Monodromy of a class of analytic generalized nilpotent systems through their Newton diagram

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Abstract

Newton diagram of a planar vector field allows to determine whether a singular point of an analytic system is a monodromic singular point. We solve the monodromy problem for the nilpotent systems and we apply our method to a wide family of systems with a degenerate singular point, so-called generalized nilpotent cubic systems

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1. Introduction.

Newton diagrams are important tool for studying singularities of maps and vector fields, see e.g. classical book [9]. We consider the planar analytic differential system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, i.e. the origin is a singular point of the system, and are interested in characterizing, through the Newton diagram of the vector field, when the origin is surrounded by orbits of the system, or, equivalently, we want to determine when the system does not have characteristic orbits at the origin, i.e. trajectories that enter or leave the origin with a fixed tangent. When this occurs the singular point is called monodromic.

From the finiteness theorem for the number of limit cycles, a monodromic point of an analytic planar vector field can be only either a focus (all trajectory by lying on a vicinity of a monodromic singular point is a spiral) or a center (the trajectories are closed orbits that surround at the origin), see Il'yashenko [16]. So, the monodromy problem is a previous step to solve the center problem

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which is one of the classical open problems in the qualitative theory of planar differential systems.

In relation to the monodromy problem, unless the differential matrix $D\mathbf{F}(\mathbf{0})$ is not identically null, the problem was completely solved by Poincaré [19], and Andreev [8] for nilpotent systems. Finally, if $D\mathbf{F}(\mathbf{0})$ is identically null, the origin is a degenerate singular point, there are only a few partial results: Algaba *et* al. [6], García *et al.* [13], Gasull *et al.* [14], Mañosa [17] and Medvedeva [18], among others. Recently, Algaba *et* al. [7] give an algorithm based on the main result of [6] and as application they solve the monodromy problem for the family

$$\dot{x} = ay^3 + cxy^2 + gx^2y + ex^5, \qquad \dot{y} = dy^3 + hxy^2 + fx^4y + bx^7,$$

with $ab(g^2 + h^2) > 0$, which extends to the family studied in Medvedeva [18]. We emphasize that all of them are families of polynomial systems.

Here, we deal with the analytic case. From [6, Theorem 3], if origin of (1) is monodromic, then the system is of the form

$$\dot{x} = ay^{2n-1} + f(x, y), \qquad \dot{y} = bx^{2m-1} + g(x, y), \tag{2}$$

with $ab < 0$ and f and g are analytic functions satisfying $\partial f/\partial y(0, 0) = \partial^j f/\partial y^j(0, 0) = 0$, for $j = 1, ..., 2n - 1$, and $\partial g/\partial x(0, 0) = \partial^j g/\partial x^j(0, 0) = 0$, for $j = 1, ..., 2m - 1$. Moreover, we can suppose that $n \leq m$.

In this paper we solve the monodromy problem for analytic nilpotent systems using the Newton diagram of the vector field, i.e. systems (2) with $n = 1$. This result is stated in Theorem 3. For systems (2) with $n = 2$, so-called generalized nilpotent cubic systems, Theorems 5, 6 and 7 characterize when the lowest degree quasi-homogeneous term of vector field determine its monodromy. Otherwise, Propositions 9 and 10 establish when the systems with invariant axis $x = 0$ and whose Newton diagram has the inner vertex $(1, 2)$, do not have any characteristic orbits different from $x = 0$, which allows us to determine the monodromy of the generalized nilpotent cubic systems, after performing a blow-up. Theorem 8 summarizes this result.

In short, we solve the monodromy problem for the analytic systems (2) with $n = 1$(solved before by Andreev) and $n = 2$. For $n \geq 3$, by using our techniques, it is possible to characterize the monodromy in the cases which the lowest-degree quasi-homogeneous term determines it. The remaining cases only we can to provide necessary conditions of monodromy. In general, it is necessary the performing of a series of blows up to be able to determine the monodromy. So, in general, the problem remains open for analytic systems.

2. Preliminaries

To show our results, we recall some concepts and results, which we use throughout the paper.

Conservative-dissipative splitting.

Let $\mathbf{t} = (t_1, t_2)$ non-null with t_1 and t_2 non-negative integer numbers without common factors. A function f of two variables is quasi-homogeneous of type $\mathbf{t}$ and degree k if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (F_1, F_2)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $F_1 \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k.

The quasi-homogeneous vector monomials can be determined by drawing the lattice $\mathbb{Z}_+^2$, and assigning each point (m,n) to the quasi-homogeneous vector fields $(x^m y^{n-1}, 0)^T$ and $(0, x^{m-1}y^n)^T$. The points with integer coordinates aligned in the straight lines perpendicular to $\mathbf{t}$, $(m-1)t_1 + (n-1)t_2 = k$, determine the quasi-homogeneous vector monomials with the same degree k.

Any vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, system (1) can be written in the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r^{\mathbf{t}}(\mathbf{x}) + \mathbf{F}_{r+1}^{\mathbf{t}}(\mathbf{x}) + \cdots = \sum_{j=0}^{\infty} \mathbf{F}_{r+j}^{\mathbf{t}}(\mathbf{x}),$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j^{\mathbf{t}} = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r^{\mathbf{t}} \not\equiv \mathbf{0}$. Such expansions are expressed as $\mathbf{F} = \mathbf{F}_r^{\mathbf{t}} + \text{q-h.h.o.t.}$

Such expansions are valuable tools for analysing the singularity, see Dumortier [12]. This concept also is used for the study of the integrability, the center problem and the existence of an inverse integrating factor of systems with a degenerate singular point, i.e. systems whose matrix of the linear part evaluated in the singular point is identically null, see [4, 5, 3].

Next, we show the splitting of a quasi-homogeneous vector field as a sum of two quasi-homogeneous vector fields, a conservative one (having zero-divergence) and a dissipative one that plays a main role in our analysis. Throughout this paper, the Hamiltonian system associated to the $\mathcal{C}^1$ function f is denoted by $\mathbf{X}_f$, i.e. $\mathbf{X}_f = (-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})^T$. Algaba et. al. [5] have proved that any quasi-homogeneous vector field $\mathbf{F}_j^{\mathbf{t}} = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_j^{\mathbf{t}} = \mathbf{X}_{h_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0^{\mathbf{t}}, \tag{3}$$

where $\mathbf{D}_0^{\mathbf{t}}(x,y) := (t_1 x, t_2 y)^T$ (a dissipative quasi-homogeneous vector field of type $\mathbf{t}$ and degree 0), $(j + |\mathbf{t}|)\mu_j := \operatorname{div}(\mathbf{F}_j^{\mathbf{t}}) \in \mathcal{P}_j^{\mathbf{t}}$ (divergence of $\mathbf{F}_j^{\mathbf{t}}$), $(j + |\mathbf{t}|)h_{j+|\mathbf{t}|} := t_1 x Q_{j+t_2} - t_2 y P_{j+t_1} \in \mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$ (wedge product of $\mathbf{D}_0^{\mathbf{t}}$ and $\mathbf{F}_j^{\mathbf{t}}$) and $|\mathbf{t}| = t_1 + t_2$.

