

Quantum Phase Transitions in Coupled Systems

An application of Catastrophe Theory

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Universidad
de Huelva

- The Lipkin model
- The two-fluid Lipkin model
 - ▶ Algebraic structure
 - ▶ The Consistent-Q-like Lipkin Hamiltonian
- Energy surface: mean field
- Phase transition and Quantum Phase Transition in a nutshell
- Phase diagram
- With a little help from my ... Catastrophe Theory

Lipkin model

The SU(2) algebra

$$J^+ = t^\dagger s, \quad J^- = s^\dagger t, \quad J^0 = \frac{1}{2}(t^\dagger t - s^\dagger s), \quad J'^0 = J^0 + N/2$$

$$[J^+, J^-] = 2J^0, \quad [J^0, J^\pm] = \pm J^\pm$$

The Hamiltonian

$$H = aJ'^0 + b(J^+ + J^-) + c(J^+ J^-) + d((J^+)^2 + (J^-)^2) \\ + e(J^+ J'^0 + J'^0 J^-) + f(J'^0)^2$$

The Consistent-Q-like Hamiltonian

$$H = xn_t + \frac{x-1}{N} Q^{(y)} Q^{(y)},$$

where $n_t = t^\dagger t$ and $Q^{(y)} = s^\dagger t + t^\dagger s + yt^\dagger t$

Two-fluid Lipkin model

The Hamiltonian

$$H = H_1 + H_2 + H_{12},$$

where

$$H_i = a_i J_i^0 + b_i (J_i^+ + J_i^-) + c_i (J_i^+ J_i^-) + d_i ((J_i^+)^2 + (J_i^-)^2) \\ + e_i (J_i^+ J_i^0 + J_i^0 J_i^-) + f_i (J_i^0)^2$$

$$H_{12} = w_1 (J_1^+ J_2^+ + J_1^- J_2^-) + w_2 (J_1^+ J_2^- + J_1^- J_2^+) + w_3 (J_1^+ J_2^0 + J_1^- J_2^0) \\ + w_4 (J_1^0 J_2^+ + J_1^0 J_2^-) + w_5 J_1^0 J_2^0,$$

being

$$J_i^0 = J_i^z + \frac{N}{2}.$$

Consistent-Q-like two-fluid Lipkin model

$$H = x(n_{t1} + n_{t2}) - \frac{1-x}{N_1 + N_2} Q^{(y_1, y_2)} \cdot Q^{(y_1, y_2)}$$

where

$$\begin{aligned}n_{ti} &= t_i^\dagger t_i \\ Q^{(y_1, y_2)} &= (Q_1^{y_1} + Q_2^{y_2}) \\ Q_i^{y_i} &= s_i^\dagger t_i + t_i^\dagger s_i + y_i (t_i^\dagger t_i)\end{aligned}$$

Mean field, energy surface

Ground state

$$|g\rangle = \frac{1}{\sqrt{N_1! N_2!}} (\Gamma_1^\dagger)^{N_1} (\Gamma_2^\dagger)^{N_2} |0\rangle$$

$$\Gamma_i^\dagger = \frac{1}{\sqrt{1 + \beta_i^2}} (s_i^\dagger + \beta_i t_i^\dagger)$$

Energy

$$\begin{aligned} \frac{E(\beta_1, \beta_2, x, y_1, y_2)}{N_1 + N_2} &= \frac{x}{2} \left(\frac{\beta_1^2}{1 + \beta_1^2} + \frac{\beta_2^2}{1 + \beta_2^2} \right) \\ &- \frac{1-x}{4} \left(\frac{1}{(1 + \beta_1^2)^2} (2\beta_1 + y_1 \beta_1^2)^2 \right. \\ &+ \frac{1}{(1 + \beta_2^2)^2} (2\beta_2 + y_2 \beta_2^2)^2 \\ &+ \left. 2 \frac{1}{(1 + \beta_1^2)} \frac{1}{(1 + \beta_2^2)} (2\beta_1 + y_1 \beta_1^2)(2\beta_2 + y_2 \beta_2^2) \right) \end{aligned}$$

