$$

Proof.

1.

$$\left[\begin{pmatrix}\mathbf{X}_g\\ 0\end{pmatrix},\mathbf{F}_r\right] = \left[\begin{pmatrix}\mathbf{X}_g\\ 0\end{pmatrix},\begin{pmatrix}\mathbf{X}_h\\ f(x,y)\end{pmatrix}\right] = D\begin{pmatrix}\mathbf{X}_g\\ 0\end{pmatrix}\begin{pmatrix}\mathbf{X}_h\\ f(x,y)\end{pmatrix}$$

$$-D\begin{pmatrix}\mathbf{X}_h\\ f(x,y)\end{pmatrix}\begin{pmatrix}\mathbf{X}_g\\ 0\end{pmatrix} = \begin{pmatrix}D_{(x,y)}\mathbf{X}_g\cdot\mathbf{X}_h + f\frac{\partial}{\partial z}\mathbf{X}_g\\ 0\end{pmatrix}$$

$$-\begin{pmatrix}D_{(x,y)}\mathbf{X}_h\cdot\mathbf{X}_g\\ \nabla f\cdot\mathbf{X}_g\end{pmatrix} = \begin{pmatrix}[\mathbf{X}_g,\mathbf{X}_h] + f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}\\ \nabla f\cdot\mathbf{X}_g\end{pmatrix}.$$

Therefore, we have obtained the following,

$$\left[\begin{pmatrix}\mathbf{X}_g\\ 0\end{pmatrix},\mathbf{F}_r\right] = \begin{pmatrix}[\mathbf{X}_g,\mathbf{X}_h] + f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}\\ 0\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\ \nabla f\cdot\mathbf{X}_g\end{pmatrix}.$$

From Lemma 3.2.72, we can write,

$$[\mathbf{X}_g,\mathbf{X}_h] + f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}} = \mathbf{X}_{\nabla g\cdot\mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}} + \tfrac{1}{r+k+|\mathbf{t}|}\left(\nabla f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}\right)\mathbf{D}_0.$$

Then consequently,

$$\left[\begin{pmatrix}\mathbf{X}_g\\ 0\end{pmatrix},\mathbf{F}_r\right] = \begin{pmatrix}\mathbf{X}_{\nabla g\cdot\mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}} + \frac{1}{r+k+|\mathbf{t}|}\left(\nabla f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}\right)\mathbf{D}_0\\ 0\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\ \nabla f\cdot\mathbf{X}_g\end{pmatrix}.$$

Let consider that,

$$\mathbf{X}_{\nabla g\cdot\mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}} + \tfrac{1}{r+k+|\mathbf{t}|}\left(\nabla f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}\right)\mathbf{D}_0 = \mathbf{X}_{\tilde{g}} + \tilde{\mu}\mathbf{D}_0 + \tilde{\lambda}\mathbf{X}_h.$$

- By one hand,

  $$\mathbf{D}_0\wedge\left(\mathbf{X}_{\nabla g\cdot\mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}} + \tfrac{1}{r+k+|\mathbf{t}|}\left(\nabla f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}\right)\mathbf{D}_0\right) =$$
  $$= (r+k+|\hat{\mathbf{t}}|)\left(\nabla g\cdot\mathbf{X}_h + \tfrac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}\right).$$

  By other hand,

  $$\mathbf{D}_0\wedge\left(\mathbf{X}_{\tilde{g}} + \tilde{\mu}\mathbf{D}_0 + \tilde{\lambda}\mathbf{X}_h\right) = (r+k+|\hat{\mathbf{t}}|)\tilde{g} + (r+|\hat{\mathbf{t}}|)\tilde{\lambda}h.$$

  From the previous two equalities it follows,

  $$\tilde{g} = \mathrm{Proj}_{\Delta_{r+k+|\hat{\mathbf{t}}|}}\left(\nabla g\cdot\mathbf{X}_h + \tfrac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}\right) \quad \text{and}$$
  $$\tilde{\lambda} = \tfrac{1}{(r+|\hat{\mathbf{t}}|)h}Proy_{h\cdot\mathfrak{P}_k^{\mathbf{t}}}(\nabla_{(x,y)}g\cdot\mathbf{X}_h + \tfrac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}).$$

- Following a similar process that in the above item,

  $$\mathrm{div}\left(\mathbf{X}_{\nabla g\cdot\mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}f\frac{\partial g}{\partial z}} + \tfrac{1}{r+k+|\mathbf{t}|}\left(\nabla f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}\right)\mathbf{D}_0\right) = \nabla f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}}.$$
  $$\mathrm{div}\left(\mathbf{X}_{\tilde{g}} + \tilde{\mu}\mathbf{D}_0 + \tilde{\lambda}\mathbf{X}_h\right) = (r+k+|\hat{\mathbf{t}}|)\tilde{\mu} + \nabla\tilde{\lambda}\cdot\mathbf{X}_h.$$

  In consequence, $\tilde{\mu} = \frac{1}{r+k+|\hat{\mathbf{t}}|}\left(\nabla f\cdot\mathbf{X}_{\frac{\partial g}{\partial z}} - \nabla\tilde{\lambda}\cdot\mathbf{X}_h\right)$.

2.

$$
\begin{aligned}
\left[\begin{pmatrix}\mu\mathbf{D}_0\\ \hline 0\end{pmatrix}, \mathbf{F}_r\right] &= \left[\begin{pmatrix}\mu\mathbf{D}_0\\ \hline 0\end{pmatrix}, \begin{pmatrix}\mathbf{X}_h\\ \hline f(x,y)\end{pmatrix}\right] = D\begin{pmatrix}\mu\mathbf{D}_0\\ \hline 0\end{pmatrix}\begin{pmatrix}\mathbf{X}_h\\ \hline f(x,y)\end{pmatrix}\\
&\quad -D\begin{pmatrix}\mathbf{X}_h\\ \hline f(x,y)\end{pmatrix}\begin{pmatrix}\mu\mathbf{D}_0\\ \hline 0\end{pmatrix} = \begin{pmatrix}D_{(x,y)}\mu\mathbf{D}_0\cdot\mathbf{X}_h + \frac{\partial\mu}{\partial z}f\mathbf{D}_0\\ \hline 0\end{pmatrix}\\
&\quad -\begin{pmatrix}D_{(x,y)}\mathbf{X}_h\cdot\mu\mathbf{D}_0\\ \hline \nabla f\cdot\mu\mathbf{D}_0\end{pmatrix} = \begin{pmatrix}[\mu\mathbf{D}_0,\mathbf{X}_h] + \frac{\partial\mu}{\partial z}f\mathbf{D}_0\\ \hline \nabla f\cdot\mu\mathbf{D}_0\end{pmatrix}\\
&= \begin{pmatrix}(\nabla\mu\cdot\mathbf{F}_r)\mathbf{D}_0 - r\mu\mathbf{X}_h\\ \hline (r+t_3)\mu f\end{pmatrix}.
\end{aligned}
$$

3.

$$
\begin{aligned}
\left[\begin{pmatrix}\lambda\mathbf{X}_h\\ \hline 0\end{pmatrix}, \mathbf{F}_r\right] &= D\begin{pmatrix}\lambda\mathbf{X}_h\\ \hline 0\end{pmatrix}\begin{pmatrix}\mathbf{X}_h\\ \hline f(x,y)\end{pmatrix} - D\begin{pmatrix}\mathbf{X}_h\\ \hline f(x,y)\end{pmatrix}\begin{pmatrix}\lambda\mathbf{X}_h\\ \hline 0\end{pmatrix}\\
&= \begin{pmatrix}D_{(x,y)}\lambda\mathbf{X}_h\cdot\mathbf{X}_h + \frac{\partial\lambda}{\partial z}f\mathbf{X}_h\\ \hline 0\end{pmatrix} - \begin{pmatrix}D_{(x,y)}\mathbf{X}_h\cdot\lambda\mathbf{X}_h\\ \hline \nabla f\cdot\lambda\mathbf{X}_h\end{pmatrix}\\
&= \begin{pmatrix}[\lambda\mathbf{X}_h,\mathbf{X}_h] + \frac{\partial\lambda}{\partial z}f\mathbf{X}_h\\ \hline \nabla f\cdot\lambda\mathbf{X}_h\end{pmatrix}\\
&= \begin{pmatrix}(\nabla\lambda\cdot\mathbf{X}_h)\mathbf{X}_h + \frac{\partial\lambda}{\partial z}f\mathbf{X}_h\\ \hline 0\end{pmatrix} + \begin{pmatrix}0\\ \hline \nabla f\cdot\lambda\mathbf{X}_h\end{pmatrix}\\
&= \begin{pmatrix}(\nabla\lambda\cdot\mathbf{F}_r)\mathbf{X}_h\\ \hline 0\end{pmatrix} + \begin{pmatrix}0\\ \hline \nabla f\cdot\lambda\mathbf{X}_h\end{pmatrix}
\end{aligned}
$$

4.

$$
\begin{aligned}
\left[\begin{pmatrix}0\\ \hline \varsigma\end{pmatrix}, \mathbf{F}_r\right] &= D\begin{pmatrix}0\\ \hline \varsigma\end{pmatrix}\cdot\begin{pmatrix}\mathbf{X}_h\\ \hline f(x,y)\end{pmatrix} - D\begin{pmatrix}\mathbf{X}_h\\ \hline f(x,y)\end{pmatrix}\cdot\begin{pmatrix}0\\ \hline \varsigma\end{pmatrix}\\
&= \begin{pmatrix}0\\ \hline \nabla_{(x,y)\varsigma\cdot\mathbf{X}_h + \frac{\partial\varsigma}{\partial z}\cdot f(x,y)}\end{pmatrix} = \begin{pmatrix}0\\ \hline \nabla\varsigma\cdot\mathbf{F}_r\end{pmatrix}.
\end{aligned}
$$

■

5.3 Normal Form under $\mathcal{C}^\infty$-equivalence.

In this section, we will describe the homological operator, under equivalence.

$$\mathcal{L}_{r+k} : \mathcal{Q}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k) \longrightarrow \mathcal{Q}_{r+k}^{\mathbf{t}},$$

where $\mathcal{L}_{r+k}(\mathbf{P}_k, \nu_k) = [\mathbf{P}_k, \mathbf{F}_r] - \nu_k \cdot \mathbf{F}_r$.

Using, Proposition 5.2.137, this operator can be written in the form,

$$\mathcal{L}_{r+k} \quad : \quad \mathcal{C}_k^{\mathbf{t}} \oplus \mathcal{D}_k^{\mathbf{t}} \oplus \mathcal{F}_k^{\mathbf{t}} \oplus \mathcal{G}_k^{\mathbf{t}} \times \mathrm{Cor}(\ell_k) \longrightarrow \mathcal{C}_{r+k}^{\mathbf{t}} \oplus \mathcal{D}_{r+k}^{\mathbf{t}} \oplus \mathcal{F}_{r+k}^{\mathbf{t}} \oplus \mathcal{G}_{r+k}^{\mathbf{t}},$$

where

$$\begin{aligned} \mathcal{L}_{r+k} \quad & \left(\left(\frac{X_g}{0}\right) + \left(\frac{\mu \mathbf{D}_0}{0}\right) + \left(\frac{\lambda \mathbf{X}_h}{0}\right) + \left(\frac{0}{\varsigma}\right), \nu_k \right) = \left[\left(\frac{X_g}{0}\right), \mathbf{F}_r \right] \\ & + \left[\left(\frac{\mu \mathbf{D}_0}{0}\right), \mathbf{F}_r \right] + \left[\left(\frac{\lambda \mathbf{X}_h}{0}\right), \mathbf{F}_r \right] + \left[\left(\frac{0}{\varsigma}\right), \mathbf{F}_r \right] - \nu_k \cdot \mathbf{F}_r. \end{aligned}$$

Taking into account Lemma 5.2.138 and that $\nu_k \cdot \mathbf{F}_r = \left(\dfrac{\nu_k \mathbf{X}_h}{0}\right) + \left(\dfrac{\mathbf{0}}{\nu_k \cdot f}\right)$, we obtain the following matricial structure for the homological operator, under equivalence,