It is a simple matter to show that any non-vanishing quasi-homogeneous polynomial of type $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 non-null, in particular $h_{j+|\mathbf{t}|}$, can be expressed as $p(x,y) = x^{k_1} y^{k_2} p_0(x^{t_2}, y^{t_1})$ with $0 \le k_1 < t_2$, $0 \le k_2 < t_1$ being p_0 a homogeneous polynomial. So, by abusing the notation, it is possible to write any quasi-homogeneous polynomial of type $\mathbf{t}$ in a compact form $p(x,y) = c \prod_{j=0}^{m} f_j^{m_j} \prod_{j=0}^{n} g_j^{n_j}$, where

$$f_j(x,y) = x,\ y \ \text{ or } y^{t_1} - \lambda_j x^{t_2}, \qquad j = 0, ..., m$$

and

$$g_j(x,y) = (y^{t_1} - a_j x^{t_2})^2 + b_j^2 x^{2t_2}, \qquad j = 0,...,n$$

with c, λ_j, a_j and b_j real numbers and λ_j, b_j non-zero, for all j.
If $h_{r+|\mathbf{t}|} \in \mathcal{P}^{\mathbf{t}}_{r+|\mathbf{t}|}$ and $\mu_r \in \mathcal{P}^{\mathbf{t}}_r$ are the polynomials associated to the lowest-degree quasi-homogeneous term of type $\mathbf{t}$ of $\mathbf{F}$, we say that a polynomial of the form $x,\ y$ or $y^{t_1} - \lambda x^{t_2},\ \lambda \neq 0$, is a *strong factor of* $\mathbf{F}$ *associated to the type* $\mathbf{t}$, or simply a *strong factor of* $h_{r+|\mathbf{t}|}$, if it satisfies one of the following conditions:

(i) it is a factor of $h_{r+|\mathbf{t}|}$ of odd multiplicity order,

(ii) it is a factor of $h_{r+|\mathbf{t}|}$ of even multiplicity order $(2m)$ and, either it is not a factor of μ_r with $\mu_r \not\equiv 0$ or is a factor of μ_r with even multiplicity order $(2n)$ with $0 < n < m$.

Otherwise, it is a non-strong factor.

Newton diagram.

We write the components of the vector field $\mathbf{F}$ in the form $P(x,y) = \sum a_{ij} x^i y^{j-1}$ and $Q(x,y) = \sum b_{ij} x^{i-1} y^j$.
The *support* of (1) and also of $\mathbf{F}$, denoted by *supp*$(\mathbf{F})$, is the set of ordered pairs (i,j) with $(a_{ij}, b_{ij}) \neq (0,0)$. The vector (a_{ij}, b_{ij}) is called the *vector coefficient* of (i,j) in the support. Consider the set $\bigcup_{(i,j)\in supp(\mathbf{F})} \left((i,j) + \mathbb{R}^2_+\right)$, where $\mathbb{R}^2_+$ is the positive quadrant and the union is taken over all points (i,j) in the support. The boundary of the convex hull of this set is made up of two open rays and a polygon, which can be just one point. Polygon together with the rays that do not lie on a coordinates axes, if they exist, is called *Newton diagram* of the vector field $\mathbf{F}$. The component parts of the Newton diagram are called *edges* and their endpoints are the *vertices* of the Newton diagram.
Unless a vertex of Newton diagram lies on a coordinates axis, then it is said to be *inner*; otherwise, it is an *exterior* vertex. Figure 1 shows two distinct Newton diagrams with two edges.

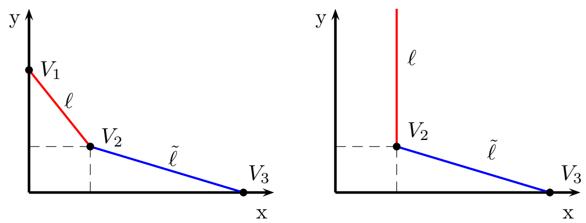

Figure 1: Newton diagram with two vertices.

Lastly, for each inner vertex V of Newton diagram of system (1) such that $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$ are the types of its upper and lower adjacent edges, respectively, i.e. $t_2/t_1 < s_2/s_1$, with $h_{r_\mathbf{t}+|\mathbf{t}|} h_{r_s+|\mathbf{s}|} \not\equiv 0$, it defines the non-null constant

$$\beta_V = c_{i_0} \tilde{c}_{j_0}, \tag{4}$$

where $i_0 = \min\{i \geq 0 \,|c_i \neq 0\}$, $j_0 = \min\{j \geq 0 \,|\tilde{c}_j \neq 0\}$ and c_i and $\tilde{c}_j$ are the coefficients of the polynomials $h_{r_{\mathbf{t}}+|\mathbf{t}|}$ and $h_{r_{\mathbf{s}}+|\mathbf{s}|}$, ordered from the highest to the lowest exponent in x and y, respectively. When $\beta_V < 0$, it says that V is a *characteristic vertex.*
Otherwise, it is a non-characteristic vertex.

Newton diagram and monodromy.

We cite the following result which provides necessary and sufficient conditions of monodromy for a singular point, see Theorems 3 and 4 of [6].

Theorem 1. *If the origin of system (1) is monodromic then Newton diagram of (1) verifies:*

1) *all its vertices have even coordinates,*
2) *it has two exterior vertices. Moreover, the vector coefficients of these vertices, $(a, 0)$ and $(0, b)$, verify $ab < 0$,*
3) *all its inner vertices are non-characteristic,*
4) *for each bounded edge, its associated Hamiltonian is non-null and does not have any strong factor.*

Reciprocally, if the Newton diagram of (1) verifies 1), 2), 3) and

4*) *for each bounded edge, its associated Hamiltonian $h_{r+|\mathbf{t}|}$ is non-null and does not have any factor of the form $y^{t_1} - \tilde{a}x^{t_2}$ with $\tilde{a}$ non-zero real,*

then the origin of system (1) is monodromic.

