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Finite time preventive maintenance optimization by using a Semi-Markov process with a degraded state. A case study for diesel engines in mining

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ABSTRACT

During the design of maintenance plans for an asset, the frequency calculation of preventive interventions is a maintenance engineer's essential task. This document presents a tool (mathematical formula) to calculate the preventive maintenance interval. The tool considers the maintenance costs and the income generated for the company by its use. Managers can benefit from its simple application to gain efficiency. Typically, business projects are executed over a period. The tool calculates the optimal interval for any given project duration. It also allows the analysis of the size of the preventive interval when the income received changes. This change may be due to asset impairment or changes in the market price of the business product. For the development of the tool, a semi-Markovian model is designed with four states representing the operation and maintenance of the asset. The asset evolves by transiting between states and accumulating costs and income in the form of returns. From the model, a system of difference equations is developed whose variable is the expected accumulated return at any transition, and the z-transform is used to solve it. Deriving its mathematical expression, the formula for the preventive interval that maximizes the expected accumulated return is obtained. Both formulae are used in a case study. Taking advantage of the degraded state, it is shown that higher income increases the interval size while lower-income decreases it. The limits of these variations are also established. Simulation and numerical methods are used to validate analytic results.

1. Introduction

Predetermined preventive maintenance (PM) is a type of maintenance activity in which the tasks are performed according to prescribed criteria or at predetermined preventive intervals. This kind of maintenance aims to assess the risk of failure occurrence, prevent a failure, or minimize the consequences of the failure (Nasrfard et al., 2022). Performing predetermined preventive maintenance tasks on a physical asset requires a comprehensive and detailed evaluation to set up its optimal interval, especially on critical assets. However, this investigation is usually not performed, and the asset owner applies the manufacturer's recommended maintenance plan. The asset has been designed for the conditions set by the manufacturer, but the owner will use it in a particular operational context to meet the needs for which it was acquired.

In the interest of improvement, the predetermined preventive interval, τ, must be adapted to these conditions. These conditions are affected by several factors. Firstly, the costs of maintenance interventions, both preventive and corrective. Also, the income generated by the operation of the asset is another factor to consider in the calculation of the interval (Zhong et al., 2019). When degradation due to use appears, the time until failure is reduced, which affects the initially expected interval. On the other hand, industrial projects have a limited duration. Many assets do not survive the project and are replaced by others. However, many others do survive the completion of the project. Some of these assets were installed at the beginning of the project, and others during the project. In these cases, when the completion of the project is a known date (finite time horizon), the preventive interval must be adapted to this date. When dealing with critical assets,

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condition-based maintenance is recommended. However, sometimes it is impossible to obtain a signal that announces the failure, or this is so close to the failure that it does not allow a reaction. In these cases, it is necessary to use the preventive interval. Due to the criticality of the asset, its optimal determination is essential. Finally, the professional who designs and applies maintenance knows that these factors influence the interval but rarely analyzes the case nor seeks optimization. Usually, it is because the available tools are complex, difficult to apply, and require hiring consultants. This paper presents a possible solution to this problem and explains it based on a real case study of a diesel engine in a mine.

Our main aim is to provide the maintenance manager with a tool to find the size of the optimal interval τ_o when degradation exists. From there, it is shown how this interval is affected when the operating income of the physical asset is modified. We could think of a mining operation where the price of metal varies monthly, up or down. Another scenario might be a manufacturing operation, where the variation in demand affects the income from the product sale, and finally, the more general case, where asset degradation affects the quality or quantity of produced items. In all these cases, the operating income from the asset changes over time or, after degradation.

In order to develop the tool, the operation and maintenance process of the asset has been represented with a mathematical model consisting of four states: operational, corrective, preventive, and degraded operational state. The asset can be in any of these states at any given time. From the beginning, the model evolves in time, according to a semi-Markovian process, with an embedded Markov chain (Lyubchenko et al., 2018). The transitions between states follow the law of probabilities as exposed by the Markov chain. However, the process is semi-Markovian because each state's sojourn time is different and not subject to exponential functions. At each transition, costs or income are accumulated and stored as returns (negative or positive). The income of the degraded state differs from that of the operational state. The goal is to find the size of the preventive interval that maximizes the expected accumulated return for each transition. During the modeling process, the transition probabilities, the sojourn times, and the returns of each state have been defined. This information has produced a mathematical expression for the expected accumulated return in the successive transitions. To that end, it was necessary to solve a difference equations system. The z-transform of the first component (that which starts from the operating state) was calculated, its value was found, and the inverse z-transform was calculated. Two types of solutions had to be chosen, one for real roots, and another for complex roots.

Subsequently, the variation of the expected accumulated return with respect to the preventive interval has been analyzed, reaching the mathematical expression of the optimal preventive interval τ_o that maximizes the expected accumulated return. As the process evolves, finding the optimal preventive interval at each transition is possible. This brings the model closer to reality, as it calculates the optimum for any point in the asset's operating time horizon. A case study on a critical asset has been presented to explain the application of this tool. Diesel mining trucks running in a mine in southern Spain provide a certain income. After an event, income is reduced, as the performance of its engine is affected. By analyzing the case and using the tool, the influence of the variation in income and the chronological timing of the event on the size of the optimal preventive interval was observed.

This article differs from previous literature in that it provides the maintenance engineer with a simple method for calculating the optimal preventive interval, considering costs and income, for situations where degradation appears, and for any moment in the time horizon of the business project (for example, the active life interval of a mining operation). When summarized in a mathematical formula, its implementation is easier and can be used for inclusion in the digital twin models of maintenance processes. Therefore, in regard to the main aim of this paper, the following research topics can be addressed.

- how a method for calculating the preventive interval can be designed when degradation occurs so that it can be easily applied by maintenance professionals.
- how income during the degraded state affects the calculation of the preventive interval.
- how the limitation of equipment use until the end of the project affects the preventive interval.

The rest of the document is organized as follows. Section 2 summarizes the literature on the issues addressed in the article. Section 3 explains the method followed in the article. Section 3.1 presents a case study and the data used to resolve it. In Section 3.2, the four-state semi-Markovian model is designed. First, the system of four difference equations for the expected accumulated return is formulated. Then, the sojourn times and transition probabilities between states and the returns of each state are calculated. Section 4 sets out the theory and the procedure for calculating the mathematical expressions. Section 4.1 calculates the expected accumulated return for the two versions, real and complex roots. Section 4.2 presents the process of optimizing the expected accumulated return and calculating the mathematical expression for the optimal preventive interval. Section 5 presents and analyzes the obtained results. Section 6 discusses the results and compares them with those obtained with the formula for the preventive interval of the three-state model without degradation and with a stochastic continuous-time simulation model (Monte Carlo simulation). Finally, Section 7 presents the conclusions of the work.

2. Literature review

The literature on maintenance policy optimization models is large and diverse. The compilation by de Jonge and Scarf (2020) is among the best. In asset maintenance, many situations occur that lead to the development of different types of tasks. For this reason, many scenarios are modeled, and various techniques are applied. One of the most widely used is predetermined preventive maintenance. The model developed in this paper is designed to obtain the optimal preventive interval. In the literature, we find many papers where this value is found in different scenarios and techniques.

Some authors look for preventive intervals using methods based on the theory of reliability and risk. Sembiring et al. (2018) consider methods used in reliability engineering and maintenance value stream mapping to calculate the preventive interval. Wang et al. (2017) propose a mathematical model that measures the reliability of a converter considering periodic preventive maintenance. From the reliability function, they determine the value of the maintenance interval and some reliability indices. Dong et al. (2019) search for the optimal preventive interval based on expected maintenance costs. They use the reliability function, which is derived analytically in terms of cumulative shock damage theory and the Laplace transform. Han et al. (2019) propose an integrated methodology to determine the maintenance interval of a specific set of safety barriers. The study addresses the trade-off between increasing and reducing the risk associated with maintenance and optimizes the maintenance cost allocation.

Other authors use specific models to arrive at their value. (Coria et al., 2015) propose a mathematical optimization method for preventive maintenance policy with minimum repair in case of failure, periodic maintenance, and replacement for systems with historical failure time data, affected by a current preventive maintenance policy. (Zhao et al., 2021) model replacement policies according to the operating unit's health status and repair history and determine replacement times by analytically minimizing cost rates. (Zhong et al., 2019) formulate a programming model with random, non-linear, multi-objective, and cost-defined constraints to obtain satisfactory schedules for wind turbine maintenance. (Liu et al., 2020) propose an integrated model of buffer stock and imperfective preventive maintenance for a production system. They consider a repairable machine subject to random failures for a production system considering buffer stock. Alvarez-Alvarado & Jayaweera (2020) propose an innovative Smart Maintenance (SM) model that provides an effective maintenance plan. The model is advanced through three main vias: Markov chains to describe the component's reliability, fuzzy logic to determine the component's operational risk, and maintenance exertion degree to define the impact of maintenance on the component's failure rate.

Other authors use more general models and techniques. Hu et al. (2020) use a Markov model with preventive replacements and imperfect repairs. They optimize the preventive interval by minimizing the long-term average cost. Semaan and Yehia (2019) develop a periodic scheduled preventive maintenance program for a helicopter using a cyclic operation network and Monte Carlo simulation techniques. Zhao et al. (2020) compare models where the component is replaced preventively and periodically versus the first and last occurrence model to find optimal periodic replacement times. Ruschel et al. (2017) present a detailed review of articles on decision-making, one area of which deals with preventive maintenance intervals.

In the paper, a semi-Markovian model has been developed to find the value of the optimal preventive interval, through a mathematical formula that is valid for any temporal scenario. Other authors also use this technique because it is well adapted to the execution characteristics of maintenance tasks. Grabski (2014) explains how to build semi-Markovian models and discusses the different parameters and reliability characteristics that can be obtained from these models. He defines the properties and theorems of the Semi-Markov Process theory. Lyubchenko et al. (2018) present an approach to evaluate recommended preventive maintenance intervals in radio devices. They use Markov process theory to mathematically describe the sequence of transitions between states and apply the semi-Markovian process model to the random process of sojourn times in states. Wang and Miao (2021) formulate an optimal preventive maintenance optimization model under a semi-Markov model for a balanced system, where each unit is subject to degradation failure and sojourn times follow an Erlang distribution.

Other authors combine semi-Markovian models with other techniques and theories. Kumar and Varghese (2018) use non-exponential failure and repair time distributions, which forces them to model from a semi-Markovian approach to assessing system availability. They obtain the preventive interval that optimizes availability using the golden section search technique. Wu et al. (2021) provide a method for solving continuous-time semi-Markovian processes using algorithms of the discrete-time case. This method is applied to a reliability problem of a system subject to sequential cyber-attacks. Two cases in which the sojourn times follow exponential and Weibull distributions are considered and calculated. Kumar et al. (2021) propose a semi-Markovian approach to analyze the degradation of complex mechanical systems constructing operating states. Wu et al. (2019) propose a competitive risk model with a constraint on transition times for multistate semi-Markovian repairable systems. The aggregate stochastic process theory is employed to obtain the formulae for the competitive risk probabilities, survival time distributions, and availability. Nasrfard et al. (2022) propose a probabilistic approach by considering some correlations and uncertainties to find the optimal inspection rates. They utilize a semi-Markov chain based on Monte Carlo simulations employing 95 percentiles of the total cost to determine the optimal inspection rates. Wang et al. (2019) consider a new optimal maintenance policy based on the condition monitoring and age information for a two-unit repairable system. The maintenance problem is formulated and solved in the semi-Markov decision process framework. A formula for the average maintenance cost is derived, and the optimal thresholds for preventive and opportunistic maintenance of the two units minimizing the long-run expected average cost are determined. Few papers like this one provide a formula for calculating the preventive interval when degradation occurs. It is the first gap that this article aims to fill.

Other authors use Markov chains to solve maintenance problems. Farahani et al. (2019) model a production system in the form of a continuous-time Markov chain. This model determines the optimal preventive maintenance interval by reducing the costs per unit of time without discarding the duration of corrective and preventive maintenance. Farhadi et al. (2022) use Markov chains to determine the optimal number of spare parts, their supplier, and the appropriate quality, leading to cost minimization. Hu et al. (2021) focus their study on a case where the evolution processes of the system degradation and the environmental conditions are governed by Markov processes, for which the transition rate matrix of degradation states varies with the environmental condition. Gu et al. (2020) built a system based on discrete-time Markov chain models to evaluate the system performance under the control limit policy in manufacturing systems consisting of multistate machines and intermediate buffers. They perform numerical analysis to demonstrate the impact of parameters such as maintenance duration. Papadopoulos et al. (2019) give models and methods for the numerical solutions of large Markov chains. In addition, the software tools are summarized.

