

# The phase diagram of the (extended) Agassi model

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Quantum Phase Transitions  
Barcelona, 5-6 April 2018

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# Why the Agassi model?

- It is a solvable many-body model that allows to **mimic the main characteristics of the pairing-plus-quadrupole model**.
- It can be exactly solved even in the case of large systems.
- Nowadays, it is used to **benchmark many-body approximations** because of its great flexibility and simplicity to be solved for large systems.
- The model (and in particular its extension) owns a very rich phase diagram.
- The model is, somehow, an extension of the **two-level Lipkin-Meshkov-Glick model that incorporates pairing interaction**.

## Dan Agassi

“Validity of the BCS and RPA approximations in the pairing-plus-monopole solvable model”, Dan Agassi, Nuclear Physics A **116**, 49 (1968).

## The original Hamiltonian

$$H = \frac{1}{2}\epsilon \sum_{m\sigma} \sigma a_{m\sigma}^\dagger a_{m\sigma} + \frac{1}{2}V \sum_{mm'\sigma} a_{m\sigma}^\dagger a_{m'\sigma}^\dagger a_{m'-\sigma} a_{m-\sigma} - \frac{1}{4}G \sum_{mm'\sigma\sigma'} a_{m\sigma}^\dagger a_{-m\sigma}^\dagger a_{-m'-\sigma'} a_{m'\sigma'}$$

$\sigma = +1, -1$  and  $m = -j, \dots, -2, -1, 1, 2, \dots, j$ . Degeneracy  $\Omega = 2j$

## The O(5) as spectrum generator algebra

$$J^+ = \sum_{m=-j}^j c_{1,m}^\dagger c_{-1,m} = (J^-)^\dagger, \quad J^0 = \frac{1}{2} \sum_{m=-j}^j (c_{1,m}^\dagger c_{1,m} - c_{-1,m}^\dagger c_{-1,m}),$$

$$A_1^\dagger = \sum_{m=1}^j c_{1,m}^\dagger c_{1,-m}^\dagger, \quad A_{-1}^\dagger = \sum_{m=1}^j c_{-1,m}^\dagger c_{-1,-m}^\dagger, \quad A_0^\dagger = \sum_{m=1}^j (c_{-1,m}^\dagger c_{1,-m}^\dagger - c_{-1,-m}^\dagger c_{1,m}^\dagger)$$

$$A_1 = \sum_{m=1}^j c_{1,-m} c_{1,m}, \quad A_{-1} = \sum_{m=1}^j c_{-1,-m} c_{-1,m}, \quad A_0 = \sum_{m=1}^j (c_{1,-m} c_{-1,m} - c_{1,m} c_{-1,-m}),$$

$$N_\sigma = \sum_{m=-j}^j c_{\sigma,m}^\dagger c_{\sigma,m}, \quad N = N_1 + N_{-1}.$$

# A pictorial view

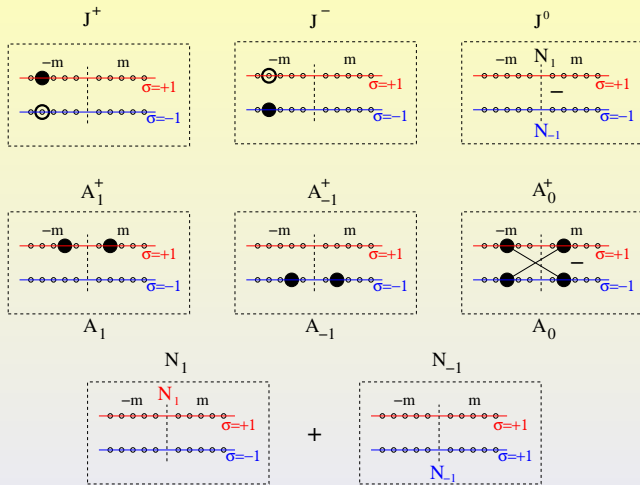


Figure: A pictorial view of the ten  $O(5)$  generators in the Agassi model Hilbert space