Lipkin model: contour plot

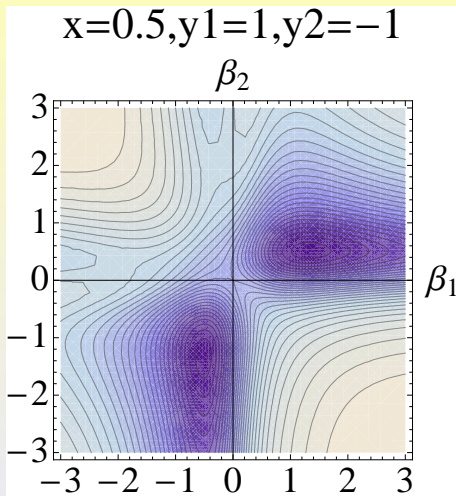


Figure: Contour plot of a selected two-fluid Lipkin Hamiltonian.

Macroscopic/Classical Phase Transitions

Definition of phase and phase transition

- **Phase:** state of matter that is uniform throughout, not only in chemical composition but also in physical properties.
- **Phase Transition:** abrupt change in one or more properties of the system.

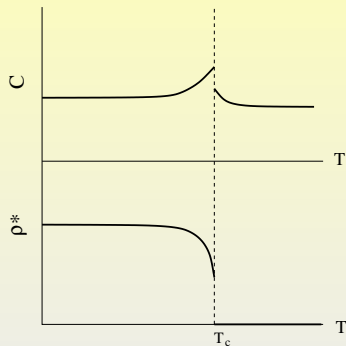
The phase of the system

- Most stable phase of matter is the one with the lowest thermodynamical potential Φ . This is a function of some parameters that are allowed to change ($F(T,V)$, $F(T,B)$; $G(T,p)$, $G(T,M)$).
- Φ is analogous to the potential energy, $V(x)$, of a particle in a one dimensional well. The system looks for the minimum energy going into the bottom of the potential.

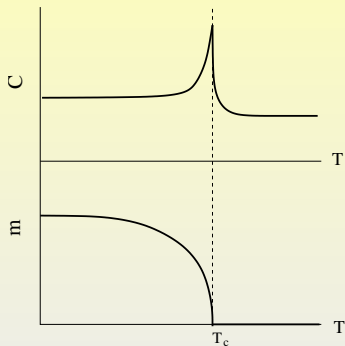
Macroscopic/Classical Phase Transitions

- **Control parameter:** variable that affects the system, it can be changed smoothly and “arbitrarily”.
- **Order parameter:** observable that changes as a function of the control parameter and that defines the “phase” of the system.
- **Ordered and disordered phases** correspond to a value of the order parameter equal and different from zero, respectively.
- **Order of a phase transition:** order of the first derivative of the Gibbs potential with respect to the control parameter that first experiences a discontinuity: first, continuous (second order).

Examples of Macroscopic Phase Transitions

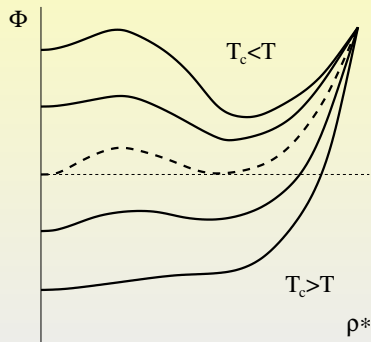


First order phase transition.
Liquid-gas

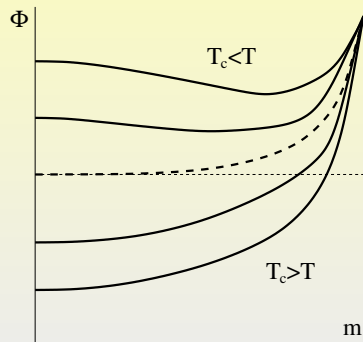


Second order phase transition.
Paramagnetic-ferromagnetic

What is happening at the phase transition point?