Summarizing, from Theorem 1, it follows that the Newton diagram of (1) determines the monodromy of the origin except the case of existing a non-strong factor of $\mathbf{F}$ different from x and y. For these cases, we deal with blow-up techniques (developed by Dumortier [12]) which consists of performing a series of changes to desingularize the point) and act of the following way:
For each non-strong factor of $h_{r+|\mathbf{t}|}$ of the form $y^{t_1} - \tilde{a}x^{t_2}$ with $\tilde{a} \neq 0$,
• if t_1 is odd, we perform the directional blow-up $x = u^{t_1}$, $y = u^{t_2}(\bar{y} + \tilde{a}^{1/t_1})$ and the reparameterization $dt = (t_1/u^r)d\tau$. System is transformed into

$$\begin{aligned} u' &= u\sum_{j=0}^{\infty} P_{r+j+t_1}(1, \bar{y} + \tilde{a}^{1/t_1})u^j, \\ \bar{y}' &= \sum_{j=0}^{\infty}(r + |\mathbf{t}| + j)h_{r+j+|\mathbf{t}|}(1, \bar{y} + \tilde{a}^{1/t_1})u^j. \end{aligned} \tag{5}$$

• if t_1 is even, we apply the directional blow-up $x = u^{t_1}(\bar{x} + \tilde{a}^{-1/t_2})$, $y = u^{t_2}$ and the reparameterization $dt = (t_2/u^r)d\tau$. We obtain the system

$$\begin{aligned} u' &= -u\sum_{j=0}^{\infty} Q_{r+j+t_2}(\bar{x} + \tilde{a}^{-1/t_2}, 1)u^j, \\ \bar{x}' &= \sum_{j=0}^{\infty}(r + |\mathbf{t}| + j)h_{r+j+|\mathbf{t}|}(\bar{x} + \tilde{a}^{-1/t_2}, 1)u^j. \end{aligned} \tag{6}$$

The axis $u = 0$ is invariant for both systems (5) and (6). On the other hand, if there exist characteristic orbits of (5) or (6) different from axis $u = 0$ lying on first or fourth quadrant, then system (1) has a characteristic orbit associated to the factor $y^{t_1} - \tilde{a}x^{t_2}$.

3. Monodromic nilpotent analytic system

Consider the systems

$$\dot{x} = y + f(x, y), \ \dot{y} = g(x, y), \tag{7}$$

where f and g are real analytic functions starting with at least quadratic terms, so-called nilpotent systems. This section is devoted to determine under which conditions the origin is monodromic. As mentioned before, the monodromy problem for singular points of this type was solved by Andreev [8]. We recall here the Andreev's theorem.

Theorem 2 (Andreev). *Let $y = \phi(x)$ be the solution of the equation $y + f(x, y) = 0$ passing through the origin. Assume that $g(x, \phi(x)) = \alpha_k x^k + O(x^{k+1})$ and $\Delta(x) = (\partial f/\partial x + \partial g/\partial y)(x, \phi(x)) = \beta_n x^n + O(x^{n+1})$ with $\alpha_k \neq 0, k \geq 2$ and $n \geq 1$. Then, the origin is monodromic if and only if k is odd, $\alpha_k < 0$ and one of the following statements hold:*

- $k = 2n + 1$ *and* $\beta_n^2 + 4\alpha_k(n + 1) < 0$,
- $k < 2n + 1$,
- $\Delta(x) \equiv 0$.

It is a result very easy and only is necessary the first terms of the expansion of the solution $y = \phi(x)$ into implementing it.

We now give our result which consists of computing a pre-normal form.

Theorem 3. *The origin of system (7) is monodromic if and only if can be transformed by means of a sequence of quasi-homogeneous degree-zero changes of variables into*

$$(\dot{x}, \dot{y}) = (y + a_0 x^\alpha, b_1 x^{\alpha-1} y + b_0 x^{2\alpha-1}) + \textit{q-h.h.o.t.}$$

with $\alpha > 1, \ b_1 = \alpha a_0$ and $b_0 < 0$.

Proof. The point $(0, 2)$ is an exterior vertex of the Newton diagram of system (7). Figure 2 shows the different Newton diagrams of these systems.

First and second diagrams correspond to systems with invariant axis $y = 0$, and the third one has an inner vertex $(\gamma, 1)$ with odd ordinate. So, if we are looking for monodromic nilpotent systems, from Theorem 1, these have the fourth Newton diagram, that is, it consists of two exterior vertices $(0, 2)$ and $(2\alpha, 0)$, and an unique bounded edge of the type $\mathbf{t} = (1, \alpha)$ with α natural number greater than one.

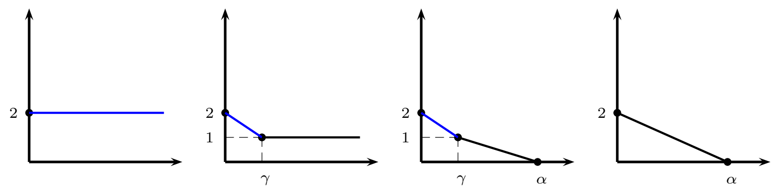

Figure 2:

Consequently, we limit our study to systems whose expansion into quasi-homogeneous vector fields respect to the type $(1,\alpha)$, type associated to the unique edge of their Newton diagram, is of the form

$$\dot{\mathbf{x}} = \mathbf{F}_{\alpha-1}^{(1,\alpha)}(\mathbf{x}) + \text{q-h.h.o.t.} \tag{8}$$

with $\mathbf{F}_{\alpha-1}^{(1,\alpha)}(x,y) = (y + a_0x^{\alpha}, b_1x^{\alpha-1}y + b_0x^{2\alpha-1})^T$ and $b_0 \neq 0$.

We claim that the change of variables $(x,y) \to (x, y - \frac{1}{2\alpha}(b_1 - \alpha a_0)x^{\alpha}) \in \mathcal{Q}_0^{(1,\alpha)}$ brings system (8) to $\dot{\mathbf{x}} = \tilde{\mathbf{F}}_{\tilde{\alpha}-1}^{(1,\tilde{\alpha})}(\mathbf{x}) + \text{q-h.h.o.t.}$ with

$$\tilde{\mathbf{F}}_{\tilde{\alpha}-1}^{(1,\tilde{\alpha})}(x,y) = (y + \tilde{a}_0x^{\tilde{\alpha}}, \tilde{b}_1x^{\tilde{\alpha}-1}y + \tilde{b}_0x^{2\tilde{\alpha}-1})^T,$$

$\tilde{b}_1 = \tilde{\alpha}\tilde{a}_0 = \frac{1}{2}(b_1 + \alpha a_0)$ and $\tilde{b}_0 = \frac{1}{4\alpha}[4\alpha b_0 + (b_1 - \alpha a_0)^2]$.