Our model includes and analyzes the behavior of the preventive interval when the income per unit of asset operating time after degradation is included in its calculation. This income received by the owner is not considered in most optimization studies. To our knowledge, there are very few cases in which this data is included in the analysis, and this paper aims to bridge that second gap. Zhu et al. (2021) investigate a preventive maintenance optimization model based on a three-stage failure process for a single-component system. The objective is to maximize profits, but unlike conventional optimization models, it uses a revenue function to correlate profits with availability and cost. Mizutani et al. (2021) use various reliability engineering techniques and tools to choose the best strategies for systems with replacements, including periodic replacements. The developed method finds the optimal preventive interval. To do so, they start from the survival function, and the values of income and costs during operation and maintenance interventions. This research is a continuation of the articles published using the income obtained from the use of assets (Sánchez Herguedas et al., 2022; Sánchez-Herguedas et al., 2021; Sánchez-Herguedas, Mena-Nieto, Rodrigo-Muñoz, et al., 2022; Sánchez-Herguedas, Crespo-Márquez, et al., 2022; Sánchez-Herguedas, Mena-Nieto, & Rodrigo-Muñoz, 2022) for the calculation of the optimal preventive interval.

Degradation data has been widely used to predict remaining useful life of a system. Most of the previous papers use a pre-specified model to capture the degradation process and focus on the degradation process without constant shocks or shock effects (Kong et al., 2021). Nasrfard et al. (2022) propose a probabilistic approach by considering the correlations and uncertainties between repair and maintenance costs and duration to find the optimal inspection rates. However, a degraded state of operation is included in this paper. The word degraded is intended to express the circumstance that the equipment follows the same failure distribution function as it started in the operational state, but the income from its operation is lower. It could, however, also be higher.

Another noteworthy aspect of the proposed model is that the equations for the expected accumulated return and the optimal preventive interval allow the calculating of these values for any finite horizon. In the literature, the finite horizon in maintenance studies is not the most common, but there are a variety of publications. Baklouti et al. (2020) develop a preventive maintenance strategy for a solar PV system that fails when its efficiency drops below a predefined threshold or any element is damaged. The proposed preventive maintenance strategy suggests systematically replacing panels for a finite operation time every certain preventive interval. This finite time affects the preventive interval. Pandey et al. (2015) propose a preventive maintenance scheduling model. The model includes a finite planning horizon and limited available resources to perform maintenance schedules. In addition to maintenance and failure costs, they have added the duration of maintenance breaks and the cost of downtime. The model determines the optimal number of periodic maintenance breaks over a finite planning horizon. Business projects are not eternal, they have a beginning and an end, and the same happens to their assets. That is why studies must abide by these temporal restrictions (Liu et al., 2021). Our article helps to fill this third gap by providing the optimal interval for any planned time horizon.

Finally, a mathematical tool has been used during the process of developing the equations, the z-transform. In industrial maintenance, there is little documentation of where this resolution technique is used, which is more often related to wave propagation and resonance (Vadalá et al., 2021) or acoustics (Mikhin, 2008). Yi et al. (2018) develop a discrete-time semi-Markovian system with a state space consisting of three subsets: working, modifiable, and failure. They use the z-transform to investigate the stochastic properties of the semi-Markovian system and the distributions of some sojourn periods. In our case, we use the z-transform to solve the system of equations for the expected accumulated return.

Analyzing the articles where the preventive interval is calculated, we can see that most of them apply algorithms. In a few cases, a mathematical formula is provided with input data available to the maintenance manager: failure data or intervention costs. The main contribution of this article is that it develops a mathematical formula that calculates the optimal preventive interval when degradation appears. This formula includes the costs of maintenance interventions carried out when predetermined preventive maintenance is applied, the corrective interventions to restore the asset in case of failure, and the preventive ones to reduce the probability of failure due to wear. It also includes the income of the asset when it works in degraded conditions. The formula can be applied to a time horizon without limitation or to cases where the use of the asset is limited, for example, by the completion of the business project. This second case is the most common in critical equipment subject to predetermined preventive maintenance. This article's contribution considers the three gaps identified in the literature review. The developed formula is easy to apply for the designer of the asset maintenance plans, and it can also be included in the designs of the digital twins of assets and maintenance processes.

3. Material and methods

The work aims to find the mathematical expression of the preventive interval that rehabilitates an asset failure mode. This preventive interval optimizes the accumulated economic return over time. Its value depends on the time of use since the last rehabilitation, the distribution of its failures, the costs of maintenance tasks, the penalty for its inactivity, and the income from its operation. The failure mode must show a clear tendency to wear out to be analyzed. Many types of industrial equipment fulfill these assumptions. Throughout the document, the process to reach the mathematical expression of the optimal preventive interval is presented and how to apply the process is also shown. With this mathematical expression, a maintenance manager can apply it to plant assets without having specialized knowledge of modeling and calculation techniques and without consuming excessive time and resources. The mathematical expression calculates the optimal interval after any transition. This is an advantage because it allows the model to be adapted to the duration of any business project. The result is more optimal than expressions calculated for an infinite value of transitions.

3.1. 3.1 Real case and information processing. Weibull distribution, and returns

In this article, a four-state model is developed and applied as an example to the case of the failure mode of the conical connection linking the timing and drive of two injector pumps in a heavy-duty diesel engine. The conical connection consists of two conical surfaces, one internal and one external. The connection is established by contact between the two surfaces. It is an intermediate element in the drive train whose task is to protect the timing elements. When one of the injection pumps is blocked, the conical connection rotates, disconnecting the timing and the pumps. The applying torque maintains this connection. The engine runs if the applied torque remains below the conical connection torque. Otherwise, the engine will stop due to a lack of synchrony.

Sometimes during engine operation or start-up, a slight movement of the conical surfaces occurs. This displacement affects the injection point of the pumps, slowing it down. Under these conditions, the engine performance drops without any apparent explanation for abnormal engine operation. However, the engine performance decreases, and so does the income it brings to the business per hour of operation. This case is very common in industrial assets and is equivalent to a system where the studio asset can be in one of four possible states. These states are: operational, degraded operational, corrective, and preventive. The asset is in operational state S_1 if it operates producing the expected income. If the income obtained from its operation is lower than expected, the asset is in the degraded state S_4. If the asset fails, it is in the corrective state S_2. Moreover, if a preventive task is performed on the asset, it is in the preventive state S_3. See Fig. 3. In this model, the effects of preventive and corrective maintenance are considered perfect.

To analyze the case of the conical connection, data from a mining activity have been selected. The data have been collected for four years, and their origin corresponds to two different types. On the one hand, data on failure times have been collected, t_i, which occurred in 16 engines. These data are collated in Table 1. The observed failure distribution function is estimated from these data. For its construction, the median rank regression method (MRR) is used, and Benard's approximation is used as the estimator, $F_i = \frac{(i-0.3)}{(N+0.4)}$ where i represents the order of failure and N the total number of failures (in this case, 48 failures) (Genschel & Meeker, 2010; Olteanu & Freeman, 2010). In its calculation, the values of the censored data have not been taken into account as a consequence of the end of the data collection process before the failure, as the values are few and much lower than the values of the failure data. The theoretical failure distribution function is estimated, $F(t)$, which best fits the pair (failure hours t_i, observed function F_i), from the failure hours data in Table 1 and the observed failure distribution function data, F_i, in Table 2. The Weibull distribution function (Assis et al., 2022) is typically fitted for the case of physical assets.

This process starts by trying to fit the observed function to the two-parameter Weibull function (α, β) using the MRR method. To do so, the points of the dual (t_i, F_i) are plotted on a graph, using the logarithmic scales, $\ln\ln(1/(1 - - F_i))$ on the vertical axis and lnt_i on the horizontal axis. The most suitable trend line for the points is then plotted, using the least-squares method. The shape parameter α of the Weibull matches with the slope of the trend line, while the scale parameter β corresponds to the negative exponential of the quotient of the ordinate at the origin and the slope. This procedure is simplified by using software such as Excel (Sánchez-Herguedas et al., 2021). In this case, the trend curve is $y = 3.5979x - 31.1$. The values obtained are $\alpha = 3.60$ and $\beta = 5675$. However, the quadratic trend curve has a downward curvature $y = -0.1734x^2 + 6.4975x - 43.2$, which points to the existence of the third parameter γ, known as the guaranteed lifetime or location parameter. In a new graph, we can plot the data again. In this new graph, the scale on the horizontal axis is $ln(t_i - \gamma)$. By giving values to γ we look for the value that converts the quadratic trend curve into a line (coefficient of x^2 equal to zero). The values of the parameters of the Weibull function that best fit the observed function are $\alpha = 3.33$, $\beta = 5368$, and $\gamma = 301$.

The failure data have also been fitted to a Weibull using other methods such as the Maximum Likelihood Estimation method (MLE) (Kang et al., 2022) and the Newton-Raphson method (Lone & Panahi, 2022). In these cases, the results obtained do not fit better than those of the MRR method. The parameters obtained when fitting by the MLE and Newton-Raphson methods coincide, and they are: $\alpha = 3.78$, and $\beta = 5,666$, in the two-parameter case, and $\alpha = 3.39$, $\beta = 5,148$, and $\gamma = 493$ in the three-parameter case. If we compare the three methods using least

Table 1
Failure time data t_i corresponding to the 48 failures (hours).

Failure time data, t_i									
6,635	4,087	4,225	3,964	6,118	3,775	4,377	6,851	2,823	6,684
6,915	1,733	5,645	4,471	4,890	5,887	5,358	4,305	6,536	6,622
4,232	3,661	7,861	5,714	7,421	4,585	4,566	4,358	6,672	4,132
3,209	4,979	6,927	4,562	2,283	7,616	6,618	3,620	3,478	2,668
7,641	3,879	5,775	7,030	4,226	3,829	5,415	6,390		

Table 2
Ordered failure times t_i and observed distribution function values for the 48 failures.

t_i	F_i	t_i	F_i	t_i	F_i	t_i	F_i	t_i	F_i
1,733	0.01446281	3,879	0.22107438	4,471	0.42768595	5,775	0.634297521	6,851	0.840909091
2,283	0.035123967	3,964	0.241735537	4,562	0.448347107	5,887	0.654958678	6,915	0.861570248
2,668	0.055785124	4,087	0.262396694	4,566	0.469008264	6,118	0.675619835	6,927	0.882231405
2,823	0.076446281	4,132	0.283057851	4,585	0.489669421	6,390	0.696280992	7,030	0.902892562
3,209	0.097107438	4,225	0.303719008	4,890	0.510330579	6,536	0.716942149	7,421	0.923553719
3,478	0.117768595	4,226	0.324380165	4,979	0.530991736	6,618	0.737603306	7,616	0.944214876
3,620	0.138429752	4,232	0.345041322	5,358	0.551652893	6,622	0.758264463	7,641	0.964876033
3,661	0.159090909	4,305	0.365702479	5,415	0.57231405	6,635	0.77892562	7,861	0.98553719
3,775	0.179752066	4,358	0.386363636	5,645	0.592975207	6,672	0.799586777		
3,829	0.200413223	4,377	0.407024793	5,714	0.613636364	6,684	0.820247934		

squares the results are slightly favorable to the MRR method as can be seen in Fig. 1 and Fig. 2.

On the other hand, the distribution functions of the repair time $F_c(t)$ and preventive time $F_p(t)$ must be estimated. In our case study, in order not to make the calculations more complex, and due to the short duration of the maintenance time compared to the operating intervals (less than 1 %), the average times of both interventions have been used as estimators. In this case, $m_2 = 72$ for the corrective task and $m_3 = 56$ for the preventive task. They could have been fitted to a Normal distribution or to another type of distribution. These data are shown in Table 3.

This model and the resulting tool have been applied to other cases: the wheels of railway equipment, a pump, or a compressor. For its application, it is required that the failure mode studied in an asset appears due to wear and tear and that due to some internal or external event, the income obtained by the company for its use varies.