$\left(\dfrac{\mathbf{X}_{\tilde{g}}}{0}\right)$	0	0	0	0	$\mathcal{C}_{r+k}^t$
$\left(\dfrac{\tilde{\eta}\mathbf{D}_0}{0}\right)$	$\left(\dfrac{(\nabla\mu \cdot \mathbf{F}_r)\mathbf{D}_0}{0}\right)$	0	0	0	$\mathcal{D}_{r+k}^t$
$\left(\dfrac{\tilde{\lambda}\mathbf{X}_h}{0}\right)$	$\left(\dfrac{r\mu\mathbf{X}_h}{0}\right)$	$\left(\dfrac{(\nabla\lambda \cdot \mathbf{F}_r)\mathbf{X}_h}{0}\right)$	$\left(\dfrac{\nu_k\mathbf{X}_h}{0}\right)$	0	$\mathcal{F}_{r+k}^t$
$\left(\dfrac{\mathbf{0}}{\nabla f \cdot \mathbf{X}_g}\right)$	$\left(\dfrac{\mathbf{0}}{(r+t_3)\mu f}\right)$	$\left(\dfrac{\mathbf{0}}{\lambda \cdot \nabla f \cdot \mathbf{X}_h}\right)$	$\left(\dfrac{\mathbf{0}}{\nu_k \cdot f}\right)$	$\left(\dfrac{\mathbf{0}}{\nabla\varsigma \cdot \mathbf{F}_r}\right)$	$\mathcal{G}_{r+k}^{\mathbf{t}}$
$\left(\dfrac{\mathbf{X}_g}{0}\right) \in \mathcal{C}_k^t$	$\left(\dfrac{\eta\mathbf{D}_0}{0}\right) \in \mathcal{D}_k^t$	$\left(\dfrac{\lambda\mathbf{X}_h}{0}\right) \in \mathcal{F}_k^t$	$\nu \in \mathrm{Cor}(\ell_k)$	$\left(\dfrac{\mathbf{0}}{c}\right) \in \mathcal{G}_k^{\mathbf{t}}$	

where

- $\tilde{g} = \mathrm{Proj}_{\Delta_{r+k+|\hat{\mathbf{t}}|}}\left(\nabla g \cdot \mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|} f \frac{\partial g}{\partial z}\right)$.
- $\tilde{\lambda} = \frac{1}{(r+|\hat{\mathbf{t}}|)h} \mathrm{Proj}_{h \cdot \mathcal{P}_k^{\mathbf{t}}}(\nabla_{(x,y)} g \cdot \mathbf{X}_h + \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|} f \frac{\partial g}{\partial z})$.
- $\tilde{\eta} = \frac{1}{r+k+|\hat{\mathbf{t}}|}(\nabla f \cdot \mathbf{X}_{\frac{\partial g}{\partial z}} - \nabla \tilde{\lambda} \mathbf{X}_h)$,

Now we define the new operator,

$$\begin{aligned} \ell_{k,A} \quad : \quad & \mathcal{P}_{k-r}^{\mathbf{t}} \longrightarrow \mathcal{P}_k^{\mathbf{t}} \\ & \mu_{k-r} \longrightarrow \nabla_{(x,y)} \mu_{k-r} \cdot \mathbf{X}_h + A \cdot f \cdot \frac{\partial \mu_{k-r}}{\partial z}. \end{aligned} \tag{5.3.3}$$

Remark 18. *Observe that $\ell_{k,A}$ is the Lie derivative respect to the field $\mathbf{F}_r + \begin{pmatrix} \mathbf{0} \\ \hline (A-1)f \end{pmatrix}$. Therefore, taking $A = 1$, we get the operator ℓ_k defined in (2.3.15).*

Taking into account this new operator we can show the following expression for the matrix of the homological operator,

$\begin{pmatrix} \mathbf{X}_{\tilde{g}} \\ \hline 0 \end{pmatrix}$	0	0	0	0	$\mathcal{C}_{r+k}^t$
$\begin{pmatrix} \tilde{\eta} \mathbf{D}_0 \\ \hline 0 \end{pmatrix}$	$\begin{pmatrix} (\ell_{r+k}(\mu)) \mathbf{D}_0 \\ \hline 0 \end{pmatrix}$	0	0	0	$\mathcal{D}_{r+k}^t$
$\begin{pmatrix} \tilde{\lambda} \mathbf{X}_h \\ \hline 0 \end{pmatrix}$	$\begin{pmatrix} r\mu \mathbf{X}_h \\ \hline 0 \end{pmatrix}$	$\begin{pmatrix} (\ell_{r+k}(\lambda)) \mathbf{X}_h \\ \hline 0 \end{pmatrix}$	$\begin{pmatrix} \nu_k \mathbf{X}_h \\ \hline 0 \end{pmatrix}$	0	$\mathcal{F}_{r+k}^t$
$\begin{pmatrix} \mathbf{0} \\ \hline \nabla f \cdot \mathbf{X}_g \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} \\ \hline (r+t_3)\mu f \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} \\ \hline \lambda \cdot \nabla f \cdot \mathbf{X}_h \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} \\ \hline \nu_k \cdot f \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} \\ \hline \ell_{r+k}(\varsigma) \end{pmatrix}$	$\mathcal{G}_{r+k}^{\mathbf{t}}$
$\begin{pmatrix} \mathbf{X}_g \\ \hline 0 \end{pmatrix} \in \mathcal{C}_k^t$	$\begin{pmatrix} \eta \mathbf{D}_0 \\ \hline 0 \end{pmatrix} \in \mathcal{D}_k^t$	$\begin{pmatrix} \lambda \mathbf{X}_h \\ \hline 0 \end{pmatrix} \in \mathcal{F}_k^t$	$\nu \in \mathrm{Cor}(\ell_k)$	$\begin{pmatrix} \mathbf{0} \\ \hline c \end{pmatrix} \in \mathcal{G}_k^{\mathbf{t}}$	

with

- $\tilde{g} = Proy_{\triangle_{r+k+|\hat{\mathbf{t}}|}}(\ell_{r+k+|\hat{\mathbf{t}}|, \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}}(g))$.
- $\tilde{\lambda} = \frac{1}{(r+|\hat{\mathbf{t}}|)h} Proy_{h \cdot \mathcal{P}_k^{\mathbf{t}}}(\ell_{r+k+|\hat{\mathbf{t}}|, \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}}(g))$.

- $\tilde{\eta} = \frac{1}{r+k+|\hat{\mathbf{t}}|}(\nabla f \cdot \mathbf{X}_{\frac{\partial g}{\partial z}} - \nabla\tilde{\lambda}\mathbf{X}_h)$.

From the structure of the above matrix is deduced the following proposition.

Proposition 5.3.139. *Consider* $\mathbf{F}_r = \begin{pmatrix} \mathbf{X}_h \\ \hline f \end{pmatrix}$ *being* $h \in \mathcal{P}^{\hat{\mathbf{t}}}_{r+|\hat{\mathbf{t}}|}$ *and* $f \in \mathcal{P}^{\hat{\mathbf{t}}}_{r+t_3}$. *Then, a complementary space of the range of the homological operator* $\mathcal{L}_{r+k}$ *described above is,*

$$Cor(\mathcal{L}_{r+k}) = \begin{pmatrix} \mathbf{X}_{Cor(\ell_{r+k,A_0}) \cap \triangle_{r+k+|\hat{\mathbf{t}}|}} \\ \hline 0 \end{pmatrix} \oplus \begin{pmatrix} Cor(\ell_{r+k}) \cdot \mathbf{D}_0 \\ \hline 0 \end{pmatrix} \oplus \begin{pmatrix} \mathbf{0} \\ \hline Cor(\ell_{r+k+t_3}) \end{pmatrix},$$

where $A_0 = \frac{k+|\hat{\mathbf{t}}|}{r+k+|\mathbf{t}|}$.

The following theorems provide a formal normal form, under $\mathcal{C}^\infty$-conjugation and $\mathcal{C}^\infty$-equivalence, for system (5.2.2). In the case of $\mathcal{C}^\infty$-conjugation, it is enough to consider, in the above matrix, $\nu \in \mathrm{Cor}(\ell_k)$ identically equal to zero, i.e., $\nu_k \equiv 0$.

Theorem 5.3.140. *A formal normal form under* $\mathcal{C}^\infty$*-conjugation for system (5.2.2) is,*

$$\mathbf{G} = \mathbf{F}_r + \sum_{j\geq 1} \left[\begin{pmatrix} \mathbf{X}_{g_j} \\ \hline 0 \end{pmatrix} + \begin{pmatrix} \mu_j \cdot \mathbf{D}_0 \\ \hline 0 \end{pmatrix} + \begin{pmatrix} \lambda_j \cdot \mathbf{X}_h \\ \hline 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \hline \varsigma_j \end{pmatrix} \right],$$

with $g_j \in Cor(\ell_{r+j+|\hat{\mathbf{t}}|,A_0}) \cap \triangle_{r+j+|\hat{\mathbf{t}}|}$, $\mu_j \in Cor(\ell_{r+j})$, $\lambda_j \in Cor(\ell_j)$ *and* $\varsigma_j \in Cor(\ell_{r+j+t_3})$.

Theorem 5.3.141. *A formal normal form under* $\mathcal{C}^\infty$*-equivalence for system (5.2.2) is,*

$$\mathbf{G} = \mathbf{F}_r + \sum_{j\geq 1} \left[\begin{pmatrix} \mathbf{X}_{g_j} \\ \hline 0 \end{pmatrix} + \begin{pmatrix} \mu_j \cdot \mathbf{D}_0 \\ \hline 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \hline \varsigma_j \end{pmatrix} \right],$$

with $g_j \in Cor(\ell_{r+k+|\hat{\mathbf{t}}|,A_0}) \cap \triangle_{r+j+|\hat{\mathbf{t}}|}$, $\mu_j \in Cor(\ell_{r+j})$ *and* $\varsigma_j \in Cor(\ell_{r+j+t_3})$.

5.3.1 The operator $\ell_{r+k,A}$

Since the co-range of the homological operator depends on the co-range of the operator $\ell_{r+k,A}$, in this section, we will study this linear operator for $A \neq 0$.

We denote by ℓ_k, the Lie derivative respect to the principal part of the vector field (5.2.2), $\mathbf{F}_r$. In the same way, we will denote $\hat{\ell}_k$, as the Lie derivative operator associated to the planar vector field $\mathbf{X}_h$.

Considering $g \in \mathcal{P}_k^{\mathbf{t}}$, this polynomial can be expressed in the form,

$$g = \sum_{i=0}^{\lfloor k/t_3 \rfloor} z^i \cdot p_{k-it_3}(x, y) \tag{5.3.4}$$

and decomposing $k = k_1 t_3 + k_2$, with $0 \leq k_2 \leq t_3$, we obtain,

$$g = \sum_{i=0}^{k_1} z^i \cdot p_{k-it_3}(x, y) = \sum_{i=0}^{k_1} z^i \cdot p_{(k_1-i)t_3+k_2}(x, y)$$

Denoting E_k a complementary subspace of $\mathrm{Ker}(\hat{\ell}_{r+k})$ in $\mathcal{P}_k^{\hat{\mathbf{t}}}$, i.e., $\mathcal{P}_k^{\hat{\mathbf{t}}} = \mathrm{E}_k \oplus \mathrm{Ker}(\hat{\ell}_{r+k})$. Supporting us in this decomposition of $\mathcal{P}_k^{\mathbf{t}}$, we can define the subspace $z^i \cdot \mathrm{E}_{k-it_3} = \tilde{\mathrm{E}}_i$ and $z^i \cdot \mathrm{Ker}(\hat{\ell}_{k-it_3}) = \tilde{\mathrm{K}}_i$, therefore,

$$\mathcal{P}_k^{\mathbf{t}} = \bigoplus_{i=0}^{k_1} z^i \cdot \mathcal{P}_{k-it_3}^{\hat{\mathbf{t}}} = \bigoplus_{i=0}^{k_1} z^i \left(\mathrm{E}_{k-it_3} \oplus \mathrm{Ker}(\ell_{r+k-it_3})\right) = \bigoplus_{i=0}^{k_1} \tilde{\mathrm{E}}_i \oplus \tilde{\mathrm{K}}_i \tag{5.3.5}$$

Hence, any $p_k \in \mathcal{P}_k^{\hat{\mathbf{t}}}$, can be expressed in the form,

$$p_k(x, y) = p_k^{(1)}(x, y) + p_k^{(2)}(x, y)$$

being $p_k^{(1)} \in \mathrm{E}_k$ and $p_k^{(2)} \in \mathrm{Ker}(\hat{\ell}_{r+k})$. Therefore, g may be decomposed as follows,

$$g = \sum_{i=0}^{k_1} z^i \cdot p_{(k_1-i)t_3+k_2}(x, y) = \sum_{i=0}^{k_1} z^i \left[p^{(1)}_{(k_1-i)t_3+k_2}(x, y) + p^{(2)}_{(k_1-i)t_3+k_2}(x, y)\right]$$

If we apply to the polynomial g, the linear operator, $\ell_{r+k,A}$, we obtain,

$$\begin{aligned}
\ell_{r+k,A}(g) &= \nabla_{(x,y)} g \cdot \mathbf{X}_h + \frac{\partial}{\partial z}(g) \cdot A \cdot f \\
&= \nabla_{(x,y)} \left(\sum_{i=0}^{k_1} z^i \left[p^{(1)}_{(k_1-i)t_3+k_2}(x,y) + p^{(2)}_{(k_1-i)t_3+k_2}(x,y) \right] \right) \cdot \mathbf{X}_h \\
&+ \frac{\partial}{\partial z} \left(\sum_{i=0}^{k_1} z^i \left[p^{(1)}_{(k_1-i)t_3+k_2}(x,y) + p^{(2)}_{(k_1-i)t_3+k_2}(x,y) \right] \right) \cdot A \cdot f \\
&= z^{k_1} \nabla_{(x,y)} p^{(1)}_{k_2}(x,y) \cdot \mathbf{X}_h + \sum_{i=0}^{k_1-1} z^i \left[\nabla_{(x,y)} p^{(1)}_{(k_1-i)t_3+k_2}(x,y) \cdot \mathbf{X}_h \right. \\
&+ \left. (i+1) \cdot z^i \cdot A \cdot f \cdot p^{(2)}_{(k_1-i-1)t_3+k_2}(x,y) \right] + \sum_{i=0}^{k_1} i \cdot z^{i-1} \cdot A \cdot f \cdot p^{(1)}_{(k_1-i)t_3+k_2}(x,y).
\end{aligned}$$