# The Hamiltonian

$$H = \varepsilon J^0 - g \sum_{\sigma\sigma'} A_{\sigma}^{\dagger} A_{\sigma'} - \frac{V}{2} \left[ (J^+)^2 + (J^-)^2 \right] - 2h A_0^{\dagger} A_0$$

For convenience

$$V = \frac{\varepsilon\chi}{2j-1}, \quad g = \frac{\varepsilon\Sigma}{2j-1}, \quad h = \frac{\varepsilon\Lambda}{2j-1}$$

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For convenience

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$$H = \varepsilon \left[ J^0 - \frac{\Sigma}{2j-1} \sum_{\sigma\sigma'} A_{\sigma}^{\dagger} A_{\sigma'} - \frac{\chi}{2(2j-1)} \left[ (J^+)^2 + (J^-)^2 \right] - 2 \frac{\Lambda}{2j-1} A_0^{\dagger} A_0 \right].$$

# Hartree-Fock-Bogoliubov transformation (I)

## The transformation

Hartree-Fock transformation

$$a_{\eta,m}^{\dagger} = \sum_{\sigma} D_{\eta\sigma} c_{\sigma,m}^{\dagger}$$

Bogoliubov transformation

$$\begin{aligned}\alpha_{\eta,m}^{\dagger} &= u_{\eta} a_{\eta,m}^{\dagger} - \text{sig}(m) v_{\eta} a_{\eta,-m}, \\ \alpha_{\eta,-m}^{\dagger} &= u_{\eta} a_{\eta,-m}^{\dagger} + \text{sig}(m) v_{\eta} a_{\eta,m},\end{aligned}$$

with the constraint (at half filling)

$$u_{-1}^2 = v_1^2, \quad u_1^2 = v_{-1}^2, \quad v_{\eta}^2 + u_{\eta}^2 = 1$$

(E.D. Davis and W.D. Weiss, J. Phys. G **12**, 805 (1986).)

# Hartree-Fock-Bogoliubov transformation (II)

## A convenient parametrization

$$D_{1,1} = D_{-1,-1} = \cos \frac{\varphi}{2}, \quad D_{-1,1} = -D_{1,-1} = \sin \frac{\varphi}{2}$$

$$v_1 = \sin \frac{\beta}{2}, \quad v_{-1} = \cos \frac{\beta}{2}.$$

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## Bogoliubov phase selection

Phase selection	Surface
$u_{-1} = v_1 = \sin \frac{\beta}{2}$ $u_1 = v_{-1} = \cos \frac{\beta}{2}$	A
$u_{-1} = v_1 = \sin \frac{\beta}{2}$ $u_1 = -v_{-1} = -\cos \frac{\beta}{2}$	B

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## The energy

$$E(\varphi, \beta) = \frac{\langle \text{HFB}(\varphi, \beta) | H | \text{HFB}(\varphi, \beta) \rangle}{\langle \text{HFB}(\varphi, \beta) | \text{HFB}(\varphi, \beta) \rangle}.$$

# The energy surfaces

$\varphi$  Hartree-Fock variational parameter.  $\beta$  Bogoliubov variational parameter

## The energy surface A

$$E_A = -\varepsilon j \cos \varphi \cos \beta - g j^2 \sin^2 \beta - V j^2 \sin^2 \varphi \cos^2 \beta$$

$$\frac{E_A}{j\varepsilon} = -\cos \varphi \cos \beta - \frac{\Sigma}{2} \sin^2 \beta - \frac{\chi}{2} \sin^2 \varphi \cos^2 \beta$$

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## The energy surface B

$$E_B = -\varepsilon j \cos \varphi \cos \beta - 2h j^2 \sin^2 \beta \sin^2 \varphi - V j^2 \sin^2 \varphi \cos^2 \beta$$

$$\frac{E_B}{j\varepsilon} = -\cos \varphi \cos \beta - \Lambda \sin^2 \beta \sin^2 \varphi - \frac{\chi}{2} \sin^2 \varphi \cos^2 \beta$$

# The phases of the system

- The spherical phase:  $\varphi = 0$  and  $\beta = 0$ .
- The Hartree-Fock deformed phase:  $\varphi \neq 0$  and  $\beta = 0$ .
- The BCS deformed phase:  $\varphi = 0$  and  $\beta \neq 0$ .
- The Hartree-Fock plus BCS deformed phase:  $\varphi = \frac{\pi}{2}$  and  $\beta = \frac{\pi}{2}$ .