3.2. 3.2 Designing a semi-Markov maintenance model with a degraded state

In order to determine the preventive interval that maximizes returns, a four-state model that evolves according to a semi-Markovian process has been designed. The designed model represents the behavior of a (asset) system when it is subjected to preventive and corrective interventions, in order to avoid wear and tear. The operation of the system in a degraded state is also considered in this case.

As a prior theoretical framework, we consider a homogeneous Markov chain $\{X_n, n \geq 0\}$ with 4 states, and transition probabilities at each step $p_{ij} = P(X_1 = j | X_0 = i)$. This chain constitutes the Markov chain embedded in a semi-Markovian process and determines the evolution of the process. On the other hand, the semi-Markovian process is characterized by sojourn times that differ in each state, which do not follow an exponential distribution (Sánchez-Herguedas, Mena-Nieto, Rodrigo-Muñoz, et al., 2022). The sojourn times and the transition between states have associated economic returns that can be positive for income or negative for costs. The variable $r_{ij}(m)$ contains the return from state S_i to state S_j in the transition m.

3.3. 3.3 Formulating the difference equation system for expected accumulated return

At m successive transitions starting from state S_i, the process accumulates returns, which added together with their respective signs make up the accumulated return in m steps from state S_i. We call this random variable $R_i(m)$. The randomness is due to the fact that, once the initial state has been selected, all of the following states are unpredictable. In m transitions, the system can establish many alternatives. For this reason,

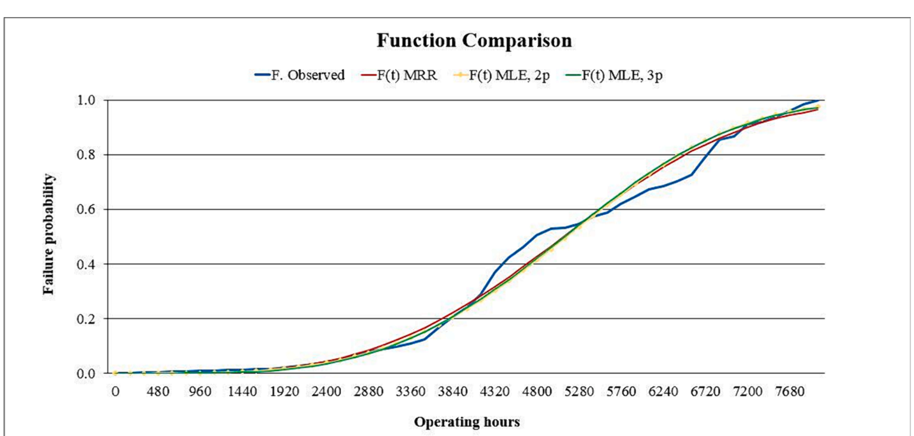

Fig. 1. Observed function and Weibull functions adjusted by the MRR and MLE methods with two and three parameters.

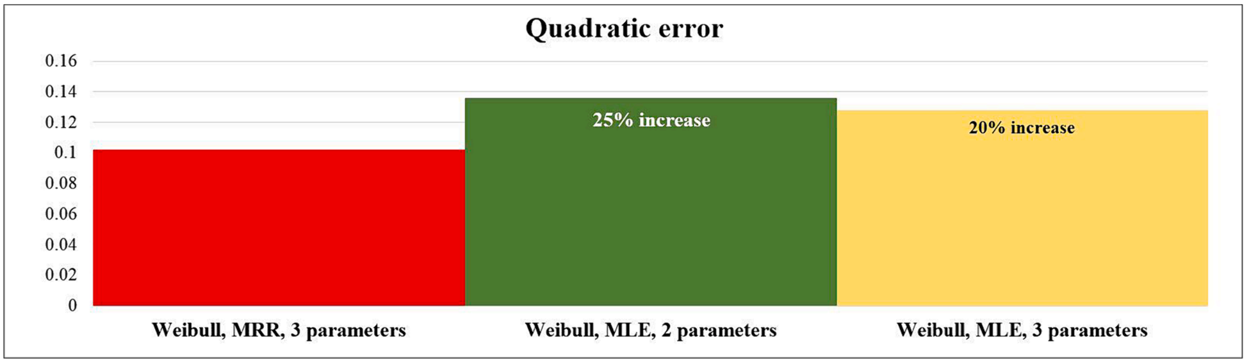

Fig. 2. Quadratic error fitting the Weibull function (two and three parameters) to the observed function using the MRR and MLE methods.

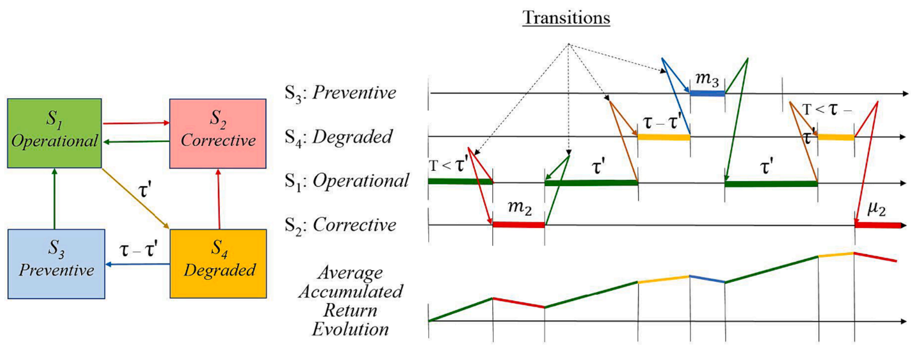

Fig. 3. Representation of states, transitions between states and accumulation of returns at each transition.

Table 3
Model input data. Failure data and average maintenance activity times (MRR method).

Failure distribution function, Weibull (α, β, γ)			Average repair time	Average preventive time
α	β	γ	m_2	m_3
3.33	5,368.00	301.00	72	56

The second type of data corresponds to the costs and income that arise from the use and maintenance of the asset. These data constitute the system returns and are associated with each state. Two types of returns are associated with the operational state S_1: the income per unit of operating time R_1, and the costs associated with the loss of operation, when the asset fails R_{12}, when the preventive is carried out R_{13}, and when the asset transitions to the degraded state R_{14}. The returns associated with the corrective state S_2 are costs. The costs per unit of time due to the repair process R_2 and the costs associated with the lack of activity due to failure are included in R_{21}. The returns associated with the preventive state S_3 are also costs. The costs per unit of time due to the preventive task R_3 and the costs associated with the lack of activity per preventive R_{31}. Two other types of returns are associated with the degraded state S_4: the income per unit of operating time R_4, which is now lower than R_1 (it could be higher in another case study), and the costs associated with the loss of operation, when the asset fails R_{42} and when it is decided to stop the asset for preventive maintenance R_{43}. All data are shown in Table 4. Values $R_1 = 5$ euros/hour and $R_4 = 4$ euros/hour are not values of this mining business. For their calculation, the economic value of the truck where the engine is installed in one hour of operation, must be considered but discounting fuel, driver, other external resources and the amortization of the purchased product.

it is impossible to determine the value of $R_i(m)$, but its expected value can be calculated. Next, the expected accumulated return will be calculated. $v_i(m) = E(R_i(m))$.

The value of $v_i(m)$ is achieved by means of a recursive equation (Sánchez Herguedas et al., 2022). The procedure consists of separating the m transitions into two stages. The first one is formed by the transition from the initial state S_i to the next S_j. As this can be any state, the return $R_i(1)$ is a random variable that can reach the values $r_{i1}(1), r_{i2}(1), r_{i3}(1), r_{i4}(1)$, with the respective probabilities, $p_{i1}, p_{i2}, p_{i3}, p_{i4}$. The expected return at the first transition can be formulated as follows:

$$v_i(1) = \sum_{j=1}^{4} r_{ij}(1)p_{ij} i = 1, 2, 3, 4 \tag{1}$$

The second stage proceeds with the $m-1$ remaining transitions. Once the first transition has been completed, the process is in a state S_j, where j takes one of the values $1, 2, 3, 4$. The expected return in the following $m-1$ transitions is a random variable that can reach the values of $v_1(m-1), v_2(m-1), v_3(m-1), v_4(m-1)$ with the respective probabilities, $p_{i1}, p_{i2}, p_{i3}, p_{i4}$. Therefore, the expected accumulated return in the second stage can be formulated as:

$$\sum_{j=1}^{4} v_j(m-1)p_{ij} i = 1, 2, 3, 4 \tag{2}$$

Finally, Eq. (3) calculates the expected accumulated return of the system for m transitions:

$$v_i(m) = v_i(1) + \sum_{j=1}^{4} v_j(m-1)p_{ij} i = 1, 2, 3, 4 \tag{3}$$

This equation can be written in matrix form as a difference equations system.

$$V(m) = V(1) + PV(m-1) \tag{4}$$

For convenience, it is rewritten with the index increased by one unit. Now, applying the z-transform to both members of Eq. (4) and reordering terms (Sánchez Herguedas et al., 2022), we have:

$$\mathscr{Z}[V(m)] = \frac{1}{z-1}\left(I - z^{-1}P\right)^{-1}V(1) + \left(I - z^{-1}P\right)^{-1}V(0) \tag{5}$$

It only remains to invert these z-transform, to solve Eq. (4).

Eq. (4) could be solved recursively for each m, but such a solution is only of interest for very small values, because the calculation becomes increasingly complicated as m increases. This happens with any recursive equation, which is why various methods are used to solve them, such as the matrix method or the z-transform. The z-transform allows us to reach an explicit solution.

3.4. 3.4 Four-state model. Calculation of sojourn times in each state

The transitions between states and the accumulation of returns are graphically expressed in Fig. 3. The four states and their possible transitions are also shown.

If the system evolves, the time it keeps running from the initial instant, under the conditions for which it was designed, is a random variable that we will call T and can take any positive value. This variable is usually called time to failure and used in operational and degraded states.

Suppose that at the initial instant the system is in the operational state S_1. It will remain in this state until one of two circumstances occurs.

a) The fault occurs and the system switches to the corrective state, S_2.
b) Although it is functioning, after time τ', the system reaches the degraded state, S_4.

The time T_0 is the time the system is in the operational state. It is the time-to-failure random variable, truncated at the interval $[0,\tau']$. It is defined as $T_0 = min\{T,\tau'\}$. If the system reaches S_2, it is because it has failed. It is repaired to return to the conditions for which it was designed. When this happens, the system returns to state S_1. The time to repair is also a random variable called T_c. But if circumstance b) occurs, the system goes to state S_4. It remains there until one of these two circumstances occurs.

c) The fault occurs and the system evolves to the corrective state, S_2.
d) It has been a while $\tau - \tau'$ where τ as well as τ' are prefixed times.

It is necessary to distinguish between the sojourn interval in the state S_4 which is $[\tau', \min\{T,\tau\}]$ and the sojourn time T_d in S_4 which is the duration of that interval.

$$T_d = min\{T,\tau\} - \tau' = min\{T-\tau',\tau-\tau'\}$$

As the system remains in the state S_4 between the instants τ' and at most τ, the sojourn time T_d is the random variable $T-\tau'$ truncated to the interval $[\tau',\tau]$. If circumstance c) occurs, the system evolves to the corrective state for repair. Once restored to the conditions for which it was designed, the system returns to state S_1. But if circumstance d) occurs, the system goes to the preventive state S_3, where it will undergo preventive action in order to return it to the conditions for which it was designed. It then returns to the operational state. The time taken for the preventive action is also a random variable T_p.

This maintenance plan behaves as a stochastic process with a four-state space. The sojourn times in each state are random variables. The objective of this work is to find the value of τ that maximizes the expected accumulated return for a value of τ'.

The τ' time depends on external circumstances (variations in the product market price) or internal circumstances of the equipment itself. In the first case, there could be a positive increase in income. In both cases, the value of τ' is initially unknown. If, after an event in τ' time, the modification of income is known (due to degradation of the internal conditions of the asset or due to variations in the product market), the value of τ' can be incorporated into the model to give a response to the new situation. Out of curiosity and knowing that it does not make sense technically, the authors have considered τ' variable and used the Nelder Mead method to numerically calculate the optimum (considering τ and τ' variables). The result is always reached for values of $\tau' = \tau$.