Ordering the above expression in powers of z, we obtain the following matrix expression for the operator linear $\ell_{r+k,A}$

$$\left(\begin{array}{c|c|c|c|c||c}
0 & 0 & 0 & 0 & 0 & \\
\cline{1-5}
\vdots & \vdots & \vdots & \vdots & \vdots & z^{k_1+1} \cdot \mathcal{P}^{\mathbf{t}}_{r+k-(k_1+1)t_3} \\
\cline{1-5}
0 & 0 & 0 & 0 & 0 & \\
\hline
d_{k_1} & 0 & 0 & 0 & 0 & z^{k_1} \cdot \mathcal{P}^{\hat{\mathbf{t}}}_{k+r-k_1t_3} \\
\hline
\vdots & d_{k_1-1} & 0 & 0 & 0 & z^{k_1-1} \cdot \mathcal{P}^{\hat{\mathbf{t}}}_{k+r-(k_1-1)t_3} \\
\hline
\vdots & \vdots & d_i & 0 & 0 & z^{k_1-2} \cdot \mathcal{P}^{\hat{\mathbf{t}}}_{k+r-(k_1-2)t_3} \\
\hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hline
\vdots & \vdots & 0 & & d_0 & z^0 \cdot \mathcal{P}^{\hat{\mathbf{t}}}_{r+k} \\
\hline\hline
\tilde{\mathrm{E}}_{k_1} & \tilde{\mathrm{E}}_{k_1-1} \oplus \tilde{\mathrm{K}}_{k_1} & \cdots & \tilde{\mathrm{E}}_1 \oplus \tilde{\mathrm{K}}_2 & \tilde{\mathrm{E}}_0 \oplus \tilde{\mathrm{K}}_1 &
\end{array}\right)$$

being,

$$d_{k_1} = z^{k_1} \cdot \nabla p^{(1)}_{k_2}(x,y) \cdot \mathbf{X}_h.$$

$$d_{k_1-1} = z^{k_1-1} \left[\nabla p^{(1)}_{t_3+k_2}(x,y) \cdot \mathbf{X}_h + k_1 \cdot A \cdot f \cdot p^{(2)}_{k_2}(x,y) \right].$$

$$\vdots$$

$$d_i = z^i \left[\nabla_{(x,y)} p^{(1)}_{(k_1-i)t_3+k_2}(x,y) \cdot \mathbf{X}_h + (i+1) \cdot A \cdot f \cdot p^{(2)}_{(k_1-i-1)t_3+k_2}(x,y) \right].$$

Therefore, from the structure of the above matrix, we can conclude:

Theorem 5.3.142. *A complementary subspace to the range of the operator* $\ell_{r+k,A}$*, is given by,*

$$\mathrm{Cor}(\ell_{r+k,A}) = z^{k_1+1}\mathcal{P}^{\mathbf{t}}_{r+k-(k_1+1)t_3} \oplus z^{k_1}\mathrm{Cor}(\hat{\ell}_{r+k_2}) \oplus \sum_{i=0}^{k_1-1} z^i \cdot \tilde{\mathrm{V}}_{r+k-it_3},$$

being $\tilde{\mathrm{V}}_{r+k}$ *a complementary subspace of* $\mathrm{V}_{r+k} = Range(\hat{\ell}_{r+k}) \oplus span\{f \cdot h^{\frac{k-t_3}{r+|\hat{\mathbf{t}}|}}\}$ *if* $k-t_3$ *is a multiple of* $r+|\hat{\mathbf{t}}|$*, or* $\mathrm{V}_{r+k} = Range(\hat{\ell}_{r+k})$ *if* $k-t_3$ *is not a multiple of* $r+|\hat{\mathbf{t}}|$*.*

Next theorem gives a complementary subspace to the range of the operator $\ell_{r+k,A}$ in the case $f \equiv 0$. More specifically, if we consider $f \equiv 0$ in (5.2.2), the system under study will take the following form,

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{X}_h \\ \hline 0 \end{pmatrix} + \cdots \qquad (5.3.6)$$

Observe that, in this case $\mathrm{V}_{r+k} = \mathrm{Range}(\hat{\ell}_{r+k})$ and consequetly, $\tilde{\mathrm{V}}_{r+k} = \mathrm{Cor}(\hat{\ell}_{r+k_2})$. The following theorem provides a co-range of the operator $\ell_{r+k,A}$ in this case.

Theorem 5.3.143. *A complementary subspace to the range of the operator* $\ell_{r+k,A}$*, in the case* $f \equiv 0$*, is given by,*

$$\mathrm{Cor}(\ell_{r+k,A}) = z^{k_1+1}\mathcal{P}^{\mathbf{t}}_{r+k-(k_1+1)t_3} \oplus \sum_{i=0}^{k_1} z^i \cdot \mathrm{Cor}(\hat{\ell}_{r+k-it_3}).$$

Remarks:

1. Notice that is possible to choose f, such that $f \in \mathrm{Cor}(\hat{\ell}_{r+t_3})$
2. We want to clarify that the subspace $span\{f \cdot h^{\frac{k-t_3}{r+|\hat{\mathbf{t}}|}}\}$ has dimension 0 or 1. In the case that this dimension is 0, $\mathrm{V}_{r+k} = \mathrm{Cor}(\hat{\ell}_{r+k})$.
3. Observe that the elements in $\tilde{\mathrm{K}}_0$ belong to $\mathrm{Ker}(\ell_k)$ are not used in the matrix.
4. Although the linear operator $\ell_{r+k,A}$ depends on A, the complementary subspace $\mathrm{Cor}(\ell_{r+k,A})$, of the above theorem does not depend on A, for $A \neq 0$. Hence, in the applications we can skip it. In others words, we can substitute $\mathrm{Cor}(\ell_{r+k,A})$ by $\mathrm{Cor}(\ell_{r+k})$

5.4 Normal Form for a Hopf-Zero singularity

In this section we calculate a normal form for a Hopf-zero singularity. Consider the system,

$$\dot{\mathbf{x}} = \mathbf{F}_0 = \begin{pmatrix} -y \\ x \\ \hline x^2+y^2 \end{pmatrix} \in \mathcal{Q}_0^{\mathbf{t}}. \tag{5.4.7}$$

respect to the type $\mathbf{t} = (1,1,2)$ (consequently, $\hat{\mathbf{t}} = (1,1)$ and $r = 0$).

In [16] is shown that $Cor(\hat{\ell}_k)$, can be expressed in the form,

$$Cor(\hat{\ell}_k) = \begin{cases} < h^{\frac{k}{2}} > & \text{if k is even,} \\ \{0\} & \text{if k is odd.} \end{cases} \tag{5.4.8}$$

Using (5.4.8) and Theorem 5.3.142, we calculate $Cor(\ell_k)$. For it, we use the decomposition of k showed in the subsection 5.3.1, i.e., $k = 2k_1 + k_2$, with $0 \le k_2 < 2$.

- If $k_2 = 0$ then $k = 2k_1$, $k_1 \in \mathbb{N}$. Then $\frac{k-t_3}{r+|\mathbf{t}|} = k_1 - 1$ and $\tilde{V}_{2k_1}$ is a complementary subspace of $V_{2k_1} = \text{Range}(\ell_{2k_1}) \oplus < h^{k_1} >$. Taking into account (5.4.8), we can conclude that $\tilde{V}_{2k_1} = \{0\}$.

  Therefore,

$$\text{Cor}(\ell_{2k_1}) = z^{k_1+1}\mathcal{P}_{-2}^{\mathbf{t}} \oplus z^{k_1}\text{Cor}(\hat{\ell}_0) \oplus \sum_{j=0}^{k_1-1} z^j \tilde{V}_{2(k_1-j)} = \left\langle z^{k_1} \right\rangle.$$

- If $k_2 = 1$ then $k = 2k_1 + 1$, $k_1 \in \mathbb{N} \cup \{0\}$ and $\frac{k-t_3}{r+|\hat{\mathbf{t}}|} = \frac{2i-1}{2}$. In this case $\tilde{V}_{2k_1+1}$ is a complementary subspace of $V_{2k_1+1} = \text{Range}(\ell_{2k_1+1})$ i.e., $\tilde{V}_{2k_1+1} = \{0\}$.

  Therefore,

$$\text{Cor}(\ell_{2k_1+1}) = z^{k_1+1}\mathcal{P}_{-1}^{\mathbf{t}} \oplus z^{k_1}\text{Cor}(\hat{\ell}_1) \oplus \sum_{j=0}^{k_1-1} z^j \tilde{V}_{2(k_1-j)+1} = \{0\}.$$

From Theorems 5.3.140 and 5.3.141 and the expressions of the above co-ranges, we give a normal form, under conjugation and equivalence, respectively, for Hopf-Zero singularity. We collect these results in the following two theorems:

Theorem 5.4.144. *A normal form for system (5.4.7), under conjugation, is given by*

$$\dot{\mathbf{x}} = \begin{pmatrix} -y \\ x \\ \overline{x^2+y^2} \end{pmatrix} + \left(\frac{g_1(z)\mathbf{D}_0 + g_3(z)\mathbf{X}_h}{g_2(z)} \right), \tag{5.4.9}$$

being $g_1(z) = \sum_{i\geq 1}^{\infty} a_i z^i$, $g_2(z) = \sum_{i\geq 2}^{\infty} b_i z^i$ *and* $g_3(z) = \sum_{i\geq 1}^{\infty} c_i z^i$.

Theorem 5.4.145. *A normal form for system (5.4.7), under equivalence, is given by*

$$\dot{\mathbf{x}} = \begin{pmatrix} -y \\ x \\ \overline{x^2+y^2} \end{pmatrix} + \left(\frac{g_1(z)\mathbf{D}_0}{g_2(z)} \right), \tag{5.4.10}$$

where $g_1(z) = \sum_{i\geq 1}^{\infty} a_i z^i$ *and* $g_2(z) = \sum_{i\geq 2}^{\infty} b_i z^i$.

5.5 Normal form for a triple-zero singularity.

Next we compute a normal form for a triple-zero singularity. Let consider the system,

$$\dot{\mathbf{x}} = \mathbf{F}_1 = \begin{pmatrix} y \\ x^2 \\ \overline{\frac{1}{6}(2x^2-3y^2)} \end{pmatrix} \in \mathcal{Q}_1^{\mathbf{t}}, \tag{5.5.11}$$

respect to the type $\mathbf{t} = (2,3,5)$ (consequently, $\hat{\mathbf{t}} = (2,3)$ and $r = 1$).

From Chapter 3, in the proof of Theorem 3.4.94, considering $l = 1$, it is possible to obtain that,

$$\mathrm{Cor}(\hat{\ell}_k) = \begin{cases} \{0\} & \text{if} \quad j \in \{1,3,4,5\}, \\ \langle h^i \rangle & \text{if} \quad j = 0, \\ \langle x\, h^i \rangle & \text{if} \quad j = 2, \end{cases}$$

where $k = 6\,i + j$ with $i \in \mathbf{N} \cup \{0\}$ and $0 \leq j < 6$.

Using (5.4.8) and Theorem 5.3.142, we calculate $Cor(\ell_{r+k})$. For this task we decompose k in the form $k = 5(6k_1 + k_2) + k_3$, where $0 \leq k_2 < 6$ and $0 \leq k_3 < 5$.

The expresión of the $Cor(\ell_{r+k})$ in this particular case is of the form,

$$Cor(\ell_{r+k}) = z^{6\,k_1+k_2+1}\mathcal{P}^{\mathbf{t}}_{k_3-4} \oplus z^{6\,k_1+k_2} Cor(\hat{\ell}_{1+k_3}) \oplus \sum_{i=0}^{6\,k_1+k_2-1} z^i\,\tilde{V}_{5(6\,k_1+k_2-i)+k_3+1}. \tag{5.5.12}$$

being $\tilde{V}_{5(6\,k_1+k_2-i)+k_3+1}$ a complementary subspace of $V_{5(6\,k_1+k_2-i)+k_3+1}$ where $V_{5(6\,k_1+k_2-i)+k_3+1}$ is the following subspace of $\mathcal{P}^{\mathbf{t}}_{5(6\,k_1+k_2-i)+k_3+1}$:

$$\begin{cases} \mathrm{Cor}(\hat{\ell}_{5(6\,k_1+k_2-i)+k_3+1}) \oplus \left\langle h^{\frac{5(6\,k_1+k_2-i)+k_3-5}{6}} \right\rangle & \text{if } 5(6\,k_1+k_2-i)+k_3-5 = \dot{6}, \\ \mathrm{Cor}(\hat{\ell}_{5(6\,k_1+k_2-i)+k_3+1}) & \text{otherwise.} \end{cases}$$

Next we calculate $Cor(\ell_{r+k})$ for the differents values of k_2 and k_3.