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In the original formulation of the Agassi model only the three first phases were present, but in the extended version of the model the four basis can be found and, moreover, **there is coexistence of some of the phases**.

# Critical points of energy surface A

I-A:  $\varphi = \beta = 0$  ( $E_A/(j\varepsilon) = -1$ ). Regardless the values of  $\Sigma$  and  $\chi$ .

- $\chi < 1$  and  $\Sigma < 1$ : it's a minimum
- $\chi > 1$  and  $\Sigma > 1$ : it's a Maximum
- $\chi > 1$  and  $\Sigma < 1$  or  $\chi < 1$  and  $\Sigma > 1$ : it's a saddle point.

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II-A:  $|\varphi| = |\beta| = \frac{\pi}{2}$  ( $E_A = -\frac{\Sigma}{2}$ )

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III-A:  $\beta = 0$ ,  $\cos \varphi = \frac{1}{\chi}$  ( $E_A/(j\varepsilon) = -\frac{\chi^2+1}{2\chi}$ ). Valid for  $\chi > 1$

- $\chi > \Sigma$ : it's a minimum.
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IV-A:  $\varphi = 0$ ,  $\cos \beta = \frac{1}{\Sigma}$  ( $E_A/(j\varepsilon) = -\frac{\Sigma^2+1}{2\Sigma}$ ) for  $\Sigma > 1$

- $\chi < \Sigma$ : it's a minimum.
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IV-A:  $\varphi = 0$ ,  $\cos \beta = \frac{1}{\Sigma}$  ( $E_A/(j\varepsilon) = -\frac{\Sigma^2+1}{2\Sigma}$ ) for  $\Sigma > 1$

- $\chi < \Sigma$ : it's a minimum.
- $\chi > \Sigma$ : it's a saddle point.

V-A: Particular case  $\chi = \Sigma$ .  $\cos \beta \cos \varphi = \frac{1}{\chi}$

This solution corresponds to a kind of closed valley in the  $\varphi - \beta$  plane.

# Critical points of energy surface B

I-B:  $\varphi = \beta = 0$  ( $E_B/(j\varepsilon) = -1$ ). Regardless the values of  $\Lambda$  and  $\chi$ .

- $\chi < 1$ : it's a minimum
- $\chi > 1$ : it's a saddle point.

# Critical points of energy surface B

I-B:  $\varphi = \beta = 0$  ( $E_B/(j\varepsilon) = -1$ ). Regardless the values of  $\Lambda$  and  $\chi$ .

- $\chi < 1$ : it's a minimum
- $\chi > 1$ : it's a saddle point.

II-B:  $\beta = 0, \cos \varphi = \frac{1}{\chi}$  ( $E_B/(j\varepsilon) = -\frac{\chi^2+1}{2\chi}$ ). Valid for  $\chi > 1$

- $\Lambda > \frac{1}{2} \frac{\chi^3}{\chi^2-1}$ : it's a saddle point.
- $\Lambda < \frac{1}{2} \frac{\chi^3}{\chi^2-1}$ : it's a minimum.