First order phase transition



Second order phase transition

Quantum Phase Transitions

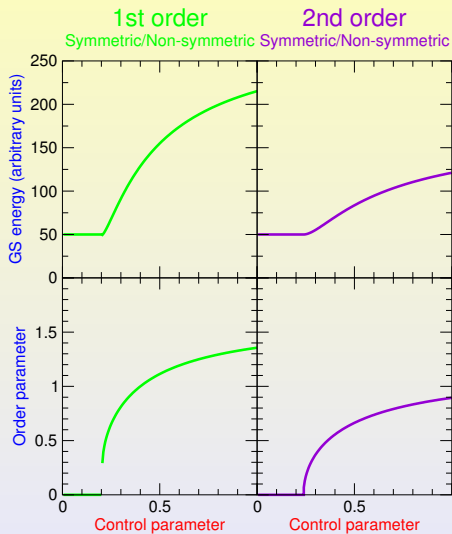
QPT occurs at some critical value, x_c , of the control parameter x that controls an interaction strength in the system's Hamiltonian $H(x)$. **It is implicit a zero temperature.**

$$\hat{H} = x \hat{H}_1 + (1 - x) \hat{H}_2$$

At the critical point:

- The ground state energy is nonanalytic.
- The gap Δ between the first excited state and the ground state vanishes.