1. Case $k_3 = 0$. In this case $k = 5(6k_1 + k_2)$.

The expression (5.5.12) is of the form,

$$Cor(\ell_{1+5(6k_1+k_2)}) = \sum_{i=0}^{6\,k_1+k_2-1} z^i\,\tilde{V}_{5(6\,k_1+k_2-i)+1}. \tag{5.5.13}$$

By taking $i = 6m + n$ with $0 \leq n < 6$ and $m \in \mathbf{N} \cup \{0\}$. The above expression can be written of the form,

$$\begin{aligned} Cor(\ell_{1+5(6k_1+k_2)}) &= \sum_{m=0}^{k_1-1}\sum_{n=0}^{5} z^{6m+n}\,\tilde{V}_{5(6\,k_1+k_2-6m-n)+1} \\ &+ \sum_{n=0}^{k_2-1} z^{6k_1+n}\tilde{V}_{5(6\,k_1+k_2-6k_1-n)+1} \\ &= \sum_{m=0}^{k_1-1}\sum_{n=0}^{5} z^{6m+n}\,\tilde{V}_{5(6(k_1-m)+k_2-n)+1} \\ &+ \sum_{n=0}^{k_2-1} z^{6k_1+n}\tilde{V}_{5(k_2-n)+1}. \end{aligned} \tag{5.5.14}$$

1. a) Case $k_2 = 0$. In this subcase we obtain that,

$$Cor(\ell_{1+5(6k_1)}) = \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)-n)+1}. \tag{5.5.15}$$

At this point, we distinguish two cases:

• If $n = 5$ then $\tilde{V}_{5(6(k_1-m)-5)+1}$ is a complementary subspace of $V_{5(6(k_1-m)-5)+1} = \mathrm{Range}(\hat{\ell}_{6[5(k_1-m)-4]}) \oplus \left\langle h^{5(k_1-m)-4} \right\rangle$. Taking into account that $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)-4]}) = \left\langle h^{5(k_1-m)-4} \right\rangle$, we can conclude that $\tilde{V}_{5(6(k_1-m)-5)+1} = \{0\}$

• If $n \neq 5$ then $\tilde{V}_{5(6(k_1-m)-n)+1}$ is a complementary subspace of $V_{5(6(k_1-m)-n)+1} = \mathrm{Range}(\hat{\ell}_{5(6(k_1-m)-n)+1})$. Therefore $\tilde{V}_{5(6(k_1-m)-n)+1} = \mathrm{Cor}(\hat{\ell}_{5(6(k_1-m)-n)+1})$ and it is easy to prove that all co-ranges are nulls, except for $n = 1$, for this case $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)-1]+2}) = \left\langle x\, h^{5(k_1-m)-1} \right\rangle$

Therefore, in this subcase $k_2 = 0$, we obtain that

$$\mathrm{Cor}(\ell_{1+30k_1}) = \bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+1} x\, h^{5(k_1-m)-1} \right\rangle.$$

1. b) Case $k_2 = 1$. In this subcase we obtain that

$$Cor(\ell_{1+5(6k_1+1)}) = z^{6k_1} \tilde{V}_6 + \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)+1-n)+1}. \tag{5.5.16}$$

Observe that $\tilde{V}_6 = \{0\}$. About $\tilde{V}_{5(6(k_1-m)+1-n)+1}$ we distinguish two cases:

• If $n = 0$ then $\tilde{V}_{5(6(k_1-m)+1)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+1)+1} = \mathrm{Range}(\hat{\ell}_{6[5(k_1-m)+1]}) \oplus \left\langle h^{5(k_1-m)+1} \right\rangle$. Taking into account that $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)+1]}) = \left\langle h^{5(k_1-m)+1} \right\rangle$, we can conclude that $\tilde{V}_{5(6(k_1-m)+1)+1} = \{0\}$.

• If $n \neq 0$ then $\tilde{V}_{5(6(k_1-m)+1-n)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+1-n)+1} = \mathrm{Range}(\hat{\ell}_{5(6(k_1-m)+1-n)+1})$. Therefore, we can affirm that $\tilde{V}_{5(6(k_1-m)+1-n)+1} = \mathrm{Cor}(\hat{\ell}_{5(6(k_1-m)+1-n)+1})$ and it is easy to prove that all co-ranges are nulls, except for $n = 2$, for this case $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)-1]+2}) = \left\langle x\, h^{5(k_1-m)-1} \right\rangle$.

Therefore, in this subcase $k_2 = 1$, we obtain that

$$\mathrm{Cor}(\ell_{6+30k_1}) = \bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+2} x\, h^{5(k_1-m)-1} \right\rangle .$$

1. c) Case $k_2 = 2$. In this subcase we obtain that,

$$\begin{aligned} Cor(\ell_{1+5(6k_1+2)}) &= \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)+2-n)+1} \\ &\quad + \sum_{n=0}^{1} z^{6k_1+n}\, \tilde{V}_{5(2-n)+1}. \end{aligned} \tag{5.5.17}$$

Observe that $\tilde{V}_{5(2-n)+1} = \{0\}$ for $n \in \{0,1\}$. About $\tilde{V}_{5(6(k_1-m)+2-n)+1}$, we distinguish two cases:

• If $n = 1$ then $\tilde{V}_{5(6(k_1-m)+1)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+1)+1} = \mathrm{Range}(\hat{\ell}_{6[5(k_1-m)+1]}) \oplus \left\langle h^{5(k_1-m)+1} \right\rangle$. Taking into account that $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)+1]}) = \left\langle h^{5(k_1-m)+1} \right\rangle$, we can conclude that $\tilde{V}_{5(6(k_1-m)+1)+1} = \{0\}$.

• If $n \neq 1$ then $\tilde{V}_{5(6(k_1-m)+2-n)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+2-n)+1} = \mathrm{Range}(\hat{\ell}_{5(6(k_1-m)+2-n)+1})$. Thus $\tilde{V}_{5(6(k_1-m)+2-n)+1} = \mathrm{Cor}(\hat{\ell}_{5(6(k_1-m)+2-n)+1})$ and it is easy to prove that all co-ranges are nulls, except for $n = 3$, for this case $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)-1]+2}) = \left\langle x\, h^{5(k_1-m)-1} \right\rangle$.

Therefore, in this subcase $k_2 = 2$, we get

$$\mathrm{Cor}(\ell_{11+30k_1}) = \bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+3} x\, h^{5(k_1-m)-1} \right\rangle .$$

1. d) Case $k_2 = 3$. In this subcase we obtain that,

$$\begin{aligned} Cor(\ell_{1+5(6k_1+3)}) &= \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)+3-n)+1} \\ &\quad + \sum_{n=0}^{2} z^{6k_1+n}\, \tilde{V}_{5(3-n)+1}. \end{aligned} \tag{5.5.18}$$

Observe that $\tilde{V}_{5(3-n)+1} = \{0\}$ if $n \in \{0,1,2\}$. About $\tilde{V}_{5(6(k_1-m)+3-n)+1}$, we distinguish two cases:

• If $n = 2$ then $\tilde{V}_{5(6(k_1-m)+1)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+1)+1} = \text{Range}(\hat{\ell}_{6[5(k_1-m)+1]}) \oplus \langle h^{5(k_1-m)+1} \rangle$. Taking into account that $\text{Cor}(\hat{\ell}_{6[5(k_1-m)+1]}) = \langle h^{5(k_1-m)+1} \rangle$, we can conclude that $\tilde{V}_{5(6(k_1-m)+1)+1} = \{0\}$.

• If $n \neq 2$ then $\tilde{V}_{5(6(k_1-m)+3-n)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+3-n)+1} = \text{Range}(\hat{\ell}_{5(6(k_1-m)+3-n)+1})$. Hence $\tilde{V}_{5(6(k_1-m)+3-n)+1} = \text{Cor}(\hat{\ell}_{5(6(k_1-m)+3-n)+1})$ and it is easy to prove that all co-ranges are nulls, except for $n = 4$, for this case $\text{Cor}(\hat{\ell}_{6[5(k_1-m)-1]+2}) = \langle x\, h^{5(k_1-m)-1} \rangle$.

Therefore, in this subcase $k_2 = 3$, we get

$$\text{Cor}(\ell_{16+30k_1}) = \bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+4} x\, h^{5(k_1-m)-1} \right\rangle.$$

1. e) Case $k_2 = 4$. In this subcase we obtain that,

$$\begin{aligned} Cor(\ell_{1+5(6k_1+4)}) &= \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)+4-n)+1} \\ &\quad + \sum_{n=0}^{3} z^{6k_1+n}\, \tilde{V}_{5(4-n)+1}. \end{aligned} \tag{5.5.19}$$

Observe that $\tilde{V}_{5(4-n)+1} = \{0\}$ if $n \in \{0,1,2,3\}$. For determining $\tilde{V}_{5(6(k_1-m)+4-n)+1}$, we distinguish two cases:

• If $n = 3$ then $\tilde{V}_{5(6(k_1-m)+1)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+1)+1} = \text{Range}(\hat{\ell}_{6[5(k_1-m)+1]}) \oplus \langle h^{5(k_1-m)+1} \rangle$. Taking into account that $\text{Cor}(\hat{\ell}_{6[5(k_1-m)+1]}) = \langle h^{5(k_1-m)+1} \rangle$, we can conclude that $\tilde{V}_{5(6(k_1-m)+1)+1} = \{0\}$.

• If $n \neq 3$ then $\tilde{V}_{5(6(k_1-m)+4-n)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+4-n)+1} = \text{Range}(\hat{\ell}_{5(6(k_1-m)+4-n)+1})$. Thus $\tilde{V}_{5(6(k_1-m)+4-n)+1} = \text{Cor}(\hat{\ell}_{5(6(k_1-m)+4-n)+1})$ and it is easy to prove that all co-ranges are nulls, except for $n = 5$, for this case $\text{Cor}(\hat{\ell}_{6[5(k_1-m)-1]+2}) = \langle x\, h^{5(k_1-m)-1} \rangle$.

Therefore, in this subcase $k_2 = 4$, we get

$$\text{Cor}(\ell_{21+30k_1}) = \bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+5} x\, h^{5(k_1-m)-1} \right\rangle.$$

1. f) Case $k_2 = 5$. In this subcase we obtain that,

$$\begin{aligned} Cor(\ell_{1+5(6k_1+5)}) &= \sum_{m=0}^{k_1-1}\sum_{n=0}^{5} z^{6m+n}\,\tilde{V}_{5(6(k_1-m)+5-n)+1} \\ &\quad + \sum_{n=0}^{4} z^{6k_1+n}\,\tilde{V}_{5(5-n)+1} \end{aligned} \tag{5.5.20}$$

Observe that $\tilde{V}_{5(5-n)+1} = \{0\}$ if $n \in \{1,2,3,4\}$, but in the case $n = 0$, $\tilde{V}_{5(5-n)+1} = \tilde{V}_{5\cdot 5+1} = \mathrm{Cor}(\hat{\ell}_{6\cdot 4+2}) = \langle x\,h^4\rangle$. About $\tilde{V}_{5(6(k_1-m)+5-n)+1}$, we distinguish two cases:

- If $n = 4$ then $\tilde{V}_{5(6(k_1-m)+1)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+1)+1} = \mathrm{Range}(\hat{\ell}_{6[5(k_1-m)+1]}) \oplus \left\langle h^{5(k_1-m)+1}\right\rangle$. Taking into account that $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)+1]}) = \left\langle h^{5(k_1-m)+1}\right\rangle$, we can conclude that $\tilde{V}_{5(6(k_1-m)+1)+1} = \{0\}$.

- If $n \neq 4$ then $\tilde{V}_{5(6(k_1-m)+5-n)+1}$ is a complementary subspace of $V_{5(6(k_1-m)+5-n)+1} = \mathrm{Range}(\hat{\ell}_{5(6(k_1-m)+5-n)+1})$. Thus $\tilde{V}_{5(6(k_1-m)+5-n)+1} = \mathrm{Cor}(\hat{\ell}_{5(6(k_1-m)+5-n)+1})$ and it is easy to prove that all co-ranges are nulls, except for $n = 0$, for this $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)+4]+2}) =$ $\mathrm{Cor}(\hat{\ell}_{6[5(k_1-m)+4]+2}) = \left\langle x\,h^{5(k_1-m)+4}\right\rangle$.