If the origin is a monodromic singular point, the Newton diagram of the new system consists of two exterior vertices $(0,2)$ and $(2\tilde{\alpha}, 0)$, and an unique bounded edge of the type $\tilde{\mathbf{t}} = (1,\tilde{\alpha})$, being $\alpha = \tilde{\alpha}$ if $\tilde{b}_0 \neq 0$, and $\alpha < \tilde{\alpha}$, otherwise.

Summarizing, we assume that a monodromic nilpotent system can be transformed by means of a sequence of quasi-homogeneous changes of variables of quasi-homogeneous degree zero into system (8) with $b_1 = \alpha a_0$ and $b_0 \neq 0$. From (3), it has that $\mathbf{F}_{\alpha-1}^{(1,\alpha)} = \mathbf{X}_{h_{2\alpha}} + \mu_{\alpha-1}\mathbf{D}_0^{(1,\alpha)}$ with

$$2\alpha h_{2\alpha}(x,y) = -\alpha y^2 + b_0x^{2\alpha},\ \mu_{\alpha-1}(x,y) = a_0x^{\alpha-1}.$$

By applying Theorem 1, it follows that if $b_0 < 0$ then the origin of system (8) is monodromic. If $b_0 > 0$, then there is a strong factor and therefore the origin is not monodromic. $\square$

We illustrate our analysis with the seven-parameter family of nilpotent systems

$$\begin{aligned} \dot{x} &= y + a_0x^2 + a_1xy, \\ \dot{y} &= b_0x^2 + b_1xy + c_0x^3 + d_0x^4 + e_0x^5. \end{aligned} \tag{9}$$

Theorem 4. *The origin of system (9) is monodromic if and only if it satisfies one of the following series of conditions:*

1) $b_0 = 0,\ c_0 < -\frac{1}{8}(b_1 - 2a_0)^2$,
2) $b_0 = 0,\ b_1 = -2a_0,\ c_0 = -2a_0^2,\ d_0 = 2a_0^2a_1, e_0 < -\frac{25}{12}a_0^2a_1^2$.

Proof. If $b_0 \neq 0$, the Newton diagram of system (9) consists of a bounded edge with two exterior vertices, $(0, 2)$ and $(3, 0)$. From [Theorem 1, item 1)], the origin of system (9) is not a monodromic singular point.
We assume that $b_0 = 0$ and distinguish two cases:
• Case $c_0 \neq 0$. Newton diagram consists of a bounded edge of type $(1, 2)$ with two exterior vertices, $(0, 2)$ and $(4, 0)$. So, system (9) can be expanded as $(\dot{x}, \dot{y})^T = \mathbf{F}_1^{(1,2)} + \mathbf{F}_2^{(1,2)} + \mathbf{F}_3^{(1,2)}$ with $\mathbf{F}_1^{(1,2)} = (y + a_0x^2, b_1xy + c_0x^3)^T$, $\mathbf{F}_2^{(1,2)} = (a_1xy, d_0x^4)^T$ and $\mathbf{F}_3^{(1,2)} = (0, e_0x^5)^T$.
If $b_1 = 2a_0$, from Theorem 3, origin of system (9) is monodromic if and only if $c_0 < 0$ (case 1) for $b_1 = 2a_0$). Otherwise, by performing the change of variables $x = x, v = y - 1/4(b_1 - 2a_0)x^2$, system (9) is transformed into

$$\begin{aligned} \dot{x} &= v + \tfrac{1}{4}(b_1 + 2a_0)x^2 + a_1xv + \tfrac{1}{4}a_1(b_1 - 2a_0)x^3, \\ \dot{v} &= \tfrac{1}{2}(b_1 + 2a_0)xv - \tfrac{1}{2}a_1(b_1 - 2a_0)x^2v + (c_0 + \tfrac{1}{8}(b_1 - 2a_0)^2)x^3 \\ &\qquad + (d_0 - \tfrac{1}{8}a_1(b_1 - 2a_0)^2x^4 + e_0x^5. \end{aligned} \tag{10}$$

If $c_0 \neq -\frac{1}{8}(b_1 - 2a_0)^2$, by Theorem 3, it follows that the origin of system (10) is monodromic if and only if $c_0 < -\frac{1}{8}(b_1 - 2a_0)^2$, (case 1)). Otherwise, $c_0 = -\frac{1}{8}(b_1 - 2a_0)^2$, it has that if $b_1 + 2a_0 \neq 0$, Newton diagram of system (10) consists of two edges and a inner vertex $(2, 1)$ associated to the vector field $(\frac{1}{4}(b_1 + 2a_0)x^2, \frac{1}{2}(b_1 + 2a_0)xv)^T$. So, by [Theorem 1, item 1)], origin is not monodromic. Otherwise, $b_1 = -2a_0$, system (10) is

$$\begin{aligned} \dot{x} &= v + a_1xv - a_1a_0x^3, \\ \dot{v} &= 2a_1a_0x^2v + (d_0 - 2a_0^2a_1)x^4 + e_0x^5. \end{aligned} \tag{11}$$

If $d_0 \neq 2a_0^2a_1$, Newton diagram of system (11) has the exterior vertex $(5, 0)$, and by [Theorem 1, item 1)], it follows that the origin is not a monodromic singular point. Otherwise, $e_0 \neq 0$ (since if $e_0 = 0$ system has an invariant axis). In such a case, Newton diagram of system (11) consists of a bounded edge of type $(1, 3)$ with two exterior vertices, $(0, 2)$ and $(6, 0)$. From Theorem 3, if a_0 or a_1 is zero, then the origin of system (11) is monodromic if and only if $e_0 < 0$. Otherwise, by performing the change of variables $x = x, w = v - 5/6a_0a_1x^3$, system (11) is transformed into

$$\begin{aligned} \dot{x} &= w - \tfrac{1}{6}a_1a_0x^3 + a_1xw + \tfrac{5}{6}a_1^2a_0x^4, \\ \dot{w} &= -\tfrac{1}{2}a_1a_0x^2w - \tfrac{5}{2}a_0a_1^2x^3w + (e_0 + \tfrac{25}{12}a_1^2a_0^2)x^5 - \tfrac{25}{12}a_0^2a_1^3x^6. \end{aligned} \tag{12}$$

Applying Theorem 3, it deduces that the origin of system (12) is a monodromic singular point if and only if $e_0 < -\frac{25}{12}a_1^2a_0^2$, (case 2)).
• Case $c_0 = 0$. The coefficient d_0 is zero since, otherwise, Newton diagram has a vertex with odd abscissa, and in such a case both a_0 and b_1 also are zero since they are associated to an inner vertex with odd ordinate. So, it arrives to case 2) for $a_0 = 0$. □

4. Analytic generalized nilpotent cubic monodromic systems.

Consider the systems

$$\dot{x} = y^3 + f(x, y),\ \ \dot{y} = g(x, y), \tag{13}$$

where f and g are real analytic functions with $f(0, y)$ starting with at least cuartic terms, i.e. $\partial f/\partial y(0, 0) = \partial^2 f/\partial y^2(0, 0) = \partial^3 f/\partial y^3(0, 0) = 0$. That is, f and g also can to have linear, quadratic and cubic terms.