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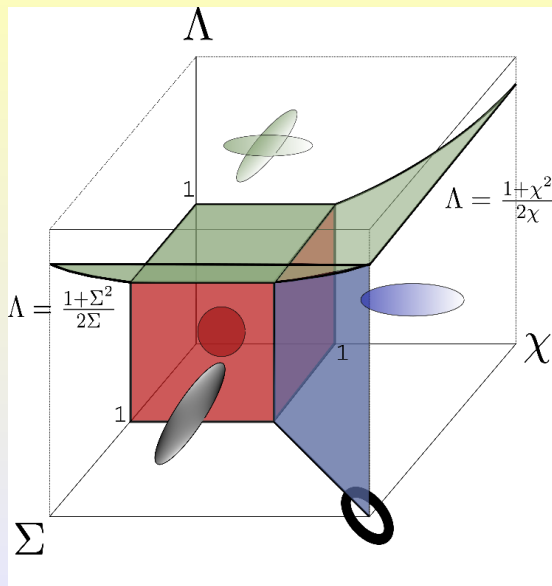
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- $\Lambda > \frac{1}{2} \frac{\chi^3}{\chi^2-1}$ : it's a saddle point.
- $\Lambda < \frac{1}{2} \frac{\chi^3}{\chi^2-1}$ : it's a minimum.

III-B:  $|\varphi| = |\beta| = \frac{\pi}{2}$  ( $E_B = -\Lambda$ )

- $\Lambda > \frac{1}{4}(\chi + \sqrt{4 + \chi^2})$ : it's a minimum.
- $\frac{1}{4}(\chi + \sqrt{4 + \chi^2}) > \Lambda > \frac{1}{4}(\chi - \sqrt{4 + \chi^2})$ : it's a saddle point.
- $\Lambda < \frac{1}{4}(\chi - \sqrt{4 + \chi^2}) < 0$ : it's a Maximum.