Therefore, in this subcase $k_2 = 5$, we obtain that

$$\mathrm{Cor}(\ell_{26+30k_1}) = \bigoplus_{m=0}^{k_1} \left\langle z^{6m} x\,h^{5(k_1-m)+4}\right\rangle.$$

The following table contains a summary of all the results obtained for the case $k_3 = 0$.

k_2	$\mathrm{Cor}(\ell_{5(6\,k_1+k_2)+1})$
0	$\bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+1} x\, h^{5(k_1-m)-1} \right\rangle$
1	$\bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+2} x\, h^{5(k_1-m)-1} \right\rangle$
2	$\bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+3} x\, h^{5(k_1-m)-1} \right\rangle$
3	$\bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+4} x\, h^{5(k_1-m)-1} \right\rangle$
4	$\bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+5} x\, h^{5(k_1-m)-1} \right\rangle$
5	$\bigoplus_{m=0}^{k_1} \left\langle z^{6m} x\, h^{5(k_1-m)+4} \right\rangle$

Table 5.1: Co-ranges for the case $k_3 = 0$.

For expressing the normal form, we write the elements of these co-ranges as follows,

$$\begin{cases} x\, z^{k_2+1} h^4 \sum_{m=0}^{k_1-1} a_m^{(k_2,k_3)} z^{6m} h^{5(k_1-1-m)} & \text{if} \quad 0 \leq k_2 \leq 4, \\ x\, h^4 \sum_{m=0}^{k_1} a_m^{(k_2,k_3)} z^{6m} h^{5(k_1-m)} & \text{if} \quad k_2 = 5. \end{cases}$$

2. Case $k_3 = 1$. In this case $k = 5(6k_1 + k_2) + 1$.

The expression (5.5.12) is of the form,

$$Cor(\ell_{2+5(6k_1+k_2)}) = \left\langle z^{6\,k_1+k_2} x \right\rangle \oplus \sum_{i=0}^{6\,k_1+k_2-1} z^i\, \tilde{V}_{5(6\,k_1+k_2-i)+2}. \tag{5.5.21}$$

Taking $i = 6m + n$ with $0 \leq n < 6$ and $m \in \mathbf{N} \cup \{0\}$. The above expression is of the form,

$$\begin{aligned} Cor(\ell_{2+5(6k_1+k_2)}) &= \left\langle z^{6\,k_1+k_2} x \right\rangle \oplus \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)+k_2-n)+2} \\ &+ \sum_{n=0}^{k_2-1} z^{6k_1+n} \tilde{V}_{5(k_2-n)+2}. \end{aligned} \tag{5.5.22}$$

In this case, and in subsequent, the calculations to determine the co-ranges are similar to those made in the previous section ($k_3 = 0$). The results obtained for different values k_2 are shown in the following table.

k_2	$\mathrm{Cor}(\ell_{5(6\,k_1+k_2)+2})$
0	$\bigoplus_{m=0}^{k_1} \langle z^{6m} x\, h^{5(k_1-m)} \rangle$
1	$\bigoplus_{m=0}^{k_1} \langle z^{6m+1} x\, h^{5(k_1-m)} \rangle$
2	$\bigoplus_{m=0}^{k_1} \langle z^{6m+2} x\, h^{5(k_1-m)} \rangle$
3	$\bigoplus_{m=0}^{k_1} \langle z^{6m+3} x\, h^{5(k_1-m)} \rangle$
4	$\bigoplus_{m=0}^{k_1} \langle z^{6m+4} x\, h^{5(k_1-m)} \rangle$
5	$\bigoplus_{m=0}^{k_1} \langle z^{6m+5} x\, h^{5(k_1-m)} \rangle$

Table 5.2: Co-ranges for the case $k_3 = 1$.

For expressing the normal form, we write the elements of these co-ranges as follows,

$$x\, z^{k_2} \sum_{m=0}^{k_1} a_m^{(k_2,k_3)} z^{6m} h^{5(k_1-m)}.$$

3. Case $k_3 = 2$. In this case $k = 5(6k_1 + k_2) + 2$.

The expression (5.5.12) is of the form,

$$Cor(\ell_{3+5(6k_1+k_2)}) = \sum_{i=0}^{6\,k_1+k_2-1} z^i\, \tilde{V}_{5(6\,k_1+k_2-i)+3}. \tag{5.5.23}$$

By decomposing $i = 6m + n$ with $0 \leq n < 6$ and $m \in \mathbf{N} \cup \{0\}$. The above expression is of the form,

$$\begin{aligned} Cor(\ell_{3+5(6k_1+k_2)}) &= \sum_{m=0}^{k_1-1}\sum_{n=0}^{5} z^{6m+n}\,\tilde{V}_{5(6(k_1-m)+k_2-n)+3} \\ &+ \sum_{n=0}^{k_2-1} z^{6k_1+n}\tilde{V}_{5(k_2-n)+3}. \end{aligned} \tag{5.5.24}$$

The results obtained for different values k_2 are shown in the following table.

k_2	$\mathrm{Cor}(\ell_{5(6\,k_1+k_2)+3})$
0	$\bigoplus_{m=0}^{k_1-1}\left\langle z^{6m+5}x\,h^{5(k_1-m)-4}\right\rangle$
1	$\bigoplus_{m=0}^{k_1}\left\langle z^{6m}x\,h^{5(k_1-m)+1}\right\rangle$
2	$\bigoplus_{m=0}^{k_1}\left\langle z^{6m+1}x\,h^{5(k_1-m)+1}\right\rangle$
3	$\bigoplus_{m=0}^{k_1}\left\langle z^{6m+2}x\,h^{5(k_1-m)+1}\right\rangle$
4	$\bigoplus_{m=0}^{k_1}\left\langle z^{6m+3}x\,h^{5(k_1-m)+1}\right\rangle$
5	$\bigoplus_{m=0}^{k_1}\left\langle z^{6m+4}x\,h^{5(k_1-m)+1}\right\rangle$

Table 5.3: Co-ranges for the case $k_3 = 2$.

The elements of these co-ranges can be written in the form,

$$\begin{cases} x\,z^{k_2-1}h\displaystyle\sum_{m=0}^{k_1} a_m^{(k_2,k_3)}z^{6m}h^{5(k_1-m)} & \text{if} \quad 1 \le k_2 \le 5, \\ x\,z^5h\displaystyle\sum_{m=0}^{k_1-1} a_m^{(k_2,k_3)}z^{6m}h^{5(k_1-1-m)} & \text{if} \quad k_2 = 0. \end{cases}$$

4. Case $k_3 = 3$. In this case $k = 5(6k_1 + k_2) + 3$.

The expression (5.5.12) is of the form,

$$Cor(\ell_{4+5(6k_1+k_2)}) = \sum_{i=0}^{6\,k_1+k_2-1} z^i\, \tilde{V}_{5(6\,k_1+k_2-i)+4}. \tag{5.5.25}$$

Taking $i = 6m+n$ with $0 \le n < 6$ and $m \in \mathbf{N} \cup \{0\}$. The above expression is of the form,

$$\begin{aligned} Cor(\ell_{4+5(6k_1+k_2)}) &= \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)+k_2-n)+4} \\ &+ \sum_{n=0}^{k_2-1} z^{6k_1+n} \tilde{V}_{5(k_2-n)+4}. \end{aligned} \tag{5.5.26}$$

The results obtained for different values k_2 are shown in the following table.

k_2	$\mathrm{Cor}(\ell_{5(6\,k_1+k_2)+4})$
0	$\bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+4} x\, h^{5(k_1-m)-3} \right\rangle$
1	$\bigoplus_{m=0}^{k_1-1} \left\langle z^{6m+5} x\, h^{5(k_1-m)-3} \right\rangle$
2	$\bigoplus_{m=0}^{k_1} \left\langle z^{6m} x\, h^{5(k_1-m)+2} \right\rangle$
3	$\bigoplus_{m=0}^{k_1} \left\langle z^{6m+1} x\, h^{5(k_1-m)+2} \right\rangle$
4	$\bigoplus_{m=0}^{k_1} \left\langle z^{6m+2} x\, h^{5(k_1-m)+2} \right\rangle$
5	$\bigoplus_{m=0}^{k_1} \left\langle z^{6m+3} x\, h^{5(k_1-m)+2} \right\rangle$

Table 5.4: Co-ranges for the case $k_3 = 3$.

The elements of this co-ranges can be written in the form,

$$\begin{cases} x\, z^{k_2+4} h^2 \displaystyle\sum_{m=0}^{k_1-1} a_m^{(k_2,k_3)} z^{6m} h^{5(k_1-1-m)} & \text{if} \quad 0 \leq k_2 \leq 1, \\ x\, z^{k_2-2} h^2 \displaystyle\sum_{m=0}^{k_1} a_m^{(k_2,k_3)} z^{6m} h^{5(k_1-m)} & \text{if} \quad 2 \leq k_2 \leq 5. \end{cases}$$

5. Case $k_3 = 4$. In this case $k = 5(6k_1 + k_2) + 4$.

The expression (5.5.12) is of the form,

$$Cor(\ell_{5+5(6k_1+k_2)}) = \left\langle z^{6k_1+k_2+1} \right\rangle + \sum_{i=0}^{6\,k_1+k_2-1} z^i\, \tilde{V}_{5(6\,k_1+k_2-i)+5}. \tag{5.5.27}$$

By taking $i = 6m + n$ with $0 \leq n < 6$ and $m \in \mathbf{N} \cup \{0\}$. The above expression is of the form,

$$\begin{aligned} Cor(\ell_{5+5(6k_1+k_2)}) &= \left\langle z^{6k_1+k_2+1} \right\rangle + \sum_{m=0}^{k_1-1} \sum_{n=0}^{5} z^{6m+n}\, \tilde{V}_{5(6(k_1-m)+k_2-n)+5} \\ &+ \sum_{n=0}^{k_2-1} z^{6k_1+n} \tilde{V}_{5(k_2-n)+5}. \end{aligned} \tag{5.5.28}$$

The results obtained for different values k_2 are shown in the following table.

k_2	$\mathrm{Cor}(\ell_{5(6\,k_1+k_2)+5})$
0	$\left\langle z^{6k_1+1}\right\rangle + \bigoplus_{m=0}^{k_1-1}\left\langle z^{6m+3}x\,h^{5(k_1-m)-2}\right\rangle$
1	$\left\langle z^{6k_1+2}\right\rangle + \bigoplus_{m=0}^{k_1-1}\left\langle z^{6m+4}x\,h^{5(k_1-m)-2}\right\rangle$
2	$\left\langle z^{6k_1+3}\right\rangle + \bigoplus_{m=0}^{k_1-1}\left\langle z^{6m+5}x\,h^{5(k_1-m)-2}\right\rangle$
3	$\left\langle z^{6k_1+4}\right\rangle + \bigoplus_{m=0}^{k_1}\left\langle z^{6m}x\,h^{5(k_1-m)+3}\right\rangle$
4	$\left\langle z^{6k_1+5}\right\rangle + \bigoplus_{m=0}^{k_1}\left\langle z^{6m+1}x\,h^{5(k_1-m)+3}\right\rangle$
5	$\left\langle z^{6(k_1+1)}\right\rangle + \bigoplus_{m=0}^{k_1}\left\langle z^{6m+2}x\,h^{5(k_1-m)+3}\right\rangle$

Table 5.5: Co-ranges for the case $k_3 = 4$.

The elements of these co-ranges can be written in the form,

$$\begin{cases} c_{k_1}^{(k_2,4)} z^{6k_1+k_2+1} + x\,z^{k_2+3}h^3 \sum_{m=0}^{k_1-1} a_m^{(k_2,4)} z^{6m} h^{5(k_1-1-m)} & \text{if} \quad 0 \leq k_2 \leq 2, \\ c_{k_1}^{(k_2,4)} z^{6k_1+k_2+1} + x\,z^{k_2-3}h^3 \sum_{m=0}^{k_1} a_m^{(k_2,4)} z^{6m} h^{5(k_1-m)} & \text{if} \quad 3 \leq k_2 \leq 5. \end{cases}$$

where $c_{k_1}^{(k_2,4)}$ and $a_m^{(k_2,4)}$ are real numbers.

From Theorems 5.3.140 and 5.3.141 and expressions of co-ranges that we have just calculated, we give the normal form, under equivalence, for triple-zero singularity. We collect this in the following theorem:

Theorem 5.5.146. *A formal normal form for system (5.5.11), under equivalence, is of the form,*

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ x^3 \\ \hline \frac{1}{6}(2x^2 - 3y^2) \end{pmatrix} + \begin{pmatrix} \mathbf{X}_g \\ \hline 0 \end{pmatrix} + \begin{pmatrix} \mu\mathbf{D}_0 \\ \hline 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \hline \varsigma \end{pmatrix} \tag{5.5.29}$$

where

$$g(x,z) = \sum_{k_1=1}^{\infty}\sum_{k_2=0}^{5}\alpha_{k_1}^{(k_2,k_3)} x\, z^{6k_1+k_2}.$$

$$\mu = zC(z) + x\sum_{j=0}^{5}\sum_{i=0}^{4} z^j h^i A^{(i,j)}(z^6, h^5).$$

$$\varsigma = z^2 D(z) + x\sum_{j=0}^{5}\sum_{i=0}^{4} z^j h^i B^{(i,j)}(z^6, h^5).$$

where $\alpha_{k_1}^{(k_2,k_3)} \in \mathbb{R}$, $A^{(i,j)}(x,y)$, $B^{(i,j)}(x,y)$ *are power series, in the variables* x *and* y*, that can be unit, except in the case* $B^{(0,0)}$*, i.e.,* $B^{(0,0)}(0,0) = 0$.