The distribution functions of T, T_c, T_p will be denoted by $F(t), F_c(t), F_p(t)$ and probability density functions $f(t), f_c(t), f_p(t)$. The random variables T_0 and T_d have truncated distributions whose distribution functions are defined from the distribution function of T.

The matrix of sojourn times in one state before moving to another is:

$$\begin{pmatrix} 0 & T_0 & 0 & \tau' \\ T_c & 0 & 0 & 0 \\ T_p & 0 & 0 & 0 \\ 0 & T_d & \tau-\tau' & 0 \end{pmatrix} \tag{6}$$

Where the i,j-th element is the sojourn time in the state S_i before jumping to state S_j.

A is the average sojourn time in the operational state S_1:

$$A = E(T_0) = \frac{1}{F(\tau')}\int_0^{\tau'} tf(t)dt = \tau' - \frac{1}{F(\tau')}\int_0^{\tau'} F(t)dt$$

B and C are the expected sojourn times in corrective and preventive states S_2 and S_3. They are the averages of the random variables T_c and T_d.

$$B = E(T_c) = \int_0^{\infty} tf_c(t)dt, C = E(T_p) = \int_0^{\infty} tf_p(t)dt$$

As discussed in section 3.1, $B = m_2$ and $C = m_3$.

The average of T_d i.e. the average sojourn time in the degraded operational state is.

$$D = E(T_d) = \frac{1}{F(\tau) - F(\tau')}\int_{\tau'}^{\tau} (t-\tau')f(t)dt \tag{7}$$

The expected sojourn times matrix is the average of matrix sojourn times (Eq. (6)), denoted by Q.

$$Q = E\left[\begin{pmatrix} 0 & T_0 & 0 & \tau' \\ T_c & 0 & 0 & 0 \\ T_p & 0 & 0 & 0 \\ 0 & T_d & \tau-\tau' & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & A & 0 & \tau' \\ B & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & D & \tau-\tau' & 0 \end{pmatrix} \tag{8}$$

Where $E[\bullet]$ means, mean value.

3.5. Transition probabilities between states

As in all random continuous-time processes, during the time our semi-Markovian process is in one state, there is no transition to the same state it is already in. It remains there until the next transition takes it to a different state. Therefore, there are no transitions from a state to itself, so the probability of transition from a state to itself is zero.

Therefore, if the system is in state S_1, it can only move to the states S_2 and S_4. The probability of moving to the state S_2 is $p_1 = P(T \le \tau') = F(\tau')$. The probability of passing to the state S_4 is $P(T > \tau') = 1 - P(T \le \tau') = 1 - F(\tau') = 1 - p_1$.

While the system remains in state S_4, it can only move to states S_2 and S_3. Being $p_2 = F(\tau)$, the probability of moving to state S_2 is:

$$P(T < \tau | T > \tau') = \frac{P[(T<\tau)\cap(T>\tau')]}{P(T>\tau')} = \frac{P(\tau' < T < \tau)}{1 - P(T<\tau')} = \frac{F(\tau) - F(\tau')}{1 - F(\tau')} = \frac{p_2 - p_1}{1 - p_1}$$

And probability of moving to the state S_3 is:

$$P(T > \tau | T > \tau') = 1 - P(T < \tau | T > \tau') = 1 - \frac{p_2 - p_1}{1 - p_1} = \frac{1 - p_2}{1 - p_1}$$

In states S_2 and S_3, the system can only move to state S_1, then the transition probabilities both take the value 1. Therefore, one can write the transition probability matrix P as:

$$P = \begin{pmatrix} 0 & p_1 & 0 & 1-p_1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{p_2-p_1}{1-p_1} & \frac{1-p_2}{1-p_1} & 0 \end{pmatrix} \tag{9}$$

It is a stochastic matrix, so it is the probability matrix of a Markov chain. The transitions of this chain occur each time the stochastic process changes state, so it is the Markov chain embedded in the process.

We have arrived at the stochastic process that models the maintenance of our system. This system has a finite state space, the sojourn times in each state are random variables with not necessarily exponential probability distributions and has an embedded Markov chain whose transition probability matrix is Eq. (9). We conclude that this is a semi-Markovian process.

3.6. Calculation of the returns contributed by each state

The returns matrix can be composed from the sojourn times given in Eq. (8) and the returns matrix can be composed with the returns expressed in Table 4. It is a matrix whose *ij* element is the sum of the return due to remaining in the *i* state, and the other due to the transition to the *j* state, Eq. (10).

$$R = \begin{pmatrix} 0 & AR_1 + R_{12} & 0 & R_1\tau' + R_{14} \\ BR_2 + R_{21} & 0 & 0 & 0 \\ CR_3 + R_{31} & 0 & 0 & 0 \\ 0 & DR_4 + R_{42} & (\tau - \tau')R_4 + R_{43} & 0 \end{pmatrix} \tag{10}$$

4. Theory and calculation

4.1. 4.1 Calculation of the expected accumulated return

Once the model is proposed, we look for a solution that covers the three gaps found in the literature: obtaining the formula for the preventive interval, at any time of asset operation, and considering the income. We start from data from which we compose the vectors and matrices and from the system of recurrent equations of the expected accumulated return of the model. It remains to solve the system of equations.

In the first step, we make the z-transformation of the system. Secondly, we solve the system, taking the expected accumulated return. Third, we perform the inverse z-transformation. Subsequently, we search by derivation for the value of the preventive interval that maximizes the expected accumulated return, resulting in the analytical formula of this variable for any moment in the time horizon of the business.

4.2. 4.2 Solving the system of difference equations for the expected accumulated return

The vector $V(1)$ is obtained from Eq. (1) for which we need the (9) and (10) matrices, i.e. the probability matrix and the returns matrix. From Eq. (1) it follows that its component i is the sum of the products of the rows i of both matrices. As can be seen, this is not a usual product of matrices.

$$\begin{pmatrix} v_1(1) \\ v_2(1) \\ v_3(1) \\ v_4(1) \end{pmatrix} = \begin{pmatrix} (AR_1 + R_{12})p_1 + (R_1\tau' + R_{14})(1-p_1) \\ BR_2 + R_{21} \\ CR_3 + R_{31} \\ (DR_4 + R_{42})\frac{p_2-p_1}{1-p_1} + ((\tau-\tau')R_4 + R_{43})\frac{1-p_2}{1-p_1} \end{pmatrix} \tag{11}$$

From the matrix P we obtain the matrix $I - z^{-1}P$,

$$I - z^{-1}P = \begin{pmatrix} 1 & -z^{-1}p_1 & 0 & -z^{-1}(1-p_1) \\ -z^{-1} & 1 & 0 & 0 \\ -z^{-1} & 0 & 1 & 0 \\ 0 & -z^{-1}\frac{p_2-p_1}{1-p_1} & -z^{-1}\frac{1-p_2}{1-p_1} & 1 \end{pmatrix} \tag{12}$$

and its inverse matrix $\left(I - z^{-1}P\right)^{-1}$,

$$\left(I - z^{-1}P\right)^{-1} = \frac{1}{\Delta}\begin{pmatrix} z^3 & z(p_1z - p_1 + p_2) & z(1-p_2) & z^2(1-p_1) \\ z^2 & z^3 + p_2 - 1 & 1-p_2 & z(1-p_1) \\ z^2 & p_1z - p_1 + p_2 & z^3 - p_1z + p_1 - p_2 & z(1-p_1) \\ z & \frac{z^2(p_1-p_2) + p_1(p_2-1)}{p_1-1} & \frac{(z^2-p_1)(p_2-1)}{p_1-1} & z(z^2-p_1) \end{pmatrix} \tag{13}$$

where $\Delta = z^3 - p_1z + p_1 - 1$. This polynomial cancels if $z = 1$ and $z = -\frac{1 \pm \sqrt{4p_1 - 3}}{2}$.

The initial vector of expected returns $V(0)$ is the null vector, since there is no prior return at the initial instant when the evolution of the chain begins. Accordingly, we write Eq. (5) as:

$$\mathscr{Z}[V(m)] = \frac{1}{z-1}\left(I - z^{-1}P\right)^{-1}V(1) = \tag{14}$$

$$\times \begin{pmatrix} (AR_1 + R_{12})p_1 + (R_1\tau' + R_{14})(1-p_1) \\ BR_2 + R_{21} \\ CR_3 + R_{31} \\ (DR_4 + R_{42})\frac{p_2-p_1}{1-p_1} + ((\tau-\tau')R_4 + R_{43})\frac{1-p_2}{1-p_1} \end{pmatrix}$$

We focus our attention on the first component of this vector.

$$\mathscr{Z}[v_1(m)] = \frac{1}{(z-1)\Delta}\left[z^3v_1(1) + z(p_1z - p_1 + p_2)v_2(1) + z(1-p_2)v_3(1) + z^2(1-p_1)v_4(1)\right]$$

If we arrange the terms in the bracket in the order of decreasing powers of z:

$$\mathscr{Z}[v_1(m)] = \frac{1}{(z-1)\Delta}\left[v_1(1)z^3 + (p_1v_2(1) + (1-p_1)v_4(1))z^2 + ((p_2 - p_1)v_2(1) + (1-p_2)v_3(1))z\right] \tag{15}$$

Recalling that $\Delta = z^3 - p_1z + p_1 - 1$, we are left with the fact that Eq. (15) is a linear combination of rational functions in z. These functions are:

Table 4
Returns of each state. With a positive sign for income and with a negative sign for costs.

S_1 Returns R_1 (€/hour)	R_{12} (€)	R_{13} (€)	S_4 Returns R_4 (€/hour)	R_{42} (€)	R_{43} (€)	S_2 Returns R_2 (€/hour)	R_{21} (€)	S_3 Returns R_3 (€/hour)	R_{31} (€)
5.0	−3,270	−1.0	4.0	−3,270	−1.0	−95.0	−360	−82.0	−360

$$\frac{z^3}{(z^3-p_1z+p_1-1)(z-1)}=\frac{z^3}{(z^2+z-p_1+1)(z-1)^2} \tag{16}$$

$$\frac{z^2}{(z^3-p_1z+p_1-1)(z-1)}=\frac{z^2}{(z^2+z-p_1+1)(z-1)^2}$$

$$\frac{z}{(z^3-p_1z+p_1-1)(z-1)}=\frac{z}{(z^2+z-p_1+1)(z-1)^2}$$

The fractions (16) have in the denominator the double root $z = 1$. Decomposing them into fractions we obtain:

$$\frac{z^3}{(z^2+z-p_1+1)(z-1)^2}=\frac{a_1z+b_1}{z^2+z-p_1+1}+\frac{c_1}{(z-1)^2}+\frac{d_1}{z-1} \tag{17}$$

$$\frac{z^2}{(z^2+z-p_1+1)(z-1)^2}=\frac{a_2z+b_2}{z^2+z-p_1+1}+\frac{c_2}{(z-1)^2}+\frac{d_2}{z-1}$$

$$\frac{z}{(z^2+z-p_1+1)(z-1)^2}=\frac{a_3z+b_3}{z^2+z-p_1+1}+\frac{c_3}{(z-1)^2}+\frac{d_3}{z-1}$$

Where

$$a_1=\frac{p_1^2-3p_1+3}{p_1^2-6p_1+9},b_1=\frac{2p_1^2-5p_1+3}{p_1^2-6p_1+9},c_1=\frac{1}{3-p_1},d_1=\frac{3(2-p_1)}{p_1^2-6p_1+9}$$

$$a_2=\frac{2p_1-3}{p_1^2-6p_1+9},b_2=\frac{p_1(p_1-1)}{p_1^2-6p_1+9},c_2=\frac{1}{3-p_1},d_2=\frac{3-2p_1}{p_1^2-6p_1+9} \tag{18}$$

$$a_3=\frac{p_1}{p_1^2-6p_1+9},b_3=\frac{3(p_1-1)}{p_1^2-6p_1+9},c_3=\frac{1}{3-p_1},d_3=\frac{-p_1}{p_1^2-6p_1+9}$$

In the decomposition into fractions, Eq. (17) the first three have in the denominator the polynomial of second degree z^2+z-p_1+1, which has two roots, then admits a decomposition into two simple fractions. The roots of this polynomial can be real or complex depending on the value of p_1. The roots are $\frac{-1\pm\sqrt{4p_1-3}}{2}$ real if $4p_1-3$ is non-negative and complex conjugate if it is negative. Thus, in Eq. (17), the three fractions of the first member, would be decomposed into four simple fractions. In case $4p_1-3<0$, two of the fractions would correspond to complex roots, and otherwise to real roots. For the sake of clarity in the exposition, the real and complex cases will be separated.