# The phase diagram (I)



## The phase diagram (II)

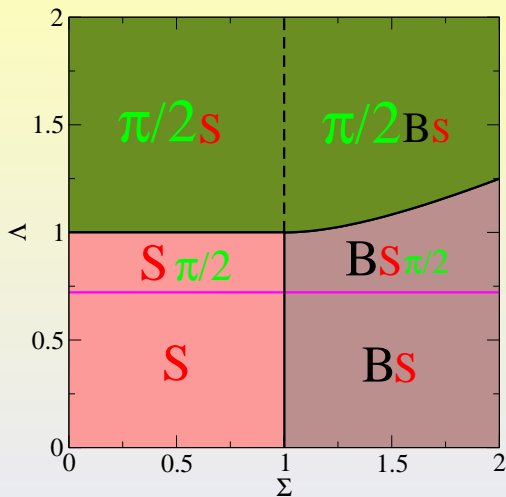


Figure: Phase diagram for the plane  $\chi = 0.75$

# The phase diagram (III)

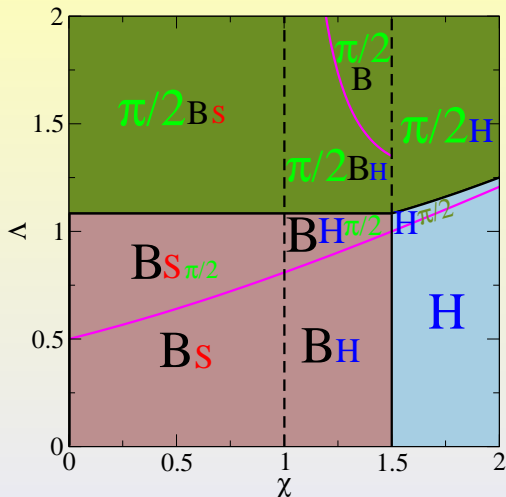


Figure: Phase diagram for the plane  $\Sigma = 1.5$

# The phase diagram (IV)

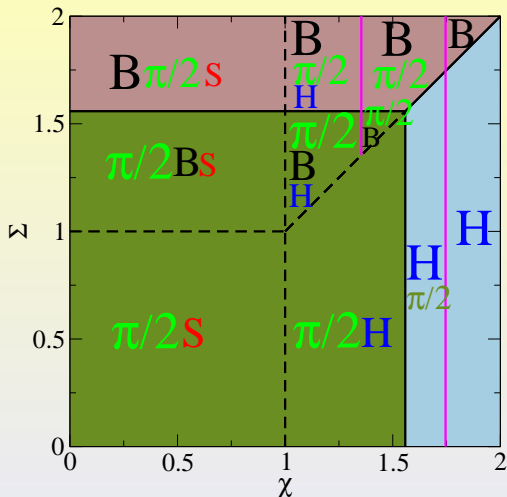
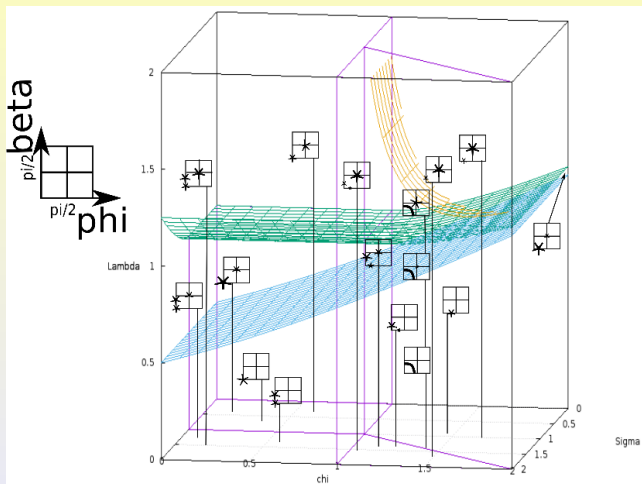


Figure: Phase diagram for the plane  $\Lambda = 1.1$

# The phase diagram (V)



## Competition of phases

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- **Spherical** ( $\varphi = 0, \beta = 0$ ), **HF-BCS deformed minimum** ( $\varphi = \pi/2, \beta = \pi/2$ ) and **BCS deformed** one ( $\varphi = 0, \beta = \arccos(1/\Sigma)$ ).
- **HF-BCS deformed minimum** ( $\varphi = \pi/2, \beta = \pi/2$ ), **HF deformed minimum** in ( $\varphi = \arccos(1/\chi), \beta = 0$ ) and **BCS deformed** one in ( $\varphi = 0, \beta = \arccos(1/\Sigma)$ ).
- **Closed valley minimum** (for  $\Sigma = \chi$ ) and the combined **HF-BCS deformed minimum** ( $\varphi = \pi/2, \beta = \pi/2$ ).

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- **HF-BCS deformed minimum** ( $\varphi = \pi/2, \beta = \pi/2$ ), **HF deformed minimum** in ( $\varphi = \arccos(1/\chi), \beta = 0$ ) and **BCS deformed** one in ( $\varphi = 0, \beta = \arccos(1/\Sigma)$ ).
- **Closed valley minimum** (for  $\Sigma = \chi$ ) and the combined **HF-BCS deformed minimum** ( $\varphi = \pi/2, \beta = \pi/2$ ).
- **HF-BCS deformed minimum** ( $\varphi = \pi/2, \beta = \pi/2$ ), **HF deformed minimum** in ( $\varphi = \arccos(1/\chi), \beta = 0$ ), **BCS deformed minimum** and the **closed valley minimum** along the line  $\Lambda = \frac{1+\chi^2}{2\chi}$  with  $\chi = \Sigma$ . All the minima are degenerated.
- **Spherical** ( $\varphi = 0, \beta = 0$ ), **HF-BCS deformed minimum** ( $\varphi = \pi/2, \beta = \pi/2$ ), **BCS deformed** one ( $\varphi = 0, \beta = \arccos(1/\Sigma)$ ), **HF deformed minimum** in ( $\varphi = \arccos(1/\chi), \beta = 0$ ), and the **closed valley minimum** at the point  $\chi = \Sigma = \Lambda = 1$ . All the minima are degenerated.