5.6 Parametric normal form

In this section we discuss the role of the parameters in the normal form calculations. Let consider a perturbation of the vector field given in (5.1.1), i.e., a family parametrized by ξ of the form,

$$\dot{\mathbf{x}} = \widetilde{\mathbf{F}}(\mathbf{x},\boldsymbol{\xi}) = \mathbf{F}(\mathbf{x}) + \mathbf{Y}(\mathbf{x},\boldsymbol{\xi}) \tag{5.6.30}$$

with $\mathbf{x} \in \mathbb{R}^n$, $\boldsymbol{\xi} \in \mathbb{R}^m$, $\mathbf{Y}(\mathbf{x},\mathbf{0}) = \mathbf{0}$ and $\mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots$ respect the type $\mathbf{t}$ and $r \in \mathbb{Z}$.

For this task, we use suspension. Respect of the type $\mathbf{t} = (t_1, t_2, \ldots, t_n, 0, 0, \ldots, 0)$ the above vector field can be expressed by,

$$\begin{aligned}\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\xi}}\end{pmatrix} = \begin{pmatrix}\widetilde{\mathbf{F}}(\mathbf{x},\boldsymbol{\xi})\\ \mathbf{0}\end{pmatrix} &= \begin{pmatrix}\widetilde{\mathbf{F}}_{r-s}(\mathbf{x},\boldsymbol{\xi})\\ \mathbf{0}\end{pmatrix} + \cdots + \begin{pmatrix}\widetilde{\mathbf{F}}_{r-2}(\mathbf{x},\boldsymbol{\xi})\\ \mathbf{0}\end{pmatrix} + \begin{pmatrix}\widetilde{\mathbf{F}}_{r-1}(\mathbf{x},\boldsymbol{\xi})\\ \mathbf{0}\end{pmatrix}\\ &+ \begin{pmatrix}\widetilde{\mathbf{F}}_{r}(\mathbf{x},\boldsymbol{\xi})\\ \mathbf{0}\end{pmatrix} + \begin{pmatrix}\widetilde{\mathbf{F}}_{r+1}(\mathbf{x},\boldsymbol{\xi})\\ \mathbf{0}\end{pmatrix} + \cdots,\end{aligned}$$

where $(x,\boldsymbol{\xi}) \in \mathbb{R}^n \times \mathbb{R}^m$. It is possible to find weights $p_1, p_2, \ldots, p_m$, so that, by ordering the system respect to the type $\boldsymbol{\tau} = (t_1, t_2, \ldots, t_n, p_1, p_2, \ldots, p_m)$, the components $\tilde{\mathbf{F}}_{r-k}(\mathbf{x},\boldsymbol{\xi}) \in \mathcal{Q}_{r-k}^{\mathbf{t}}$, i.e., belong to the vectorial space of the quasi-homogeneous polynomials of degree $r+l$, $l \geq 0$ and type $\boldsymbol{\tau}$. In this case, the vector field can be expressed in the form,

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\xi}}\end{pmatrix} = \begin{pmatrix}\widetilde{\mathbf{F}}(\mathbf{x},\boldsymbol{\xi})\\ 0\end{pmatrix} = \begin{pmatrix}\widehat{\mathbf{F}}_{r}(\mathbf{x},\boldsymbol{\xi})\\ 0\end{pmatrix} + \begin{pmatrix}\widehat{\mathbf{F}}_{r+1}(\mathbf{x},\boldsymbol{\xi})\\ 0\end{pmatrix} + \begin{pmatrix}\widehat{\mathbf{F}}_{r+2}(\mathbf{x},\boldsymbol{\xi})\\ 0\end{pmatrix} + \cdots \tag{5.6.31}$$

being $\widehat{\mathbf{F}}_r = \widetilde{\mathbf{F}}_r$.

Fisty we apply a change in the time of the form

$$\frac{dt}{dT} = 1 - \mu_k(\mathbf{y}, \boldsymbol{\nu}), \text{ with } \mu_k \in \mathcal{P}_k^{\boldsymbol{\tau}} \tag{5.6.32}$$

and following a coordinate transformation, of the form

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ \boldsymbol{\nu} \end{pmatrix} + \begin{pmatrix} \mathbf{P}_k(\mathbf{y}, \boldsymbol{\nu}) \\ \mathbf{0} \end{pmatrix}, \text{ with } \mathbf{P}_k \in \mathcal{Q}_k^{\boldsymbol{\tau}} \tag{5.6.33}$$

These changes transform system (5.6.31), for $\mathbf{y}$ sufficiently small, into,

$$\begin{pmatrix} \dot{\mathbf{y}} \\ \dot{\boldsymbol{\nu}} \end{pmatrix} = \left(I + \begin{pmatrix} D\mathbf{P}_k(\mathbf{y}, \boldsymbol{\nu}) \\ \mathbf{0} \end{pmatrix}^{-1} \right) \sum_{i \le r} \begin{pmatrix} \mathbf{F}_i(\mathbf{y} + P_k(\mathbf{y}, \boldsymbol{\nu})) \\ \mathbf{0} \end{pmatrix} + \cdots \tag{5.6.34}$$

and, ordering in quasi-homogeneous terms,

$$\begin{pmatrix} \dot{\mathbf{y}} \\ \dot{\boldsymbol{\nu}} \end{pmatrix} = \begin{pmatrix} \mathbf{G}_r(\mathbf{y}, \boldsymbol{\nu}) \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{G}_{r+1}(\mathbf{y}, \boldsymbol{\nu}) \\ \mathbf{0} \end{pmatrix} + \cdots \tag{5.6.35}$$

being $\mathbf{G}_j(\mathbf{y}, \boldsymbol{\nu}) = \widehat{\mathbf{F}}_j(\mathbf{y}, \boldsymbol{\nu})$ for all $j = r, r+1, \cdots, r+k-1$ and $\mathbf{G}_{r+k}(\mathbf{y}, \boldsymbol{\nu}) = \widehat{\mathbf{F}}_{r+k}(\mathbf{y}, \boldsymbol{\nu}) - D\mathbf{P}_k(\mathbf{y}, \boldsymbol{\nu})\widehat{\mathbf{F}}_r(\mathbf{y}, \boldsymbol{\nu}) - D\mathbf{P}_k(\mathbf{y}, \boldsymbol{\nu})\widehat{\mathbf{F}}_r(\mathbf{y}, \boldsymbol{\nu}) - \mu_k\widehat{\mathbf{F}}_r(\mathbf{y}, \boldsymbol{\nu})$.

In terms of the Lie bracket, $\mathbf{G}_{r+k}(\mathbf{y}, \boldsymbol{\nu}) = \widehat{\mathbf{F}}_{r+k}(\mathbf{y}, \boldsymbol{\nu}) - [\widehat{\mathbf{F}}_r(\mathbf{y}, \boldsymbol{\nu}), \mathbf{P}_k(\mathbf{y}, \boldsymbol{\nu})]_{\mathbf{y}} - \mu_k\widehat{\mathbf{F}}_r(\mathbf{y}, \boldsymbol{\nu})$.

This prove the following result:

Proposition 5.6.147. *Let consider the vector field given in (5.6.31). If we apply the change in the time (5.6.32) and the change in the state of variables (5.6.33), the system is transformed into (5.6.35) where,* $\mathbf{G}_r = \widehat{\mathbf{F}}_r$, $\dots$, $\mathbf{G}_{r+k-1} = \widehat{\mathbf{F}}_{r+k-1}$ *and* $\mathbf{G}_{r+k} = \widehat{\mathbf{F}}_{r+k} - [\widehat{\mathbf{F}}_r, \mathbf{P}_k]_{\mathbf{y}} - \mu_k\widehat{\mathbf{F}}_r$.

Our goal in this section, it is to show a parametric normal form for specific families of vector fields. Works in this sense can be found in several articles, some of them, already mentioned in the introduction to this chapter, as [53]. In the work of Gao & Zhang [48] also is described the process to calculate a parametric normal form using changes of variables on the modules of homogeneous polynomials over the ring of all continuous functions of ξ. In both works, the authors start from a versal deformation of the linear part of the system (5.6.30). In this section, we study both questions. In first time we calculate a versal deformation of the components of the quasi-homogeneous vector whose degree are equals or less than r. After obtaining such versal deformation, we will remove terms of higher degree than r.

In the calculate of a versal deformation of a nilpotent vector field. We will make this process using change of variables, in a first time with change of variables with possitive degrees (using suspension) and after, using change of variables with negative degrees (using Lie bracket).

5.6.1 An illustrative example.

In this subsection, we consider the nilpotent case, that is, a family with the form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \xi), \text{ with } \mathbf{x} \in \mathbb{R}^2 \text{ and } \xi \in \mathbb{R}^m \ ,$$

where the quasi-homogeneous lower degree term, $\mathbf{F}_r$ is of the form, $\mathbf{F}_1 = \begin{pmatrix} y \\ x^2 \end{pmatrix}$, respect to the type $\mathbf{t} = (t_1, t_2) = (2, 3)$.

Our goal is to compute a versal deformation for this singularity. To do this, we perturb the field by adding all possible parameters to the quasi-homogeneous components whose degrees are less than or equal to $r = 1$, i.e.,

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 \\ \xi_7 \end{pmatrix} + \begin{pmatrix} \xi_6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \xi_5 x \end{pmatrix} + \begin{pmatrix} \xi_3 x \\ \xi_4 y \end{pmatrix} + \begin{pmatrix} (\xi_1 + 1) y \\ (\xi_2 + 1) x^2 \end{pmatrix}. \tag{5.6.36}$$

For eliminating term in $\mathbf{F}_1$ we use change of variables of degree 0, that is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix} \text{ with } \alpha = (\xi_1 + 1)(\xi_2 + 1) \text{ and } \beta = (\xi_1 + 1)^2 (\xi_2 + 1).$$

Applying this change to system (5.6.36), we obtain,

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 \\ \tilde{\xi}_7 \end{pmatrix} + \begin{pmatrix} \tilde{\xi}_6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\xi}_5 x \end{pmatrix} + \begin{pmatrix} \tilde{\xi}_3 x \\ \tilde{\xi}_4 y \end{pmatrix} + \begin{pmatrix} y \\ x^2 \end{pmatrix}. \tag{5.6.37}$$

This vector field can be expressed in the form,

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} y \\ x^2 \\ \hline 0 \end{pmatrix} + \begin{pmatrix} \tilde{\xi}_1 x \\ \tilde{\xi}_2 y \\ \hline 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\xi}_3 x \\ \hline 0 \end{pmatrix} + \begin{pmatrix} \tilde{\xi}_4 \\ 0 \\ \hline 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{\xi}_5 \\ \hline 0 \end{pmatrix}, \tag{5.6.38}$$

being $(\mathbf{x}, \tilde{\boldsymbol{\xi}}) \in \mathbb{R}^2 \times \mathbb{R}^5$ and $\boldsymbol{\tau} = (2,3,2,2,4,6,8)$. Respect of this type, the quasi-homogeneous components of system (5.6.38) are of degrees 1, 2, 3, 4 and 5 respectively. It should be noted that, in the choice of the change of variables $\mathbf{P}_i$ we will consider only those involving only parameters or state variables multiplied by these. We use conjugation, i.e., we consider $\mu_k \equiv 0$ in (5.6.32).

- Consider $\mathbf{P}_1 = \left(\begin{array}{c} 0 \\ \gamma\tilde{\xi}_1 x \\ \hline 0 \end{array}\right)$. In this case the homological operator is of the form $\mathbf{F}_2 - [\mathbf{P}_1, \mathbf{F}_1] = \left(\begin{array}{c} (\tilde{\xi}_1 + \gamma\tilde{\xi}_1)x \\ (\tilde{\xi}_2 - \gamma\tilde{\xi}_1)y \\ \hline 0 \end{array}\right)$ and considering $\gamma = -1$ we obtain the transformed vector field to 2nd-order is $\mathbf{G}_2 = \left(\begin{array}{c} 0 \\ (\tilde{\xi}_2 + \tilde{\xi}_1)y \\ \hline 0 \end{array}\right)$.

- Consider $\mathbf{P}_2 = \left(\begin{array}{c} \delta\tilde{\xi}_3 \\ 0 \\ \hline 0 \end{array}\right)$. In this case the homological operator is of the form $\mathbf{F}_3 - [\mathbf{P}_2, \mathbf{F}_1] = \left(\begin{array}{c} 0 \\ (\tilde{\xi}_3 + 2\delta\tilde{\xi}_3)x \\ \hline 0 \end{array}\right)$ and considering $\delta = -1/2$ it is possible to eliminate all terms of 3th-degree.