Case 1. Real roots

If the roots of the polynomial (z^2+z-p_1+1) are real, call them by simplifying:

$$r_1=\frac{-1+\sqrt{4p_1-3}}{2},r_2=\frac{-1-\sqrt{4p_1-3}}{2},$$

we obtain the decomposition:

$$\frac{a_jz+b_j}{z^2+z-p_1+1}=\frac{1}{\sqrt{4p_1-3}}\left[\frac{a_jr_1+b_j}{z-r_1}-\frac{a_jr_2+b_j}{z-r_2}\right],j=1,2,3. \tag{19}$$

We introduce the Eq. (19) into Eq. (17) and these in turn into Eq. (15), which is as follows:

$$\mathscr{Z}[v_1(m)]=v_1(1)\left[\frac{1}{\sqrt{4p_1-3}}\left(\frac{a_1r_1+b_1}{z-r_1}-\frac{a_1r_2+b_1}{z-r_2}\right)+\frac{c_1}{(z-1)^2}+\frac{d_1}{z-1}\right]$$

$$+(p_1v_2(1)+(1-p_1)v_4(1))\left[\frac{1}{\sqrt{4p_1-3}}\left(\frac{a_2r_1+b_2}{z-r_1}-\frac{a_2r_2+b_2}{z-r_2}\right)+\frac{c_2}{(z-1)^2}+\frac{d_2}{z-1}\right]$$

$$+((p_2-p_1)v_2(1)+(1-p_2)v_3(1))\left[\frac{1}{\sqrt{4p_1-3}}\left(\frac{a_3r_1+b_3}{z-r_1}-\frac{a_3r_2+b_3}{z-r_2}\right)+\frac{c_3}{(z-1)^2}+\frac{d_3}{z-1}\right] \tag{20}$$

Regrouping the terms of the second member of Eq. (20) and taking out as common factors the simple fractions $\frac{1}{z-r_1},\frac{1}{z-r_2},\frac{1}{(z-1)^2},\frac{1}{z-1}$, the equation is as follows:

$$\mathscr{Z}[v_1(m)]=A_1\frac{1}{z-r_1}+A_2\frac{1}{z-r_2}+A_3\frac{1}{(z-1)^2}+A_4\frac{1}{z-1} \tag{21}$$

Where:

$$A_1=\frac{1}{\sqrt{4p_1-3}}[v_1(1)(a_1r_1+b_1)+(p_1v_2(1)+(1-p_1)v_4(1))(a_2r_1+b_2)+((p_2-p_1)v_2(1)+(1-p_2)v_3(1))(a_3r_1+b_3)], \tag{22}$$

$$A_2=-\frac{1}{\sqrt{4p_1-3}}[v_1(1)(a_1r_2+b_1)+(p_1v_2(1)+(1-p_1)v_4(1))(a_2r_2+b_2)+((p_2-p_1)v_2(1)+(1-p_2)v_3(1))(a_3r_2+b_3)],$$

$$A_3=c_1[v_1(1)+(1-p_1)v_4(1)+p_2v_2(1)+(1-p_2)v_3(1)]$$

$$A_4=v_1(1)d_1+(p_1v_2(1)+(1-p_1)v_4(1))d_2+((p_2-p_1)v_2(1)+(1-p_2)v_3(1))d_3.$$

The next step is to calculate the Laurent developments of each simple fraction of Eq. (21):

$$\frac{1}{z-r_j}=\frac{1}{z}\cdot\frac{1}{1-(r_j/z)}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{r_j}{z}\right)^n=\sum_{n=0}^{\infty}\frac{r_j^n}{z^{n+1}},\qquad j=1,\ 2, \tag{23}$$

$$\frac{1}{(z-1)^2}=-\frac{d}{dz}\left(\frac{1}{z-1}\right)=-\frac{d}{dz}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}=-\sum_{n=0}^{\infty}\frac{d}{dz}\left(\frac{1}{z^{n+1}}\right)=\sum_{n=0}^{\infty}\frac{n+1}{z^{n+2}}.$$

$$\frac{1}{z-1}=\frac{1}{z}\cdot\frac{1}{1-(1/z)}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Substituting the developments (23) in Eq. (21) we obtain:

$$\mathscr{Z}[v_1(m)]=A_1\sum_{n=0}^{\infty}\frac{{r_1}^n}{z^{n+1}}+A_2\sum_{n=0}^{\infty}\frac{{r_2}^n}{z^{n+1}}+A_3\sum_{n=0}^{\infty}\frac{n+1}{z^{n+2}}+A_4\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}= \tag{24}$$

$$=A_1\sum_{n=1}^{\infty}\frac{{r_1}^{n-1}}{z^n}+A_2\sum_{n=1}^{\infty}\frac{{r_2}^{n-1}}{z^n}+A_3\sum_{n=2}^{\infty}\frac{n-1}{z^n}+A_4\sum_{n=1}^{\infty}\frac{1}{z^n}=$$

$$=\frac{A_1+A_2+A_4}{z}+\sum_{n=2}^{\infty}\frac{A_1{r_1}^{n-1}+A_2{r_2}^{n-1}+A_3(n-1)+A_4}{z^n}$$

The coefficients of the last series of Eq. (24) form a sequence, which is the inverse transform of $\mathscr{Z}[v_1(m)]$, that is to say, this sequence is $v_1(m)$:

$$v_1(0)=0, \tag{25}$$

$$v_1(m)=A_1{r_1}^{m-1}+A_2{r_2}^{m-1}+A_3(m-1)+A_4,m=1,2,3,\cdots$$

Case 2. Complex roots

In the case of complex roots, a development like the case of real roots is followed. For simplicity, complex roots are expressed as follows:

$$\mu+i\sigma=\frac{-1+i\sqrt{3-4p_1}}{2},\mu-i\sigma=\frac{-1-i\sqrt{3-4p_1}}{2}$$

the decomposition for $j=1,2,3$ en:

$$\frac{a_j z + b_j}{z^2 - z - p_1 + 1} = \frac{1}{2}\left[\left(a_j - \frac{a_j\mu + b_j}{\sigma}i\right)\frac{1}{z-(\mu+\sigma i)} + \left(a_j + \frac{a_j\mu + b_j}{\sigma}i\right)\frac{1}{z-(\mu-\sigma i)}\right] \tag{26}$$

We substitute the second member of Eq. (26) in Eq. (17) and these in turn into Eq. (15), which is as follows:

$$\begin{aligned}\mathcal{Z}[v_1(m)] =& v_1(1)\left[\frac{1}{2}\left[\left(a_1 - \frac{a_1\mu+b_1}{\sigma}i\right)\frac{1}{z-(\mu+\sigma i)} + \left(a_1 + \frac{a_1\mu+b_1}{\sigma}i\right)\frac{1}{z-(\mu-\sigma i)}\right] + \frac{c_1}{(z-1)^2} + \frac{d_1}{z-1}\right]\\ &+ (p_1v_2(1)+(1-p_1)v_4(1))\left[\frac{1}{2}\left[\left(a_2 - \frac{a_2\mu+b_2}{\sigma}i\right)\frac{1}{z-(\mu+\sigma i)} + \left(a_2 + \frac{a_2\mu+b_2}{\sigma}i\right)\frac{1}{z-(\mu-\sigma i)}\right] + \frac{c_2}{(z-1)^2} + \frac{d_2}{z-1}\right]\\ &+ ((p_2-p_1)v_2(1)+(1-p_2)v_3(1))\left[\frac{1}{2}\left[\left(a_3 - \frac{a_3\mu+b_3}{\sigma}i\right)\frac{1}{z-(\mu+\sigma i)} + \left(a_3 + \frac{a_3\mu+b_3}{\sigma}i\right)\frac{1}{z-(\mu-\sigma i)}\right] + \frac{c_3}{(z-1)^2} + \frac{d_3}{z-1}\right].\end{aligned} \tag{27}$$

We regroup the terms of the second member of Eq. (27) by taking out as common factors the simple fractions $\frac{1}{z-(\mu+\sigma i)}, \frac{1}{z-(\mu-\sigma i)}, \frac{1}{(z-1)^2}, \frac{1}{z+1}$, so that:

$$\mathcal{Z}[v_1(m)] = B_1\frac{1}{z-(\mu+\sigma i)} + B_2\frac{1}{z-(\mu-\sigma i)} + A_3\frac{1}{(z-1)^2} + A_4\frac{1}{z-1} \tag{28}$$

Where:

$$B_1 = \frac{1}{2}\left[v_1(1)\left(a_1 - \frac{a_1\mu+b_1}{\sigma}i\right) + (p_1v_2(1)+(1-p_1)v_4(1))\left(a_2 - \frac{a_2\mu+b_2}{\sigma}i\right) + ((p_2-p_1)v_2(1)+(1-p_2)v_3(1))\left(a_3 - \frac{a_3\mu+b_3}{\sigma}i\right)\right] \tag{29}$$

$$B_2 = \frac{1}{2}\left[v_1(1)\left(a_1 + \frac{a_1\mu+b_1}{\sigma}i\right) + (p_1v_2(1)+(1-p_1)v_4(1))\left(a_2 + \frac{a_2\mu+b_2}{\sigma}i\right) + ((p_2-p_1)v_2(1)+(1-p_2)v_3(1))\left(a_3 + \frac{a_3\mu+b_3}{\sigma}i\right)\right]$$

$$A_3 = c_1[v_1(1) + (1-p_1)v_4(1) + p_2v_2(1) + (1-p_2)v_3(1)]$$

$$A_4 = v_1(1)d_1 + (p_1v_2(1)+(1-p_1)v_4(1))d_2 + ((p_2-p_1)v_2(1)+(1-p_2)v_3(1))d_3$$

Note that B_1 and B_2 are conjugate complexes and that A_3 and A_4 are the same as in Eq. (22). The Laurent series developments of the simple fractions of Eq. (28) are:

$$\frac{1}{z-(\mu\pm\sigma i)} = \frac{1}{z}\cdot\frac{1}{1-\frac{\mu\pm\sigma i}{z}} = \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{\mu\pm\sigma i}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(\mu\pm\sigma i)^n}{z^{n+1}} \tag{30}$$

$$\frac{1}{(z-1)^2} = -\frac{d}{dz}\left(\frac{1}{z-1}\right) = -\frac{d}{dz}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{n=0}^{\infty}\frac{d}{dz}\left(\frac{1}{z^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{n+1}{z^{n+2}}.$$

$$\frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-(1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Now we insert the first development of Eq. (30) into the first two summands of Eq. (28):

$$B_1\frac{1}{z-(\mu+\sigma i)} + B_2\frac{1}{z-(\mu-\sigma i)} = \sum_{n=0}^{\infty}\frac{B_1(\mu+\sigma i)^n}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{B_2(\mu-\sigma i)^n}{z^{n+1}} = \sum_{n=0}^{\infty}\frac{B_1(\mu+\sigma i)^n + B_2(\mu-\sigma i)^n}{z^{n+1}}. \tag{31}$$

With the notation Re(•), e Im(•) denotes real part and imaginary part. Then the numerator of the general term of the series (31) is as follows:

$$\begin{aligned}B_1(\mu+\sigma i)^n + B_2(\mu-\sigma i)^n &= 2\mathrm{Re}[B_1(\mu+\sigma i)^n]\\ &= 2\mathrm{Re}\ (B_1)\cdot\mathrm{Re}(\mu+\sigma i)^n - 2\mathrm{Im}(B_1)\cdot\mathrm{Im}(\mu+\sigma i)^n\end{aligned}$$

$$B_1(\mu+\sigma i)^n + B_2(\mu-\sigma i)^n = 2r^n(\mathrm{Re}(B_1)\cos n\omega - \mathrm{Im}(B_1)\sin n\omega)$$

It is now possible to write in simplified form Eq. (28) using the developments of Eq. (30):

$$\begin{aligned}\mathcal{Z}[v_1(m)] &= 2\sum_{n=0}^{\infty}\frac{r^n(\mathrm{Re}(B_1)\cos n\omega - \mathrm{Im}(B_1)\sin n\omega)}{z^{n+1}} + A_3\sum_{n=0}^{\infty}\frac{n+1}{z^{n+2}} + A_4\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\\ &= \frac{2\mathrm{Re}(B_1)+A_4}{z} + \sum_{n=2}^{\infty}\frac{2r^{n-1}(\mathrm{Re}(B_1)\cos(n-1)\omega - \mathrm{Im}(B_1)\sin(n-1)\omega) + A_3(n-1) + A_4}{z^n}\end{aligned} \tag{32}$$

The inverse transform of $\mathcal{Z}[v_1(m)]$ is the sequence whose terms are the coefficients of the series Eq. (32):

$$v_1(0) = 0 \tag{33}$$

$$v_1(m) = 2r^{m-1}(\mathrm{Re}(B_1)\cos(m-1)\omega - \mathrm{Im}(B_1)\sin(m-1)\omega) + A_3(m-1) + A_4 m = 1, 2, 3, \cdots$$

4.3. 4.3 Optimization of the expected accumulated return

In order to calculate the preventive interval that maximizes the expected accumulated return, this return is derived with respect to the preventive interval τ. In the same way as when calculating the expected accumulated return, the case of real roots and the case of complex roots must be considered separately.