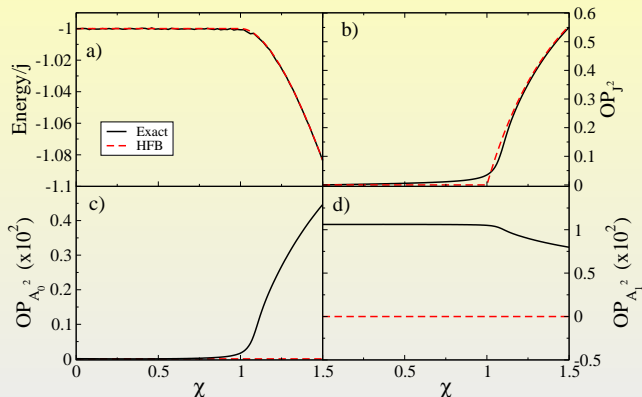
## Order parameters in the laboratory frame

$$OP_{J^2} = \frac{\langle J_+^2 \rangle + \langle J_-^2 \rangle}{2j^2} \rightarrow \sin^2 \varphi \cos^2 \beta (\sin^2 \varphi \cos^2 \beta)$$

$$OP_{A_0^2} = \frac{\langle A_0^+ A_0 \rangle}{j^2} \rightarrow 0 (\sin^2 \beta \sin^2 \varphi)$$

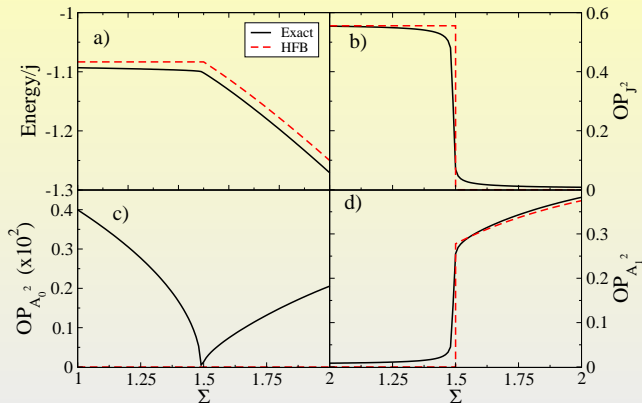
$$OP_{A_1^2} = \frac{\langle A_1^+ A_1 \rangle + \langle A_{-1}^+ A_{-1} \rangle}{2j^2} \rightarrow \frac{1}{2} \sin^2 \beta (0)$$

# Numerical calculations



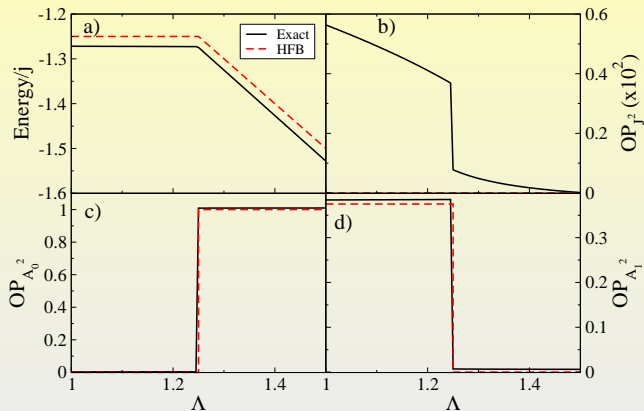
**Figure:** Comparison of HFB and exact results.  $j = 100$  and Hamiltonian parameters  $\Sigma = 0.5$ ,  $\Lambda = 0$ .

# Numerical calculations



**Figure:** Comparison of HFB and exact results.  $j = 100$  and Hamiltonian parameters  $\chi = 1.5$ ,  $\Lambda = 0.5$ , as a function of  $\Sigma$ .

# Numerical calculations



**Figure:** Comparison of HFB and exact results.  $j = 100$  and Hamiltonian parameters  $\chi = 1.5$ ,  $\Sigma = 2$ .

- We have presented an extended version of the Agassi model.
- We have obtained its phase diagram.
- The phase diagram of the present extended Agassi model shows a rich variety of phases.
- Phase coexistence is present in extended areas of the parameter space.
- The existence of coexisting phases is expected to have a strong influence on Excited-state Quantum Phase Transitions.

Thank you for your attention