- Consider $\mathbf{P}_3 = \left(\begin{array}{c} 0 \\ \alpha\tilde{\xi}_4 \\ \hline 0 \end{array}\right)$. In this case the homological operator is of the form $\mathbf{F}_4 - [\mathbf{P}_3, \mathbf{F}_1] = \left(\begin{array}{c} 0 \\ \tilde{\xi}_4 + \alpha\tilde{\xi}_4 \\ \hline 0 \end{array}\right)$ and considering $\alpha = -1$ it is possible to eliminate all terms of 4th-degree.

- Consider $\mathbf{P}_4 = \left(\begin{array}{c} \beta_1 x^2\tilde{\xi}_1 + \beta_2 x^2\tilde{\xi}_2 + \beta_3\tilde{\xi}_1^3 + \beta_4\tilde{\xi}_2^3 + \beta_5 x\tilde{\xi}_4 + \beta_6\tilde{\xi}_4 \\ \beta_7 y\tilde{\xi}_1^2 + \beta_8 y\tilde{\xi}_2^2 + \beta_9 y\tilde{\xi}_3 \\ \hline 0 \end{array}\right)$. In this case the homological operator is of the form $\mathbf{F}_5 - [\mathbf{P}_4, \mathbf{F}_1] =$

$$\begin{pmatrix} (2\beta_1\tilde{\xi}_1 - 2\beta_2\tilde{\xi}_2)xy + (\beta_5\tilde{\xi}_4 - \beta_7\tilde{\xi}_1^2 + \beta_8\tilde{\xi}_2^2 + \beta_9\tilde{\xi}_3)y \\ (2\beta_1\tilde{\xi}_1 + 2\beta_2\tilde{\xi}_2)x^3 + (\beta_7\tilde{\xi}_1^2 - \beta_8\tilde{\xi}_2^2 - \beta_9\tilde{\xi}_3 + 2\beta_5\tilde{\xi}_4)x^2 + (2\beta_6\tilde{\xi}_4 + 2\beta_3\tilde{\xi}_1^3 + 2\beta_4\tilde{\xi}_2^3)x + \tilde{\xi}_5 \\ \hline 0 \end{pmatrix}.$$

In this case it is impossible to remove any term.

Renaming parameters, the versal deformation obtained is,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ \varepsilon_1 + \varepsilon_2 y + x^2 \end{pmatrix}. \tag{5.6.39}$$

Remark 19. *Note that the changes, P_i, that we have made, affect not only to the components of the vector field with degree i, also it affects to the rest of the components.*

This process of removing terms of degree less than 1 it is possible to do it from another ways, for example, to use the Lie bracket with changes of variables of negative degrees. Before to show it , we will define the homological operator for this case. The homological operator is defined in (2.3.13), here we generalize it when we use change of variables with negative degrees. The homological operator, under $\mathcal{C}^\infty$-conjugation, is given by

$$\mathbf{L}_{r-k} \quad : \quad \mathcal{Q}^{\mathbf{t}}_{-k} \longrightarrow \mathcal{Q}^{\mathbf{t}}_{r-k}\ ,$$

where

$$\mathbf{L}_{r-k}(\mathbf{P}_{-k}) \quad = \quad [\mathbf{P}_{-k}, \mathbf{F}_r]. \tag{5.6.40}$$

We apply the following proposition:

Proposition 5.6.148. *Let $\mathbf{F}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{j=s}^{0} \mathbf{F}_{r-j}(\mathbf{x}, \boldsymbol{\xi})$ be, such that $\mathbf{F}_r(\mathbf{x}, \mathbf{0}) = \mathbf{F}_r(\mathbf{x})$ and $\mathbf{F}_{r-j} \in \mathcal{Q}^{\mathbf{t}}_{r-j}$ if we apply the change of variables $\mathbf{x} = \mathbf{y} + \mathbf{P}_{-k}(\mathbf{y})$ the above system is transformed into $\mathbf{G} = \sum_{j=s}^{0} \mathbf{G}_{r-j}$ where, $\mathbf{G}_r = \mathbf{F}_r$, $\mathbf{G}_{r-1} = \mathbf{F}_{r-1}$, $\cdots$, $\mathbf{G}_{r-k+1} = \mathbf{F}_{r-k+1}$ and $\mathbf{G}_{r-k} = \mathbf{F}_{r-k} - \mathbf{L}_{k-r}(\mathbf{P}_{-k})$.*

Next we apply these concepts to our example, system (5.6.37). We denote parameters $\tilde{\xi}_i = \xi^{(i)}$, $3 \le i \le 7$.

- Consider $\mathbf{P}_{-1} = \begin{pmatrix} 0 \\ \gamma x \end{pmatrix}$.

  The homological operator is of the form, $\mathbf{F}_0 - [\mathbf{P}_{-1}, \mathbf{F}_1] = \begin{pmatrix} (\xi_3^{(0)} + \gamma)x \\ (\xi_4^{(0)} - \gamma)y \end{pmatrix}$, and considering $\gamma = -\xi_3^{(0)}$, the transformed vector field is $\mathbf{G}^{(1)} = \mathbf{G}_{-3}^{(1)} + \mathbf{G}_{-2}^{(1)} + \mathbf{G}_{-1}^{(1)} + \mathbf{G}_0^{(1)}$ where

  $\mathbf{G}_{-3}^{(1)} = \begin{pmatrix} 0 \\ \xi_7^{(1)} \end{pmatrix}$, $\mathbf{G}_{-2}^{(1)} = \begin{pmatrix} \xi_6^{(1)} \\ 0 \end{pmatrix}$, $\mathbf{G}_{-1}^{(1)} = \begin{pmatrix} 0 \\ \xi_5^{(1)} x \end{pmatrix}$ and $\mathbf{G}_0^{(1)} = \begin{pmatrix} 0 \\ \xi_4^{(1)} y \end{pmatrix}$ with $\xi_4^{(1)} = \xi_4^{(0)} - \xi_3^{(0)}$.

- Consider $\mathbf{P}_{-2} = \begin{pmatrix} \delta \\ 0 \end{pmatrix}$.

  The homological operator is of the form, $\mathbf{G}_{-1}^{(1)} - [\mathbf{P}_{-2}, \mathbf{F}_1] = \begin{pmatrix} 0 \\ (\xi_5^{(1)} - 2\delta)x \end{pmatrix}$, and considering $\delta = \frac{\xi_5^{(1)}}{2}$, we eliminate all terms to -1-th order. Thus, the vector fields transformed is $\mathbf{G}^{(2)} = \mathbf{G}_{-3}^{(2)} + \mathbf{G}_{-2}^{(2)} + \mathbf{G}_{-1}^{(2)} + \mathbf{G}_0^{(2)}$, where

  $\mathbf{G}_{-3}^{(2)} = \begin{pmatrix} 0 \\ \xi_7^{(2)} \end{pmatrix}$, $\mathbf{G}_{-2}^{(2)} = \begin{pmatrix} \xi_6^{(2)} \\ 0 \end{pmatrix}$, $\mathbf{G}_{-1}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{G}_0^{(2)} = \begin{pmatrix} 0 \\ \xi_4^{(2)} y \end{pmatrix}$ with $\xi_4^{(2)} = \xi_4^{(1)}$.

- Consider $\mathbf{P}_{-3} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$.

  The homological operator is of the form, $\mathbf{G}_{-2}^{(2)} - [\mathbf{P}_{-2}, \mathbf{F}_1] = \begin{pmatrix} \xi_6^{(2)} - \eta \\ 0 \end{pmatrix}$, and considering $\eta = \xi_6^{(2)}$, we eliminate all terms to -2-th order. So, the vector field transformed is $\mathbf{G}^{(2)} = \mathbf{G}_{-3}^{(3)} + \mathbf{G}_{-2}^{(3)} + \mathbf{G}_{-1}^{(3)} + \mathbf{G}_0^{(3)}$ where

  $\mathbf{G}_{-3}^{(3)} = \begin{pmatrix} 0 \\ \xi_7^{(3)} \end{pmatrix}$, $\mathbf{G}_{-2}^{(3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{G}_{-1}^{(3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{G}_0^{(3)} = \begin{pmatrix} 0 \\ \xi_4^{(3)} y \end{pmatrix}$ with $\xi_4^{(3)} = \xi_4^{(2)}$.

- The -3-th order terms can not eliminate because it is necessary a change of variables with degree equal to -4 and there is no change of this degree.

Therefore, renaming parameters, we obtain the following versal deformation,

$$\dot{\mathbf{x}} = \begin{pmatrix} y \\ \varepsilon_1 + \varepsilon_2 y + x^2 \end{pmatrix}. \qquad (5.6.41)$$

This versal deformation is used in Guckenheimer & Holmes (see [60]).

5.6.2 A Hopf-zero singularity

We consider

$$\dot{\mathbf{x}} = \mathbf{F}_0 = \begin{pmatrix} \mu_1 x + (\mu_2 - 1)y \\ (\mu_3 + 1)x + \mu_4 y \\ (\mu_5 + 1)x^2 + \mu_6 xy + (\mu_7 + 1)y^2 + \mu_8 z \end{pmatrix}, \qquad (5.6.42)$$

respect to the type $\mathbf{t} = (1, 1, 2)$.

Firstly, we will calculate a versal deformation of $\mathbf{F}_0$ using change of variables of degree zero.

In first time, we use the change $u = x + \alpha y$, $v = y$, $w = z$, with $\alpha = \frac{\mu_4 - \mu_1}{2(\mu_3+1)}$. This change transforms (5.6.42) into,

$$\dot{\mathbf{x}} = \mathbf{F}_0 = \begin{pmatrix} \varepsilon_1 x - (1 + \varepsilon_2)y \\ (1 + \varepsilon_3)x + \varepsilon_1 y \\ (1 + \varepsilon_4)x^2 + \varepsilon_5 xy + (1 + \varepsilon_6)y^2 + \varepsilon_7 z \end{pmatrix}, \qquad (5.6.43)$$

being $\varepsilon_1 = \frac{1}{2}(\mu_1 + \mu_4)$, $\varepsilon_2 = -(\mu_2 + \frac{(\mu_1-\mu_4)^2}{4(1+\mu_3)})$, $\varepsilon_3 = \mu_3$, $\varepsilon_4 = \mu_5$, $\varepsilon_5 = \frac{\mu_6(\mu_3+1)+(\mu_5+1)(\mu_1-\mu_4)}{\mu_3+1}$, $\varepsilon_6 = \mu_7 + \frac{(-\mu_4+\mu_1)^2(\mu_5+1)+2\mu_6(\mu_3+1)(-\mu_4+\mu_1)}{4(\mu_3+1)^2}$ and $\varepsilon_7 = \mu_8$.

Next, we use the change of variable $u = \beta x$, $v = y$, $w = z$, with $\beta = \frac{\sqrt{(1+\varepsilon_2)(1+\varepsilon_3)}}{1+\varepsilon_2}$. This change transforms (5.6.43) into,

$$\dot{\mathbf{x}} = \mathbf{F}_0 = \begin{pmatrix} \eta_1 x - \eta_2 y \\ \eta_2 x + \eta_1 y \\ (1 + \eta_3)x^2 + \eta_4 xy + (1 + \eta_5)y^2 + \eta_6 z \end{pmatrix}, \qquad (5.6.44)$$

being $\eta_1 = \varepsilon_1$, $\eta_2 = \sqrt{(1+\varepsilon_2)(1+\varepsilon_3)}$, $\eta_3 = \frac{\varepsilon_3\varepsilon_2 - 1}{1+\varepsilon_3}$, $\eta_4 = \frac{\varepsilon_5(1+\varepsilon_2)}{\sqrt{(1+\varepsilon_2)(1+\varepsilon_3)}}$, $\eta_5 = \varepsilon_6$ and $\eta_6 = \varepsilon_7$.

Now, we use the following reparametrization in the time, $u = y$, $v = y$, $w = z$ and $T = \eta_2 t$. This change transforms system (5.6.44) into,

$$\dot{\mathbf{x}} = \mathbf{F}_0 = \begin{pmatrix} \xi_1 x - y \\ x + \xi_1 y \\ (1+\xi_2)x^2 + \xi_3 xy + (1+\xi_4)y^2 + \xi_5 z \end{pmatrix}, \tag{5.6.45}$$

being $\xi_1 = \frac{\eta_1}{\eta_2}$, $\xi_2 = \frac{1+\eta_3-\eta_2}{\eta_2}$, $\xi_3 = \frac{\eta_4}{\eta_2}$, $\xi_4 = \frac{1+\eta_5-\eta_2}{\eta_2}$ and $\xi_5 = \frac{\eta_6}{\eta_2}$.