The variation of the order parameter



Phase diagram

Phase diagram

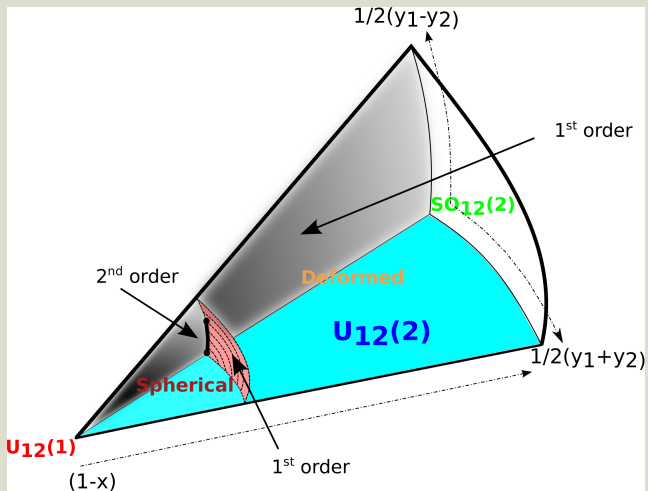


Figure: Phase diagram of the Q-like two-fluid Lipkin model

Numerical results 1

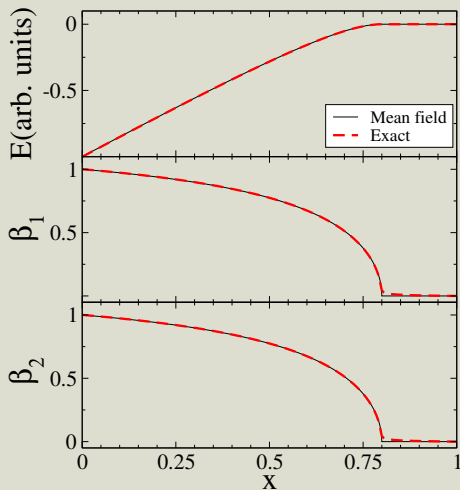


Figure: $y_1 = 0, y_2 = 0$

Numerical results 2

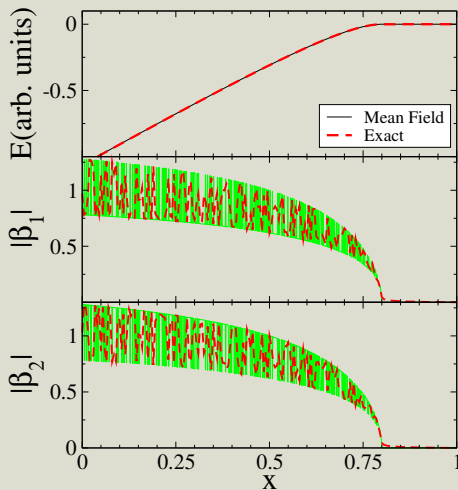


Figure: $y_1 = 0.5$, $y_2 = -0.5$

Numerical results 3

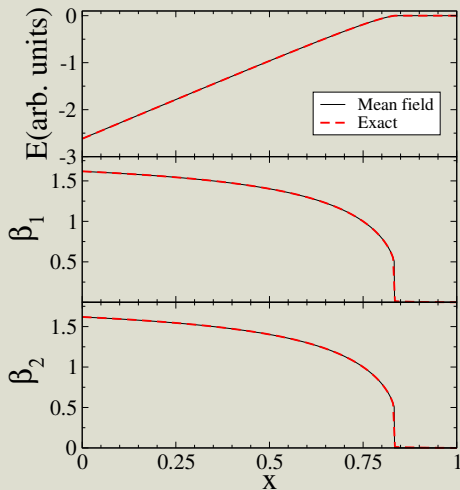


Figure: $y_1 = 1, y_2 = 1$

Numerical results 4

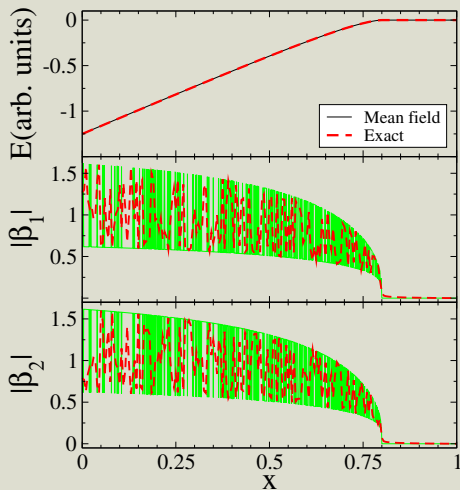


Figure: $y_1 = 1, y_2 = -1$

Numerical results 4b

Numerical results 4b

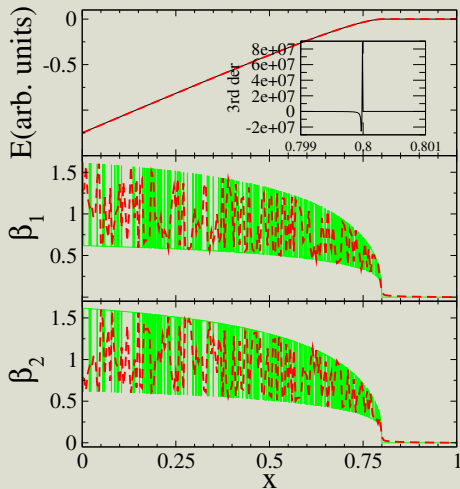


Figure: $y_1 = 1, y_2 = -1$

What is for Catastrophe Theory?

Some notes

- First reference: René Thom, *Stabilité Structurelle et Morphogénèse* (1972).
- Catastrophe theory (CT) is framed in the theory of **singularities for differentiable mappings** and in the **theory of bifurcations**, therefore it deals with singularities of smooth real-valued functions and tries to classify such singularities.
- CT attempts to study how the qualitative nature of the solutions of equations depends on the parameters that appear in the equations (Gilmore 1981).
- CT explains how the state of a system can change suddenly under a smooth change in the control variables.

- Let us assume a system described by a real family of potentials:

$$V(\mathbf{x}, \lambda) \in \mathfrak{R}$$

where $\mathbf{x} \in \mathfrak{R}^n$ are the state (order) variables and $\lambda \in \mathfrak{R}^r$ are the control parameters.

- In this family one can find three types of points:
 - ▶ Regular points: $\nabla V \neq 0$.
 - ▶ Morse points (isolated critical points):
 $\nabla V = 0$ and $|\mathcal{H}_{ij}| \neq 0$.
 - ▶ Non-Morse points (degenerated critical points):
 $\nabla V = 0$ and $|\mathcal{H}_{ij}| = 0$.

- Morse lemma for isolated critical points.

$$V(\mathbf{x}) \rightarrow \mathbf{x}^2$$

- Thom theorem for degenerated critical points.

$$V(\mathbf{x}) \rightarrow g(\mathbf{x}) + \text{unfolding}$$

- Splitting lemma for potential with several variables.

$$V(\mathbf{x}, \mathbf{y}, \mathbf{z}) \rightarrow g(\mathbf{x}) + \text{unfolding} + \mathbf{y}^2 - \mathbf{z}^2$$

The physical way

- Substitution of $V(\mathbf{x}, \lambda)$ by a truncated Taylor expansion $V(\mathbf{x}, \lambda)_{pol}$, being the germ the higher order term (the order of the Taylor expansion is the **determinacy**...).
- Establish the mapping between $V(\mathbf{x}, \lambda)_{pol}$ and a canonical form through a nonlinear change of variables.
- Work out $V(\mathbf{x}, \lambda)_{pol}$ for getting the bifurcation and the Maxwell set.