4.4. 4.4 Optimization of the expected accumulated return for the case of real roots

From the explicit expression of $v_1(m)$, Eq. (25), we proceed to its optimization for values of τ. To do so, we calculate the derivative:

$$\frac{dv_1(m)}{d\tau} = {r_1}^{m-1}\frac{dA_1}{d\tau} + {r_2}^{m-1}\frac{dA_2}{d\tau} + (m-1)\frac{dA_3}{d\tau} + \frac{dA_4}{d\tau} \tag{34}$$

Only p_2 depends on τ, so it will be necessary to derive those expressions where p_2 appears and those where τ appears explicitly. According to this observation:

$$\frac{dA_1}{d\tau}=\frac{1}{\sqrt{4p_1-3}}\left[(1-p_1)(a_2r_1+b_2)\frac{dv_4(1)}{d\tau}+(v_2(1)-v_3(1))(a_3r_1+b_3)\frac{dp_2}{d\tau}\right]. \tag{35}$$

$$\frac{dA_2}{d\tau}=\frac{-1}{\sqrt{4p_1-3}}\left[(1-p_1)(a_2r_2+b_2)\frac{dv_4(1)}{d\tau}+(v_2(1)-v_3(1))(a_3r_2+b_3)\frac{dp_2}{d\tau}\right].$$

$$\frac{dA_3}{d\tau}=c_1\left[(1-p_1)\frac{dv_4(1)}{d\tau}+(v_2(1)-v_3(1))\frac{dp_2}{d\tau}\right].$$

$$\frac{dA_4}{d\tau}=d_2(1-p_1)\frac{dv_4(1)}{d\tau}+d_3(v_2(1)-v_3(1))\frac{dp_2}{d\tau}.$$

From Eq. (11) we take $v_4(1)$, simplify it, substitute D by Eq. (7), p_1 by $F(\tau')$, and p_2 by $F(\tau)$, so that $v_4(1)$ is left:

$$v_4(1)=R_{42}+\frac{1}{1-p_1}\left[R_4\int_{\tau'}^{\tau}(t-\tau')f(t)dt+(1-F(\tau))((\tau-\tau')R_4+R_{43}-R_{42})\right] \tag{36}$$

And its derivative:

$$\frac{dv_4(1)}{d\tau}=\frac{1}{1-p_1}[f(\tau)(R_{42}-R_{43})+(1-F(\tau))R_4] \tag{37}$$

We can now incorporate Eq. (37) to each of the expressions (35) and add them together to obtain Eq. (34). This is done two by two summands. The first two:

$$r_1^{m-1}\frac{dA_1}{d\tau}+r_2^{m-1}\frac{dA_2}{d\tau}=\frac{1}{\sqrt{4p_1-3}}\left((1-F(\tau))R_4\left(a_2(r_1^{m}-r_2^{m})+b_2\left(r_1^{m-1}-r_2^{m-1}\right)\right)+\right. \tag{38}$$

$$+f(\tau)\left((R_{42}-R_{43})\left(a_2(r_1^{m}-r_2^{m})+b_2\left(r_1^{m-1}-r_2^{m-1}\right)\right)+\right.$$

$$\left.\left.+(v_2(1)-v_3(1))\left(a_3(r_1^{m}-r_2^{m})+b_3\left(r_1^{m-1}-r_2^{m-1}\right)\right)\right)\right)$$

And the last two:

$$(m-1)\frac{dA_3}{d\tau}+\frac{dA_4}{d\tau}=(1-F(\tau))R_4\left((m-1)c_1+d_2\right)+ \tag{39}$$

$$+f(\tau)((R_{42}-R_{43})((m-1)c_1+d_2)+((m-1)c_1+d_3)(v_2(1)-v_3(1)))$$

To write (38) and (39), $\frac{dp_2}{d\tau}$ have been replaced by $f(\tau)$ and p_2 by $F(\tau)$. In addition, to simplify further calculations, in both expressions, we have taken the common factors $1-F(\tau)$ and $f(\tau)$.

Substituting (38) and (39) in the derivative (34), we write:

$$\frac{dv_1(m)}{d\tau}=M_1(1-F(\tau))+M_2f(\tau). \tag{40}$$

Where the coefficients M_1 and M_2 are:

$$M_1=R_4\left[\frac{1}{\sqrt{4p_1-3}}\left(a_2(r_1^{m}-r_2^{m})+b_2\left(r_1^{m-1}-r_2^{m-1}\right)\right)+(m-1)c_1+d_2\right], \tag{41}$$

$$\begin{aligned}M_2=&(R_{42}-R_{43})\left[\frac{1}{\sqrt{4p_1-3}}\left(a_2(r_1^{m}-r_2^{m})+b_2\left(r_1^{m-1}-r_2^{m-1}\right)\right)+(m-1)c_1+d_2\right]\\&+(v_2(1)-v_3(1))\left[\frac{1}{\sqrt{4p_1-3}}\left(a_3(r_1^{m}-r_2^{m})+b_3\left(r_1^{m-1}-r_2^{m-1}\right)\right)+(m-1)c_1+d_3\right]\end{aligned}$$

4.5. 4.5 Optimization of the expected accumulated return for the case of complex roots

We proceed as in the real case, deriving, in this case, Eq. (33):

$$\frac{dv_1(m)}{d\tau}=2r^{m-1}\left(\cos(m-1)\omega\frac{d\mathrm{Re}(B_1)}{d\tau}-\sin(m-1)\omega\frac{d\mathrm{Im}(B_1)}{d\tau}\right)+(m-1)\frac{dA_3}{d\tau}+\frac{dA_4}{d\tau}. \tag{42}$$

At (42), only p_2 depends on τ, so it will be necessary to derive those terms in which it appears p_2 and those in which τ appears explicitly. Then, using the expression of B_1 at (29), we derive:

$$\frac{d\mathrm{Re}(B_1)}{d\tau}=\frac{1}{2}\left(a_2(1-p_1)\frac{dv_4(1)}{d\tau}+a_3(v_2(1)-v_3(1))\frac{dp_2}{d\tau}\right), \tag{43}$$

$$\frac{d\mathrm{Im}(B_1)}{d\tau}=-\frac{1}{2}\left(\frac{a_2\mu+b_2}{\sigma}(1-p_1)\frac{dv_4(1)}{d\tau}+\frac{a_3\mu+b_3}{\sigma}(v_2(1)-v_3(1))\frac{dp_2}{d\tau}\right).$$

The last two summands of Eq. (42), which contain the derivatives of A_3 and A_4 were calculated in Eq. (39). The first summand is calculated using Eq. (43). The derivative of $v_4(1)$ is taken from Eq. (37). Where p_2 appears $F(\tau)$ is written and where its derivative appears, we write $f(\tau)$. Thus, the first summand of Eq. (42) is as follows:

$$2r^{m-1}\left(\cos(m-1)\omega\frac{d\mathrm{Re}(B_1)}{d\tau}+\sin(m-1)\omega\frac{d\mathrm{Im}(B_1)}{d\tau}\right)= \tag{44}$$

$$\begin{aligned}=r^{m-1}\Bigg(&f(\tau)\Bigg[\cos(m-1)\omega(a_2(R_{42}-R_{43})+a_3(v_2(1)-v_3(1)))\\&+\sin(m-1)\omega\left(\frac{a_2\mu+b_2}{\sigma}(\tau' R_4+R_{42}-R_{43})+\frac{a_3\mu+b_3}{\sigma}(v_2(1)-v_3(1))\right)\Bigg]+(1-F(\tau))\Bigg[a_2R_4\cos(m-1)\omega\\&+\frac{a_2\mu+b_2}{\sigma}R_4\sin(m-1)\omega\Bigg]\Bigg).\end{aligned}$$

Where we have taken the common factors $f(\tau)$ and $1-F(\tau)$. Now we can write the Eq. (42) by adding Eqs. (44) and (39):

$$\begin{aligned}\frac{dv_1(m)}{d\tau}=r^{m-1}\Bigg(&f(\tau)\Bigg[\cos(m-1)\omega(a_2(R_{42}-R_{43})+a_3(v_2(1)-v_3(1)))\\&+\sin(m-1)\omega\left(\frac{a_2\mu+b_2}{\sigma}(R_{42}-R_{43})+\frac{a_3\mu+b_3}{\sigma}(v_2(1)-v_3(1))\right)\Bigg]\\&+(1-F(\tau))\left[a_2R_4\cos(m-1)\omega+\frac{a_2\mu+b_2}{\sigma}R_4\sin(m-1)\omega\right]\Bigg)\\&+(1-F(\tau))R_4\left((m-1)c_1+d_2\right)\\&+f(\tau)((R_{42}-R_{43})((m-1)c_1+d_2)\\&+((m-1)c_1+d_3)(v_2(1)-v_3(1))).\end{aligned} \tag{45}$$

As was the case in the real case, $f(\tau)$ and $1-F(\tau)$ are common factors so that Eq. (45) can be written as a linear combination of them, the coefficients of which do not depend on τ:

$$\frac{dv_1(m)}{d\tau} = M_1(1-F(\tau)) + M_2 f(\tau). \tag{46}$$

The coefficients M_1 and M_2 in the complex case are:

$$M_1 = R_4\left((a_2\cos(m-1)\omega + \frac{a_2\mu+b_2}{\sigma}\sin(m-1)\omega)\cdot r^{m-1} + (m-1)c_1 + d_2\right) \tag{47}$$

$$M_2 = r^{m-1}\Big[\cos(m-1)\omega(a_2(R_{42}-R_{43}) + a_3(v_2(1) - v_3(1))) + \sin(m-1)\omega\left(\frac{a_2\mu+b_2}{\sigma}(R_{42}-R_{43}) + \frac{a_3\mu+b_3}{\sigma}(v_2(1) - v_3(1))\right)\Big] + ((R_{42}-R_{43})((m-1)c_1+d_2) + (v_2(1)-v_3(1))((m-1)c_1+d_3))$$

4.6. 4.6 Calculation of the mathematical expression of the optimal preventive interval

So far, no assumptions have been made about the failure time distribution. Now it is time to do so. Suppose it has a Weibull distribution with shape parameter α, scale parameter β, and guaranteed lifetime (location) γ. Then the probability density functions $f(t)$ and distribution $F(t)$ are the Weibull functions:

$$f(t) = \frac{\alpha}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\alpha-1} e^{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}}, F(t) = 1 - e^{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}}, \alpha, \beta > 0.$$

When substituted in Eq. (46) or in Eq. (40) it results:

$$\frac{dv_1(m)}{d\tau} = M_1 e^{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}} + M_2\frac{\alpha}{\beta}\left(\frac{\tau-\gamma}{\beta}\right)^{\alpha-1} e^{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}} \tag{48}$$

Equating Eq. (48) to zero, the non-zero common factor $e^{-(\tau-\gamma/\beta)^{\alpha}}$ is eliminated, so that it will be:

$$M_1 + M_2\frac{\alpha}{\beta}\left(\frac{\tau-\gamma}{\beta}\right)^{\alpha-1} = 0 \tag{49}$$

If $M_2 \neq 0$ can be written, the equation defining the optimal preventive interval can be written:

$$\tau_o = \left(-\frac{\beta^{\alpha}}{\alpha}\bullet\frac{M_1}{M_2}\right)^{\frac{1}{\alpha-1}} + \gamma \tag{50}$$

The range of values that can be taken τ_o is the interval $(0,\infty)$ if $0 < \alpha < 1$ and the interval $[0,\infty)$ if $\alpha \geq 1$. In both cases, Eq. (50) will have a non-zero solution, which is also unique, if and only if M_1 and M_2 are non-zero with different signs. Although not impossible, given the complexity of M_2 it is extremely unlikely that it will cancel out.