These systems are called generalized nilpotent cubic systems. The center and integrability problems of such systems is studied in [15].

Our proposal is to characterize the systems whose origin is monodromic.

In our analysis, we assume that $\partial g/\partial x(0, 0) = 0$, since, otherwise, the system is clearly a nilpotent system, which has been study before.

Newton diagrams of such systems verifying the conditions 1) and 2) of Theorem 1 have two exterior vertex $(0, 4)$ and $(2\alpha, 0)$ and either can have the inner vertex $(2\gamma, 2)$, $2\gamma < \alpha$, or does not have any inner vertex. Moreover, in the last case, if α is odd at most there is one support point on the bounded edge and if α is even, there are three support point on the edge at most, see figure 3.

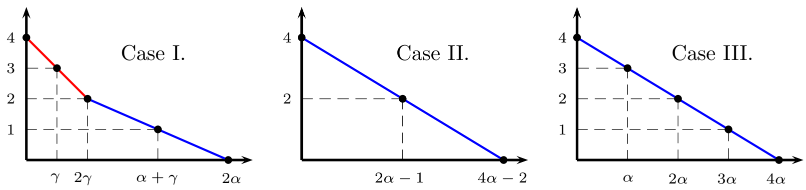

Figure 3: Newton diagrams of the monodromic systems with $(0, 4)$ exterior vertex.

We devote our study to finding conditions of monodromy. We analyze each case separately.

Case I. The expansions into quasi-homogeneous vector fields of these systems, respect to the types associated to the edges of their Newton diagram, $(1, \gamma)$ and $(1, \alpha - \gamma)$, are, respectively,

$$\begin{aligned} \dot{\mathbf{x}} \ &= \mathbf{F}(\mathbf{x}) &&= \mathbf{F}_{3\gamma-1}^{(1,\gamma)}(\mathbf{x}) + \text{q-h.h.o.t.} \\ & &&= \mathbf{F}_{\alpha+\gamma-1}^{(1,\alpha-\gamma)}(\mathbf{x}) + \text{q-h.h.o.t.} \end{aligned} \tag{14}$$

with $2\gamma < \alpha$, $(b_2, a_1) \neq (0,0)$ and

$$\begin{aligned}
\mathbf{F}^{(1,\gamma)}_{3\gamma-1}(x,y) &= (y^3 + a_2 x^{\gamma} y^2 + a_1 x^{2\gamma} y, b_3 x^{\gamma-1} y^3 + b_2 x^{2\gamma-1} y^2)^T, \\
\mathbf{F}^{(1,\alpha-\gamma)}_{\alpha+\gamma-1}(x,y) &= (a_1 x^{2\gamma} y + a_0 x^{\alpha+\gamma}, b_2 x^{2\gamma-1} y^2 + b_1 x^{\alpha+\gamma-1} y - x^{2\alpha-1})^T.
\end{aligned}$$

If the origin is monodromic, by a change of variables, it is possible to assume that $b_1 = (\alpha-\gamma)a_0$. Indeed, the associated Hamiltonian function of $\mathbf{F}^{(1,\alpha-\gamma)}_{\alpha+\gamma-1}$ is given by

$$2\alpha h_{2\alpha}(x,y) \quad = \quad (b_2 - (\alpha-\gamma)a_1) x^{2\gamma} y^2 + (b_1 - (\alpha-\gamma)a_0) x^{\gamma+\alpha} y - x^{2\alpha}.$$

If $b_2 - (\alpha-\gamma)a_1 \neq 0$, the change of variables $(x,y) \to (x, y + \frac{b_1-(\alpha-\gamma)a_0}{2(b_2-(\alpha-\gamma)a_1)} x^{\alpha-\gamma}) \in \mathcal{Q}^{(1,\alpha-\gamma)}_0$ transforms system (14) into a new system ,whose Newton diagram is also a diagram of type case I and $b_1 = (\alpha-\gamma)a_0$. On the other hand, if $b_2 - (\alpha-\gamma)a_1 = 0$, then the origin is monodromic if $b_1 - (\alpha-\gamma)a_0 = 0$ since, otherwise, $x^{\alpha-\gamma} - (b_1 - (\alpha-\gamma)a_0)y$ is a simple factor of $h_{2\alpha}$, that is, from Theorem 1 it follows that the origin of (14) is not monodromic.

The following result states the monodromy of (14).

Theorem 5. *If the origin of system (14) with $2\gamma < \alpha$ and $b_1 = (\alpha-\gamma)a_0$ is monodromic, then one of the following series of conditions is satisfied:*

1) $b_2 - (\alpha-\gamma)a_1 \leq 0$, $(b_3 - \gamma a_2)^2 + 4\gamma(b_2 - \gamma a_1) < 0$,
2) $b_2 = \gamma a_1$, $b_3 = \gamma a_2$, $a_1 > 0$.
3) $4\gamma^2 a_1 = (\gamma a_2 - b_3)(b_3 + \gamma a_2)$, $2\gamma b_2 = b_3(\gamma a_2 - b_3)$, $(b_3 - \gamma a_2)((\alpha - 3\gamma)b_3 + \gamma(\alpha-\gamma)a_2) \leq 0$.

Moreover, if 1) or 2) is satisfied, then the origin of system (14) is monodromic.