A local Taylor expansion

- Let us make a Taylor expansion around $\beta_1 = 0$ and $\beta_2 = 0$.
- Moreover we will take as variables the eigenvectors of the Hessian matrix.

$$\mathcal{H} = \begin{pmatrix} \partial^2 E / \partial \beta_1^2 & \partial^2 E / \partial \beta_1 \partial \beta_2 \\ \partial^2 E / \partial \beta_2 \partial \beta_1 & \partial^2 E / \partial \beta_2^2 \end{pmatrix} = \begin{pmatrix} 3x - 2 & 2x - 2 \\ 2x - 2 & 3x - 2 \end{pmatrix}.$$

The two eigenvalues are $5x - 4$ and x , and the corresponding eigenvectors are,

$$\beta_a = \frac{1}{2} (\beta_1 + \beta_2),$$
$$\beta_b = \frac{1}{2} (\beta_1 - \beta_2).$$

A different view of the energy surface

- Let us make a Taylor expansion around $\beta_1 = 0$ and $\beta_2 = 0$.
- Moreover we will take as variables the eigenvectors of the Hessian matrix.

$$\frac{E(x, y, y', \beta_a, \beta_b)}{N} = (5x - 4)\beta_a^2 + 4(x - 1)y\beta_a^3 + (8 - 9x + y^2(x - 1))\beta_a^4 + \Theta(\beta_a^5) + x\beta_b^2 + \Theta(\beta_a\beta_b^2, \beta_b\beta_a^2)$$

with $y = (y_1 + y_2)/2$ and $y' = (y_1 - y_2)/2$.

In order to cancel the higher order ($\beta_a^i\beta_b^j$ with $j > 1$) we have to implement a nonlinear transformation in β_b

$$\tilde{\beta}_b = \beta_b + \sum_{i+j>1} a_{ij}\beta_a^i\beta_b^j$$

β_a becomes the essential variable while β_b the non-essential one.

A different view of the energy surface

For $y' = 0$

$$\begin{aligned} \frac{E(x, y, \beta_a, \beta_b)}{N} &= (5x - 4)\beta_a^2 + 4(x - 1)y\beta_a^3 \\ &+ \left(y^2(x - 1) + 8 - 9x\right)\beta_a^4 + O(\beta_a^5) + x\beta_b^2 \end{aligned}$$

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For $y = 0$

$$\begin{aligned}\frac{E(x, y', \beta_a, \beta_b)}{N} &= (5x - 4)\beta_a^2 + \left(8 - 9x - \frac{16(x - 1)^2 y'^2}{x}\right)\beta_a^4 \\ &+ \frac{1}{2} \left(26x - 24 + \frac{128(x - 1)^3 y'^2 (y'^2 - 6)}{x^2}\right)\beta_a^6 \\ &+ O(\beta_a^7) + x\beta_b^2\end{aligned}$$

A different view of the energy surface

For the general Hamiltonian

$$\begin{aligned}\frac{E(x, y, y', \beta_a, \beta_b)}{N} &= (5x - 4)\beta_a^2 + 4(x - 1)y\beta_a^3 + \left(8 - 9x + y^2(x - 1) + \frac{16y'^2}{x}(2x - x^2 - 1)\right)\beta_a^4 \\ &+ \frac{8(x - 1)y((6y'^2 - 1)x^2 - 14y'^2x + 8y'^2)}{x^2}\beta_a^5 \\ &+ \frac{1}{x^3}((64y'^4 - 384y'^2 - 2y^2(82y'^2 + 1) + 13)x^4 + 2(-96y'^4 + 576y'^2 \\ &+ y^2(372y'^2 + 1) - 6)x^3 + 4y'^2(48(y'^2 - 6) - 313y^2)x^2 \\ &+ 32y'^2(29y^2 - 2y'^2 + 12)x - 256y^2y'^2)\beta_a^6 \\ &+ O(\beta_a^7) + x\beta_b^2\end{aligned}$$

A different view of the energy surface

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The function is **6 – determined**. The relevant elementary catastrophe of this model will be the *butterfly* (A_{+5}). Note that we only have 3 control parameters, though the universal unfolding of the *butterfly* has 4 parameters.