This optimum is a maximum since the second derivative of $v_1(m)$ as the same sign as M_2 which always takes negative values.

5. Analysis and results

In sections 3 and 4 the model has been designed and a tool (mathematical formula), easy to use for the maintenance engineer, has been developed. The tool calculates the optimal preventive interval (first gap) for a failure mode of a wear-prone asset. The calculation incorporates the income (second gap) earned from the operation of the asset. The tool also allows the calculation of the interval at each decision point (each transition), and results can be obtained for the time horizon of the project considered by the company (third gap). In addition, the mathematical formula of the expected accumulated return is obtained during its development. This economic variable makes it possible to decide between different maintenance policies. Both formulae are easy to integrate and use into the modern digital twin designs of maintenance workshops to optimize asset maintenance plans.

After analyzing the general issues, we use the formulae to solve the case study. From the values of the variables involved in the analysis, which are listed in Table 3 and Table 4, we proceed to apply the mathematical equations that have been designed. The value of the time for which the transition to the degraded state occurs, τ' is an experimental value to be included in the model. To make a comparison exercise between results, different values of τ', between 1,000 h and 8,000 h, were taken at intervals of 1,000 h. For these τ' values, the type of solution (with real or complex roots) must be identified. If the value of $4p_1 - 3$ is non-negative, the equations of the real case must be used, but if the value of $4p_1 - 3$ is negative, the equations of the complex case must be used. See Table 5.

The values of the preventive interval that maximizes the expected accumulated return for the transition $m = 10$ for different values of the time to degradation τ' are listed in the third row of Table 6. The values in the table corresponding to $\tau' \geq 7,000$ are not significant since the preventive maintenance time would be reached before the time to degradation. Nevertheless, the values given by the formulae, in this case for real roots, are shown in italics.

If larger values of m are taken, the τ_o values decrease as m increases until it reaches a value of 6.040 when m is very large. Table 7 expresses this trend for $m = 60$. The speed at which these data reach this limiting value depends on the input data to the model.

If we use the three-state model formulation (no degraded state) (Sánchez Herguedas et al., 2022), using the same data, we obtain the results given in Table 8. The same results are also obtained when in the four-state model, we equal the values of $R_1 = R_4$. The mathematical expression for the optimal preventive interval τ_o in the three-state case is:

$$(\tau_o - \gamma)^{\alpha-1} = \frac{\beta^{\alpha}}{\alpha}\bullet\frac{-R_1}{R_{12} - R_{13} + \frac{2m-1-(-1)^{m-1}}{2m+1+(-1)^{m-1}}(R_2B + R_{21} - R_3C - R_{31})}$$

The values in Table 8 explain the behavior of the value of τ_o in Table 6, from 6.042 to 6.146. For a large number of transitions, the values of τ_o are close to 6.040, which is the value that is reached when $R_1 = R_4 = 4$.

The transition between the complex and real models occurs between the values of $\tau' = 6,222$ and $\tau' = 6,223$ because $p_1 = F(\tau')$ in the complex

Table 5
Time to degradation values and equations of the real or complex case to use.

τ'(hours)	1,000	2,000	3,000	4,000	5,000	6,000	7,000	8,000
$p_1 = F(\tau')$	0.0011	0.0215	0.0963	0,2513	0.4737	0.7049	0.8764	0.9640
$4p_1 - 3$	−2.995	−2.914	−2.615	−1.995	−1.105	−0.18	0.5057	0.8558
Case	Complex	Complex	Complex	Complex	Complex	Complex	Real	Real

From Eq. (50), with the information in Table 5 and with its coefficients M_1 and M_2 from Eq. (47) if it is the complex case, or the coefficients of Eq. (41) in the real case, we obtain the values of Table 6, when the number of transitions is $m = 10$.

Table 6
Values of the optimal preventive interval τ_o and expected accumulated return $v_1(10)$ for different time intervals until degradation τ', at the transition $m = 10$.

τ'(hours)	1,000	2,000	3,000	4,000	5,000	6,000	*7,000*	*8,000*
$v_1(10)$(€)	39,364	47,744	55,695	61,412	66,996	74,656	*78,585*	*78,166*
τ_o(hours)	6,042	6,061	6,115	6,164	6,159	6,146	*6,184*	*6,229*

Table 7
Values of the optimal preventive interval τ_o and expected accumulated return $v_1(60)$ for different time intervals until degradation τ', at the transition $m = 60$.

τ' (hours)	1,000	2,000	3,000	4,000	5,000	6,000	*7,000*	*8,000*
$v_1(60)$(€)	228,956	252,718	283,371	318,087	361,095	407,152	*444,591*	*464,327*
τ_o (hours)	6,040	6,043	6,056	6,057	6,057	6,057	*6,057*	*6,059*

Table 8
Values obtained for the optimal preventive interval τ_o, and expected accumulated return $v_1(10)$, when there is no degradation. Three-state model.

R_1 (€/hora)	**R_4 (€/hora)**	$v_1(10)$ (€)	**τ_o (horas)**
5 €/hour	5 €/hour	76,747	6,617 h
4 €/hour	4 €/hour	52,216	6,040 h

case takes the value 0.75 and in the real case takes the value 0.7502.

It is also possible to know the behavior of the τ_o as the transitions m are increased for a given τ'. Specifically, in Fig. 4, the values of the optimal preventive interval τ_o are plotted against the values of m between 1 and 60, for the value of $\tau' = 4,000$ hours. This figure also shows the value of τ_o when there is no degraded state (3 states). If the values of τ_o are represented for other values of τ', as the value of τ' increases, the curve would shift upwards. On the contrary, as the value of τ' decreases, the curve will shift downwards. The upper limit of the possible displacement is set at the value of the τ_o when there is no degradation (three states), 6.617. On the other hand, the lower limit is set at the value of the τ_o value when there is no degradation, but the income of the operating state coincides with the income value when there is degradation $R_4 = R_1$. This value is 6.040 h. This evidence can also be verified with the data in Table 6, and Table 7.

The expected sojourn times in each state are calculated according to the expression: $t_i = \sum_{j=1}^{4} p_{ij} r_{ij}$. In the case of $\tau' = 4,000$ and $\tau_o = 6,040$, the mean values of the stay in each state (transition size) are given in Table 9.

As a summary of this section, after the experiment with the case study, we can expose the following results:

- The value of income during the degradation phase R_4 is decisive in the economic optimization of the preventive interval. So far, few businesses consider this fact when scheduling their maintenance activities. In addition, the value of income before the degradation R_1 was also in the case of the three-state model (Sánchez Herguedas et al., 2022).
- Once degradation is detected, the asset can continue to be used, although the initially set preventive interval needs to be reduced.
  - The new optimal preventive interval value approaches the value of the preventive interval if the income R_1 is reduced until it approaches the income of the degraded state.
  - The new optimal preventive interval value would coincide with the value of the preventive interval if the income coincides with the income of the degraded state, when we perform many transitions, high values of m, close to an infinite horizon.

Table 9
Average length of stays in each state, for $\tau' = 4,000 and \tau_o = 6,040$.

Size of stay in state 1	Size of stay in state 2	Size of stay in state 3	Size of stay in state 4
3,772 h	72 h	56 h	1,417 h

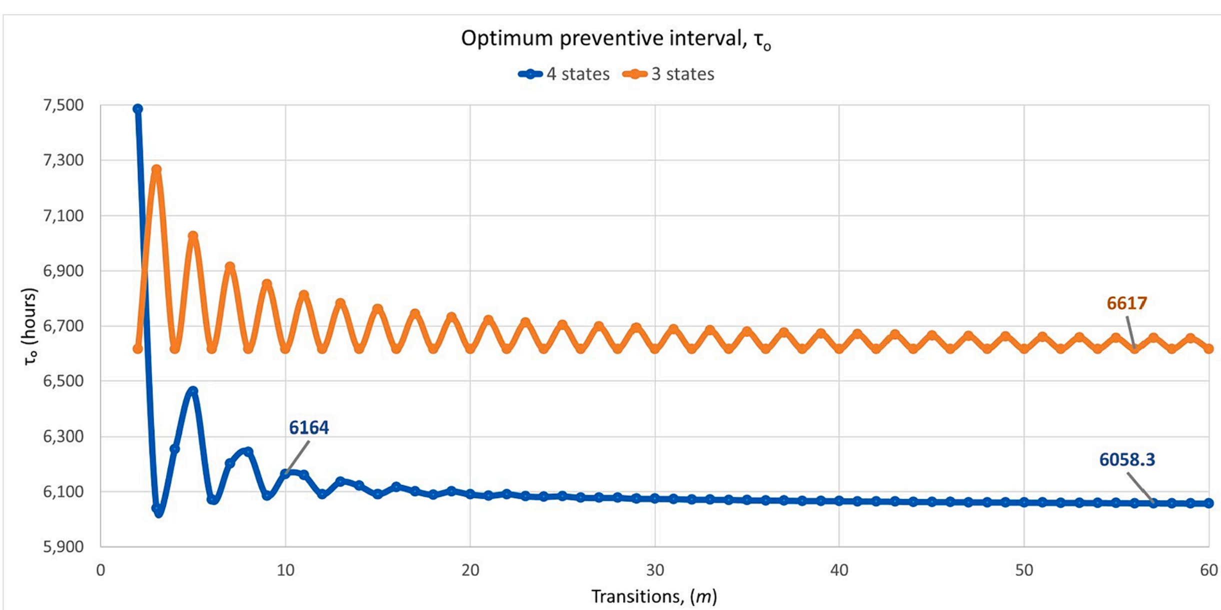

Fig. 4. Representation of the optimal preventive interval at each transition. 3-state case with, and 4-state case with.

Table 10
Representation of the values of $\nu_1(10)$ obtained from different τ and τ'. The best values have been framed.

τ(hours)	$\nu_1(10)$ (€)					
6,000	39,363.2	47,741.1	55,684.9	61,390.0	66,973.7	
6,020	39,364.1	47,742.6	55,688.1	61,395.1	66,979.1	74,640.6
6,040	$\underline{39,364.5}$	47,743.4	55,690.6	61,399.5	66,983.6	74,645.0
6,060	39,364.2	$\underline{47,743.7}$	55,692.6	61,403.1	66,987.4	74,648.7
6,080	39,363.5	47,743.5	55,693.9	61,406.1	66,990.5	74,651.5
6,100	39,362.2	47,742.7	55,694.6	61,408.4	66,992.8	74,653.6
6,120	39,360.3	47,741.4	$\underline{55,694.7}$	61,410.1	66,994.4	74,654.9
6,140	39,358.0	47,739.5	55,694.3	61,411.1	66,995.4	$\underline{74,655.5}$
6,160	39,355.2	47,737.1	55,693.3	$\underline{61,411.6}$	$\underline{66,995.7}$	74,655.4
6,180	39,351.9	47,734.3	55,691.8	61,411.4	66,995.3	74,654.5
6,200	39,348.1	47,731.0	55,689.8	61,410.7	66,994.3	74,653.0
τ'(hours)	1,000	2,000	3,000	4,000	5,000	6,000
τ_o(hours)	6,042.0	6,061.0	6,115.0	6,164.0	6,159.0	6,146.0

- Usually, many transitions cannot be considered in the execution of maintenance, so the formula developed is fundamental, as it allows the optimum to be calculated when there is a small number of transitions or a specific transition (finite horizon).
- In market situations (think of the value of metal produced in a mine), they could change the value of the income from the operation, without the need for an asset-related event to have occurred. In these cases, the income could increase and consequently the preventive interval.
- Any other failure mode could have been considered, if it tends to wear.
- Any change in costs or income, or in the conditions affecting them, influences the size of the preventive interval.