Proof. According to the types chosen, the lowest-degree quasi-homogeneous term of $\mathbf{F}$ are $\mathbf{F}^{(1,\alpha-\gamma)}_{\alpha+\gamma-1}$ and $\mathbf{F}^{(1,\gamma)}_{3\gamma-1}$. By (3), these have the form $\mathbf{F}^{(1,\alpha-\gamma)}_{\alpha+\gamma-1} = \mathbf{X}_{h_{2\alpha}} + \mu_{\alpha+\gamma-1}\mathbf{D}^{(1,\alpha-\gamma)}_0$, with

$$\begin{aligned}
2\alpha h_{2\alpha}(x,y) &= x^{2\gamma}\left[(b_2 - (\alpha-\gamma)a_1) y^2 - x^{2(\alpha-\gamma)}\right], \\
2\alpha \mu_{\alpha+\gamma-1}(x,y) &= x^{2\gamma-1}\left[2(b_2 + \gamma a_1) y + 2\alpha a_0 x^{\alpha-\gamma}\right],
\end{aligned}$$

and $\mathbf{F}^{(1,\gamma)}_{3\gamma-1} = \mathbf{X}_{h_{4\gamma}} + \mu_{3\gamma-1}\mathbf{D}^{(1,\gamma)}_0$ with

$$\begin{aligned}
4\gamma h_{4\gamma}(x,y) &= (b_2 - \gamma a_1) x^{2\gamma} y^2 + (b_3 - \gamma a_2) x^{\gamma} y^3 - \gamma y^4 \\
&= -\gamma y^2 \left[\left(y - \tfrac{b_3 - \gamma a_2}{2\gamma} x^{\gamma}\right)^2 - \tfrac{\Delta_u}{4\gamma^2} x^{2\gamma}\right], \\
4\gamma \mu_{3\gamma-1}(x,y) &= x^{\gamma-1} y \left[(3b_3 + \gamma a_2) y + 2(b_2 + \gamma a_1) x^{\gamma}\right],
\end{aligned}$$

where $\Delta_u = (b_3 - \gamma a_2)^2 + 4\gamma(b_2 - \gamma a_1)$.

Note that $b_2 - (\alpha-\gamma)a_1$ and Δ_u are the discriminants of $h_{2\alpha}$ and $h_{4\gamma}$, respectively, and therefore, we can analyze their factors by according the sign of them:

a) If $b_2-(\alpha-\gamma)a_1>0$ or $\Delta_u>0$, from Theorem 1, the origin of system (14) is not monodromic since $h_{2\alpha}$ or $h_{4\gamma}$ has simple factors.
b) We assume that $\Delta_u<0$. in such a case $b_2<\gamma a_1$. If $b_2-(\alpha-\gamma)a_1<0$, then the inner vertex $V=(2\gamma,2)$ is not characteristic since $\beta_V>0$; thus, by applying Theorem 1, the origin is monodromic (case 1)). If $b_2-(\alpha-\gamma)a_1=0$, it has that $h_{2\alpha}=-x^{2\alpha}$, i.e. $\beta_V=\gamma a_1-b_2>0$. Again, the origin is a monodromic point (case 1)).
c) We assume that $\Delta_u=0$. If $b_2-\gamma a_1=0$, we claim that $b_3-\gamma a_2=0$ and $b_2-(\alpha-\gamma)a_1=(2\gamma-\alpha)a_1$. If $a_1<0$, then the origin is not monodromic. If $a_1>0$, it has that $h_{4\gamma}(x,y)=-\gamma y^4$, $\mu_u=(a_2y+a_1x^\gamma)x^{\gamma-1}y\not\equiv 0$ and $\beta_V=-\gamma(2\gamma-\alpha)a_1>0$. Thus, the conditions of Theorem 1 are satisfied (case 2)).
d) If $\Delta_u=0$, $b_2-(\alpha-\gamma)a_1\leq 0$ and $b_2-\gamma a_1<0$, then $y-\frac{b_3-\gamma a_2}{2\gamma}x^\gamma$ with $b_3-\gamma a_2\neq 0$, is a factor of $h_{4\gamma}$ whose multiplicity order is two. The origin will be monodromic if either the divergence of the vector field $\mathbf{F}^{(1,\gamma)}_{3\gamma-1}$ is identically zero or the above factor is a factor of the divergence of the vector field. So,

$$(b_3-\gamma a_2)(3b_3+\gamma a_2)+4\gamma(b_2+\gamma a_1)=0.$$

It is easy to check that this one and above conditions arrive to 3). □

Remark 1. Under the assumption of case 3) of Theorem 5, the polynomial $y-\frac{b_3-\gamma a_2}{2\gamma}x^\gamma$ with $b_3-\gamma a_2\neq 0$, is a non-strong factor of $h_{4\gamma}$. The first term of quasi-homogeneous expansion of type $(1,\gamma)$ does not determine the monodromy of the origin. So, for determining if the origin is monodromic, we perform the blow-up $x=u$, $y=u^\gamma\left(\bar{y}+\frac{b_3-\gamma a_2}{2\gamma}\right)$, $dt=\gamma\frac{d\tau}{u^{3\gamma-1}}$. System (14) is transformed into a new system with invariant axis $u=0$ and the ordered pair $(1,2)$ in the support of the vector field whose vector coefficient is $(a_1,b_2-(\alpha-\gamma)a_1)\neq(0,0)$.

Newton diagrams having to $(1,2)$ as an inner vertex and whose associated system has one invariant axis, $u=0$, are drawn in figure 4.

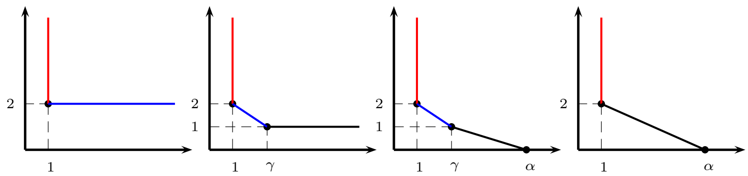

Figure 4:

The two firsts diagrams correspond to systems with invariant axis $y=0$, and the third one has an inner vertex with odd ordinate. We emphasize that Algaba *et al.* [6] give a result for the system with invariant axis $u=0$ and without characteristic orbits different from $u=0$ similar to Theorem 1. Speaking robustly, the difference lies in clearing property 2) of Theorem 1 and changing property 1) of Theorem 1 by "all its inner vertices have even ordinate".

So, if we are looking for systems whose Newton diagram has the inner vertex $V = (1, 2)$ and without characteristic orbits different from $u = 0$, these have the fourth Newton diagram, that is, it consists of the inner vertex V associated to the vector field $(A_1uy, B_2y^2)^T$, one exterior vertex $\bar{V} = (\alpha + 1, 0)$ associated to $(0, b_0u^\alpha)^T$ and an unique bounded edge of type $(2, \alpha)$. The expansion into quasi-homogeneous terms of these systems is $\dot{\mathbf{u}} = \mathbf{F}_\alpha^{(2,\alpha)}(\mathbf{u}) + \cdots$

Propositions 9 and 10 characterize these systems without characteristic orbits different from $u = 0$, see Appendix 1. Therefore, the monodromy problem for the system (14) is solved.