- We have presented the Consistent-Q like double Lipkin Hamiltonian.

Summary and conclusions

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Summary and conclusions

- We have presented the Consistent-Q like double Lipkin Hamiltonian.
- We have analyzed the appearance of phase transitions in this model.
- We used Catastrophe Theory to disentangle the order of the transitions.
- We found that the appropriated catastrophe is the *butterfly* (A_{+5}).

Thank you!

Numerical results 5

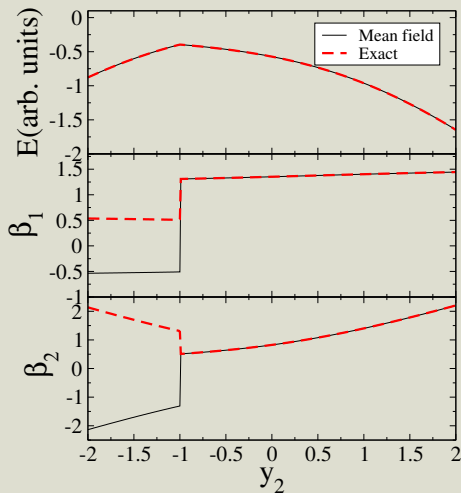


Figure: $x = 0.5, y_1 = 1$

The mathematical way (*Margalef-Roig, et al*)

- Definition of $h(\mathbf{x}, \lambda) = V(\mathbf{x} + \mathbf{x}^0, \lambda + \lambda^0) - V(\mathbf{x}^0, \lambda^0)$, where $(\mathbf{x}^0, \lambda^0)$ correspond to a degenerated critical point.
- Definition of the **germ**: $g(\mathbf{x}) = h(\mathbf{x}, \mathbf{0})$.
- Calculation of the **determinacy and the codimension** of $g(\mathbf{x})$ through the k-jet of $g(\mathbf{x})$ (truncated Taylor expansion with k term).
- Study the **k-transversality** of $g(\mathbf{x})$ in order to establish the isomorphism between $h(\mathbf{x}, \lambda)$ and a canonical unfolding of $g(\mathbf{x})$.
- Note that it is only possible to prove the existence of an isomorphism but this DOES NOT provides the necessary change of coordinates.

%

QPT: Lipkin model

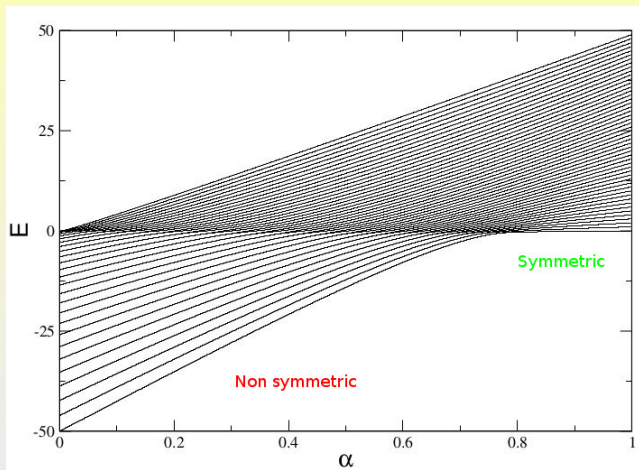


Figure: Energy spectrum (per boson) for a Lipkin model with $N = 50$, $\alpha = x$ and $y = 0$.

The coordinates

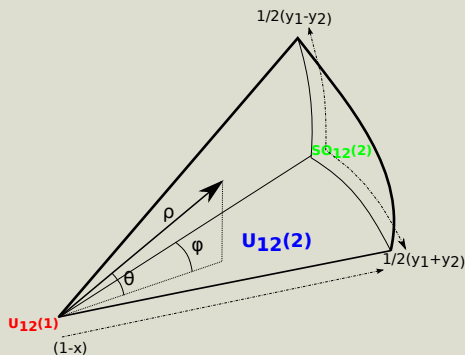


Figure: Coordinates of the Q-like two-fluid Lipkin space.

$$H = x(n_{t1} + n_{t2}) - \frac{1-x}{N_1 + N_2} Q^{(y_1, y_2)} \cdot Q^{(y_1, y_2)}$$