6. Discussion

The mathematical Eqs. (50), (41), and (47) found for the optimal preventive interval have been verified by four different methods. On the one hand, for a given value of τ', the expected accumulated return has been calculated $\nu_1(m)$ for each transition m and for each τ. Among all the values of τ, the value that maximized the expected accumulated return was chosen. This value always coincided with the value offered by Eq. (50) in both the complex case and the real case. As an example, Table 10 shows some values of $\nu_1(10)$ for different values of τ and τ', marking the best values.

On the other hand, the values obtained for the optimal preventive interval have been compared with those obtained with the equation generated for the three-state model without a degraded state (Sánchez Herguedas et al., 2022). These values always coincide when the same values are used, and $R_4 = R_1$, $R_{42} = R_{12}$, and $R_{43} = R_{13}$ are matched.

Furthermore, the results obtained with the method presented in the paper have been compared with the utilization of Monte Carlo simulation. To that end, a dynamic simulation model has been built using continuous time stochastic simulation. States and possible transitions have been defined, as well as the associated returns to the transitions

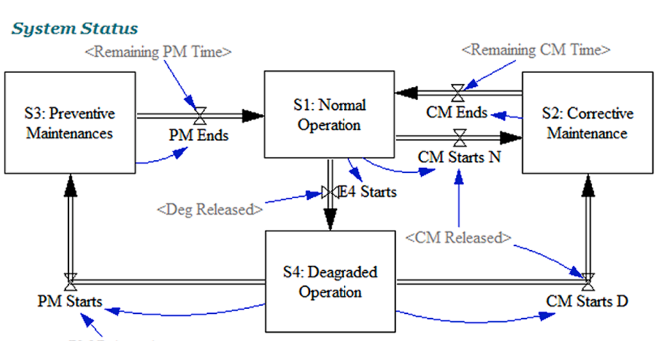

Fig. 5. Possible states and system transitions.

and the permanence in the different states (see Fig. 5 and Fig. 6).

In order to govern transitions, maintenance control must be modeled, including the conditions that must be fulfilled for a given maintenance activity to be released (see Fig. 7) or for the degraded operation state to be reached.

The methodology used can be consulted in (Crespo Márquez, 2010). The variability is introduced by three seeds (see Fig. 7), one for the preventive time (Seed 1), a second for the corrective time (Seed 2), and a third for the random number generating failures according to the Weibull distribution (Seed 3). Difference equations are used in the model and the simulation time step considered is one hour.

200 simulations are done per scenario and six of the many scenarios studied are referred to as presented in Table 11 (with a variation of τ' from 1,000 to 6,000 h) and a selection of the optimum τ as obtained with the model. Each simulation has a time horizon of 30,000 h.

The following Table 11 checks the results obtained in Table 5. The reader can compare how the mean values of the sensitivity results are very close to those obtained in Table 5 for returns after 10 transitions.

In Fig. 8, a graph containing the sensitivity results for one of the scenarios is presented. The selected variable is returned after 10 transitions, and the confidence bounds illustrate the percentile values for the total 200 simulations over the time horizon simulated (30,000 h). The mean value plot is included in red (ending in 60,747, as presented in Table 11).

In this work, only the expected accumulated return has been calculated when the process starts from the operational state, $\nu_1(m)$. With its mathematical expression, the optimal preventive interval has been calculated. In the same way, if other starting states are considered, the equations for $\nu_2(m)$, $\nu_3(m)$ and $\nu_4(m)$ could be determined. The optimal preventive intervals for these starting points can also be calculated, from these expressions.

Finally, Eq. (25) and Eq. (32) have been solved numerically using the Nelder-Mead method. In all cases studied (e.g., with the data in Table 6 and Table 7) the results agree with those obtained by these equations and by Eq. (50).

7. Conclusions

Most equipment installed on ships, vehicles, or industrial plants is subject to wear and tear. The optimal size of the maintenance interval is essential to minimize business maintenance costs. One of the data to be taken into account for its calculation is the income per operating hour. When there is a variation in this value (degraded state), the optimal preventive interval is affected. The engineer responsible for maintenance must detect this fact and modify the preventive interval.

Three gaps were found in the literature, i) few authors design a formula for calculating the preventive interval for cases such as the one exposed; ii) the income during the degraded state is barely considered for its calculation; iii) also, usually, the calculation is not made for a certain period of time. Research has been carried out in search of find-

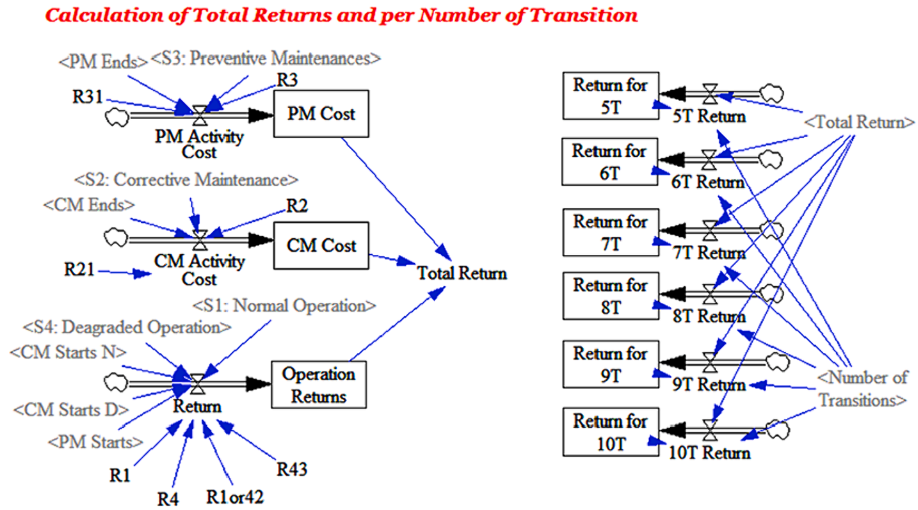

Fig. 6. Modeling returns according to number of transitions.

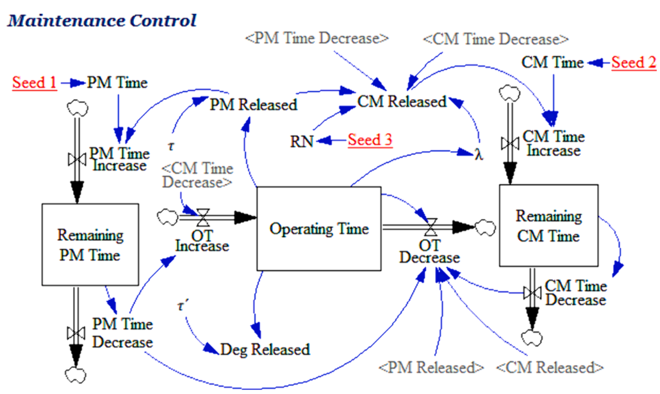

Fig. 7. Modeling operating time and maintenance release.

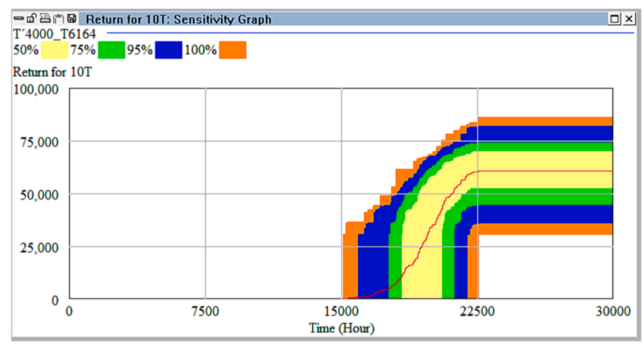

Fig. 8. Sample sensitivity results for the scenario. Confidence bounds in colors and mean value plot in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 11
Monte Carlo simulation. Returns values $v_1(10)$, simulation variable: Return after 10 transitions, for different values of τ and τ'.

Scenario τ' and τ (hours)	Count	Min (€)	Max (€)	Mean (€)	Median (€)	StDev (€)
$\tau' = 6,000$ and $\tau = 6,146$	200	34,210	106,736	75,184	74,397	14,517
$\tau' = 5,000$ and $\tau = 6,159$	200	34,210	93,901	67,427	67,537	11,914
$\tau' = 4,000$ and $\tau = 6,164$	200	30,568	85,901	60,747	61,358	11,869
$\tau' = 3,000$ and $\tau = 6,115$	200	23,716	83,167	55,357	57,067	10,806
$\tau' = 2,000$ and $\tau = 6,061$	200	19,526	74,191	47,232	48,343	10,439
$\tau' = 1,000$ and $\tau = 6,042$	200	11,526	66,021	39,220	40,429	10,627

ings to help improve these gaps. This article develops a mathematical tool or formula to find the optimal preventive interval (first finding) when degradation affects income, or income changes because of the market (second finding). For this purpose, a semi-Markovian model is designed that uses the income and costs that appear because of the stays and transitions in the different states. Any change in costs or income, or in the conditions affecting them, influences the size of the preventive interval. The expected accumulated return is the variable that accounts for them. This variable depends on the transition probabilities, the sojourn times, and the returns of each state. By using z-transforms, it is possible to solve the system of difference equations generated by modeling a four-state system: operational, corrective, preventive, and degraded operational. The system evolves between the states according to a Markov chain embedded in a semi-Markovian process, which determines the sojourn time in each state. The expected accumulated return is influenced by the number of times the preventive task is performed, there being a value of preventive interval τ_o that maximizes the expected accumulated return. The mathematical process developed has two distinct parts that depend on the probability of system failure when the degradation occurs (transition from the S_1 to S_4), presenting one format for real roots and another for complex roots. All roots have a modulus of less than unity.

The two expressions reached depends on the income in the degraded state R_4 and the costs in the degraded state before falling to other states R_{42} and R_{43}. It also depends on the probability of system failure when the degradation occurs p_1 and the costs of a failure $v_2(1)$ and a preventive intervention $v_3(1)$.

Usually, many transitions cannot be considered in the execution of maintenance, so the formulae developed are fundamental, as they allow the optimum to be calculated when there is a small number of transitions or a specific transition (third finding).

Once the tool was obtained, it was applied to a case study. Following an event, a degradation is observed that reduces income. After analyzing the case, the following observations were made:

- The value of income during the degradation phase R_4 is decisive in the economic optimization of the preventive interval. So far, few businesses consider this fact when scheduling their maintenance activities. In addition, the value of income before degradation R_1 was also in the case of the three-states model.
- When the asset degrades, it can still be used, although the initially set preventive interval needs to be reduced.
    - The new optimal preventive interval value approaches the value of the preventive interval if the income R_1 is reduced until it approaches the income of the degraded state.
    - The new optimal preventive interval value will coincide with the value of the preventive interval if the income coincides with the income of the degraded state when we perform many transitions, high values of m.
- In market situations (for example, when the value of metal produced in a mine fluctuates), they could change the value of the income from the operation, without the need for an asset-related event to have occurred. In these cases, the income could increase and, consequently, the preventive interval.
- Any other failure mode could have been considered if it tends to wear.

In today's information-intensive context, the maintenance manager must have the tools to calculate and update the preventive interval of the tasks of his/her physical assets. These tools are easy to apply because they are the same for all assets and only require giving values to the variables. Moreover, it does not require modeling the different types of equipment one by one, nor searching for an optimal solution with complex methods of numerical resolution. These tools are easy to apply when designing digital twins.

Our next work focuses on the possibility of changing the failure distribution function when degradation occurs. Economic data has been used for this development, but other objective functions related to availability, safety, environment, emissions, or fuel consumption could have been used. These tools also must develop in their application to condition-based maintenance. In this case, the states would determine each degradation level. The objective would be to find the sojourn times in each state, which would relate to inspections or monitoring and the appropriate timing of preventive replacement.

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CRediT authorship contribution statement

Antonio Sánchez-Herguedas: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing – original draft, Writing – review & editing. **Angel Mena-Nieto:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Project administration, Resources, Supervision, Writing – review & editing, Methodology. **Adolfo Crespo-Márquez:** Formal analysis, Resources, Supervision, Validation, Writing – review & editing. **Francisco Rodrigo-Muñoz:** Conceptualization, Data curation, Methodology, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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