Case II. The expansion into quasi-homogeneous polynomials of these systems respect to the type $(2, 2\alpha-1)$, type associated to the unique edge of their Newton diagram, is

$$\dot{\mathbf{x}} = \mathbf{F}_{6\alpha-5}^{(2,2\alpha-1)}(\mathbf{x}) + \text{q-h.h.o.t.} \tag{15}$$

with

$$\mathbf{F}_{6\alpha-5}^{(2,2\alpha-1)}(x, y) = (y^3 + a_1x^{2\alpha-1}y, b_2x^{2\alpha-2}y^2 - x^{4\alpha-3})^T.$$

Theorem 6. *If the origin of system (15) is monodromic, then one of the following series of conditions is satisfied:*

1) $(2b_2 - (2\alpha - 1)a_1)^2 < 8(2\alpha - 1)$,
2) $b_2 = -\frac{1}{4}, \quad a_1 = \frac{1}{2(2\alpha-1)}$.

Moreover, if 1) is satisfied, then the origin of system (15) is monodromic.

Proof. The lowest-degree quasi-homogeneous term is $\mathbf{F}_{6\alpha-5}^{(2,2\alpha-1)} = \mathbf{X}_{h_{8\alpha-4}} + \mu_{6\alpha-5}\mathbf{D}_0^{(2,2\alpha-1)}$, with

$$\begin{aligned} (8\alpha - 4)h_{8\alpha-4}(x, y) &= -(2\alpha - 1)y^4 + (2b_2 - (2\alpha - 1)a_1)x^{2\alpha-1}y^2 - 2x^{4\alpha-2}, \\ &= -2\left[\left(x^{2\alpha-1} - \tfrac{2b_2-(2\alpha-1)a_1}{4}y^2\right)^2 - \tfrac{\Delta_1}{16}y^4\right], \\ (8\alpha - 4)\mu_{6\alpha-5}(x, y) &= (2b_2 + (2\alpha - 1)a_1)x^{2\alpha-2}y \end{aligned}$$

being $\Delta_1 := (2b_2 - (2\alpha - 1)a_1)^2 - 8(2\alpha - 1)$ the discriminant of $h_{8\alpha-4}$. We distinguish the following cases:

a) If $\Delta_1 > 0$, origin is not monodromic since $h_{8\alpha-4}$ has simple factors.
b) If $\Delta_1 < 0$, $h_{8\alpha-4}$ does not have characteristic factor. It follows that the origin is monodromic (case 1)).
c) If $\Delta_1 = 0$, then $h_{8\alpha-4}$ has the double factor $x^{2\alpha-1} - \frac{2b_2-(2\alpha-1)a_1}{4}y^2$ with $2b_2 - (2\alpha - 1)a_1 \neq 0$. The origin of (15) is a monodromic singular point if the factor is not a strong factor. So, we claim that $\mu_{6\alpha-5} \equiv 0$, i.e. $2b_2 + (2\alpha - 1)a_1 = 0$. This fact arrives to $b_2 = -\frac{1}{4}$ and $a_1 = \frac{1}{2(2\alpha-1)}$, case 2). □

Remark 2. In case of 2) of Theorem 6, by performing the blow-up and the reparameterization of the time

$$x = u^2\left(\bar{x} - \left(\frac{(2\alpha-1)a_1}{2}\right)^{-\frac{1}{2\alpha-1}}\right), \quad y = u^{2\alpha-1}, \qquad dt = \frac{d\tau}{u^{6\alpha-5}},$$

system (15) is transformed into a new system with invariant axis $u = 0$ and the pair $(1, 2)$ in the support of the vector field, whose vector coefficient is $(1, -2)$. So, Propositions 9 and 10 allow us to characterize the monodromy of the origin in such case.

Case III. The expansion into quasi-homogeneous polynomials of these systems respect to $(1, \alpha)$, type associated to the unique edge of their Newton diagram, is

$$\dot{\mathbf{x}} \;\; = \mathbf{F}^{(1,\alpha)}_{3\alpha-1}(\mathbf{x}) + \text{q-h.h.o.t.} \qquad (16)$$

From the splitting (3) of a quasi-homogeneous vector field, $\mathbf{F}^{(1,\alpha)}_{3\alpha-1}$ is given by

$$\mathbf{F}^{(1,\alpha)}_{3\alpha-1}(x, y) = \mathbf{X}_{g_{4\alpha}} + \sigma_{3\alpha-1}\mathbf{D}^{(1,\alpha)}_0$$

where $g_{4\alpha} = -1/4[(y-ax^\alpha)^2+b^2x^{2\alpha}][(y-cx^\alpha)^2+d^2x^{2\alpha}]$ and $\sigma_{3\alpha-1} = x^{\alpha-1}(Ay^2+Byx^\alpha + Cx^{2\alpha})$ with a, b, c, d, A, B, C real numbers.

Theorem 7. *If origin of system (16) is monodromic, then one of the following series of conditions is satisfied:*

1) $bd \neq 0$,
2) $d = 0,\ b \neq 0,\ A \neq 0,\ (4Ac + 2B)^2 = B^2 - 4AC > 0$,
3) $d = 0,\ b \neq 0,\ A = 0,\ c = -BC$,
4) $b = 0,\ d = 0,\ A \neq 0,\ a \neq c,\ a + c = -B/A,\ ac = C/A$,
5) $b = 0,\ d = 0,\ A \neq 0,\ a = c,\ (4Ac + 2B)^2 = B^2 - 4AC > 0$.

Moreover, if 1) is satisfied, then origin of system (16) is monodromic.