Now we apply the following change, $u = x$, $v = y$, $w = x^2 + \gamma xy + \delta y^2 + z$ with $\gamma = -\frac{2\xi_2 - \xi_5\xi_3 + 2\xi_1\xi_3 - 2\xi_4}{(\xi_5 - 2\xi_1)^2 + 4}$ and $\delta = \frac{4 + 4\xi_1^2 - 4\xi_1\xi_5 + \xi_5^2 + 2\xi_1\xi_2 - 2\xi_1\xi_4 - 2\xi_3 - \xi_5\xi_2 + \xi_5\xi_4}{(\xi_5 - 2\xi_1)^2 + 4}$. This change transforms system (5.6.45) into,

$$\dot{\mathbf{x}} = \mathbf{F}_0 = \begin{pmatrix} \nu_1 x - y \\ x + \nu_1 y \\ (1+\nu_2)x^2 + (1+\nu_2)y^2 + \nu_3 z \end{pmatrix}, \tag{5.6.46}$$

being

$\nu_1 = \xi_1$,

$\nu_2 = \frac{4 - 4\xi_1\xi_5\xi_2 - 4\xi_1\xi_5 - 4\xi_5 + 2\xi_4 + 2\xi_2 + 8\xi_1 - 2\xi_1\xi_3 + \xi_5\xi_3 + 8\xi_1^3 - \xi_5^3 + 4\xi_1^2 + \xi_5^2 + \xi_5^2\xi_2 - 12\xi_1^2\xi_5 + 6\xi_1\xi_5^2 + 4\xi_1^2\xi_2}{(\xi_5 - 2\xi_1)^2 + 4}$,

$\nu_3 = \xi_5$.

Finally, we apply the reparametrization, $u = x$, $v = y$, $w = \mu z$ with $\mu = \frac{1}{1+\nu_2}$ that transforms system (5.6.46) into,

$$\dot{\mathbf{x}} = \mathbf{F}_0 = \begin{pmatrix} \mu_1 x - y \\ x + \mu_1 y \\ x^2 + y^2 + \mu_2 z \end{pmatrix}. \tag{5.6.47}$$

Now we will remove the lower order terms of $\mathbf{F}_0$. For it we start from the following vector field,

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \\ \mu_7 \end{pmatrix} + \begin{pmatrix} \mu_3 \\ \mu_4 \\ \mu_5 x + \mu_6 y \end{pmatrix} + \begin{pmatrix} \mu_1 x - y \\ x + \mu_1 y \\ x^2 + y^2 + \mu_2 z \end{pmatrix}. \tag{5.6.48}$$

To eliminate terms in the above quasi-homogeneous vector field, we apply a translations on it, i.e., we consider, $u = x+\alpha$, $v = y+\beta$ and $w = z+\varepsilon x+\delta y$ being α, β, ε and δ the followting,

$$\begin{aligned}
\alpha &= \frac{\mu_1\mu_3+\mu_4}{1+\mu_1^2}.\\
\beta &= \frac{\mu_1\mu_4+\mu_3}{1+\mu_1^2}.\\
\delta &= \frac{-(\mu_6\mu_1^3+\mu_5\mu_1^2-\mu_2\mu_6\mu_1^2-2\mu_1^2\mu_4+\mu_6\mu_1]+2\mu_2\mu_1\mu_4+\mu_5-2\mu_4-2\mu_2\mu_3-\mu_2\mu_6)}{\mu_1^4-2\mu_2\mu_1^3+2\mu_1^2+\mu_2^2\mu_]^2-2\mu_2\mu_1+1+\mu_2^2}.\\
\varepsilon &= -\frac{\mu_5\mu_1^3-\mu_2\mu_5\mu_1^2-\mu_6\mu_1^2-2\mu_1^2\mu_3+\mu_1\mu_5+2\mu_1\mu_2\mu_3-2\mu_3-\mu_6-\mu_2\mu_5+2\mu_2\mu_4}{(1+\mu_1^2)(1+(\mu_1-\mu_2)^2)}.
\end{aligned}$$

This translation transforms system (5.6.48) into,

$$\dot{\mathbf{x}} = \begin{pmatrix} \alpha_1 x - y \\ x+\alpha_1 y \\ x^2+y^2+\alpha_2+\alpha_3 z \end{pmatrix}, \tag{5.6.49}$$

being α_1, α_2 and α_3 the followting,

$$\begin{aligned}
\alpha_1 &= \mu_1.\\
\alpha_2 &= \frac{1}{(1+\mu_1^2)(1+(\mu_1-\mu_2)^2)}(\mu_7+\mu_3^2+\mu_6\mu_3+\mu_6\mu_1^2\mu_3-\mu_6\mu_1\mu_4-\mu_6\mu_1^3\mu_4\\
&\quad -\mu_5\mu_1\mu_3-\mu_5\mu_1^3\mu_3-\mu_5\mu_1^2\mu_4+\mu_1^2\mu_3^2+\mu_4^2+2\mu_7\mu_1^2+\mu_7\mu_1^4-\mu_5\mu_4+\mu_1^2\mu_4^2\\
&\quad +\mu_2\mu_4\mu_6\mu_1^2+\mu_2\mu_4\mu_6+\mu_5\mu_1^2\mu_3\mu_2-2\mu_7\mu_1^3\mu_2+\mu_7\mu_1^2\mu_2^2-2\mu_7\mu_2\mu_1+\mu_2\mu_3\mu_5\\
&\quad -\mu_2^2\mu_3^2-\mu_2^2\mu_4^2+\mu_7\mu_2^2).\\
\alpha_3 &= \mu_2.
\end{aligned}$$

Next, by making a new translation of the form $u = x$, $v = y$ and $w = z+\alpha$, and considering the normal form truncated until second degree, the term in z can be eliminated in (5.6.49). Thus, the quasi-homogeneous principal component is transformed into,

$$\dot{\mathbf{x}} = \begin{pmatrix} \eta_1 x - y \\ x+\eta_1 y \\ x^2+y^2+\eta_2 \end{pmatrix}. \tag{5.6.50}$$

Therefore, in generic coonditions, a versal deformation for the Hopf-zero singularity is given by (5.6.50). This versal deformation is used by Guckenheimer and Holmes (see [60]) for studying an unfolding of the Hopf-zero singularity. Next, we describe this procces.

Considering the following system,

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{pmatrix}\alpha_1 x - y\\ x+\alpha_1 y\\ x^2+y^2+\alpha_2+\alpha_3 z\end{pmatrix}+\begin{pmatrix}a_2^{(0)}zx+a_2^{(1)}(x^2+y^2)x\\ a_2^{(0)}zy+a_2^{(1)}(x^2+y^2)y\\ \hline 0\end{pmatrix}\\ &+\begin{pmatrix}0\\0\\ \hline b_4^{(0)}z^2+b_4^{(1)}(x^2+y^2)z+b_4^{(2)}(x^2+y^2)^2\end{pmatrix},\ \text{with}\quad b_4^{(0)}\neq 0.\end{aligned} \tag{5.6.51}$$

Now, we consider the translation of the form $u=x$, $v=y$ and $w=z+\alpha$ with $\alpha=\frac{\mu_3}{2b_4^{(0)}}$, (at this point it is necessary to suposse that the coefficient $b_4^{(0)}\neq 0$). Thus, system (5.6.51) is transformed into,

$$\begin{aligned}\begin{pmatrix}\dot u\\ \dot v\\ \dot w\end{pmatrix} &= \begin{pmatrix}\eta_1 u - v\\ u+\eta_1 v\\ \eta_3(u^2+v^2)+\eta_2\end{pmatrix}+\begin{pmatrix}a_2^{(0)}wu+a_2^{(1)}(u^2+v^2)u\\ a_2^{(0)}wv+a_2^{(1)}(u^2+v^2)v\\ \hline 0\end{pmatrix}\\ &+\begin{pmatrix}0\\0\\ \hline b_4^{(0)}w^2+b_4^{(1)}(u^2+v^2)w+b_4^{(2)}(u^2+v^2)^2\end{pmatrix},\end{aligned} \tag{5.6.52}$$

being $\eta_1=\frac{2\alpha_1 b_4^{(0)}+a_2^{(0)}\alpha_3)}{2b_4^{(0)}}$, $\eta_2=\frac{2b_4^{(0)}\alpha_2-\alpha_3^2}{2b_4^{(0)}}$ and $\eta_3=1-\frac{\alpha_3 b_4^{(1)}}{4b_4^{(0)}}$.

Now we will apply Proposition 5.6.147 for the calculation again of a versal deformation. We start newly of the system (5.6.48). We will remove terms with degrees -1 and -2 respectively.

- For removing term of degree -1 we use $\mathbf{P}_{-1}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3 x+\alpha_4 y\end{pmatrix}$.

In this case the homological operator is of the form $\mathbf{F}_{-1} - [\mathbf{P}_{-1}, \mathbf{F}_0] = \begin{pmatrix} \mu_1\alpha_1 - \alpha_2 \\ \alpha_1 + \mu_1\alpha_2 \\ (\mu_2\alpha_3 + 2\alpha_1 - \alpha_3\mu_1 - \alpha_4)x + (\alpha_3 + 2\alpha_2 + \mu_2\alpha_4 - \alpha_4\mu_1)y \end{pmatrix}$.

Taking

$$\begin{aligned}
\alpha_1 &= \frac{\mu_4 + \mu_1\mu_3}{1+\mu_1^2}. \\
\alpha_2 &= \frac{-\mu_3 + \mu_1(\mu_4 + \mu_1\mu_3]}{1+\mu_1^2}. \\
\alpha_3 &= \frac{1}{\mu_1^4 - 2\mu_2\mu_1^3 + 2\mu_1^2 + \mu_2^2\mu_1^2 - 2\mu_2\mu_1 + \mu_2^2 + 1}(-2\mu_2\mu_4 - \mu_1^3\mu_5 + 2\mu_3 \\
&\quad +2\mu_1^2\mu_3 - \mu_1\mu_5 - 2\mu_2\mu_1\mu_3 + \mu_6\mu_1^2 + \mu_2\mu_5\mu_1^2 + \mu_2\mu_5 + \mu_6). \\
\alpha_4 &= -\frac{1}{(1+\mu_1^2)(\mu_1^2 - 2\mu_2\mu_1 + \mu_2^2 + 1)}(\mu_6\mu_1^3 - 2\mu_4\mu_1^2 + \mu_5\mu_1^2 - \mu_2\mu_6\mu_1^2 \\
&\quad +2\mu_4\mu_2\mu_1 + \mu_1\mu_6 - 2\mu_4 + \mu_5 - \mu_2\mu_6 - 2\mu_2\mu_3).
\end{aligned}$$

it is possible to eliminate all terms of -1-th degree.

- For removing term of degree -2 we use $\mathbf{P}_{-2} = \begin{pmatrix} 0 \\ 0 \\ \alpha_5 \end{pmatrix}$.

  In this case the homological operator is of the form $\mathbf{F}_{-2} - [\mathbf{P}_{-2}, \mathbf{F}_0] = \begin{pmatrix} 0 \\ 0 \\ \mu_7 - \mu_2\alpha_7 \end{pmatrix}$ and, in this case, it is not posible to remove any term.

In consequence, the versal deformation obtained, after renaming parameters, is given in (5.6.50).

5.6.3 The Lie triangle with change of variables of negative degrees

The process described to obtain the versal deformation given in (5.6.49), considering the first quasi-homogeneous term $\mathbf{F}_r$, can also be described using the Lie triangle with change of variables of negative degrees. The process is similar to which presented in (2.5.27) and (2.5.28). Next, we develop this method.

The triangle that involves negative degrees is represented in the following figure.

$$
\begin{array}{|c|c|c|c|c|}
\hline
\mathbf{V}_{r-s,0} & \mathbf{V}_{r-s,1} & \mathbf{V}_{r-s,2} & \cdots & \mathbf{V}_{r-s,s} \\
\hline
\vdots & \vdots & \vdots & \ddots & \\
\cline{1-4}
\mathbf{V}_{r-2,0} & \mathbf{V}_{r-2,1} & \mathbf{V}_{r-2,2} & & \\
\cline{1-3}
\mathbf{V}_{r-1,0} & \mathbf{V}_{r-1,1} & & & \\
\cline{1-2}
\mathbf{V}_{r,0} & & & & \\
\cline{1-1}
\end{array}
\tag{5.6.53}
$$

Figure 5.1: Negative part of the Lie triangle.

The functions $\{\mathbf{V}_{r-k,l}\}$ are defined, in recursive form, as shown below:

$$
\begin{aligned}
\mathbf{V}_{r-k,0} &= \mathbf{F}_{r-k}, \quad 0 \le k \le -s, \\
\mathbf{V}_{r-k,l} &= \sum_{j=-1}^{-k-1+l} [\mathbf{V}_{r-k-j,l-1}\,, \mathbf{U}_j], \text{ for } l = 1, \cdots, s.
\end{aligned}
\tag{5.6.54}
$$

The expression of the transformed vector field is given by,

$$
\begin{aligned}
\mathbf{G}_{r-k} &= \sum_{l=0}^{k} \frac{1}{l!} \mathbf{V}_{r-k,l} = \mathbf{V}_{r-k,0} + \mathbf{V}_{r-k,1} + \sum_{l=2}^{k} \frac{1}{l!} \mathbf{V}_{r-k,l} \\
&= \mathbf{F}_{r-k} + \mathbf{V}_{r-k,1} + \sum_{l=2}^{k} \frac{1}{l!} \mathbf{V}_{r-k,l}
\end{aligned}
$$

Remark 20. *We would note that, the process described to remove terms with lower degrees than $\mathbf{F}_r$ is a finite process.*

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