Proof. The polynomial $g_{4\alpha}$ does not have a strong factor if it satisfies one of the following conditions:

a) If both b and d are different from zero, the factors of $g_{4\alpha}$ are actually complex factors. Thus, the origin is a monodromic singular point, case 1).
b) If $d \neq 0$ and $b = 0$, then $g_{4\alpha}$ has the factor $y - cx^\alpha$ with multiplicity order two. Thus, the origin is monodromic if $y - cx^\alpha$ is a factor of $\sigma_{3\alpha-1}$.
So, if $A \neq 0$, then c is a root of $\sigma_{3\alpha-1}$, that is, $c = \frac{1}{4A}(-2B + \sqrt{B^2 - 4AC})$ or $c = \frac{1}{4A}(-2B - \sqrt{B^2 - 4AC})$, case 2).
If $A = 0$, it arrives to $c = -BC$, case 3).
c) If $b = d = 0$ and $a \neq c$, then $g_{4\alpha}$ has two real double factors. The origin is monodromic if both factors are factors of $\sigma_{3\alpha-1}$. So, it has that $A \neq 0,\ a + c = -B/A,\ ac = C/A$, case 4).

d) Last on, for $b = d = 0$ and $a = c$, for non-monodromy, $y - cx^\alpha$ must be a factor of $\sigma_{3\alpha-1}$.
So, $A \neq 0$ and c is a root of $\sigma_{3\alpha-1}$. This one arrives to $c = \frac{1}{4A}(-2B + \sqrt{B^2 - 4AC})$ or $c = \frac{1}{4A}(-2B - \sqrt{B^2 - 4AC})$, case 5). □

Remark 3. We note that for cases 2-5), by means of the change $u = x$, $v = y - cx^\alpha$, the system (16) is transfomed into a new system of type case I.

Summarizing,

Theorem 8. *The origin of the system (13) is monodromic if can be transformed by means of a sequence of quasi-homogeneous degree-zero changes of variables into one of the systems given by Theorems 5, 6 or 7.*
Furthermore, for the cases where the first quasi-homogeneous term does not determine the monodromy, Propositions 9 and 10 solve the problem.

Appendix 1

We consider the systems whose expansion into quasi-homogeneous terms is

$$\dot{\mathbf{x}} = \mathbf{F}_\alpha^{(2,\alpha)}(\mathbf{x}) + \text{q-h.h.o.t.}, \tag{17}$$

According to evenness of α, we distinguish two cases.

Proposition 9. *Consider system (17) with* $\mathbf{F}_\alpha^{(2,\alpha)}(x, y) = (A_1xy, B_2y^2 + b_0x^\alpha)^T$, *$\alpha$ is odd number,* $b_0 \neq 0$ *and* $(A_1, B_2) \neq (0, 0)$. *System (17) does not have any characteristic orbits different from axis* $x = 0$ *if and only if* $2B_2 = \alpha A_1 \neq 0$ *and* $b_0B_2 > 0$.

Proof. Hamiltonian associated to the bounded edge is

$$(2\alpha + 2)h_{2\alpha+2} = x\left[(2B_2 - \alpha A_1)y^2 + 2b_0x^\alpha\right].$$

If $2B_2 - \alpha A_1 \neq 0$, then $h_{2\alpha+2}$ has a simple factor and, therefore, the system has a characteristic orbit different from $x = 0$. Otherwise, $2B_2 - \alpha A_1 = 0$, then $(\alpha + 1)h_{2\alpha+2} = b_0x^{\alpha+1}$ (there are not any strong factors) and the divergence of $\mathbf{F}_\alpha^{(2,\alpha)}$ is $\mu_\alpha = \frac{1}{2}A_1y \not\equiv 0$. As $\beta_V = b_0B_2$, if $b_0B_2 < 0$, then vertex V is characteristic. From Theorem 1, it deduces that there are characteristic orbits different from $x = 0$ and if both b_0 and B_2 have same sign, there are not any characteristic orbits. □

Lastly, we analyze the systems (17) with α even, $\alpha = 2\beta$.

Proposition 10. *Consider system*

$$\dot{\mathbf{x}} = \mathbf{F}_\beta^{(1,\beta)}(\mathbf{x}) + \textit{q-h.h.o.t.}, \tag{18}$$

with $\mathbf{F}_\beta^{(1,\beta)}(x,y) = (A_1xy + a_0x^{\beta+1}, B_2y^2 + b_1x^\beta y + b_0x^{2\beta})^T$, $b_0 \neq 0$ *and* $(A_1, B_2) \neq (0,0)$. *System (18) does not have any characteristic orbits different from axis* $x = 0$ *if and only if one of the following series of conditions is satisfied:*

1) $b_1 = \beta a_0$, $B_2 = \beta A_1 \neq 0$ *and* $b_0B_2 > 0$,
2) $B_2 \neq 0$, $\Delta < 0$ *and* $A_1 = 0$,
3) $B_2 \neq 0$, $\Delta < 0$, $A_1 \neq 0$ *and* $B_2/A_0 \notin [0, \beta]$,

with $\Delta = (b_1 - \beta a_0)^2 - 4b_0(B_2 - \beta A_0)$.

Proof. By (3), it has that $\mathbf{F}_\beta^{(1,\beta)} = \mathbf{X}_{h_{2\beta+1}} + \mu_\beta \mathbf{D}_0^{(1,\beta)}$ with

$$\begin{aligned}(2\beta+1)h_{2\beta+1}(x,y) &= x\left[(B_2 - \beta A_0)y^2 + (b_1 - \beta a_0)x^\beta y + b_0x^{2\beta}\right]\\ (2\beta+1)\mu_\beta(x,y) &= (A_1 + 2B_2)y + [(\beta+1)a_0 + b_1]x^\beta\end{aligned}$$

If $\Delta > 0$, then $h_{2\beta+1}$ has two strong factor.
If $\Delta = 0$ and $b_1 = \beta a_0$, then $h_{2\beta+1}$ does not have strong factor since $\mu_\beta = A_1y + a_0x^\beta \not\equiv 0$. Also, $B_2 = \beta A_1 \neq 0$ and hence $\beta_V = b_0B_2$. From Theorem 1, if both b_0 and B_2 have the same sign, then there is not characteristic orbits different from $x = 0$, case 1).

If $\Delta = 0$ and $b_1 \neq \beta a_0$, then $B_2 \neq \beta A_1 \neq 0$ and $(2\beta+1)h_{2\beta+1} = x^{\beta+1}\left[(b_1 - \beta a_0)y + b_0x^\beta\right]$ with $a_0 \neq 0$, that is, there is a strong factor and consequently there are characteristic orbits different from $x = 0$.

If $\Delta < 0$, $h_{2\beta+1}$ does not have any factors of the form $y - \lambda x^\beta$, $\lambda \neq 0$, and $\beta_V = (B_2 - \beta A_1)B_2$. So, if $0 < B_2/A_0 < \beta$, it has that $\beta_{V_0} < 0$, it is a characteristic vertex, and thus there are characteristic orbits. And if $A_1 = 0$ or $B_2/A_0 \notin [0, \beta]$ there are not any characteristic orbits different from $x = 0$, cases 2) and 3). □

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